Reliability analysis for the fractional-order circuit system subject to the uncertain random fractional-order model with Caputo type

Ting Jin [a], Shangce Gao [b,*], Hongxuan Xia [c], Hui Ding [d]

[a] School of Science, Nanjing Forestry University, Nanjing, 210037 Jiangsu, China
[b] Faculty of Engineering, University of Toyama, Toyama-shi 930-8555, Japan
[c] College of International Education, Nanjing Forestry University, Nanjing, 210037 Jiangsu, China
[d] College of Economics and Management, Nanjing Forestry University, Nanjing, 210037 Jiangsu, China

HIGHLIGHTS

- We investigate the independent competing failure process of a RC system in an uncertain random environment.
- Continuous degradation of the system is subject to an uncertain fractional process.
- External shocks obey a random distribution given by real data.
- Two shock models that lead to hard failure are considered.
- Analytical express the system reliability and perform numerical simulations.

GRAPHICAL ABSTRACT
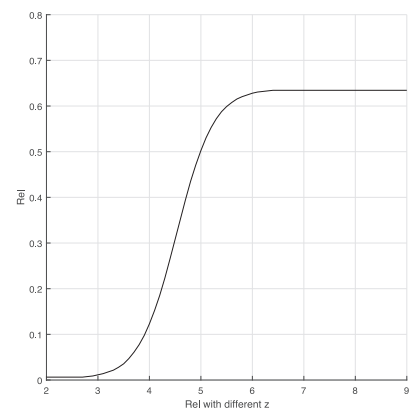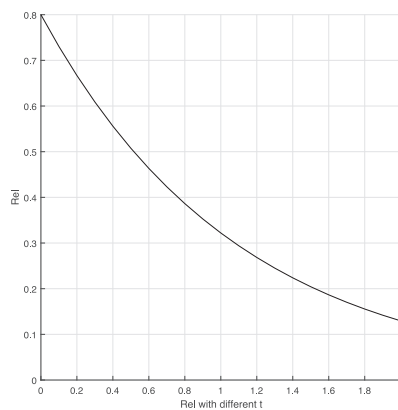
By calculating with the predictor-corrector algorithm, we pioneer the reliability index for an uncertain fractional-order circuit system, which is positively correlated with the operating time t and negatively correlated with the failure threshold z. We also find that z is decisive for the ultimate reliability of the system in a particular interval.

ARTICLE INFO

ABSTRACT

Introduction: According to the competing failure theorem, the fractional-order Resistor Capacitance (RC) circuit system suffers not only from internal degradation but also from external shocks. However, due to the general differences of each failure type in the data availability and cognitive uncertainty, a better model is needed to describe the degradation process within the system. Also, a new reliability analysis method is needed for the circuit system under internal degradation and external shocks.
Objectives: To demonstrate this problem, this paper proposes a novel class of Caputo-type uncertain random fractional-order model that focuses on the reliability analysis of a fractional-order RC circuit system.
Methods: First, an uncertain Liu process is used to describe the internal degradation of soft faults and a stochastic process is used to describe the external random shocks of hard faults. Secondly, taking into account the correlation and competition among the fault types, an extreme shock model and a cumulative shock model are constructed, and chance theory is introduced to further explore the fault mechanisms, from which the corresponding reliability indices are derived. Finally, the predictor–corrector method is applied and numerical examples are given.

Results: This paper presents a reliability analysis of a fractional-order RC circuit system with internal failure obeying an uncertain process and external failure obeying a stochastic process, and gives the calculation of the reliability indexes for different cases and the corresponding numerical simulations.
Conclusion: A new competing failure model for a fractional-order RC circuit system is presented and analyzed for reliability, which is proved to be of practical importance by numerical simulations.

1. Introduction

Reliability is one of the vital essential features of a system to perform the required functions. System reliability can be affected by a variety of factors that can be categorized as internal (wear, corrosion, fatigue, etc.) and external mutations (impact, stress, etc.). Uncertain internal shocks, also known as soft failure, can lead to degradation of the system's performance, which introduces the risk of system damage. On the other hand, A sudden system collapse caused by random external shocks is called a hard failure. As the service life increases, the system can fail due to competition between internal degradation (soft faults) and external shocks (hard faults), which can even lead to safety incidents and economic losses. As a result, the failure process has received close attention from those who want to optimize system reliability. As the service life increases, systems can fail due to a combination of internal degradation (soft failures) and external shocks (hard failures), which may even lead to safety incidents and economic losses, the failure process thus has received close attention from those who want to optimize system reliability.

The controversy and difficulty of reliability research lie in the description of system failure processes. In 2009, Frosting and Kenzin [1] used Markov processes to describe the effects of the external environment and Poisson processes to describe the degradation of the cumulative effects and constructed a model of system maintenance. The same year, Lehmann [2] argued that the system failure is an abrupt stochastic Poisson process and thus built a degradation threshold shock model. In 2010, Wang [3] divided the degradation failure process of a system into three modes and considered a reliability assessment model based on the aging structure. Next year, Jiang [4] translated the effect of external shocks on degradation items into lower failure thresholds and proposed a reliability model based on the associated degradation and shocks. Raifiee et al. [5] built a generalized mixed-shock model based on repairable degraded equipment in 2015. Qiu and Cui [6] considered the degradation process of extreme shocks and gave the corresponding reliability formulae in 2018.

On the one hand, with its superior ability to describe complex phenomena, systems and dynamic processes, fractional-order calculus has become a powerful modeling tool. The concept of fractional-order calculus was introduced almost simultaneously with the concept of ordinary-order calculus [7]. However, the lack of an intuitively clear physical significance of fractional-order calculus has kept the study of it in the theoretical mathematical field for a long time. It was not until the 1970s that Mandelbrot[8] introduced fractal theory, pointing out the widespread existence of fractional dimension in nature and in science, that is, having self-similarity. He also pointed out that there is a close connection between fractional-order Brownian motion and fractional-order calculus. Since then, scholars have come to realize the unique advantages of fractional-order calculus in describing self-similarity, memory properties and genetic phenomena [9]. In electrical engineering, capacitance and resistance defined in traditional circuit theory are ideal components with no losses, which contradicts the non-conservative characteristics of actual devices. Scholars usually choose to characterize losses by connecting resistors in series or parallel. However, this approach only holds under certain conditions and thus fails to provide an accurate description of the characteristics of the circuit system. By analogy with electrical and mechanical models, Westerlund [10] proposed a fractional-order differential relationship between the voltage flowing through a capacitor and its terminals, a conclusion that was then proved by experiments on more than 10,000 capacitors. Later, Westerlund and Ekstam [11] demonstrated the memory properties of dielectric materials from a quantum mechanical perspective, emphasizing the importance of describing circuit systems using fractional-order calculus. Since then, fractional-order components have come to researchers' attention in the field of electrical engineering. Based on memory-resistive elements and memory systems, Petras and Chen [12] discussed fractional-order memory-capacitor systems, and fractional-order memory-inductor systems. Francisco et al. [13] presented a fractional-order differential equation for RC circuits using the Caputo definition, introduces parameters into the equation that characterize the fractional structure of the circuit and investigates the effect of the fractional-order parameters on the circuit characteristics. Guia et al.[14] gave Time-domain solution and Frequency-domain solution for fractional-order RC circuits. For more details on fractional-order circuit systems in practice, see [15–17].

In addition to the description of the global relevance, the modelling of fractional-order circuits faces a lack of data on non-determinants. The traditional models are usually based on stochastic theory, dealing with the large amounts of historical data with statistical methods to describe the variation of parameters. However, due to the lack of financial, material, and technical resources there is difficulty in obtaining sufficient historical data for analysis. Therefore, some scholars turn to experts in the field for suggestions. When describing competitive failure processes in complex systems, experienced experts can offer advice closer to the real situation than the results obtained by statistical methods. From this perspective, Liu [18] pioneered the dynamic system uncertainty analysis method based on his novel uncertainty theory, a model with almost no failure data. Liu [19] proposed the uncertainty theory related to the degree of belief of the expert for cases that there is not enough data to obtain the frequency distribution of event occurrences. Liu [20] also proposed the definition of uncertainty process that differs from random processes used in probability theory. In 2013, Zeng et al. [21] proposed reliability indicators that can assess product reliability through uncertainty theory. Zeng et al. [22] also gave a numerical evaluation method based on the minimal cut set to calculate the reliability index of the uncertain system. Furthermore, for complex systems with a mixture of uncertainty and randomness, Liu [23,24] introduced the concept of uncertain random variables, which can be broadly understood as measurable functions from the probability space to the set of uncertain variables. Yao and Gao [25] proposed a law of large numbers for uncertain random variables, which stated that the mean of an uncertain random variable converges on an uncertain variable in the distribution. Zhou et al. [26] proposed the uncertain random multi-objective programming, a type of uncertain stochastic optimisation for decision systems. In 2016, Wen and Kang [27] extended the analytical approach to system reliability by adopting

chance theory. Zhang et al. [28] searched for a more rational approach to reliability analysis in the field of engineering and proposed the concept of confidence reliability. Liu et al. [29] performed reliability analysis on uncertain stochastic systems with independent degradation processes and shocks. Gao and Yao [30] supplemented some basic important formulas on which reliability analysis depends. Liu et al. [31] analyzed the reliability of systems subjected to uncertain random shocks and independent degradation processes. Zhu [32] ulteriorly combined uncertainty theory and fractional-order differential equations to introduce the concept of uncertain fractional-order differential equations (UFDEs) and define two types of UFDEs: the Riemann–Liouville type and the Caputo type. Subsequently, according to Lipschitz continuous and linear growth condition, Zhu [33] proved the uniqueness of the solution of UFDEs. Then, Lu [34] proposed α-path to obtain methods for solving the numerical solutions of UFDEs. More and more scholars cast their light on the extension and application of UFDEs, and some remarkable results have been achieved. Jin and Zhu [35] were the first to study the extremes of Caputo-type UFDE solutions, and thus explored their application to American option pricing. In addition, Jin and Zhu [36] discussed the time integral problem of UFDEs solutions and addressed the problem of pricing zero-coupon bonds. Jin and Zhu [37] also presented first-hitting time theorems for UFDEs, which had an application for the fractional risk index in 2020.

However, to the best of our knowledge, there is a gap in the analysis of competing failure model based on uncertain random fractional-order systems. With this consideration in mind, in this paper, we will discuss a reliability model that internally obeys an uncertain degradation process and externally obeys two different kinds of random shocks.

The other parts are arranged as follows: in Section 2, some necessary theories and conclusions required for reliability research are proposed. Section 3 explores the laws, characteristics and failure mechanisms of system degradation failure and sudden failure respectively. Section 4 further analyzes the competitive failure process, which is subject to the extreme shock model or cumulative shock model, and the corresponding reliability index are derived. In Section 5, the change of reliability index is studied by numerical examples. Finally, the results of the above studies are summarized in Section 6.

2. Preliminary

This section mainly introduces the related definitions and results of UFDEs, first hitting time, and chance theory, which lays a sufficient theoretical foundation for the follow-up research on the reliability analysis of complex systems under indeterministic environments.

2.1. Uncertain analysis

Uncertain theory, which was proposed by Liu [19] in 2007, has been widely used in uncertain analysis due to its excellent practicability in dealing with uncertain information and cognitive uncertainty.

Definition 1 (*Liu [19]*). **Uncertainty measure**. Represent the triplet $(\mathcal{Y}, \mathcal{L}, \mathcal{M})$ as an uncertain space, $\mathcal{Y}$ is a non-empty set, and $\mathcal{L}$ is the σ-algebra over it. The uncertainty measure $\mathcal{M}$ is a set function from $\mathcal{L} \times (\mathcal{Y}, \mathcal{L}, \mathcal{M})$ to the set of real numbers, which satisfies the axioms of normality, duality and subadditivity as follows.

1) For the universal set $\mathcal{Y}$, there exists $\mathcal{M}\{\mathcal{Y}\} = 1$;
2) For any event Λ_i, there exists $\mathcal{M}\{\Lambda_i\} + \mathcal{M}\{\Lambda_i^c\} = 1$;
3) For countable event columns $\Lambda_1, \Lambda_2, \cdots$, there exists

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leqslant \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Subsequently, in 2009, Liu [38] supplemented the product axioms to study uncertainty measures on product spaces, that is,
4) For arbitrarily selected $\Lambda_k \in \mathcal{L}_k$, there exists

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}\{\mathcal{L}_k\},$$

where $\mathcal{L}_1 \times \mathcal{L}_2 \times \mathcal{L}_3 \times \cdots$ is the product σ-algebra for $k = 1, 2, 3, \cdots$, and Λ_k is any event in $\mathcal{L}_k$.

Definition 2 (*Liu [39]*). **Uncertain process**. An uncertain process is a function $X_t(\gamma)$ from $\mathcal{L} \times (\mathcal{Y}, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B, in which T is a totally ordered set. Furthermore, C_t is a Liu canonical process which satisfies that:

1) $C_0 = 0$ and almost all sample paths are Lipschitz continuous;
2) C_t has stationary and independent increments;
3) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

2.2. UFDE with the Caputo type

Uncertain fractional-order differential equations of the Riemann–Liouville and Caputo types have been proposed before. Since Ford and Simpson [40], as well as Diethelm et al. [41] indicated that the fractional-order derivative in Caputo sense has more advantages than Riemann–Liouville sense for modelling the real dynamic process, we choose only UFDEs of Caputo type for discussion in this paper. Without considering other cases, we assume that the real number p satisfies the condition that $0 < n - 1 < p \leqslant n$.

Definition 3 (*Zhu [32]*). **UFDEs**. Let F and G be two given functions, C_t is a canonical Liu process, and X_t is an uncertain process, then the Caputo type uncertain fractional differential equations go

$$\begin{cases} {}^cD^pX_t = F(t, X_t) + G(t, X_t)\frac{dC_t}{dt} \\ X_{t^{(k)}}|_{t=0} = x_k, k = 0, 1, \cdots, n-1, \end{cases} \tag{1}$$

where X_t denotes the solution of the UFDE derived by Lu and Zhu [34],

$$X_t = \sum_{k=0}^{n-1}\frac{x_k t^k}{\Gamma(k+1)} + \frac{1}{\Gamma(p)}\int_0^1 (t-s)^{p-1}F(s, X_s)ds + \frac{1}{\Gamma(p)}$$
$$\times \int_0^t G(s, X_s)(t-s)^{p-1}dC_s, \tag{2}$$

where $\Gamma(p) = \int_0^\infty t^{p-1}\exp(-t)dt$ is the gamma function.

Particularly, both functions $F(t, X_t)$ and $G(t, X_t)$ are continuous functions that satisfy the Lipschitz condition and the linear growth condition. In other words, on the interval $[0, \infty]$, the solution X_t of the UFDE exists exclusively.

Definition 4 (*Lu and Zhu [42]*). Let $0 < \alpha < 1$, the solution X_t of the corresponding UFDE would have an α-path X_t^α as long as X_t satisfies the below fractional differential equation.

$$\begin{cases} {}^cD^p X_t^\alpha = F(t, X_t^\alpha) + |G(t, X_t^\alpha)|\Phi^{-1}(\alpha) \\ X_{t^{(k)}}|_{t=0} = x_k, k = 0, 1, \cdots, n-1. \end{cases} \quad (3)$$

where the inverse uncertain distribution (IUD) $\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}$.

Theorem 1 (*Lu and Zhu [42]*). X_t and X_t^α are solutions of UFDEs and FDEs, respectively. Then,the relationship between X_t and X_t^α is that

$$\begin{cases} \mathcal{M}\{X_t \leqslant X_t^\alpha, \forall t \in [0, T]\} = \alpha, \\ \mathcal{M}\{X_t > X_t^\alpha, \forall t \in [0, T]\} = 1 - \alpha. \end{cases} \quad (4)$$

Let $\psi_s^{-1}(\alpha)$ denote the IUD of X_t then $\psi_s^{-1}(\alpha) = X_t^\alpha$.

2.3. First-hitting time

Even for a highly reliable and long-lived system, once a shock causes damage that exceeds the system threshold, the normal operation of the system will be disturbed, resulting in a failure. The system lifetime, which is the first-hitting time (FHT) when the threshold is reached, was proposed by Liu.

Definition 5 (*Liu [38]*). **First-hitting time**. Assuming that changes to the system state X_t obey an uncertain process with a failure threshold z, then the life of the system can be expressed as

$$\tau_z = \inf\{t \geqslant 0 | X_t = z\}. \quad (5)$$

Subsequently, Jin and Zhu [35] further investigated the extreme value theorem and they [43] also investigated the FHT for solutions of UFDEs via α-path method, and confirmed that the FHT when X_t reached the threshold satisfied the IUD as follows.

Theorem 2 (*Jin and Zhu [43]*). For the UFDE (1), when $J(X_t)$ increases strictly, the FHT τ_z that $J(X_t)$ hits the threshold z takes the uncertainty distribution

$$U(s) = \begin{cases} 1 - \inf\left\{\alpha \in (0, 1) \middle| \sup_{0 \leqslant t \leqslant s} J(X_t^\alpha) \geqslant z\right\}, & if z > J(x_0), \\ \sup\left\{\alpha \in (0, 1) \middle| \inf_{0 \leqslant t \leqslant s} J(X_t^\alpha) \leqslant z\right\}, & if z < J(x_0). \end{cases} \quad (6)$$

When $J(X_t)$ decreases strictly, the FHT τ_z that $J(X_t)$ hits the threshold z takes the uncertainty distribution

$$\overline{U}(s) = \begin{cases} \sup\left\{\alpha \in (0, 1) \middle| \sup_{0 \leqslant t \leqslant s} J(X_t^\alpha) \geqslant z\right\}, & if z > J(x_0), \\ 1 - \inf\left\{\alpha \in (0, 1) \middle| \inf_{0 \leqslant t \leqslant s} J(X_t^\alpha) \leqslant z\right\}, & if z < J(x_0). \end{cases} \quad (7)$$

2.4. Chance theory

Chance theory, proposed by Liu [24] in 2013, has been widely used in the modeling of complex systems for its superiority in the comprehensive interpretation of uncertainty and randomness.

Definition 6 (*Liu [24]*). **Chance measure**. $(\Omega, \text{Å}, Pr)$ represents the probability space and $(\mathcal{Y}, \mathcal{L}, \mathcal{M})$ denotes the uncertain space, Respectively. Let $(\mathcal{Y}, \mathcal{L}, \mathcal{M}) \times (\Omega, \text{Å}, Pr)$ be a chance space and $\Theta \in \mathcal{L} \times \text{Å}$ be an uncertain random event. In that way, the chance measure Ch of Θ is defined as

$$Ch\{\Theta\} = \int_0^1 Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \mathcal{Y} | (\gamma, \omega) \in \Theta\} \geqslant r\} dr, \quad (8)$$

where γ is an uncertain variable and ω is a random variable. Moreover, for any $\Lambda \in \mathcal{L}$ and $A \in \text{Å}$, there exists

$$Ch\{\Lambda \times A\} = \mathcal{M}\{\Lambda\} \times Pr\{A\}. \quad (9)$$

It is worth adding that the chance measure Ch is a monotonically increasing function of Θ, similar to the uncertainty measure with self-duality and subadditivity.

Definition 7 (*Liu [24]*). **Uncertain random variable**. Define the measurable function ξ as an uncertain random variable from the chance space $(\mathcal{Y}, \mathcal{L}, \mathcal{M}) \times (\Omega, \text{Å}, Pr)$ to the real number set, then for any Borel set $B, \{\xi = B\}$ is an event.

3. Analysis for RC circuit system failure

Generally, RC circuit failure is a combination of two independent failure process, namely soft failure and hard failure. That means, either of such failure results in system failure. Among them, soft failure due to internal continuous degradation process carries out the whole system lifetime, such as the ageing of wires, the wear of insulating layers and the fatigue of metal components. Hard failure due to external random failure process is persistent or transient, such as the instantaneous burning of the protective fuse and the continuous heating induced by equipment overload. In this section, the rules, characteristics, and failure mechanisms of each failure are discussed separately.

3.1. Soft failure

Soft failure is a gradual failure of equipment caused by internal degradation. Since the emergence of soft failure has a certain degree of uncertainty and instability, accurate monitoring and data acquisition with current technology are still unattainable. Hence, human empirical judgment is still the commonly adopted and recognized method in engineering applications.Considering the lack of data and cognitive uncertainty, an uncertain fraction differential equation is introduced to describe the degradation process for the RC circuit system, that is,

$$RCdX_s + X_s ds = \omega ds + \sigma dC_s, \quad (10)$$

where R is the resistance, C is the capacitance, X_s represents the internal degradation process, ω is the input voltage, and $\sigma > 0$ is the diffusion coefficient respectively. C_s is a Liu process satisfying Lipschitz continuous and linear growth conditions.However, compared with the ordinary differential Eq. (10), fractional-order calculus, thanks to its special genetic and memory properties, can show higher superiority in describing the essential properties and dynamic behaviour of physical processes. Hence, the degradation process of soft failure can be further derived as

$$\begin{cases} {}^cD^p X_s = -\frac{X_s}{RC} + \frac{\omega}{RC} + \frac{\sigma}{RC}\frac{dC_s}{ds}, \\ X^{(k)}(0) = x_k, \quad k = 0, 1, \ldots, n-1. \end{cases} \quad (11)$$

According to the failure mechanism, only when the degradation processes reach a pre-given threshold z soft failure can occur. Let the corresponding first-hitting time be

$$\tau = \{t > 0 | X_s = z\},$$

from which the uncertainty measure that soft failure does not induce the system collapse by time t can be derived as $\mathcal{M}\{\tau > t\}$.

Theorem 3. Assuming that the degradation process which can occur soft failure is X_s, the threshold is z, and the degradation process is subject to the UFDEs (11). Denote τ as the first-hitting time of the degradation reaches the threshold. Then, the uncertainty measure of no soft failure by time t is that

$$\overline{F}(t) = \inf \left\{ \alpha \in (0,1) \,\middle|\, \sup_{0 \leqslant s \leqslant t} X_s^\alpha \geqslant z \right\}. \tag{12}$$

Proof: Since the belief degree of the uncertain event $\mathcal{M}\{X_s < z\}$ equals to $\mathcal{M}\{\tau > t\}$, the uncertainty measure of the system with no failure is defined as

$$\overline{F}(t) = \mathcal{M}\{X_s < z\} = \mathcal{M}\{\tau > t\}, \tag{13}$$

where $X_0 < z$. Meanwhile, it has been mentioned in Definition 4 that the solution of Eq. (11) has a corresponding α-path X_s^α.

Owing to the FHT has the uncertain inverse distribution mentioned in Theorem 2, we can obtain that

$$\overline{F}(t) = \inf \left\{ \alpha \in (0,1) \,\middle|\, \sup_{0 \leqslant s \leqslant t} J(X_s^\alpha) \geqslant z \right\}. \tag{14}$$

Theorem 3 is proved. □

3.2. Hard failure

Hard failure is a failure process of equipment caused by external random shocks. Generally, the traditional description of sudden failure process takes the life data of the system as a premise. Suppose Y_k is the size of the k-th external shock, denoting the influence of the k-th shock on the system and are i.i.d random variables. $N(t)$ counts the number of external shocks that have occurred by time t, thus subjects to a typical Poisson process, which is often used to count random independent events.

According to failure mechanism, only when the external shock's magnitude Y_k reaches the sudden failure threshold D, hard failure immediately shows dominant characteristics, and then interfere with the normal operation of the system. Only the system hard failures induced by extreme or cumulative shocks are discussed here, from which the probability measure that hard failure do not occur can be derived as follows.

3.2.1. Extreme shock model

Suppose that an RC circuit system is subject to extreme shocks, whose collapse risk stems from a single shock strength Y_k exceeding the sudden failure threshold D. In this case, the probability measure that hard failure of the system does not occur by time t is that

$$Pr \left\{ \bigcap_{k=0}^{N(t)} (Y_k < D) \right\}. \tag{15}$$

3.2.2. Cumulative shock model

Different from the extreme model, considering the failure mechanism of the system subjected to cumulative shocks, once the cumulative shock strength Y_k exceeds the sudden failure threshold D, the system cannot maintain the normal operation. In this case, the probability measure of the system of no hard failure by time t can be expressed as

$$Pr \left\{ \sum_{k=0}^{N(t)} Y_k < D \right\}. \tag{16}$$

4. Reliability index for RC circuit system

Due to the uncertainty and stochastic nature of complex systems, this section uses chance theory to analyze the failure mechanisms of RC circuit systems. Based on the analysis of the internal degradation process, the extreme shocks and cumulative shocks are further considered, and the corresponding reliability index is also derived in this section, respectively.

4.1. Competing failure process involving the extreme shocks

The system is assumed to be subjected not only to internal degradation processes but also to extreme external shocks that follow UFDEs and random failure processes, respectively. Any fault with priority reaching the threshold can induce system failure. Then, combined with the chance measure, the reliability index of the system can be expressed as follows.

Theorem 4. The criteria for the RC system are the degradation failure threshold z and the fatal failure threshold D. Denote X_t as the internal degradation process, which is subject to the UFDE (11). Y_k is the shock strength, which is subjected to a random failure process. $N(t)$ is a Poisson process having intensity λ and ϕ is the probability distribution of the shock load Y_k. Then, the reliability index of the system whose collapse may be caused by internal degradation and external extreme shocks by time t is that

$$Rel = \overline{F}(t) \left(e^{-\lambda t} + \sum_{k=1}^{\infty} \left(\frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \phi^k(D) \right) \right). \tag{17}$$

Proof: Firstly, adopt chance measures to determine the reliability of the system, then it can be easily obtained that

$$Rel = Ch \left\{ \sup_{0 \leqslant s \leqslant t} X_s^\alpha \leqslant z, \bigcap_{k=1}^{N(t)} (Y_k < D) \right\}. \tag{18}$$

According to Definition 6, Eq. (18) can be furthered as follows,

$$\begin{aligned}
&Ch \left\{ \sup_{0 \leqslant s \leqslant t} X_s^\alpha \leqslant z, \bigcap_{k=1}^{N(t)} (Y_k < D) \right\} \\
&= Ch \left\{ \tau > t, \bigcap_{k=0}^{N(t)} (Y_k < D) \right\} \\
&= \mathcal{M}\{\tau > t\} \times Pr \left\{ \bigcap_{k=0}^{N(t)} (Y_k < D) \right\}.
\end{aligned} \tag{19}$$

Meanwhile, as mentioned in the Theorem 3, the uncertain measure $\mathcal{M}\{\tau > t\}$ has the uncertain inverse distribution that $\overline{F}(t) = 1 - U(t)$, where $U(t)$ indicates the uncertainty distribution of τ for internal degradation X_s.

Secondly, since the events $Pr \left\{ \bigcap_{k=0}^{N(t)} (Y_k < D) \right\}$ is equivalent to

$$Pr \left\{ \bigcup_{k=0}^{\infty} (N(t) = k) \cap \left\{ \bigcap_{i=1}^{k} (Y_i < D) \right\} \right\}. \tag{20}$$

Then, its corresponding measure can be further transformed into

$$\begin{aligned}
&Pr \left\{ \bigcap_{k=0}^{N(t)} (Y_k < D) \right\} \\
&= Pr \left\{ \bigcup_{k=0}^{\infty} (N(t) = k) \cap \left\{ \bigcap_{i=1}^{k} (Y_i < D) \right\} \right\} \\
&= Pr\{N(t) = 0\} + Pr \left\{ \bigcup_{k=1}^{\infty} (N(t) = k) \cap \left\{ \bigcap_{i=1}^{k} (Y_i < D) \right\} \right\} \\
&= e^{-\lambda t} + \sum_{k=1}^{\infty} \left(\frac{(\lambda t)^k}{k!} e^{-\lambda t} \prod_{i=1}^{k} Pr(Y_i < D) \right) \\
&= e^{-\lambda t} + \sum_{k=1}^{\infty} \left(\frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \phi^k(D) \right).
\end{aligned} \tag{21}$$

Hence, we can safely derive that the reliability index of system whose failure due to internal degradation and external extreme shock by time t satisfy

$$Rel = \mathcal{M}\{\tau > t\} \times Pr\left\{\bigcap_{k=0}^{N(t)}(Y_k < D)\right\}$$

$$= \bar{F}(t)\left(e^{-\lambda t} + \sum_{k=1}^{\infty}\left(\frac{(\lambda t)^k}{k!}e^{-\lambda t} \cdot \phi^k(D)\right)\right). \tag{22}$$

The proof is completed. □

4.2. Competing failure process involving the cumulative shocks

Similarly, suppose that the RC circuit system not only suffers from the internal degradation but also the external cumulative shock. Any fault with priority reaching the threshold can induce system failure. Hence, combined with the chance measure, the reliability index of the RC circuit system can be expressed as follows.

Theorem 5. The criteria for RC system are the degradation failure threshold z and the fatal failure threshold D. Denote X_s as the cumulative degradation, which is subject to the UFDE (11). Y_k as the loss caused by the k-th shock, which is subjected to a random failure process. $N(t)$ is a Poisson process having intensity λ and ϕ is the probability distribution of the shock load Y_k. Then, the reliability index of the system whose failure due to internal degradation and external cumulative shock by time t is

$$Rel = \bar{F}(t)\left(e^{-\lambda t} + \sum_{k=1}^{\infty}\left(\frac{(\lambda t)^k}{k!}e^{-\lambda t}\sum_{h=0}^{D}\left(\sum_{x_n}\cdots\sum_{x_2}\sum_{x_1}\phi(x_1)\phi(x_2)\right.\right.\right.$$
$$\left.\left.\left.\cdots\phi(x_n)\phi(h - x_1 - x_2 - \cdots - x_n)\right)\right)\right). \tag{23}$$

Lemma 1 (*Multiple convolution theorem*). Let $Y_1, Y_2, \cdots, Y_n$ be n mutually independent random variables with probability densities of $f_{Y_i}(x_i)(i = 1, 2, \cdots, n)$, then the probability density of $\eta = \sum_{i=1}^{k}Y_i$ is

$$f_\eta(h) = f_{Y_1} \times f_{Y_2} \times \ldots \times f_{Y_n}$$
$$= \sum_{x_n=-\infty}^{+\infty}\cdots\sum_{x_2=-\infty}^{+\infty}\sum_{x_1=-\infty}^{+\infty}f_{Y_1}(x_1)f_{Y_2}(x_2)$$
$$\cdots f_{Y_n}(x_n)f_{Y_i}(h - x_1 - x_2 - \cdots - x_n).$$

Proof: Firstly, with the chance measure adopted to determine the reliability of the system, it can be easily obtained that

$$Rel = Ch\left\{\sup_{0 \leqslant s \leqslant t}X_s^\alpha \leqslant z, \sum_{k=1}^{N(t)}Y_k < D\right\}. \tag{24}$$

According to Definition 6, Eq. (24) can be furthered as follows,

$$Ch\left\{\sup_{0 \leqslant s \leqslant t}X_s^\alpha \leqslant z, \sum_{k=1}^{N(t)}Y_k < D\right\}$$
$$= Ch\left\{\tau > t, \sum_{k=0}^{N(t)}Y_k < D\right\} \tag{25}$$
$$= \mathcal{M}\{\tau > t\} \times Pr\left\{\sum_{k=0}^{N(t)}Y_k < D\right\}.$$

Meanwhile, as mentioned in the Theorem 3, the uncertain measure $\mathcal{M}\{\tau > t\}$ has the uncertain inverse distribution that $\bar{F}(t) = 1 - U(t)$, where $U(t)$ indicates the uncertainty distribution of τ for internal degradation X_s.

Secondly, since the events $Pr\left\{\sum_{k=0}^{N(t)}Y_k < D\right\}$ is equivalent to

$$Pr\left\{\bigcup_{k=0}^{\infty}(N(t) = k) \cap \left\{\sum_{i=1}^{k}Y_i < D\right\}\right\}.$$

Then, its corresponding measure can be further transformed into

$$Pr\left\{\sum_{k=0}^{N(t)}Y_k < D\right\}$$
$$= Pr\left\{\bigcup_{k=0}^{\infty}(N(t) = k) \cap \left\{\sum_{i=1}^{k}Y_i < D\right\}\right\}$$
$$= Pr\{N(t) = 0\} + Pr\left\{\bigcup_{k=1}^{\infty}(N(t) = k) \cap \left\{\sum_{i=1}^{k}Y_i < D\right\}\right\} \tag{27}$$
$$= e^{-\lambda t} + \sum_{k=1}^{\infty}\left(\frac{(\lambda t)^k}{k!}e^{-\lambda t}Pr\left(\sum_{i=1}^{k}Y_i < D\right)\right).$$

According to the Lemma 1, the measure of event $Pr\left(\sum_{i=1}^{k}Y_i < D\right)$ is derived as

$$Pr\left(\sum_{i=1}^{k}Y_i < D\right) = \sum_{h=0}^{D}f_\eta(z)$$
$$= \sum_{h=0}^{D}\left(\sum_{x_n=-\infty}^{+\infty}\cdots\sum_{x_2=-\infty}^{+\infty}\sum_{x_1=-\infty}^{+\infty}f_{Y_1}(x_1)f_{Y_2}(x_2)\cdots f_{Y_n}(x_n)f_{Y_i}\right.$$
$$\left.(h - x_1 - x_2 - \cdots - x_n)\right)$$
$$= \sum_{h=0}^{D}\left(\sum_{x_n=-\infty}^{+\infty}\cdots\sum_{x_2=-\infty}^{+\infty}\sum_{x_1=-\infty}^{+\infty}\phi(x_1)\phi(x_2)\cdots\phi(x_n)\right.$$
$$\left.\phi\left(h - x_1 - x_2 - \cdots - x_n\right)\right).$$

Hence, we can safely derive that the reliability index of the system whose failure due to internal cumulative degradation and external cumulative shock by time t satisfy

$$Rel$$
$$= \mathcal{M}\{\tau > t\} \times Pr\left\{\sum_{k=0}^{N(t)}Y_kD\right\} \tag{28}$$
$$= \bar{F}(t)\left(e^{-\lambda t} + \sum_{k=1}^{\infty}\left(\frac{(\lambda t)^k}{k!}e^{-\lambda t}\sum_{h=0}^{D}\left(\sum_{x_n}\cdots\sum_{x_2}\sum_{x_1}\phi(x_1)\phi(x_2)\cdots\right.\right.\right.$$
$$\left.\left.\left.\phi(x_n)\phi(h - x_1 - x_2 - \cdots - x_n)\right)\right)\right). \tag{29}$$

The proof is completed. □

Lemma 2 (*Central limit theorem*). Let $Y_1, Y_2, \cdots, Y_k$ be a randomly selected sample from population, whose distribution is unknown, but whose average and variance are namely given as μ and σ (finite and not zero). Therefore, when the sample size k is sufficiently large we have

$$\frac{Y_1 + Y_2 + \cdots + Y_k - k\mu}{\sqrt{n}\sigma} \sim \mathcal{N}(0, 1). \tag{30}$$

Remark 1. When k is large enough, we can more easily get the stochastic distribution of $\sum_{i=1}^{k}Y_i$ according to Lemma 2. Therefore it can be obtained that

$$\sum_{i=1}^{k}Y_i \sim \mathcal{N}(n\mu, n\sigma^2). \tag{31}$$

Make

$$\Phi(x) = \frac{1}{\sqrt{2\pi n}\sigma}\int_{-\infty}^{x}e^{-\frac{(x-n\mu)^2}{2n\sigma^2}}dx$$

the distribution function of $\sum_{i=1}^{k}Y_i$, which means that

$$Pr\left(\sum_{i=1}^{k} Y_i < D\right) = \Phi(D).$$

Hence,

$$Pr\left\{\sum_{k=0}^{N(t)} Y_k < D\right\} = e^{-\lambda t} + \sum_{k=1}^{\infty}\left(\frac{(\lambda t)^k}{k!}e^{-\lambda t}\Phi(D)\right), \qquad (32)$$

Which means when k is sufficiently large, Rel for the fractional-order RC circuit system is that

$$
\begin{aligned}
Rel &= Ch\left\{\sup_{0\leqslant s\leqslant t}J(X_s^\alpha) \leqslant z, \bigcap_{k=1}^{N(t)}(Y_k < D)\right\} \\
&= \overline{F}(t)Pr\left\{\bigcap_{k=1}^{N(t)}(Y_k < D)\right\} \\
&= \overline{F}(t)\left\{e^{-\lambda t} + \sum_{k=1}^{\infty}\left(\frac{(\lambda t)^k}{k!}e^{-\lambda t}\Phi(D)\right)\right\}.
\end{aligned} \qquad (33)
$$

5. Numerical simulation

In this section, in order to verify the validity of the uncertain random fractional-order model with Caputo type, the fault data of the RC circuit system is selected to analyze the influence of its various parameters on the reliability index. Assume that the internal degradation X_t follows the UFDE, and the external random shock Y_k follows an empirical distribution function which is fitted according to the real data of the failure system in the past year provided by State grid Nanjing maintenance branch. Then, two kinds of numerical simulations are discussed in this section.

5.1. Ordinary-order simulation

Specifically, when derivative order $p = 1$, the fractional-order state-equation degenerates into an ordinary-order RC circuit equation,

$$RCdX_s + X_s dt = wds + \sigma dC_s, X_0 = x_0, \qquad (34)$$

Obviously, $X_{s,\omega}^\alpha$ associated to the UDE (34) is obtained as follows

$$X_{s,\omega}^\alpha = \left((\omega - x_0) + \sigma\Phi^{-1}(\alpha)\right)(1 - \exp(-s/RC)) + x_0. \qquad (35)$$

Note that, in this particular case, we can obtain an exact analytical solution for the distribution function for τ_z, which will be expanded upon in more detail below.

Theorem 6. Under $p = 1$, for $z > 0$, the first hitting time τ at which X_s reaches z has a distribution function

$$U(t) = \left(1 + \exp\left(\frac{\pi}{\sqrt{3}\sigma}\left(\frac{z - x_0}{1 - \exp\left(\frac{t}{RC}\right)} - \omega + x_0\right)\right)\right)^{-1}. \qquad (36)$$

Proof: To get the analytical formula for the distribution function of the first-hitting time τ when $J(X_s)$ reaches z, it is required to pick out α that satisfies $\sup_{0\leqslant s\leqslant t} X_s^\alpha \geqslant z$.

Let $(0,1) = I_1 \bigcup I_2$, where

$$I_1 = \left\{\alpha \in (0,1)| -\omega + x_0 - \sigma\Phi^{-1}(\alpha) \geqslant 0\right\},$$

$$I_2 = \left\{\alpha \in (0,1)| -\omega + x_0 - \sigma\Phi^{-1}(\alpha) < 0\right\}.$$

It is obvious that $(0.5, 1) \subset I_2$. Thus $I_2 \neq \emptyset$. When $\alpha \in I_1$,

$$\inf_{0\leqslant s\leqslant t}J(X_s^\alpha) = J(X_0^\alpha) = x_0 \geqslant z,$$

Hence, according to Theorem 2,

$$
\begin{aligned}
\alpha_0 &= \inf\left\{\alpha \in (0,1)| \sup_{0\leqslant s\leqslant t}X_s^\alpha \geqslant z\right\} \\
&= \inf\left\{\alpha \in I_2|\left(\omega - x_0 + \sigma\Phi^{-1}(\alpha)\right)\left(1 - \exp\left(\frac{t}{RC}\right)\right) + x_0 \geqslant z\right\} \\
&= \inf\left\{\alpha| \ln\frac{\alpha}{1-\alpha} \geqslant \frac{\pi}{\sqrt{3}\sigma}\left(\frac{z-x_0}{1-\exp\left(\frac{t}{RC}\right)} - \omega + x_0\right)\right\} \\
&= \frac{\exp\left(\frac{\pi}{\sqrt{3}\sigma}\left(\frac{z-x_0}{1-\exp\left(\frac{t}{RC}\right)}-\omega+x_0\right)\right)}{1+\exp\left(\frac{\pi}{\sqrt{3}\sigma}\left(\frac{z-x_0}{1-\exp\left(\frac{t}{RC}\right)}-\omega+x_0\right)\right)}.
\end{aligned} \qquad (37)
$$

$$
\begin{aligned}
U(t) &= 1 - \alpha_0 \\
&= 1 - \inf\left\{\alpha \in (0,1)| \sup_{0\leqslant s\leqslant t}J(X_s^\alpha) \geqslant z\right\} \\
&= \left(1 + \exp\left(\frac{\pi}{\sqrt{3}\sigma}\left(\frac{z-x_0}{1-\exp\left(\frac{t}{RC}\right)} - \omega + x_0\right)\right)\right)^{-1}.
\end{aligned} \qquad (38)
$$

The proof is completed. □

5.1.1. Competing failure process involving the extreme shocks

Theorem 7. For ordinary-order circuit systems (34) under external extreme shocks, the reliability index by time t can be expressed as

$$
\begin{aligned}
Rel = &\left[1 - \left(1 + \exp\left(\frac{\pi}{\sqrt{3}\sigma}\left(\frac{z - x_0}{1 - \exp\left(\frac{t}{RC}\right)} - \omega + x_0\right)\right)\right)^{-1}\right] \\
&\left(e^{-\lambda t} + \sum_{k=1}^{\infty}\left(\frac{(\lambda t)^k}{k!}e^{-\lambda t} \cdot \phi^k(D)\right)\right).
\end{aligned} \qquad (39)
$$

Proof: According to Theorem 4, we have

$$Rel = \overline{F}(t)\left(e^{-\lambda t} + \sum_{k=1}^{\infty}\left(\frac{(\lambda t)^k}{k!}e^{-\lambda t} \cdot \phi^k(D)\right)\right). \qquad (40)$$

Then by Theorem 6,

$$U(t) = \left(1 + \exp\left(\frac{\pi}{\sqrt{3}\sigma}\left(\frac{z - x_0}{1 - \exp\left(\frac{t}{RC}\right)} - \omega + x_0\right)\right)\right)^{-1},$$

substituting which into the $\overline{F}(t)$ in Eq. (40). The reliability index (39) can be easily obtained. The theorem is proved. □

Based on this analytic formula, we give a numerical example of ordinary-order failure process under extreme shocks to measure its reliability.

Example 1. Assume an uncertain ordinary-order RC circuit model (34) faced with extreme shocks has current voltage $x_0 = 2$, resistance $R = 1$, capacitance $C = 2$, input voltage $\omega = 6$. Furthermore, the log-diffusion $\sigma = 1$ and the derivative order $p = 1$. For soft failure, consider the pre-given level $z = 4$. According to the real data of the failure system in the past year provided by State grid Nanjing maintenance branch, the real $N(t)$ counting shocks obeys a Poisson distribution with parameter $\lambda = 1.5699$ and the critical point of the external shock to be $D = 9$.

Fig. 1 shows that the variation of reliability index *Rel* with differential parameters is still in line with the law summarized in Example 3 that the reliability of the system is positively correlated with the operating time t and negatively correlated with the failure threshold z. When z is in the interval $(3.5, 5.5)$, the growth rate of the reliability index *Rel* is significantly larger, which means that the reliability is more sensitive to z in this region, that is, the right z in this interval is decisive for the the final reliability of the system.

We move on to the ordinary-order cumulative shock circuit model below.

5.1.2. Competing failure process involving the cumulative shocks

Theorem 8. For ordinary-order circuit systems (34) under cumulative shocks, the reliability index by time t can be expressed as

$$Rel = \left[1 - \left(1 + \exp\left(\frac{\pi}{\sqrt{3}\sigma} \left(\frac{z - x_0}{1 - \exp\left(\frac{t}{RC}\right)} - \omega + x_0 \right) \right) \right)^{-1} \right]$$
$$\left(e^{-\lambda t} + \sum_{k=1}^{\infty} \left(\frac{(\lambda t)^k}{k!} e^{-\lambda t} \sum_{h=0}^{D} \left(\sum_{x_n} \cdots \sum_{x_2} \sum_{x_1} \phi(x_1)\phi(x_2) \cdots \right. \right. \right.$$
$$\left. \left. \left. \phi(x_n)\phi(h - x_1 - x_2 - \cdots - x_n) \right) \right) \right). \tag{41}$$

Proof: According to Theorem 5, we have

$$Rel = \bar{F}(t) \left(e^{-\lambda t} + \sum_{k=1}^{\infty} \left(\frac{(\lambda t)^k}{k!} e^{-\lambda t} \sum_{h=0}^{D} \left(\sum_{x_n} \cdots \sum_{x_2} \sum_{x_1} \phi(x_1)\phi(x_2) \right. \right. \right.$$
$$\left. \left. \left. \cdots \phi(x_n)\phi(h - x_1 - x_2 - \cdots - x_n) \right) \right) \right). \tag{42}$$

Then by Theorem 6,

$$U(t) = \left(1 + \exp\left(\frac{\pi}{\sqrt{3}\sigma} \left(\frac{z - x_0}{1 - \exp\left(\frac{t}{RC}\right)} - \omega + x_0 \right) \right) \right)^{-1},$$

substituting which into the $\bar{F}(t)$ in Eq. (42). The reliability index (41) can be easily obtained. The theorem is proved. □

Based on this analytic formula, we give a numerical example of ordinary-order failure process under cumulative shocks to measure its reliability.

Example 2. Assume an uncertain fractional-order RC circuit model (34) faced with cumulative shocks features current input voltage $x_0 = 2$, resistance $R = 1$, capacitance $C = 2$, voltage $\omega = 6$. Furthermore, the log-diffusion $\sigma = 1$ and the derivative order $p = 1$. For soft failure, consider the pre-given level $z = 5$. According to the real data of the failure system in the past year provided by State grid Nanjing maintenance branch, the real $N(t)$ counting shocks obeys a Poisson distribution with parameter $\lambda = 1.5699$ and the critical point of the external shock to be $D = 9$.

The data shown in Fig. 2 shows that the variation of reliability index with differential parameters is still in line with the law summarized in Example 4, i.e. the reliability of the system is positively correlated with the operating time t and negatively correlated with the failure threshold z.

When z is in the interval $(4, 4.5)$, the growth rate of the reliability index *Rel* is significantly larger, which means that the reliability is more sensitive to z in this region, that is, the right z in this interval is decisive for the final reliability of the system.

Remark 2. It is interesting to note that, as can be seen in Figs. 2(b) and 1(b), when all other parameters are the same and the reliability index *Rel* is smoothed out at $z = 7$, a system that faces the threat of extreme failure (where *Rel* ends in 0.6344) is more reliable than one that suffers a cumulative failure(where *Rel* ends in 0.5553), which indicates that a system under extreme failure without cumulative failure is superior to a system under cumulative failure without extreme failure.

5.2. Fraction-order simulation

For the more general fractional-order circuit case (11), we will obtain numerical expression of the reliability index through the predicator–corrector algorithm, which has been confirmed by Jin et al. [35] that it has been a useful as well as effective scheme to solve FDEs. Again, we start with the failure process under extreme shocks.
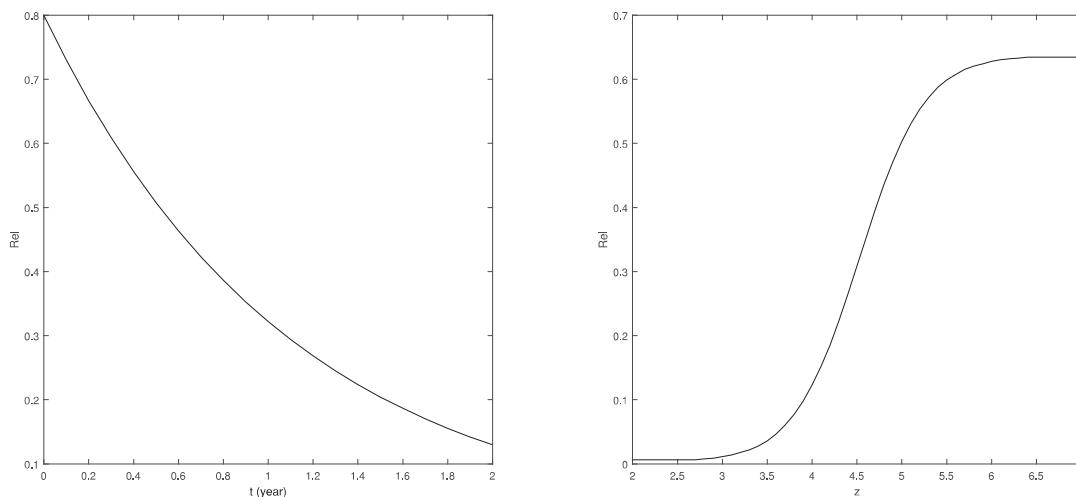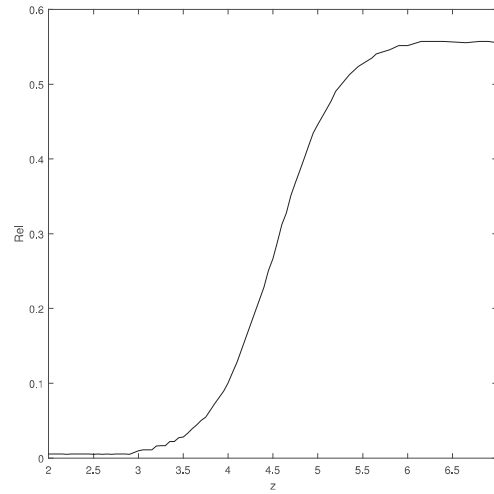

(a) Reliability index for different t with $z = 4$ (b) Reliability index for different z with $T = 2$

Fig. 1. Reliability index for systems under extreme shocks with $p = 1$.
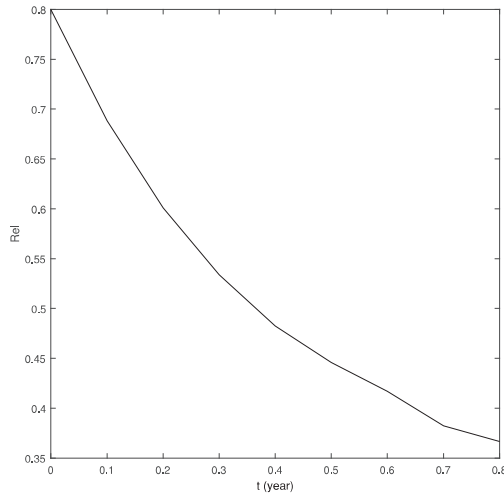
(a) Reliability index for different t with $z = 5$ (b) Reliability index for different z with $T = 2$

Fig. 2. Reliability index for systems under cumulative shocks with $p = 1$.

5.2.1. Competing failure process involving the extreme shocks

Example 3. Assume an uncertain fractional-order RC circuit model (11) faced with extreme shocks has initial voltage $x_0 = 2$, resistance $R = 1$, capacitance $C = 2$, voltage $\omega = 6$. Furthermore, the log-diffusion $\sigma = 1$ and the derivative order $p = 0.5$. For soft failure, consider the pre-given level $z = 4$. According to the real data of the failure system in the past year provided by State grid Nanjing maintenance branch, the real $N(t)$ counting shocks obeys a Poisson distribution with parameter $\lambda = 1.5699$ and the critical point of the external shock to be $D = 9$.

We can get that

$$X_s^\alpha = \sum_{k=0}^{n-1} x_k \cdot s^k E_{0.5,k+1}\left(-\frac{s^{0.5}}{2}\right) + \left(3 + \frac{1}{2}\Phi^{-1}(\alpha)\right) s^{0.5} E_{0.5,1.5}\left(-\frac{s^{0.5}}{2}\right).$$

Then, according to Theorem 4, the reliability index is that

$$Rel = \left(\inf\left\{\alpha \in (0,1) \mid \sup_{0 \leqslant s \leqslant t} \sum_{k=0}^{n-1} x_k \cdot s^k E_{0.5,k+1}\left(-\frac{s^{0.5}}{2}\right) \right.\right.$$
$$+ \left(3 + \frac{1}{2}\Phi^{-1}(\alpha)\right) s^{0.5} E_{0.5,1.5}\left(-\frac{s^{0.5}}{2}\right) \geqslant 4\right\}\right) \qquad (43)$$
$$\times \left(e^{-1.5699t} + \sum_{k=1}^{\infty}\left(\frac{(1.5699t)^k}{k!} e^{-1.5699t} \cdot \phi^k(D)\right)\right).$$

Analyze the reliability index corresponding to different parameters. According to the numerical analysis results of Fig. 3, the reliability of the system is positively correlated with the running time t and negatively correlated with the failure threshold z. The reasons are as follows: the increase of running time means that the cumulative degradation within the system is increasing, which in turn reducing the system reliability; the increase of the failure threshold means that the ability of the system to withstand internal and external shocks increases, which in turn improving the system reliability. When z is in the interval $(3.5, 4.5)$, the growth rate of the reliability index Rel is significantly larger, which means that the reliability is more sensitive to z in this region, that is, the right z in this interval is decisive for the final reliability of the system.

Remark 3. It is easy to see that when $p = 0.5$, the system begins with $Rel = 0.59$ at $t = 0$, which is lower than $Rel = 0.8$ in the case of

$p = 1$. This is because the first-order model is ideal than the real situation. Therefore, it is more reasonable to use a fractional-order model to guide the operation and maintenance of the system.

Then, discuss the reliability index corresponding to different order p. According to the results of the data provided by the Table 1, it is not difficult to find that the reliability index of the system suffering from extreme shocks and cumulative degradation is negatively related to the change of order p.

Remark 4. Easily seen, the reliability index *Rel* drops a bit at $p = 1$, which is consistent with the fractional calculus theory.

Next is the case of failure process under cumulative shocks.

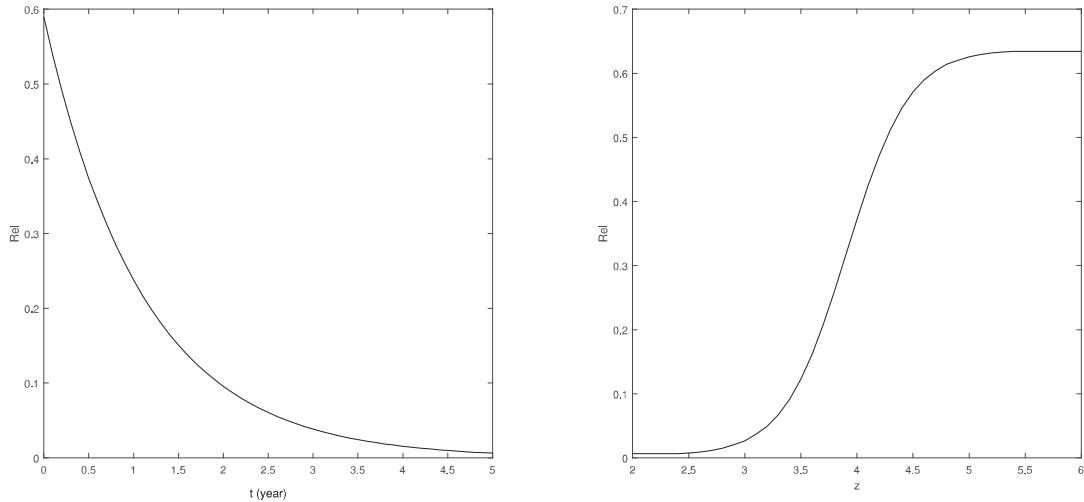*5.2.2. Competing failure process involving the cumulative shocks*

Example 4. Assume an uncertain fractional-order RC circuit model (11) faced with cumulative shocks has initial voltage $x_0 = 2$, resistance $R = 1$, capacitance $C = 2$, voltage $\omega = 6$. Furthermore, the log-diffusion $\sigma = 1$ and the derivative order $p = 0.5$. For soft failure, consider the pre-given level $z = 5$. According to the real data of the failure system in the past year provided by State grid Nanjing maintenance branch, the real $N(t)$ counting shocks obeys a Poisson distribution with parameter $\lambda = 1.5699$ and the critical point of the external shock to be $D = 9$.

We can get that

$$X_s^\alpha = \sum_{k=0}^{n-1} x_k \cdot s^k E_{0.5,k+1}\left(-\frac{s^{0.5}}{2}\right) + \left(3 + \frac{1}{2}\Phi^{-1}(\alpha)\right) s^{0.5} E_{0.5,1.5}\left(-\frac{s^{0.5}}{2}\right).$$

Then, according to Theorem 5, the reliability index is that

$$Rel = \left(\inf\left\{\alpha \in (0,1) \mid \sup_{0 \leqslant s \leqslant t} \sum_{k=0}^{n-1} x_k \cdot s^k E_{0.5,k+1}\left(-\frac{s^{0.5}}{2}\right)\right.\right.$$
$$+ \left(3 + \frac{1}{2}\Phi^{-1}(\alpha)\right) s^{0.5} E_{0.5,1.5}\left(-\frac{s^{0.5}}{2}\right) \geqslant 4\right\}\right)$$
$$\cdot \left(e^{-1.5699t} + \sum_{k=1}^{\infty}\left(\frac{(1.5699t)^k}{k!} e^{-1.5699t} \cdot \sum_{h=0}^{D}\left(\sum_{x_n}\cdots\sum_{x_2}\sum_{x_1} \phi(x_1)\phi(x_2)\right.\right.\right.$$
$$\left.\left.\left.\left.\cdots \phi(x_n)\phi(h - x_1 - x_2 - \cdots - x_n)\right)\right)\right)\right). \qquad (44)$$
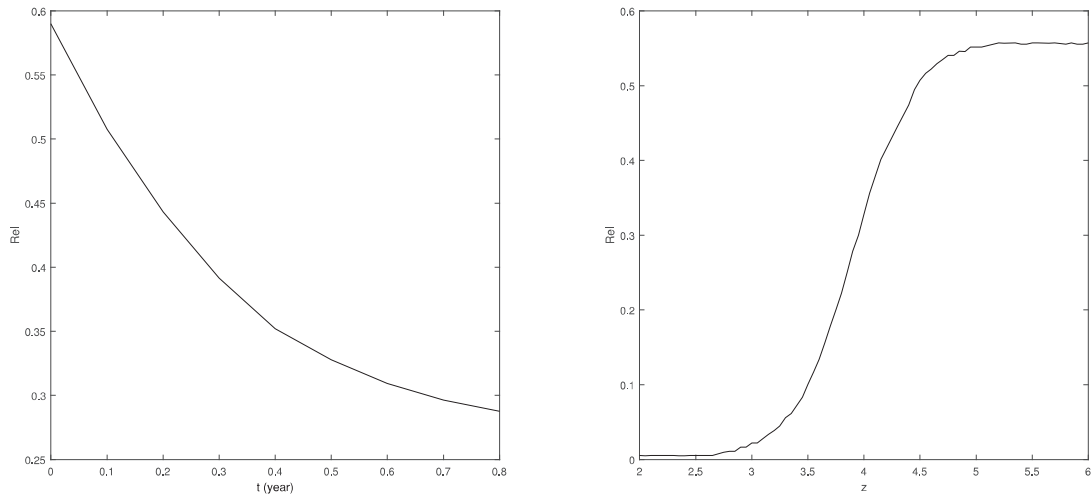
(a) Reliability index for different t with $z = 4$ (b) Reliability index for different z with $T = 2$

Fig. 3. Reliability index for systems under extreme shocks with $p = 0.5$.

Table 1
Reliability index with different p for systems under extreme shocks.

p	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Rel	0.8169	0.7919	0.7669	0.7169	0.6669	0.2667	0.2167	0.1751	0.1500	0.1250

(a) Reliability index for different t with $z = 5$ (b) Reliability index for different z with $T = 2$

Fig. 4. Reliability index for systems under cumulative shocks with $p = 0.5$.

Analyze the reliability index corresponding to different parameters. According to the numerical analysis results of Fig. 4, the reliability of the system is positively correlated with the operating time t and negatively correlated with the failure threshold z. The reasons are as follows: the increase of operating time means that the cumulative degradation caused by internal and external shocks of the system is increasing, which in turn reducing the system reliability;

the increase of the failure threshold means that the ability of the system to withstand internal and external shocks increases, which in turn improving the system reliability.

When z is in the interval $(3, 5)$, the growth rate of the reliability index Rel is significantly larger, which means that the reliability is more sensitive to z in this region, that is, the right z in this interval is decisive for the final reliability of the system.

Table 2
Reliability index with different p for systems under cumulative shocks.

p	0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4	1.5
Rel	0.7363	0.7137	0.6912	0.6461	0.6009	0.2400	0.1950	0.1578	0.1352	0.1125

Remark 5. It is interesting to note that, as can be seen in Figs. 4(b) and 3(b), when all other parameters are the same and the reliability index *Rel* is smoothed out at $z = 6$, a system that faces the threat of extreme failure (where *Rel* ends in 0.6312) is more reliable than one that suffers a cumulative failure(where *Rel* ends in 0.5574), which indicates that a system under extreme failure without cumulative failure is superior to a system under cumulative failure without extreme failure.

Then, discuss the reliability index corresponding to different order p. According to the results of the data provided by the Table 2, it is not difficult to find that the reliability index of the system suffering from cumulative shocks and degradation is negatively related to the change of order p.

Remark 6. Easily seen, the reliability index *Rel* drops a bit at $p = 1$, which is consistent with the fractional calculus theory.

6. Conclusion

The traditional uncertain physical models are limited to the ordinary derivative, whereas in real dynamic process, the state variable is not only rely on the current state but also on the past state. Uncertain fractional-order differential equations (UFDEs) have non-locality features to reflect memory and hereditary characteristics for the state variable changes, thus is more suitable to model the real uncertain process. This paper mainly seeked a feasible reliability analysis method, introduced the chance theory, deeply analyzed the mechanism of RC circuit competition failure process, and established a novel type of uncertain random fractional-order model with Caputo type. Related theories and characteristics of reliability modelling were summarized, and its applicability under different engineering backgrounds was studied. Two types of failure types were discussed, namely sudden failure and degradation failure, under the premise of a given failure threshold, the corresponding life distribution and failure mechanism were explored. Numerical expression of the reliability index using the predicator–corrector scheme and corresponding monotonicity were discussed. The research content of this paper only considers the establishment of the reliability model and neglects the formulation of maintenance strategy. However, the maintenance strategy can have high research value in the actual engineering application, so the maintenance time, maintenance cost and other issues can be further explored in the following research.

References

[1] Frostig E, Kenzin M. Availability of inspected systems subject to shocks - a matrix algorithmic approach. Eur J Oper Res 2009;193(1):168–83.

[2] Lehmann A. Joint modeling of degradation and failure time data. J Statal Plan Inference 2009;139(5):1693–706.

[3] Wang X. Nonparametric estimation of the shape function in a gamma process for degradation data. Canad J Stats 2010;37(1):102–18.

[4] Jiang L, Feng Q, Coit DW. Reliability analysis for dependent failure processes and dependent failure threshold. In: 2011 International Conference on Quality, Reliability, Risk, Maintenance, and Safety Engineering. IEEE; 2011. p. 30–4.

[5] Rafiee K, Feng Q, Coit DW. Condition-based maintenance for repairable deteriorating systems subject to a generalized mixed shock model. IEEE Trans Reliab 2015:1–11.

[6] Qiu Q, Cui L. "Reliability evaluation based on a dependent two-stage failure process with competing failures." Appl Math Model 2018;vol. 64(no. DEC) p. 699–712.

[7] Jerome S, Oldham K. The Fractional Calculus - Theory and Applications of Differentiation and Integration to Arbitrary Order, vol. 111 of Mathematics in Science and Engineering. Elsevier; 1974.

[8] Mandelbrot BB. !!

[20] Liu B. Fuzzy process, hybrid process and uncertain process. J Uncertain Syst 2008;2(1):3–16.

[21] Zeng Z, Wen M, Kang R. Belief reliability: A new metrics for products reliability. Fuzzy Optim Decis Making 2013;12(1):15–27.

[22] Zeng Z, Kang R, Wen M, Zio E. Uncertainty theory as a basis for belief reliability. Inf Sci 2018;429:26–36.

[23] Liu Y. Uncertain random variables: a mixture of uncertainty and randomness. Springer-Verlag; 2013.

[24] Liu Y. Uncertain random programming with applications. Kluwer Academic Publishers; 2013.

[25] Yao K, Gao J. Law of large numbers for uncertain random variables. IEEE Trans Fuzzy Syst 2015;24(3):615–21.

[26] Zhou J, Yang F, Wang K. Multi-objective optimization in uncertain random environments. Fuzzy Optim Decis Making 2014;13(4):397–413.

[27] Wen, Meilin, Kang, and Rui, "Reliability analysis in uncertain random system," Fuzzy Optimization Decision Making A Journal of Modeling Computation Under Uncertainty, 2016.

[28] Zhang Q, Kang R, Wen M. "Belief reliability for uncertain random systems." IEEE Trans Fuzzy Syst 2018; vol. PP, pp. 1–1.

[29] Liu B, Zhang Z, Wen Y. Reliability analysis for devices subject to competing failure processes based on chance theory. Appl Math Model 2019;75:398–413.

[30] Gao R, Yao K. Importance index of components in uncertain random systems. Knowl-Based Syst 2016.

[31] Liu B, Zhang Z, Wen Y. "Reliability analysis for devices subject to competing failure processes based on chance theory." Appl Math Model 2019; vol. 75, no. Nov., p. 398–413.

[32] Zhu Y. "Uncertain fractional differential equations and an interest rate model." Math Methods Appl 2015; p. 3359–68.

[33] Zhu Y. "Existence and uniqueness of the solution to uncertain fractional differential equation." J Uncertain Anal Appl; 2015.

[34] Lu Z, Yan H, Zhu Y. European option pricing model based on uncertain fractional differential equation. Fuzzy Optim Decis Making 2019.

[35] Jin T, Sun Y, Zhu Y. Extreme values for solution to uncertain fractional differential equation and application to american option pricing model. Phys A 2019;534:122357.

[36] Jin T, Sun Y, Zhu Y. Time integral about solution of an uncertain fractional order differential equation and application to zero-coupon bond model. Appl Math Comput 2020;372:124991.

[37] Jin T, Zhu Y. First hitting time about solution for an uncertain fractional differential equation and application to an uncertain risk index model. Chaos, Solitons & Fractals 2020;137:109836.

[38] Liu B. Theory and Practice of Uncertain Programming. Berlin Heidelberg: Springer; 2009.

[39] Liu B. Some Research Problems in Uncertainty Theory PhD thesis. Berlin Heidelberg: Springer; 2008.

[40] Ford NJ, Simpson AC. The numerical solution of fractional differential equations: speed versus accuracy. Numer Algorithms 2001;26(4):333–46.

[41] Diethelm K, Ford NJ, Freed AD. A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dyn 2002;29(1–4):3–22.

[42] Lu Z, Zhu Y. Numerical approach for solution to an uncertain fractional differential equation. Appl Math Comput 2019;343:137–48.

[43] Jin T, Zhu Y. First hitting time about solution for an uncertain fractional differential equation and application to an uncertain risk index model. Chaos, Solitons & Fractals 2020;137:109836.

Ting Jin received her Ph.D. degree in mathematics from Nanjing University of Science and Technology, Nanjing, China. She is currently a Lecturer with the School of Science, Nanjing Forestry University, Nanjing, China. She has authored or co-authored on several journals, including Physica A: Statistical Mechanics and its Applications, Applied Mathematics and Computation and Chaos, Solitons & Fractals. Her current research interests include reliability analysis, decision making, fractional-order differential equation, and their applications.
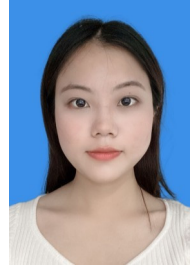

Hongxuan Xia pursuing the bachelor's degree from the Nanjing Forestry University, Nanjing, China. Her current research interests include machine learning reliability analysis, and fractional-order differential equation.
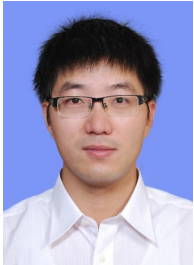
Shangce Gao received his Ph.D. degree in innovative life science from University of Toyama, Toyama, Japan in 2011. He is currently an Associate Professor with the Faculty of Engineering, University of Toyama, Japan. His current research interests include nature-inspired technologies, mobile computing, machine learning, and neural networks for real-world applications. Dr. Gao was a recipient of a Best Paper Award at IEEE International Conference on Progress in Informatics and Computing, the Shanghai Rising-Star Scientist award, the Chen-Guang Scholar of Shanghai award, the Outstanding Academic Performance Award of IEICE, and the Outstanding Academic Achievement Award of IPSJ. He is a Senior Member of IEEE, and serves as an Associate Editor for many international journals such as IEEE Access, IEEE/CAA Journal of Automatica Sinica, etc. He also serves as a Secretary for Hokuriku Section, IEICE and has served on the program committees for several international professional conferences.

Hui Ding pursuing the bachelors degree from the Nanjing Forestry University, Nanjing. China.
Her current research interests include financial option pricing, reliability analysis and fractional differential equations