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Adomian decomposition sumudu transform method for solving a solid and porous fin with temperature dependent internal heat generation

Trushit Patel and Ramakanta Meher[*]

*Correspondence:
meher_ramakanta@yahoo.com
Department of Applied
Mathematics and Humanities,
SVNIT, Icchanath Circle,
Surat 395007, India

Abstract

In this paper, Adomian decomposition sumudu transform method is introduced and used to solve the temperature distribution in a solid and porous fin with the temperature dependent internal heat generation for a fractional order energy balance equation. In this study, we assume heat generation as a variable of fin temperature for solid and porous fin and the heat transfer through porous media is simulated by using Darcy's model. The results are presented for the temperature distribution for the range of values of parameters appeared in the mathematical formulation and also compared with numerical solutions in order to verify the accuracy of the proposed method. It is found that the proposed method is in good agreement with direct numerical solution.

Keywords: Adomian decomposition sumudu transform method, Fractional order differential equation, Caputo fractional derivative, Porous fin, Temperature dependent thermal conductivity, Internal heat generation, Thermal analysis

Background

Fins are commonly used to facilitate the dissipation of heat from a heated wall to the surrounding environment. Examples of fin are the radiator in vehicles and heat exchangers in power plants. In electrical devices like motors and transformers, the generated heat can be efficiently transferred. In the study of heat transfer, fin is a surface, made by metallic material which is used to increase the rate of heat transfer to the environment. The rate of heat transfer depends on the surface area of the fin. Fins are extensively used to improve the rate of heat dissipation from a hot surface, especially in thermal engineering applications (Bergman et al.; Nield and Bejan).

In many thermal engineering applications, convective flow through porous media is mandatory for the investigation. Several numerical and analytical revisions has been conducted so far to afford a profound understanding of the transport system of the heat transfer inside the porous medium. Generally, high thermal conductivity porous substrates are employed to improve the rate of forced convection heat transfer in many engineering applications such as reactor heat exchangers, solar collectors and in cooling process (Alkam and Al-Nimr 1999). However, heat transfer in porous fins has attracted

a lot of attention of researchers with a wide range of it's applications, especially in recent years. Kiwan and Al-Nimr (2001) was the first person who introduced the concept of fins made of porous materials by introducing Darcy's model (Kiwan 2007; Kiwan and Zeitoun 2008).

Now a days, Heat exchanger industries are looking for more compact and cost–effective heat exchanger manufacturing techniques which leads to use porous fins in enhance heat transfer (Kiwan 2007). The heat-transfer enhancement between two parallel-plate channels was investigated by adding porous fin through the channel (Hamdan and Moh'd 2010) and by adding porous insert to one side of the duct walls (Hamdan et al. 2000). Alkam et al. (2002) investigated the thermal analysis of natural convection porous fins. They studied all the geometric flow parameters that influence the temperature distribution in to a single parameter specified S_h. They considered three cases: the infinite fin, a finite fin with an insulated tip and a finite fin with uninsulated tip. Similarly Gorla and Bakier (2011) discussed the thermal analysis of natural convection and radiation in the porous fin and showed that the radiation transfers more heat than a similar model without radiation. Hatami et al. (2013) studied the heat transfer through porous fin with different porous material and compared their results with the Differential Transform Method, Collocation Method and Least Square Method. They Hatami and Ganji (2013) also studied the thermal performance of circular convective–radiative porous fins with different section, shapes and materials. Ghasemi et al. (2014) used the Differential Transform Method for solving the nonlinear temperature distribution in solid and porous fin with temperature dependent internal heat generation. Patel and Meher (2015a, b) studied the fractional solution of longitudinal porous fin for the case of temperature distribution, efficiency and effectiveness and also analysed the variation of temperature distribution for a straight rectangular fin with power-law temperature dependent surface heat flux by using Adomian decomposition sumudu transform method.

It is revealed that, the concept of fractional derivative is more suitable for modeling real world problem than the local derivative. Many researchers have devoted their attention in developing new definition of fractional derivative (Atangana 2016). Baskonus and Bulut (2015) applied the fractional Adams–Bashforth–Moulton Method for obtaining the numerical solutions of some linear and nonlinear fractional ordinary differential equations. Baskonus and Bulut (2015) studied it to obtain some new analytical solutions to the $(1 + 1)$-dimensional nonlinear Dispersive Modified Benjamin Bona Mahony equation by using modified exp-function method. Roshid et al. (2014) studied solitary wave solutions for vakhnenko-parkes equation via exp-function and $\text{Exp}\,(-\phi(\xi))$—expansion method. Also they Roshid et al. (2014) studied traveling wave solutions of nonlinear partial differential equation via new extended (G'/G)—expansion method.

In the present study, we fractionalize the energy balance equation in order to understand the anomalous behavior of this system and to find the temperature distribution in solid and porous fin by using Adomian decomposition sumudu transform method.

Preliminaries

Definitions of Caputo fractional derivative

In this part of the paper it would be useful to introduce some definitions and properties of the fractional calculus theory. There are several definitions of fractional derivatives of

order $\alpha > 0$ (Miller and Ross; Srivastava et al. 2014). The two most commonly used definitions are Riemann-Liouville and Caputo.

Definition The Riemann-Liouville fractional integral of fractional order is defined as Miller and Ross

$$J^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad \alpha \in R^+ \tag{1}$$

where R^+ is the set of positive real numbers and $\Gamma(\cdot)$ is the gamma function.

Definition The fractional derivative of $f(t)$ in the Caputo sense is defined by

$$\begin{aligned} D_t^{\alpha}f(t) &= J^{m-\alpha}D^m f(t) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{(m-\alpha-1)}\frac{d^m f(\tau)}{d\tau^m}, & if \ m-1 < \alpha < m, \ m \in N \\ \frac{d^m f(t)}{dt^m} & if \ \alpha = m, \ m \in N \end{cases} \end{aligned} \tag{2}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. Here N is the set of natural numbers. In this paper only real and positive α will be considered. The properties underpinning the use of the Caputo derivative can be found in Atangana (2016), Atangana and Alqahtani (2016) and Atangana and Baleanu (2016).

Basics of sumudu transform method

The sumudu transform is a new integral transform (Kadem and Baleanu 2012; Atangana and Baleanu 2013; Jarad et al. 2012) which is a little known and not widely used whose defined for the functions of exponential order.

Definition The sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(u)$, defined by Watugala (1993)

$$F(u) = S[f(t)] = \int_0^{\infty} \frac{1}{u}f(t)\, e^{-\left(\frac{t}{u}\right)}dt, \quad u \in (-\tau_1, \ \tau_2) \tag{3}$$

Definition Sumudu transform of function derivatives is defined as Belgacem and Karaballi (2006)

$$S[f^{(n)}(t)] = \frac{S[f(t)]}{u^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{u^{n-k}} \tag{4}$$

Definition Sumudu transform of the Caputo fractional derivative is defined as Belgacem and Karaballi (2006)

$$S\left[D_t^{\alpha}f(t)\right] = u^{-\alpha}S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k}f^{(k)}(0+), \quad m-1 < \alpha \leq m \tag{5}$$

Theorem *Let $G(u)$ be the sumudu transform of $f(t)$ such that*

1. $(G(1/s)/s)$ is a meromorphic function, with singularities having $\mathrm{Re}[s] \leq \gamma$;
2. There exist a circular region γ with radius R and positive constants M and K with $|G(1/s)/s| < MR^{-K}$, then the function $f(t)$ is given by

$$
S^{-1}[F(s)] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F\left(\frac{1}{s}\right) \frac{ds}{s}
$$

$$
= \sum resdidues \left[\frac{e^{st} F\left(\frac{1}{s}\right)}{s} \right]
\tag{6}
$$

For the proof see Belgacem and Karaballi (2006).

Formulation of Adomian decomposition sumudu transform method (ADSTM)

Consider a general fractional nonlinear nonhomogeneous differential equation with satisfying the initial condition of the form, as,

$$
\frac{d^\alpha \theta(\zeta)}{d\zeta^\alpha} + R\theta(\zeta) + N\theta(\zeta) = g(\zeta)
\tag{7}
$$

subject to the initial condition

$$
\theta(0) = K
\tag{8}
$$

where $\frac{d^\alpha}{d\zeta^\alpha}$ denotes without loss of generality the Caputo fraction derivative operator, R is the linear differential operator, N represents the nonlinear differential operator, and $g(\zeta)$ is the source term.

On applying the sumudu transform and the Caputo fractional derivative in Eq. (7), it obtains

$$
S[\theta(\zeta)] = \theta(0) + u^\alpha S[g(\zeta)] - u^\alpha S[R\theta(\zeta) + N\theta(\zeta)]
\tag{9}
$$

Sumudu inverse transform of Eq. (9), gives

$$
\theta(\zeta) = K + S^{-1}[u^\alpha S[g(\zeta) - R\theta(\zeta) - N\theta(\zeta)]]
\tag{10}
$$

where $\theta(0) = K$ is prescribed initial conditions.

Using Adomian decomposition method which obtains the approximate solution of Eq. (10) in a series form as Adomian

$$
\theta(\zeta) = \sum_{n=0}^{\infty} \lambda^n \theta_n(\zeta)
\tag{11}
$$

and the nonlinear term can be expressed as a sum of Adomian Polynomials

$$
N\theta(\zeta) = \sum_{n=0}^{\infty} \lambda^n A_n(\theta(\zeta))
\tag{12}
$$

where the Adomian polynomials $A_n(\theta)$, depends upon the solution components $\theta_0, \theta_1, \theta_2, \ldots, \theta_n$, can be defined as follows

$$A_n(\theta_0, \theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \left[N \left(\sum_{i=0}^{\infty} \lambda^i \theta_i \right) \right]_{\lambda=0}, \quad for \ \ n = 0, 1, 2, \ldots \tag{13}$$

On substituting Eqs. (11) and (12) in Eq. (10), it obtains

$$\sum_{n=0}^{\infty} \lambda^n \theta_n(\zeta) = K + \lambda \left[S^{-1} \left[u^\alpha S \left[g(\zeta) - R \sum_{n=0}^{\infty} \lambda^n \theta_n(\zeta) - \sum_{n=0}^{\infty} \lambda^n A_n(\theta) \right] \right] \right] \tag{14}$$

The resulting Eq. (14) is the coupling of the Adomian decomposition method and the sumudu transform. On comparing the coefficients of like powers of λ, the recursive relation of Eq (14) can be written as

$$\begin{aligned}
\theta_0(\zeta) &= K \\
\theta_1(\zeta) &= S^{-1}[u^\alpha S[g(\zeta) - R\theta_0(\zeta) - A_0(\zeta)]] \\
\theta_{n+1}(\zeta) &= -S^{-1}[u^\alpha S[R\theta_n(\zeta) + A_n(\zeta)]] \quad n = 1, 2, 3, \ldots
\end{aligned} \tag{15}$$

Since $\sum_{n=0}^{\infty} \theta_n(\zeta)$ is a rapidly converging series, the partial sum $\phi_m = \sum_{i=0}^{m-1} \theta_i(\zeta)$ is our approximant to the solution (Adomian).

Problem description

Here we considered two cases, namely (1) solid fin and (2) porous fin to study the fin temperature distribution in longitudinal fin with rectangular profile.

Case 1: solid fin with temperature dependent internal heat generation and constant thermal conductivity

Consider a longitudinal fin with a constant rectangular profile, section area A, length L, perimeter P, thermal conductivity k, and heat generation $\bar{Q}$. Fin is attached to a surface with constant temperature T_b and losses heat to the surrounding medium with temperature T_∞ through a constant convective heat transfer coefficient h. Here we assumed that the temperature variation in the transfer direction is negligible, so heat conduction occurs only in the longitudinal direction (x direction). A schematic diagram of the described fin is shown in Fig. 1.

The governing differential equation and boundary condition for this problem can be written as Ghasemi et al. (2014)

$$\frac{d^2 T}{dx^2} - \frac{hP}{kA}(T - T_\infty) + \frac{\bar{Q}}{k} = 0 \tag{16}$$

$$\left[\frac{dT}{dx}\right]_{x=0} = 0 \tag{17}$$

$$[T]_{x=L} = T_b \tag{18}$$

Here it is assumed that the temperature heat generation in the solid fin varies with temperature T_∞, that can be defined as

$$\bar{Q} = \bar{Q}_\infty(1 + I(T - T_\infty)) \tag{19}$$

Fig. 1 Schematic of fin geometry with the internal heat generation source
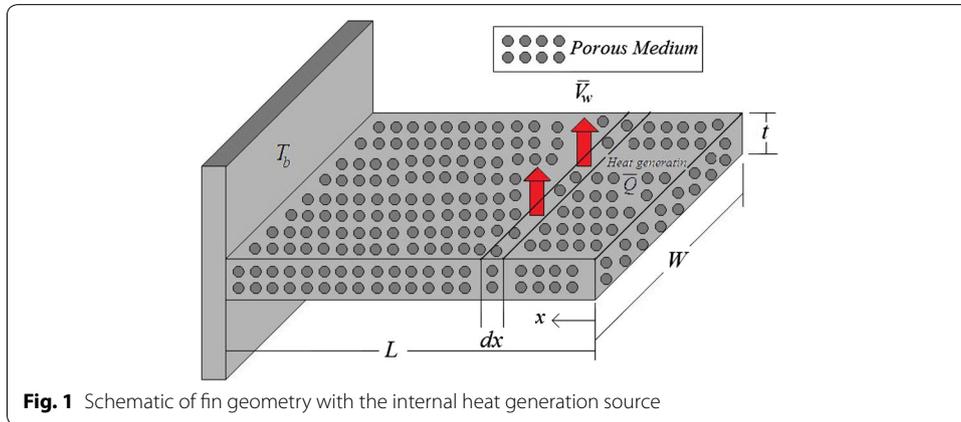
Where $\bar{Q}_\infty$ is the internal heat generation at temperature T_∞. On introducing the dimensionless variables

$$\theta = \frac{(T - T_\infty)}{(T_b - T_\infty)}, \quad \zeta = \frac{x}{L}, \quad M^2 = \frac{hPL^2}{k_0 A},$$

$$G = \frac{\bar{Q}_\infty}{hP(T_b - T_\infty)}, \quad I_G = I(T_b - T_\infty) \tag{20}$$

The dimensionless form of Eqs. (16)–(18) can be written as

$$\frac{d^2\theta}{d\zeta^2} - M^2\theta + M^2 G(1 + I_G\theta) = 0 \tag{21}$$

$$\zeta = 0, \quad \frac{d\theta}{d\zeta} = 0 \tag{22}$$

$$\zeta = 1, \quad \theta = 1 \tag{23}$$

To understand the anomalous behavior of this system, we fractionalize the energy balance Eq. (21) into fractional order ($\alpha > 0$) in order to find the fin temperature in solid fins as,

$$\frac{d^\alpha\theta}{d\zeta^\alpha} - M^2\theta + M^2 G(1 + I_G\theta) = 0, \quad 1 < \alpha \leq 2 \ and \ 0 \leq \zeta \leq 1 \tag{24}$$

With boundary conditions

$$\frac{d\theta}{d\zeta}\bigg|_{\zeta=0} = 0 \quad and \quad \theta|_{\zeta=1} = 1. \tag{25}$$

Now, applying sumudu transform on both sides of Eq. (24), we obtain

$$S[\theta(\zeta)] = K + u^\alpha S\left[(M^2 - M^2 G I_G)\theta - M^2 G\right] \tag{26}$$

taking inverse sumudu transform on both side of Eq. (26), we get

$$\theta(\zeta) = K + S^{-1}\left[u^\alpha S\left[(M^2 - M^2 G I_G)\theta - M^2 G\right]\right] \tag{27}$$

Now on applying Adomian decomposition method it obtains,

$$\sum_{n=0}^{\infty} \lambda^n \theta_n(\zeta) = K + \lambda\left[S^{-1}\left[u^\alpha S\left[\sum_{n=0}^{\infty} \lambda^n (M^2 - M^2 G I_G)\theta_n(\zeta) - M^2 G\right]\right]\right] \tag{28}$$

Comparing the coefficients of λ in Eq. (28), we have

$$\theta_0(\zeta) = K$$

$$\theta_1(\zeta) = S^{-1}\left[u^\alpha S\left[\theta_0(\zeta)\left(M^2 - M^2 G I_G\right) - M^2\theta_0(\zeta)\right]\right]$$

$$= \frac{KM^2(1 - GI_G)\,\zeta^\alpha}{\Gamma(\alpha+1)} - \frac{M^2 G\,\zeta^\alpha}{\Gamma(\alpha+1)}$$

$$\theta_2(\zeta) = S^{-1}\left[u^\alpha S\left[\theta_1(\zeta)\left(M^2 - M^2 G I_G\right)\right]\right]$$

$$= \frac{KM^4(1 - GI_G)^2\zeta^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{M^4(1 - GI_G)G\zeta^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$\theta_3(\zeta) = S^{-1}\left[u^\alpha S\left[\theta_2(\zeta)\left(M^2 - M^2 G I_G\right)\right]\right]$$

$$= \frac{KM^6(1 - GI_G)^3\zeta^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{M^6(1 - GI_G)^2 G\zeta^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\theta_4(\zeta) = S^{-1}\left[u^\alpha S\left[\theta_3(\zeta)\left(M^2 - M^2 G I_G\right)\right]\right]$$

$$= \frac{KM^8(1 - GI_G)^4\zeta^{4\alpha}}{\Gamma(4\alpha+1)} - \frac{M^8(1 - GI_G)^3 G\zeta^{4\alpha}}{\Gamma(4\alpha+1)}$$

$$\vdots$$

Summing these terms, the final temperature field $\theta(\zeta)$, is calculated up to five terms as follows

$$\begin{aligned}
\theta(\zeta) \cong {} & K - \frac{M^2 G\,\zeta^\alpha}{\Gamma(\alpha+1)} + \frac{KM^2(1 - GI_G)\,\zeta^\alpha}{\Gamma(\alpha+1)} - \frac{M^4(1 - GI_G)G\zeta^{2\alpha}}{\Gamma(2\alpha+1)} \\
& + \frac{KM^4(1 - GI_G)^2\zeta^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{M^6(1 - GI_G)^2 G\zeta^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{KM^6(1 - GI_G)^3\zeta^{3\alpha}}{\Gamma(3\alpha+1)} \\
& - \frac{M^8(1 - GI_G)^3 G\zeta^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{KM^8(1 - GI_G)^4\zeta^{4\alpha}}{\Gamma(4\alpha+1)}
\end{aligned} \tag{29}$$

Equation (29) represents the expression for finding the temperature distribution in solid fin of fractional order energy balance Eq. (24), where the coefficient K denotes the fin tip temperature, and it can be determined at the boundary condition $\theta|_{\zeta=1} = 1$.

Case 2: porous fin with the temperature dependent internal heat generation

Here the energy balance equation for rectangular porous profile fin with temperature dependent internal heat generation can be written as

$$Q(x) - Q(x + \Delta x) + \bar{Q} \cdot A \cdot \Delta x = \bar{m} \cdot c_p [T(x) - T_\infty] \\ + h(p \cdot \Delta x)[T(x) - T_\infty] \tag{30}$$

The mass flow rate of the fluid passing through the porous material can be written as

$$\bar{m} = \rho \cdot \bar{V}_w \cdot \Delta x \cdot w \tag{31}$$

From Darcy's model, the passage velocity (Hatami et al. 2013) is

$$\bar{V}_w = \frac{g \cdot k \cdot \beta \cdot (T(x) - T_\infty)}{\nu} \tag{32}$$

On substituting Eqs. (31) and (32) into Eq. (30), it gives

$$\frac{Q(x) - Q(x + \Delta x)}{\Delta x} + \bar{Q} \cdot A = \frac{\rho \cdot g \cdot k \cdot \beta \cdot (T(x) - T_\infty)^2 \cdot w \cdot c_p}{\nu} \\ + h \cdot p(T(x) - T_\infty) \tag{33}$$

As $\Delta x \to 0$, Eq. (33) becomes

$$\frac{dQ}{dx} + \bar{Q} \cdot A = \frac{\rho \cdot g \cdot k \cdot \beta \cdot (T(x) - T_\infty)^2 \cdot w \cdot c_p}{\nu} + h \cdot p(T(x) - T_\infty) \tag{34}$$

Also, from Fourier's law of conduction

$$Q = -k_{eff} A \frac{dT}{dx} \tag{35}$$

where k_{eff} is the effective thermal conductivity of the porous fin, that can be obtained from the following equation (Hatami et al. 2013)

$$k_{eff} = \psi k_f + (1 - \psi)k_s \tag{36}$$

Where ψ is the porosity of the porous fin. On substituting Eq. (35) into Eq. (34), it gives

$$\frac{d^2 T}{dx^2} + \frac{\bar{Q}}{k_{eff}} = \frac{\rho \cdot g \cdot k \cdot \beta \cdot (T(x) - T_\infty)^2 \cdot w \cdot c_p}{\nu} + \frac{h \cdot p}{k_{eff} A}(T(x) - T_\infty) \tag{37}$$

On introducing the dimensionless variables and numbers

$$\theta = \frac{(T - T_\infty)}{(T_b - T_\infty)}, \quad \zeta = \frac{x}{L}, \ M^2 = \frac{hPL^2}{k_0 A}, \quad G = \frac{\bar{Q}_\infty}{hP(T_b - T_\infty)},$$
$$I_G = I(T_b - T_\infty), \quad \xi = \frac{Da \cdot x \cdot Ra}{k_r} \left(\frac{L}{t}\right)^2 \tag{38}$$

The dimensionless form of Eq. (37) can be written as

$$\frac{d^2 \theta}{d\zeta^2} - M^2 \theta + M^2 G(1 + I_G \theta) - \xi \theta^2 = 0 \tag{39}$$

with its boundary condition

$$\frac{d\theta}{d\zeta}\bigg|_{\zeta=0} = 0, \quad and \quad \theta|_{\zeta=1} = 1. \tag{40}$$

where M is a convection parameter that indicates the effect of surface convecting of the fin and ξ is a porous parameter that indicates the effect of the permeability of the porous medium as well as the buoyancy effect so higher the value of ξ indicates higher permeability of the porous medium or higher buoyancy forces.

To understand the anomalous behaviour of this system, we fractionalize the energy balance Eq. (39) into fractional order ($\alpha > 0$) in order to find the fin temperature in porous fins as,

$$\frac{d^\alpha \theta}{d\zeta^\alpha} - M^2\theta + M^2 G(1 + I_G\theta) - \xi\theta^2 = 0, \qquad 1 < \alpha \leq 2 \text{ and } 0 \leq \zeta \leq 1 \tag{41}$$

with boundary conditions

$$\frac{d\theta}{d\zeta}\bigg|_{\zeta=0} = 0 \quad and \quad \theta|_{\zeta=1} = 1. \tag{42}$$

Now, again applying sumudu transform on both sides of Eq. (41), it obtains

$$S[\theta(\zeta)] = \theta_0(\zeta) + u^\alpha S\left[\left(M^2 - M^2 GI_G\right)\theta(\zeta) + \xi A_n(\theta) - M^2 G\right] \tag{43}$$

Applying Inverse sumudu transfer on both sides, we get

$$[\theta(\zeta)] = \theta_0(\zeta) + S^{-1}\left[u^\alpha S\left[\left(M^2 - M^2 GI_G\right)\theta(\zeta) + \xi A_n(\theta) - M^2 G\right]\right] \tag{44}$$

by applying Adomian Decomposition Method, it obtains the following equation

$$\sum_{n=0}^{\infty} \lambda^n \theta_n(\zeta) = K - \lambda\left[S^{-1}\left[u^\alpha\left(M^2 G\right)\right]\right]$$
$$+ S^{-1}\left[u^\alpha S\left[\sum_{n=0}^{\infty} \lambda^n\left(\theta_n(\zeta)\left(M^2 - M^2 GI_G\right) + \sum_{n=0}^{\infty} \lambda^n \xi A_n(\theta)\right)\right]\right] \tag{45}$$

where $A_n(\theta)$ is the nonlinear term which can be determined by using Eq. (13). The first few components of the adomian's polynomial for corrosponding nonlinear terms are given by

$A_0 = (\theta_0(\zeta))^2$
$A_1 = 2\theta_0(\zeta)\theta_1(\zeta)$
$A_2 = 2\theta_0(\zeta)\theta_2(\zeta) + (\theta_1(\zeta))^2$
$A_3 = 2\theta_0(\zeta)\theta_3(\zeta) + 2\theta_1(\zeta)\theta_2(\zeta)$
$\vdots$

On comparing the coefficients of like powers of λ in Eq. (45), we get decomposition components as

$$\theta_0(\zeta) = K$$
$$\theta_1(\zeta) = S^{-1}\left[u^\alpha S\left[\left(M^2 - M^2 GI_G\right)\theta_0(\zeta) + \xi A_0(\theta) - M^2 G\right]\right]$$
$$\theta_{n+1}(\zeta) = S^{-1}\left[u^\alpha S\left[\left(M^2 - M^2 GI_G\right)\theta_n(\zeta) + \xi A_n(\theta)\right]\right], \quad n > 1. \tag{46}$$

By solving above equations, we get

$$\theta_0(\zeta) = K$$

$$\theta_1(\zeta) = \frac{\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^\alpha}{\Gamma(\alpha + 1)}$$

$$\theta_2(\zeta) = \frac{(M^2 - M^2 GI_G)\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$+ \frac{2K\xi\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$\theta_3(\zeta) = \frac{(M^2 - M^2 GI_G)^2\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$+ \frac{\xi\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)^2 \Gamma(2\alpha + 1)X^{3\alpha}}{(\Gamma(\alpha + 1))^2 \Gamma(3\alpha + 1)}$$

$$+ \frac{4K\xi\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)(M^2 - M^2 GI_G)X^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$+ \frac{4K^2\xi^2\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$\vdots$$

The approximate solution of $\theta(\zeta)$ up to four terms becomes

$$\theta(\zeta) = K + \frac{\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^\alpha}{\Gamma(\alpha + 1)}$$

$$+ \frac{(M^2 - M^2 GI_G)\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$+ \frac{2K\xi\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^{2\alpha}}{\Gamma(2\alpha + 1)}$$

$$+ \frac{(M^2 - M^2 GI_G)^2\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$+ \frac{\xi\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)^2 \Gamma(2\alpha + 1)X^{3\alpha}}{(\Gamma(\alpha + 1))^2 \Gamma(3\alpha + 1)}$$

$$+ \frac{4K\xi\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)(M^2 - M^2 GI_G)X^{3\alpha}}{\Gamma(3\alpha + 1)}$$

$$+ \frac{4K^2\xi^2\left((M^2 - M^2 GI_G)K + \xi K^2 - M^2 G\right)X^{3\alpha}}{\Gamma(3\alpha + 1)} \tag{47}$$

where value of K can be determined at the boundary condition $\theta|_{\zeta=1} = 1$ using Eq. (47). Since a constant K is assumed as an initial guess, it automatically satisfies the given boundary condition.

Stability analysis via fixed point theorem

Let $(X, \|\cdot\|)$ be a Banach space and H a self-map of X. Let $\theta_{n+1} = f(H, \theta_n)$ be particular recursive procedure. Suppose that $F(H)$ the fixed point set of H has at lease one element and that θ_n converges to a point $p \in F(H)$. Let $\theta_n \subseteq X$ and define $e_n = \|\theta_{n+1} - f(H, \theta_n)\|$. If $\lim_{n\to\infty} \theta^n = p$, then the iteration $\theta_{n+1} = f(H, \theta_n)$ is said to be H-stable. Without any loss

of generality, we must assume that, our sequence θ_n has an upper boundary; otherwise we cannot expect the possibility of convergence. If all these conditions are satisfied for $\theta_{n+1} = H\theta_n$ which is known as Picard's iteration, then the iteration will be *H*-Stable. Now we state the following theorem (Atangana 2016).

Theorem 1   *Let $(X, \|\,\|)$ be a Banach space and H a self-map of X satisfying*

$$\left\| H_x - H_y \right\| \leq C\|x - H_x\| + c\|x - y\|, \tag{48}$$

for all x, y in X where $0 \leq C$. $0 \leq c < 1$. Then H is Picard H-Stable.

Let the following succession correlate to the nonlinear fractional order energy balance equation (Eq. 46),

$$\theta_{n+1}(\zeta) = S^{-1}\left[u^\alpha S\left[\theta_n(\zeta)\left(M^2 - M^2 GI_G \right) + \xi(A_n(\theta)) \right] \right], \quad n > 1.$$

Theorem   *Let T be a self-map defined as*

$$\begin{aligned}
T(\theta_n(\zeta)) &= \theta_{n+1}(\zeta) \\
&= S^{-1}\left[u^\alpha S\left[\theta_n(\zeta)\left(M^2 - M^2 GI_G \right) + \xi(A_n(\theta)) \right] \right]
\end{aligned} \tag{49}$$

is T-stable in $L^2(a, b)$ if $\{\beta_1 + \beta_1 \kappa\} < 1$.

Proof   The first step of the proof will consist on showing that T has a fixed point. To achieve this, we evaluate the following for all $(n, k) \in N \times N$

$$\|T(\theta_n(\zeta)) - T(\theta_k(\zeta))\| = \left\| \begin{array}{c} S^{-1}\left[u^\alpha S\left[(M^2 - M^2 GI_G)\theta_n(\zeta) + \xi(\theta_n^2(\zeta)) \right] \right] \\ -S^{-1}\left[u^\alpha S\left[(M^2 - M^2 GI_G)\theta_k(\zeta) + \xi(\theta_k^2(\zeta)) \right] \right] \end{array} \right\| \tag{50}$$

Using the linearity property of the inverse sumudu transform, we obtain

$$\|T(\theta_n(\zeta)) - T(\theta_k(\zeta))\| = \left\| S^{-1}\left[u^\alpha S\left\{ \begin{array}{c} (M^2 - M^2 GI_G)(\theta_n(\zeta) - \theta_k(\zeta)) \\ +\xi(\theta_n^2(\zeta) - \theta_k^2(\zeta)) \end{array} \right\} \right] \right\| \tag{51}$$

Using the triangle inequality for the norm, we get

$$\begin{aligned}
\|T(\theta_n(\zeta)) - T(\theta_k(\zeta))\| &\leq \left\| S^{-1}\left[u^\alpha S\left\{ \left(M^2 - M^2 GI_G \right)(\theta_n(\zeta) - \theta_k(\zeta)) \right\} \right] \right\| \\
&\quad + \left\| S^{-1}\left[u^\alpha S\left\{ \xi(\theta_n^2(\zeta) - \theta_k^2(\zeta)) \right\} \right] \right\|
\end{aligned} \tag{52}$$

The above can be further be transformed using the property of norm and integral as follows

$$\begin{aligned}
\|T(\theta_n(\zeta)) - T(\theta_k(\zeta))\| &\leq S^{-1}\left[u^\alpha S\left\{ \left\| \left\{ \left(M^2 - M^2 GI_G \right)(\theta_n(\zeta) - \theta_k(\zeta)) \right\} \right\| \right\} \right] \\
&\quad + S^{-1}\left[u^\alpha S\left\{ \left\| \left\{ \xi(\theta_n^2(\zeta) - \theta_k^2(\zeta)) \right\} \right\| \right\} \right]
\end{aligned} \tag{53}$$

The evaluation of Eq. (53) can be done as follows

$$\left\| \left(M^2 - M^2 GI_G \right) \{\theta_n(\zeta) - \theta_k(\zeta)\} \right\| \leq \left\| \left(M^2 - M^2 GI_G \right) \right\| \|\theta_n(\zeta) - \theta_k(\zeta)\|$$
$$\leq \beta_1 \|\theta_n(\zeta) - \theta_k(\zeta)\| \tag{54}$$

and

$$\left\| \xi \left\{ \theta_n^2(\zeta) - \theta_k^2(\zeta) \right\} \right\| \leq \|\xi\| \left\| \theta_n^2(\zeta) - \theta_k^2(\zeta) \right\|$$
$$\leq \beta_2 \|\theta_n(\zeta) + \theta_k(\zeta)\| \|\theta_n(\zeta) - \theta_k(\zeta)\|$$
$$\leq \beta_2 \kappa \|\theta_n(\zeta) - \theta_k(\zeta)\| \tag{55}$$

where $\|\xi\|\beta_2$. Now putting together Eqs. (54) and (55) into Eq. (53), we obtain the following

$$\|T(\theta_n(\zeta)) - T(\theta_k(\zeta))\| \leq \left\{ \beta_1 + \beta_2 k \right\} \|\theta_n(\zeta) - \theta_k(\zeta)\| \tag{56}$$

with $\{\beta_1 + \beta_2 \kappa\} < 1, \forall \beta_1 \beta_2$. Hence, the nonlinear T-self mapping has a fixed point. This completes the proof. Further we show that, T satisfies the condition in Theorem 1, Now for

$$C = 0, \ c = \left\{ \beta_1 + \beta_2 k \right\} \tag{57}$$

shows that conditions of this theorem holds for the nonlinear mapping T. Since all condition in Theorem 1 hold for the defined non-linear mapping T. Hence, T is Picard's T-stable. This completes the proof of this theorem. □

Results and discussion

The purpose of this work is to observe the simultaneous effects of the governing parameters and the different fractional values α on solid and porous fins. The range of thermal and physical parameters selected the present work is shown in Table 1.

Solid fin with temperature dependent internal heat generation and constant thermal conductivity

Temperature distribution for this case (temperature dependent heat generation and constant thermal conductivity) is shown in Figs. 2 and 3 where M = 1 that is common in fin design.

Figure 2 shows the temperature distribution for this state and $I_G = G = 0.2$, $I_G = G = 0.4$ and $I_G = 0.4$, $G = 0.6$. This choice of parameters represents a fin with moderate temperature dependent heat generation and the thermal conductivity variation of 20 % between the base and the surrounding coolant temperatures that are often used in nuclear rods.

It is shown in Figs. 2 and 3 that temperature of the fin increases by increasing the value of I_G and G because of increasing in heat generation. The comparison of obtained results with numerical results reveals that ADSTM has good efficiency and accuracy.

Table 1 Range of values for physical and thermal parameters

Parameters	M	G	I_G	ξ
Values	0.5, 1 and 5	0.1–0.9	0.1–0.9	0.1–0.9

Fig. 2 Temperature variation obtained by ADSTM in comparison with the numerical solution for $\alpha = 2$ and $M = 1$

Fig. 3 Temperature distribution in solid fins for $M = 1$ and **a** $\alpha = 1.75$, **b** $\alpha = 1.5$, **c** $\alpha = 1.25$
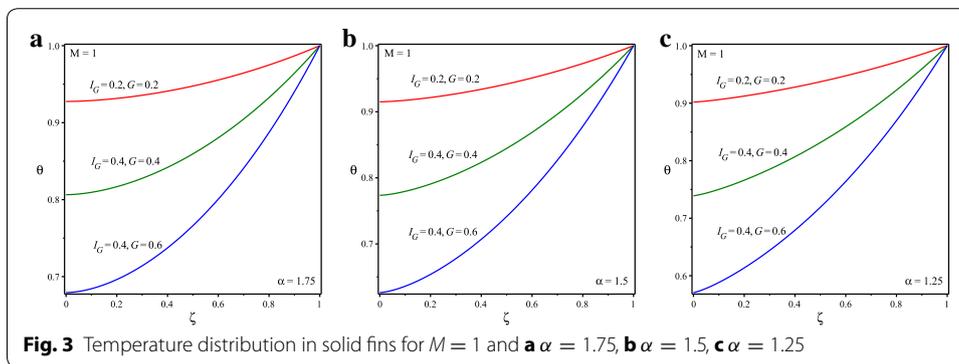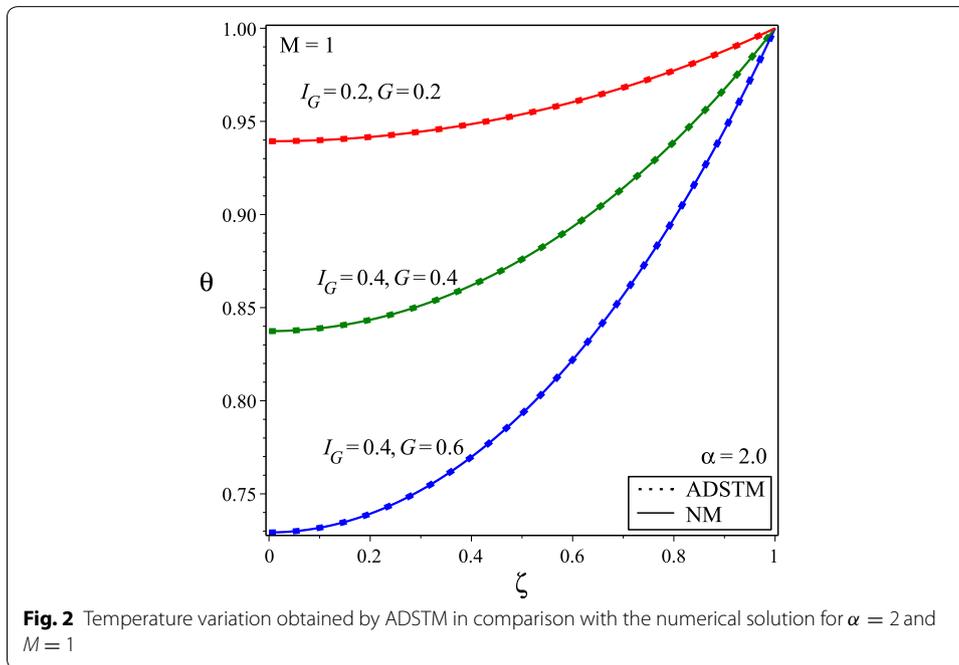
Figure 3 shows the temperature distribution for this state and $I_G = G = 0.2$, $I_G = G = 0.4$ and $I_G = 0.4$, $G = 0.6$ and for the different fractional order value of $\alpha = 1.75, 1.5$ and 1.25. Further, the nature of the graphs depicts that, considered value of α represents the point of convergence under the given range of interval between 1 and 2.

Porous fin with temperature dependent internal heat generation

Figure 4 shows the comparison between the numerical solution and the ADSTM solution for temperature distribution when $M = 1$, $G = 0.4$, $I_G = 0.6$ and for the different value of ξ and $\alpha = 2$. When the porous parameter ξ increases, it can be noticed from Fig. 5 that, a declines in fin temperature which causes stronger cooling results decrease in temperature distribution. Figure 5 shows the variation of temperature distribution of porous fin with temperature dependent internal heat generation when $M = 1$, $G = 0.4$, $I_G = 0.6$ and for the different value of ξ and for $\alpha = 1.75, 1.5, 1.25$.
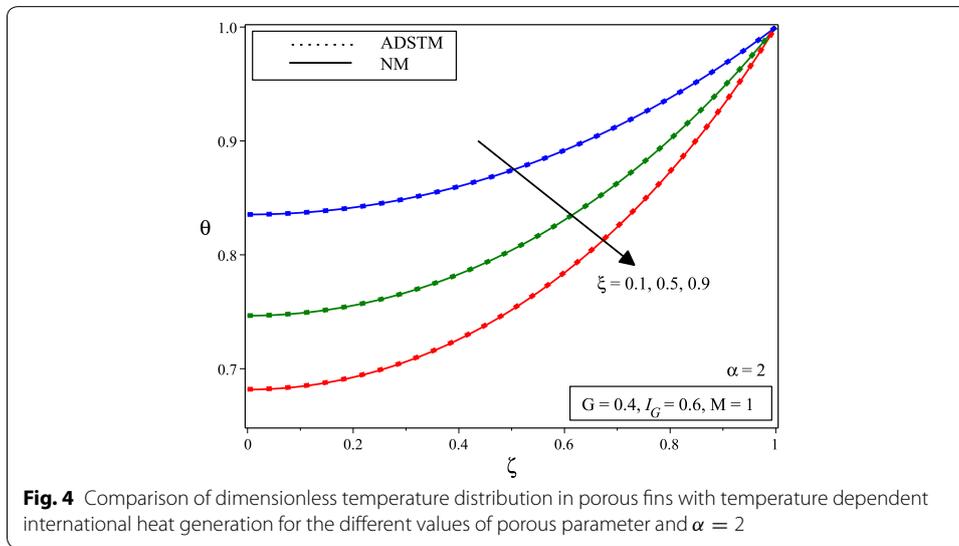
Fig. 4 Comparison of dimensionless temperature distribution in porous fins with temperature dependent international heat generation for the different values of porous parameter and $\alpha = 2$
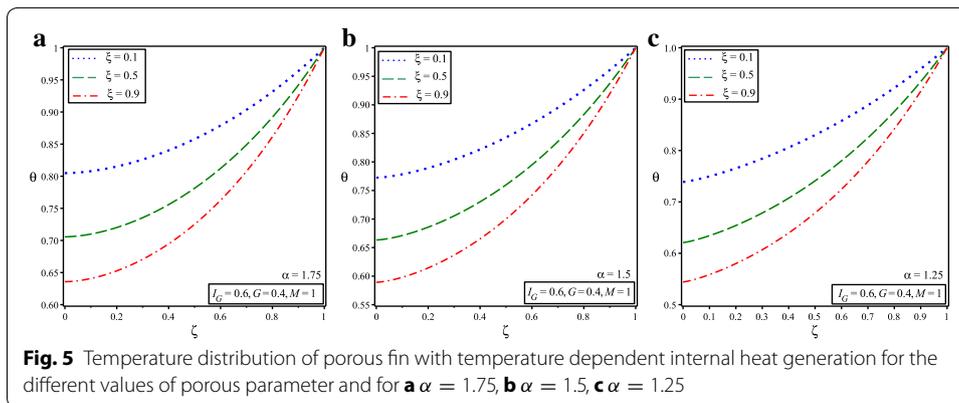
Fig. 5 Temperature distribution of porous fin with temperature dependent internal heat generation for the different values of porous parameter and for **a** $\alpha = 1.75$, **b** $\alpha = 1.5$, **c** $\alpha = 1.25$

The variation of the temperature distribution along the fin for the different values of internal heat generation, I_G when $M = 1$, $G = 0.4$, $\zeta = 0.4$ and for $\alpha = 2$ is illustrated in Fig. 6. It is observed from figure that, if the internal heat generation increases then fin temperature be increases. Figure 7 shows the variation of temperature distribution of porous fin with temperature dependent internal heat generation when $M = 1$, $G = 0.4$, $\xi = 0.4$ and for the different value of I_G and for $\alpha = 1.75, 1.5, 1.25$.

Figure 8 shows the comparison between the numerical solution and the ADSTM solution for temperature distribution when $M = 1$, $I_G = 0.4, \xi = 0.4$ and for the different value of G and for $\alpha = 2$.

Figure 9 shows the variation of temperature distribution of porous fin with temperature dependent internal heat generation when $M = 1$, $I_G = 0.4, \xi = 0.4$ and for the different value of G and for $\alpha = 1.75, 1.5, 1.25$.

Figure 10 shows the comparison between the numerical solution and ADSTM solution for temperature distribution when $G = 0.4$, $I_G = 0.6, \xi = 0.4$ and for the different value of M and for $\alpha = 2$. At last, it can be concluded that the analytical results correspond

Fig. 6 Comparison of dimensionless temperature distribution in porous fins with temperature dependent international heat generation for the different values of internal heat generation parameter and $\alpha = 2$

Fig. 7 Temperature distribution of porous fin with temperature dependent internal heat generation for the different values of internal heat generation parameter and for **a** $\alpha = 1.75$, **b** $\alpha = 1.5$, **c** $\alpha = 1.25$

Fig. 8 Comparison of dimensionless temperature distribution in porous fins with temperature dependent international heat generation for the different values of heat generation parameter and $\alpha = 2$

Fig. 9 Temperature distribution of porous fin with temperature dependent internal heat generation for the different values of heat generation parameter and for **a** $\alpha = 1.75$, **b** $\alpha = 1.5$, **c** $\alpha = 1.25$

Fig. 10 Comparison of dimensionless temperature distribution in porous fins with temperature dependent international heat generation for the different values of convective parameter and $\alpha = 2$

exactly with the numerical results. This means that the ADSTM has a high aptitude in solving highly nonlinear initial and boundary value problems without involving linearization. Figure 11 shows the variation of temperature distribution of porous fin with temperature dependent internal heat generation when $G = 0.4$, $I_G = 0.6$, $\xi = 0.4$ and for the different value of M and for $\alpha = 1.75, 1.5, 1.25$.

Fig. 11 Temperature distribution of porous fin with temperature dependent internal heat generation for the different values of convective parameter and for **a** $\alpha = 1.75$, **b** $\alpha = 1.5$, **c** $\alpha = 1.25$
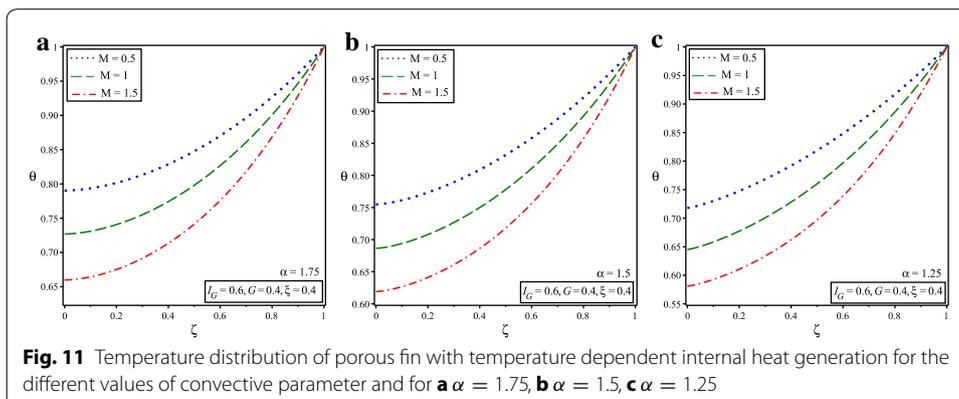
Conclusion

In this study, the heat transfer in rectangular solid and porous fin with the temperature dependent internal heat generation is analyzed by using ADSTM and used the concept of *T*-stable mapping and the fixed point theorem to prove the stability of ADSTM. Here, it is shown that, ADSTM provide a simple, accurate and appropriate technique for simulating the heat transfer in solid and porous fin in a fractional order energy balance equation. The results shows that the temperature distribution strongly depends on different parameter in solid fin as well as on Darcy's number in porous fin and also on the fractional parameter.

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