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Locality and entanglement of indistinguishable particles

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Entanglement is one of the strongest quantum correlation, and is a key ingredient in fundamental aspects of quantum mechanics and a resource for quantum technologies. While entanglement theory is well settled for distinguishable particles, there are five inequivalent approaches to entanglement of indistinguishable particles. We analyse the different definitions of indistinguishable particle entanglement in the light of the locality notion. This notion is specified by two steps: (i) the identification of subsystems by means of their local operators; (ii) the requirement that entanglement represent correlations between the above subsets of operators. We prove that three of the aforementioned five entanglement definitions are incompatible with any locality notion defined as above.

Entanglement is one of the strongest form of quantum correlations and is crucial for the understanding of Bell's inequalities¹, quantum communications², quantum metrology^{3,4}, and quantum computation^{5,6}. Fundamental aspects of quantum theory and the quantum information framework suggest that entanglement is a notion derived from the definition of subsystems that can be correlated or, in other words, from the notion of *locality*. For instance, any entangled pure state of two distinguishable particles violates a Bell's inequality, and thus prove quantum non-locality^{7,8}, and the resourcefulness of Bell non-locality is a special instance of entanglement theory⁹. On the other hand, several quantum technologies consist of parties independently manipulating subsystems, and entanglement among subsystems is a fundamental resource that allows to overcome the limitations of local operations on subsystems and classical communications between them¹⁰.

Entanglement theory is very well understood for distinguishable particles: two-particle separable, namely non-entangled, pure states are of the form $|\psi_1\rangle \otimes |\psi_2\rangle$, and each subsystem is implicitly assumed to be a particle. This definition can be reformulated by describing each subsystem with operators, termed *local* and acting non-trivially only on it, i.e. $A = O_1 \otimes \mathbb{1}_2$ for the first particle and $B = \mathbb{1}_1 \otimes O_2$ for the second one. Therefore, a pure state is separable if and only if its expectations do not show correlations between single-particle operators: $\langle AB \rangle = \langle A \rangle \langle B \rangle$ for any A, B as above¹¹.

The choice of the operators A and B naturally arises from the experimental ability to individually address particles. Nevertheless, correlations between operators of different form have been proven to correspond to entanglement between more general subsystems^{12–16}, already in the framework of distinguishable particles.

Although distinguishable particles are a very useful paradigm in many physical systems, e.g. spin models where particles are localised at different positions, Nature is made of several kinds of indistinguishable particles, like electrons, atoms and photons. Particle indistinguishability requires that pure states and operators be invariant under particle permutations. In particular, the aforementioned single-particle operators are no longer allowed. These considerations challenge the notion of particle as a natural subsystem, whenever indistinguishability cannot be neglected, e.g. if particles are not spatially separated (see Fig. 1 for an illustration of particles that are progressively less separated and lose their distinguishability).

In this context, five alternative definitions of entanglement in systems of indistinguishable particles have been proposed: see¹⁷ for a review. Nevertheless, the identification of subsystems and of local operators, whose correlations correspond to entanglement, is not always addressed. This identification indeed allows us to systematically describe the advantage that indistinguishable particle entanglement provides over local operations and classical communications in quantum technologies. Other resource theories can characterize resourceful states that overcome the limitations of operations defined by lifting the locality requirement. The characterisation of local operators also enables us to understand the overlap of a resource theory with entanglement theory, and the cost of resource conversion in terms of resourceful operations of both theories.

We say that a definition of entanglement is compatible with a notion of locality if entanglement corresponds to correlations between commuting subsets of operators \mathcal{A} and \mathcal{B} : $\langle AB \rangle = \langle A \rangle \langle B \rangle$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ for pure separable states (see^{11,17} for the generalization to mixed states). These operator subsets formally and

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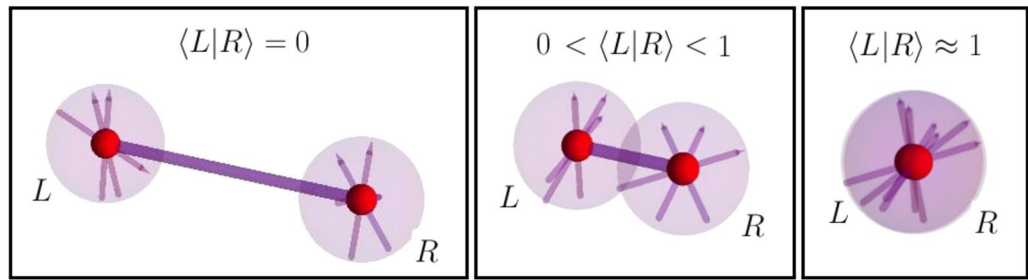


Figure 1. Pictorial representation of two identical particles with an internal degree of freedom and a spatial degree of freedom, when the spatial wave function do not overlap $\langle L|R \rangle = 0$ (effectively distinguishable particles), partially overlap $0 < \langle L|R \rangle < 1$, and overlap almost completely $\langle L|R \rangle \approx 1$.

operationally define subsystems as they consist of observables individually measurable and transformations individually implementable without mutual disturbance due to their commutativity $[\mathcal{A}, \mathcal{B}] = 0$. A preliminary comparative analysis among the existing entanglement approaches shows that only one of them is fully consistent¹⁷. In particular, three of these approaches cannot be interpreted as entanglement of particles, because they are not consistent with the correlations between subsets made of permutationally invariant single-particle operators. The physical intuition is that truly indistinguishable particles cannot be individually addressed (see the sketch in Fig. 1). This issue does not apply to the other approaches, since one, the so-called superselection rule (SSR) entanglement, is restricted to physical situations of particles that can be effectively distinguished by means of certain degrees of freedom^{18–20}, and the other accounts for more general correlations between modes in a second quantised description that also recovers particle correlations for effectively distinguishable particles^{21–30}. The latter notion, i.e. mode-entanglement, is also routinely applied for entanglement detection, manipulations and measures^{17,31–33}.

In this paper, we focus on the aforementioned entanglement definitions that do not correspond to particle correlations, and investigate if they rather represent correlations between subsystems identified by general subsets \mathcal{A} and \mathcal{B} . For each definition, we define the set of separable pure states, namely SEP, and look for candidates for the subsets \mathcal{A} and \mathcal{B} .

Before sketching the general scheme of our analysis, it is crucial to notice that the linearity of quantum mechanics implies that the set of operators of a system, as well as subsystems, is an algebra^{34,35}. An algebra is a linear space closed under a multiplication between its elements and under a conjugation operation (hermitian conjugation in our case). Therefore, the subsets \mathcal{A} and \mathcal{B} that comprise all operators acting on each subsystem are algebras, as happens in the standard case of distinguishable particles. Nevertheless, we have used neither the algebra structure nor commutativity in our main results which therefore hold also for more general subsets of operators, as those considered in³⁶.

A necessary condition for operators $A \in \mathcal{A}$ and $B \in \mathcal{B}$ is that they do not generate entanglement, because subsystems cannot be correlated by local operators, like A , B , and AB . This requirement is formulated in full generality as $A \cdot \text{SEP} = \text{SEP}$ and $B \cdot \text{SEP} = \text{SEP}$, or

$$A|\Psi\rangle, B|\Psi\rangle \in \text{SEP}, \quad \forall |\Psi\rangle \in \text{SEP} \quad (1)$$

relaxing the normalisation conditions, such that $\langle \Psi|\Psi \rangle > 0$, $\langle A|\Psi \rangle > 0$, $\langle B|\Psi \rangle > 0$.

Operators in each subset, \mathcal{A} and \mathcal{B} , are chosen from the above ones, and those belonging to different subsets must commute with each other. Thus, we check the factorisation condition

$$\frac{\langle \Psi|AB|\Psi \rangle}{\langle \Psi|\Psi \rangle} = \frac{\langle \Psi|A|\Psi \rangle \langle \Psi|B|\Psi \rangle}{\langle \Psi|\Psi \rangle \langle \Psi|\Psi \rangle}, \quad \forall A \in \mathcal{A}, B \in \mathcal{B}, |\Psi\rangle \in \text{SEP} \quad (2)$$

for any subset, \mathcal{A} and \mathcal{B} , of operators that do not generate entanglement.

Results

We are now ready to go through the different entanglement definitions and to look for possible subsets, \mathcal{A} and \mathcal{B} , of local operators. We shall prove that it is impossible to find subsets whose correlations correspond to the following notions of entanglement. Specifically, we find that for any possible subset bipartition there are separable states that do not fulfil the factorisation of local expectations (2).

The first quantisation formalism is most familiar to the entanglement definitions analysed here. For completeness, we provide some definitions both in first and second quantisation, using the tilde to distinguish the second quantization formalism. However, our computations rely on matrix representations of the Hilbert space that are equivalent to matrix representations of Fock space sectors, after rewriting the basis states from first to second quantisation. Therefore, our conclusions remain valid in both formalisms, and we often consider the symbols without the tilde for simplifying the notation.

Moreover, we shall present our results in the form of lemmas and theorems in order to emphasise the key results in the statements.

Entanglement-I. For the sake of concreteness, let us focus on two bosonic two-level particles. Define a basis of the single particle Hilbert space \mathbb{C}^2 , namely $|0\rangle$ and $|1\rangle$, and that of the symmetrised two-particle Hilbert space $S(\mathbb{C}^2 \otimes \mathbb{C}^2)$ where S is the symmetrisation projector,

$$|\Phi_0\rangle = |0\rangle|0\rangle, \quad |\Phi_1\rangle = |1\rangle|1\rangle, \quad |\Phi_2\rangle = \frac{|0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{2}}. \tag{3}$$

In second quantisation the above states read

$$|\tilde{\Phi}_0\rangle = \frac{(a_0^\dagger)^2}{\sqrt{2}}|\text{vac}\rangle, \quad |\tilde{\Phi}_1\rangle = \frac{(a_1^\dagger)^2}{\sqrt{2}}|\text{vac}\rangle, \quad |\tilde{\Phi}_2\rangle = a_0^\dagger a_1^\dagger |\text{vac}\rangle, \tag{4}$$

where $a_{0,1}^\dagger$ are creation operators of a single particle in the state $|0\rangle$ or $|1\rangle$ respectively, with $[a_i, a_j^\dagger] = \delta_{ij}$, and $|\text{vac}\rangle$ is the vacuum.

In this case, the first entanglement definition is

Definition 1 (*Entanglement-I*³⁷⁻⁴⁰) The set of pure separable-I states is, in first and second quantization respectively,

$$\text{SEP}_I = \{(c_0|0\rangle + c_1|1\rangle)^{\otimes 2}\}_{c_0, c_1 \in \mathbb{C}} = \{c_0^2|\Phi_0\rangle + c_1^2|\Phi_1\rangle + \sqrt{2}c_0c_1|\Phi_2\rangle\}_{c_0, c_1 \in \mathbb{C}}, \tag{5}$$

$$\text{SEP}_I = \left\{ \frac{1}{\sqrt{2}}(c_0 a_0^\dagger + c_1 a_1^\dagger)^2 |\text{vac}\rangle \right\}_{c_0, c_1 \in \mathbb{C}} = \{c_0^2|\tilde{\Phi}_0\rangle + c_1^2|\tilde{\Phi}_1\rangle + \sqrt{2}c_0c_1|\tilde{\Phi}_2\rangle\}_{c_0, c_1 \in \mathbb{C}}. \tag{6}$$

All other pure states are entangled-I.

Practical realisations of the above single particle states $|0\rangle$ and $|1\rangle$ are, e.g. in cold atoms, spatial localisation in separated wells of a lattice, or hyperfine energy levels. Therefore, separable-I states are those with particles in the same superposition of localised or energy states.

In the following theorem, we characterise operators that do not generate entanglement-I.

Theorem 1 *Operators that leave SEP_I invariant are represented in first quantisation as $O \otimes O$ on the enlarged Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ with $O = O^\dagger$, and in second quantisation as*

$$\frac{1}{2} \sum_{i,j,k,l=0,1} \langle i|O|j\rangle \langle k|O|l\rangle a_i^\dagger a_k^\dagger a_j a_l. \tag{7}$$

Proof Any operator A that does not generate entanglement-I fulfils Eq. (1) which implies, from Definition 1,

$$\langle \Phi_2|A|\Psi\rangle^2 = 2 \langle \Phi_0|A|\Psi\rangle \langle \Phi_1|A|\Psi\rangle. \tag{8}$$

where $|\Psi\rangle \in \text{SEP}_I$. Note that only the projection SAS onto the symmetric space $S(\mathbb{C}^2 \otimes \mathbb{C}^2)$ contribute to Eq. (9) since we considered bosonic states. Equalities (8) for $|\Psi\rangle = |\Phi_{0,1}\rangle$ imply

$$SAS = \begin{pmatrix} a_{00}^2 & a_{10}^2 & x \\ a_{01}^2 & a_{11}^2 & y \\ \sqrt{2} a_{00} a_{01} & \sqrt{2} a_{10} a_{11} & z \end{pmatrix}, \tag{9}$$

in the basis (3), or (4), with $a_{00,01,10,11}, x, y, z \in \mathbb{C}$.

Conditions for x and y are found by requiring Eq. (1) for arbitrary $|\Psi\rangle \in \text{SEP}_I$. This requirement implies, after plugging into (8) the explicit form of $|\Psi\rangle$ in Eq. (5), that the coefficients of each monomial $c_0^n c_1^m$ vanish. We obtain

$$z^2 + 2 a_{00} a_{01} a_{10} a_{11} = a_{00}^2 a_{11}^2 + a_{01}^2 a_{10}^2 + 2xy, \tag{10}$$

$$\sqrt{2} a_{00} a_{01} z = a_{00}^2 y + a_{01}^2 x, \tag{11}$$

$$\sqrt{2} a_{10} a_{11} z = a_{10}^2 y + a_{11}^2 x. \tag{12}$$

The solutions of the above equations are

$$x = \pm \sqrt{2} a_{00} a_{10}, \tag{13}$$

$$y = \pm \sqrt{2} a_{01} a_{11}, \tag{14}$$

$$z = \pm a_{00} a_{11} \pm a_{01} a_{10}, \tag{15}$$

with either all plus signs or all minus signs.

The two cases, e.g. plus and minus signs, are equivalent as they result in the same operator by redefining $a'_{00} = -a_{00}$ and $a'_{01} = -a_{01}$. Therefore, we found

$$\text{SAS} = \begin{pmatrix} a_{00}^2 & a_{10}^2 & \sqrt{2} a_{00} a_{10} \\ a_{01}^2 & a_{11}^2 & \sqrt{2} a_{01} a_{11} \\ \sqrt{2} a_{00} a_{01} & \sqrt{2} a_{10} a_{11} & a_{00} a_{11} + a_{01} a_{10} \end{pmatrix}. \tag{16}$$

In first quantisation, the above matrix is exactly the projection onto the symmetric subspace of the operator $O \otimes O$ on the larger Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$, with

$$O = \begin{pmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \end{pmatrix}, \tag{17}$$

in the basis $\{|0\rangle, |1\rangle\}$. In second quantization, Eq. (16) is the matrix representation of Eq. (7). This concludes the proof. \square

The factorisation condition can be checked for arbitrary operators, $A = O \otimes O$ and $B = Q \otimes Q$, that could form subsets of local operators. Note that the subsets are not necessarily commuting in the next theorem. We have therefore proven a more general result than what we need in entanglement analysis.

Theorem 2 Given two subsets \mathcal{A} and \mathcal{B} of operators that leave $\mathcal{S}_{\text{sep}}^I$ invariant, the factorisation condition (2) and Definition 1 imply that either \mathcal{A} or \mathcal{B} consists only of operators proportional to the identity.

Proof Consider a separable-I state $|\Psi\rangle \in \text{SEP}_I$ as in Definition 1, and two operators, $A \in \mathcal{A}$ and $B \in \mathcal{B}$ as in Theorem 1. The factorisation condition (2) reads

$$\left(\sum_{j=0,1} |c_j|^2 \right)^2 \left(\sum_{l,k=0,1} \langle l|OQ|k\rangle \bar{c}_l c_k \right)^2 = \left(\sum_{i,j,l,k=0,1} \langle i|O|j\rangle \langle l|Q|k\rangle \bar{c}_i c_j \bar{c}_l c_k \right)^2. \tag{18}$$

Both the left and the right hand sides of Eq. (18) are eight order polynomials in c_0 and c_1 . Nevertheless, there are monomials in the right-hand-side that do not appear in the left-hand-side. Since the factorisation condition must hold for any separable-I state, thus for any c_0 and c_1 , the monomials that appear only in the right-hand-side, i.e. with $i \neq j$ and $l \neq k$, must be multiplied by vanishing coefficients, i.e. $\langle i|O|j\rangle \langle l|Q|k\rangle = 0$. This consideration implies that either O or Q are diagonal in the basis $\{|0\rangle, |1\rangle\}$. The latter basis is arbitrary and the diagonality requirement must hold for any basis rotation. In conclusion, either O or Q must be proportional to the identity. \square

Therefore, entanglement-I is incompatible with any locality notions where each subsystem is characterised by subsets of local operators. We have not used the commutativity of subsets \mathcal{A} and \mathcal{B} in Theorem 2. In the Supplementary Information, we provide a characterisation of commuting subsets of operators that do not generate entanglement-I, a second proof of Theorem 2 using $[\mathcal{A}, \mathcal{B}] = 0$, and some examples of the proven properties.

Entanglement-II. Considering again two bosonic two-level particles, the second entanglement definition is

Definition 2 [Entanglement-II^{37-39,41-46}] The set of pure separable-II states is, in first and second quantization respectively,

$$\begin{aligned} \text{SEP}_{II} &= \text{SEP}_I \cup \left\{ \mathcal{S}(|c_0|0\rangle + c_1|1\rangle) \otimes (|\bar{c}_1|0\rangle - \bar{c}_0|1\rangle) \right\}_{c_0,1 \in \mathbb{C}} \\ &= \text{SEP}_I \cup \left\{ c_0 \bar{c}_1 |\Phi_0\rangle - c_1 \bar{c}_0 |\Phi_1\rangle + \frac{|c_1|^2 - |c_0|^2}{\sqrt{2}} |\Phi_2\rangle \right\}_{c_0,1 \in \mathbb{C}} \end{aligned} \tag{19}$$

$$\widetilde{\text{SEP}}_{II} = \widetilde{\text{SEP}}_I \cup \left\{ (c_0 a_0^\dagger + c_1 a_1^\dagger) (\bar{c}_1 a_0^\dagger - \bar{c}_0 a_1^\dagger) |\text{vac}\rangle \right\}_{c_0,1 \in \mathbb{C}} = \widetilde{\text{SEP}}_I \cup \left\{ c_0 \bar{c}_1 |\tilde{\Phi}_0\rangle - c_1 \bar{c}_0 |\tilde{\Phi}_1\rangle + \frac{|c_1|^2 - |c_0|^2}{\sqrt{2}} |\tilde{\Phi}_2\rangle \right\}_{c_0,1 \in \mathbb{C}} \tag{20}$$

All other pure states are entangled-II.

Keeping in mind the same physical realisations mentioned after Definition 1, two particles are separable-II if they are either in the same or in orthogonal superpositions of spatial localisations or of hyperfine levels.

Theorem 3 Any operator that leaves SEP_{II} invariant also leaves SEP_I invariant.

Proof Consider an operator A that does not generate entanglement-II, and therefore leaves SEP_{II} invariant, and a state $|\Psi\rangle \in \text{SEP}_I \subset \text{SEP}_{II}$. Either $A|\Psi\rangle \in \text{SEP}_I$ or $A|\Psi\rangle \in \text{SEP}_{II} \setminus \text{SEP}_I$. A necessary and sufficient condition for $A|\Psi\rangle \in \text{SEP}_I$ is Eq. (8), namely

$$P_A(c_0, c_1) := \langle \Phi_2 | A | \Psi \rangle^2 - 2 \langle \Phi_0 | A | \Psi \rangle \langle \Phi_1 | A | \Psi \rangle = 0. \tag{21}$$

Instead, a necessary and sufficient condition for $A|\Psi\rangle \in \text{SEP}_{\parallel} \setminus \text{SEP}_{\perp}$, in the case of two bosonic two-level particles, is⁴⁷

$$Q_A(c_0, \bar{c}_0, c_1, \bar{c}_1) = \text{Tr}_1 \left(\text{Tr}_2 \frac{A|\Psi\rangle\langle\Psi|A^\dagger}{\langle\Psi|A^\dagger A|\Psi\rangle} \right)^2 - \frac{1}{2} = 0, \tag{22}$$

where Tr_j is the standard partial trace over the j -th particle Hilbert space. This partial trace does not have a straightforward connection with entanglement when indistinguishability cannot be neglected due to the symmetrisation postulate, as discussed here and in¹⁷. Nevertheless, it is a mathematical operation that still provides information on the structure of states.

$P_A(c_0, c_1)$ is a fourth order homogeneous polynomial in c_0 and c_1 but does not depend on their complex conjugates. The fundamental theorem of algebra states that, for any fixed c_0 , Eq. (21) either has at least one up to four distinct solutions, $c_1 = c_1(c_0)$, or is a tautology.

$Q_A(c_0, \bar{c}_0, c_1, \bar{c}_1)$ is an eighth order polynomial that depends also on the complex conjugates $\bar{c}_{0,1}$, so that the fundamental theorem of algebra does not apply. Equation (22) can be a tautology, have at most eight solutions, say $\text{Re } c_1, \text{Im } c_1$ for fixed c_0 and $\text{Im } c_1$, but can also have no solutions. If Eq. (22) were a tautology, all states in SEP_{\perp} are transformed into states in $\text{SEP}_{\parallel} \setminus \text{SEP}_{\perp}$. Nevertheless, Eq. (21) has at least a solution, and so at least one state in SEP_{\perp} is transformed into a state in SEP_{\perp} but also in $\text{SEP}_{\parallel} \setminus \text{SEP}_{\perp}$. This result is impossible because of the linearity of A , unless $A|\Psi\rangle = 0$ for all $|\Psi\rangle$ that solve Eq. (21). The latter case implies $A = 0$, because the solutions of Eq. (21) span the full Hilbert space $\mathcal{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, as proven in the Lemma 1 in Methods. Thus, Eq. (22) has at most eight solutions $\text{Re } c_1, \text{Im } c_1$ for fixed c_0 and $\text{Im } c_1$.

The above argument shows that, fixing c_0 and $\text{Im } c_1$, there are at most a finite number of states $|\Psi\rangle \in \text{SEP}_{\perp}$ (corresponding to coefficients (c_0, c_1)) that are sent to $\text{SEP}_{\parallel} \setminus \text{SEP}_{\perp}$. If Eq. (21) has at most four solutions $c_1(c_0)$ for fixed c_0 , there is also at most a finite number of states $|\Psi\rangle \in \text{SEP}_{\perp}$ that are sent to SEP_{\perp} . Remind, however, that we have relaxed the normalisation condition $\langle\Psi|\Psi\rangle = 1$, and so $\text{Re } c_1$ can assume a continuity of values. Therefore, there are states $|\Psi\rangle \in \text{SEP}_{\perp}$ that are sent out of SEP_{\parallel} , and this contradicts the hypothesis of Theorem 3.

In conclusion, the only possibility is that Eq. (21) is a tautology and Eq. (22) has no solutions. This is equivalent to the statement of Theorem 3. \square

The impossibility to define subsets of local operators, whose correlations correspond to entanglement-II, follows from Theorem 3 and from the analogous result for entanglement-I.

Theorem 4 Given two subsets \mathcal{A} and \mathcal{B} of operators that leave SEP_{\parallel} invariant, the factorisation condition (2) and Definition 2 imply that either \mathcal{A} or \mathcal{B} consists only of operators proportional to the identity.

Proof Theorem 3 implies that operators that generate subsets \mathcal{A} and \mathcal{B} are special cases of those considered in Theorem 1 and in Theorem 2. The statements in Theorem 1 and in Theorem 2 therefore holds also for the operators considered here. Moreover, separable-I states are also separable-II, so that Theorem 2 proves the inconsistency also between separability-II and locality. \square

Note that also the proof of Theorem 4 does not require the commutativity of the operator subsets. In the Supplementary Information, we prove a complete characterisation of operators that do not generate entanglement-II, and prove that their subsets do not form an algebra. This provides an alternative proof of Theorem 4 if the operator subsets \mathcal{A} and \mathcal{B} are algebras. We also show some examples in the Supplementary Information.

Entanglement-III. The last definition of entanglement is more elaborated than the previous ones. The simplest physical system, where entanglement-III exists, is made of two bosons with a spatial degree of freedom, $\text{span}\{|L\rangle, |R\rangle\}$, and an internal one, $\text{span}\{|0\rangle, |1\rangle\}$. This definition is relative to the projection onto a subspace of the single particle Hilbert space. We assume that such subspace is $\mathcal{K} = \text{span}\{|L, \sigma\rangle\}_{\sigma=0,1}$, as often assumed within this approach⁴⁸. The general definition, shown in the Supplementary Information, leads in our case to the following.

Definition 3 (Entanglement-III⁴⁸) The set of pure separable-III states is, in first and second quantization respectively,

$$\text{SEP}_{\text{III}} = \left\{ a_{\sigma} |L, \sigma\rangle^{\otimes 2} + b_{\sigma, \sigma'} \mathbf{S} |L, \sigma\rangle \otimes |R, \sigma'\rangle + \sum_{\sigma_1, \sigma_2=0,1} c_{\sigma_1, \sigma_2} \mathbf{S} |R, \sigma_1\rangle \otimes |R, \sigma_2\rangle \right\}_{\substack{|\sigma\rangle, |\sigma'\rangle \in \mathbb{C}^2 \\ a_{\sigma}, b_{\sigma, \sigma'}, c_{\sigma_1, \sigma_2} \in \mathbb{C}}} \tag{23}$$

$$\widetilde{\text{SEP}}_{\text{III}} = \left\{ \left(a_{\sigma} \frac{(\mathbf{a}_{L, \sigma}^{\dagger})^2}{\sqrt{2}} + b_{\sigma, \sigma'} \mathbf{a}_{L, \sigma}^{\dagger} \mathbf{a}_{R, \sigma'}^{\dagger} + \sum_{\sigma_1, \sigma_2=0,1} c_{\sigma_1, \sigma_2} \mathbf{a}_{R, \sigma_1}^{\dagger} \mathbf{a}_{R, \sigma_2}^{\dagger} \right) |\text{vac}\rangle \right\}_{\substack{|\sigma\rangle, |\sigma'\rangle \in \mathbb{C}^2 \\ a_{\sigma}, b_{\sigma, \sigma'}, c_{\sigma_1, \sigma_2} \in \mathbb{C}}} \tag{24}$$

All other pure states are entangled-III.

Let us introduce the following projectors in first quantisation

$$P_X = \sum_{\sigma_1, \sigma_2=0,1} (2 - \delta_{X_1, X_2} \delta_{\sigma_1, \sigma_2}) \mathbf{S}(|X_1, \sigma_1\rangle\langle X_1, \sigma_1| \otimes |X_2, \sigma_2\rangle\langle X_2, \sigma_2|) \mathbf{S}, \tag{25}$$

or in second quantisation

$$\tilde{P}_X = \frac{1}{2} \sum_{\sigma_1, \sigma_2=0,1} a_{X_1, \sigma_1}^\dagger a_{X_2, \sigma_2}^\dagger a_{X_2, \sigma_2} a_{X_1, \sigma_1} \tag{26}$$

with $X = X_1 X_2$, and $X_{1,2} \in \{L, R\}$, and define $A_{X,Y} = P_X A P_Y$ for any operator A . The support of P_{LL} is isomorphic to $\mathbf{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, and separable-III states in $P_{LL} \cdot \mathbf{SEP}_{III}$, namely $|L, \sigma\rangle^{\otimes 2}$, are in one-to-one correspondence with separable-I states \mathbf{SEP}_I in Eq. (5). In particular, all the results for entanglement-I hold for entanglement-III constrained to the support of P_{LL} . The support of P_{LR} is isomorphic to $\mathbb{C}^2 \otimes \mathbb{C}^{2^{17,41,49-52}}$, therefore, separable-III states in $P_{LR} \cdot \mathbf{SEP}_{III}$, i.e. $\mathbf{S} |L, \sigma\rangle \otimes |R, \sigma'\rangle$, are in one-to-one correspondence with two distinguishable two-level particles, $|\sigma\rangle \otimes |\sigma'\rangle = (c_0|0\rangle + c_1|1\rangle) \otimes (d_0|0\rangle + d_1|1\rangle)$. Finally, $P_{RR} \cdot \mathbf{SEP}_{III}$ is the whole support of P_{RR} and isomorphic to $\mathbf{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

The search for subsets of local operators requires a detailed analysis of contributions to the factorisation condition from each sector identified by the projectors (25). Nevertheless, the identification of operators that leave \mathbf{SEP}_{III} is not needed to prove the following Theorem.

Theorem 5 Given two subsets of operators, \mathcal{A} and \mathcal{B} , the factorisation condition (2) and Definition 3 imply that either \mathcal{A} or \mathcal{B} consists only of operators proportional to the identity.

Proof Consider separable-III states $|\Psi\rangle = c_X |\Psi_X\rangle + c_Y |\Psi_Y\rangle$, with $|\Psi_X\rangle \in P_X \cdot \mathbf{SEP}_{III}$, $|\Psi_Y\rangle \in P_Y \cdot \mathbf{SEP}_{III}$, and $X, Y \in \{LL, LR, RR\}$. The factorisation condition (2) reads

$$\begin{aligned} & \sum_{\substack{T, W, W' \in \{X, Y\} \\ Z \in \{LL, LR, RR\}}} |c_T|^2 \bar{c}_W c_{W'} \langle \Psi_T | \Psi_T \rangle \langle \Psi_W | A_{W,Z} B_{Z,W'} | \Psi_{W'} \rangle \\ &= \sum_{T, T', W, W' \in \{X, Y\}} \bar{c}_T c_{T'} \bar{c}_W c_{W'} \langle \Psi_T | A_{T,T'} | \Psi_{T'} \rangle \langle \Psi_W | B_{W,W'} | \Psi_{W'} \rangle. \end{aligned} \tag{27}$$

Since c_X and c_Y in Eq. (27) are arbitrary, the coefficient of each monomial $\bar{c}_T c_{T'} \bar{c}_W c_{W'}$ must vanish whenever $T \neq T'$ and $W \neq W'$, or $T \neq W'$ and $W \neq T'$. Thus,

$$\langle \Psi_X | A_{X,Y} | \Psi_Y \rangle \langle \Psi_X | B_{X,Y} | \Psi_Y \rangle = 0. \tag{28}$$

Apply now Lemma 2 in Methods where the functions therein are $\langle \Psi_X | A_{X,Y} | \Psi_Y \rangle$ and $\langle \Psi_X | B_{X,Y} | \Psi_Y \rangle$, and the variables are the coefficients used to parametrise states $|\Psi_X\rangle$ and $|\Psi_Y\rangle$ (see the discussion after Definition 3). Lemma 2 in Methods entails that at least one factor in Eq. (28) vanish for all separable-III states, i.e. either $\langle \Psi_X | A_{X,Y} | \Psi_Y \rangle = 0$ or $\langle \Psi_X | B_{X,Y} | \Psi_Y \rangle = 0$ for all $|\Psi_X\rangle \in P_X \cdot \mathbf{SEP}_{III}$ and $|\Psi_Y\rangle \in P_Y \cdot \mathbf{SEP}_{III}$. Because $|\Psi_X\rangle$ and $|\Psi_Y\rangle$ span the support of P_X and P_Y respectively, then either $A_{X,Y} = 0$ or $B_{X,Y} = 0$.

Assume that $A_{X,Y} = 0$, as the other case is analogous. Comparing the coefficients of the monomial $|c_T|^2 \bar{c}_X c_Y$ in the right and the left hand sides of Eq. (27), we obtain

$$\langle \Psi_T | \Psi_T \rangle \langle \Psi_X | A_{X,X} B_{X,Y} | \Psi_Y \rangle = \langle \Psi_T | A_{T,T} | \Psi_T \rangle \langle \Psi_X | B_{X,Y} | \Psi_Y \rangle,$$

for both $T = X$ and $T = Y$. The difference between the two cases ($T = X$ and $T = Y$) of the above equation is

$$\langle \Psi_X | B_{X,Y} | \Psi_Y \rangle \left(\frac{\langle \Psi_X | A_{X,X} | \Psi_X \rangle}{\langle \Psi_X | \Psi_X \rangle} - \frac{\langle \Psi_Y | A_{Y,Y} | \Psi_Y \rangle}{\langle \Psi_Y | \Psi_Y \rangle} \right) = 0. \tag{29}$$

Lemma 2 in Methods implies that at least one of the factors in Eq. (29) must be identically zero. If the term in brackets vanishes then $A_{X,X} = \alpha P_X$ and $A_{Y,Y} = \alpha P_Y$ due to the arbitrariness of the separable-III states $|\Psi_X\rangle$ and $|\Psi_Y\rangle$ (see Lemma 3 in Methods). Therefore, $A = \alpha \mathbb{1}$ from the arbitrariness of $X, Y \in \{LL, LR, RR\}$, and we have proven Theorem 5. The case $\langle \Psi_X | B_{X,Y} | \Psi_Y \rangle = 0$ implies $B_{X,Y} = 0$, because separable-III states $|\Psi_X\rangle$ and $|\Psi_Y\rangle$ span the support of P_X and P_Y respectively. At this point, we have proven that $A_{X,Y} = B_{X,Y} = 0$ for any $X \neq Y$.

Choose separable-III states $\{|\Psi_Y^j\rangle\}_j$ that form an orthonormal basis of the support of P_Y . Equation (28) implies

$$0 = \sum_j \langle \Psi_X | A_{X,Y} | \Psi_Y^j \rangle \overline{\langle \Psi_X | B_{X,Y} | \Psi_Y^j \rangle} = \langle \Psi_X | A_{X,Y} B_{Y,X} | \Psi_X \rangle, \quad \forall Y \neq X. \tag{30}$$

Using Eq. (30) in the comparison between the coefficients of the monomial $|c_X|^4$ in the left and right hand side of Eq. (27), we obtain for any $X \in \{LL, LR, RR\}$

$$\langle \Psi_X | \Psi_X \rangle \langle \Psi_X | A_{X,X} B_{X,X} | \Psi_X \rangle = \langle \Psi_X | A_{X,X} | \Psi_X \rangle \langle \Psi_X | B_{X,X} | \Psi_X \rangle, \tag{31}$$

After plugging the above equation in the factorisation (27), and matching the coefficients of $|c_X|^2 |c_Y|^2$, we get

$$\left(\frac{\langle \Psi_X | A_{X,X} | \Psi_X \rangle}{\langle \Psi_X | \Psi_X \rangle} - \frac{\langle \Psi_Y | A_{Y,Y} | \Psi_Y \rangle}{\langle \Psi_Y | \Psi_Y \rangle} \right) \left(\frac{\langle \Psi_X | B_{X,X} | \Psi_X \rangle}{\langle \Psi_X | \Psi_X \rangle} - \frac{\langle \Psi_Y | B_{Y,Y} | \Psi_Y \rangle}{\langle \Psi_Y | \Psi_Y \rangle} \right) = 0. \tag{32}$$

As above, Lemma 2 in Methods proves that at least one of the factors must identically vanish. According to Lemma 3 in Methods, the arbitrariness of separable-III states $|\Psi_X\rangle$ and $|\Psi_Y\rangle$ further implies that either $A_{X,X} = \alpha P_X$ or $B_{X,X} = \beta P_X$ for any $X \in \{LL, LR, RR\}$. This conclusion, together with the property $A_{X,Y} = B_{X,Y} = 0$ for $X \neq Y$ proved above, entails Theorem 5. \square

Therefore, also entanglement-III is incompatible with any locality notion as sketched in the Introduction: expectations of products of local operators pertaining to different subsystems must factorise for all non-entangled state. We stress that commutativity between the operator subsets \mathcal{A} and \mathcal{B} have not been used in Theorem 5. This makes Theorem 5 a stronger result than analogous theorems for entanglement-I and entanglement-II. We show some examples in the Supplementary Information.

Discussion

For completeness, we report the other entanglement definitions that correspond to correlations between suitably defined subsystem operators¹⁷, generalising thus the Werner’s formulation to indistinguishable particles. These definitions are called mode-entanglement and SSR-entanglement. Mode-entanglement for N bosonic two-level particles is defined by the following.

Definition 4 (*Mode-entanglement*^{21–24}) The set of pure mode-separable states is, in first and second quantization respectively,

$$\text{SEP}_{\text{mode}} = \left\{ \mathbf{S}|0\rangle^{\otimes k} \otimes |1\rangle^{\otimes(N-k)} \right\}_{k=0,1,\dots,N} \tag{33}$$

$$\widetilde{\text{SEP}}_{\text{mode}} = \left\{ \frac{(a_0^\dagger)^k}{\sqrt{k!}} \frac{(a_1^\dagger)^{N-k}}{\sqrt{(N-k)!}} |\text{vac}\rangle \right\}_{k=0,1,\dots,N} \tag{34}$$

All other pure states are mode-entangled.

This definition can be generalised to fermions substituting the symmetrization projector \mathbf{S} with the antisymmetrization projector \mathbf{A} or considering anticommuting creation operators: see^{17,28,53,54} for a complete discussion of the fermionic case. If $N = 2$, Definition 4 reduces to $\text{SEP}_{\text{mode}} = \{|\Phi_0\rangle, |\Phi_1\rangle, |\Phi_2\rangle\}$. Mode-entanglement depends on the choice of the mode basis $\{a_0, a_1\}$ as it accounts for quantum correlations between modes in the second quantization formalism, and is ubiquitous in quantum optics and quantum field theories, and also applied in several atomic and condensed matter systems (see references in the review¹⁷). The factorisation condition (2) is fulfilled if and only if the state $|\Psi\rangle$ therein is mode-separable¹⁷ when \mathcal{A} consists of all functions of a_0 and a_0^\dagger and \mathcal{B} of all functions of a_1 and a_1^\dagger . Indeed, operators $A \in \mathcal{A}$, $B \in \mathcal{B}$ and AB do not generate mode-entanglement and are the local operators of this theory.

Non-vanishing SSR-entanglement requires that the single particle Hilbert space has at least four dimensions, and so we focus on the same system described for entanglement-III.

Definition 5 (*SSR-entanglement*^{18–20}) The set of pure SSR-separable states is, in first and second quantization respectively,

$$\text{SEP}_{\text{SSR}} = \left\{ \sum_{\sigma,\sigma'=0}^1 a_{\sigma,\sigma'} \mathbf{S}|L,\sigma\rangle \otimes |L,\sigma'\rangle + \mathbf{S} \sum_{\sigma=0}^1 b_\sigma |L,\sigma\rangle \otimes \sum_{\sigma=0}^1 c_\sigma |R,\sigma\rangle + \sum_{\sigma,\sigma'=0}^1 d_{\sigma,\sigma'} \mathbf{S}|R,\sigma\rangle \otimes |R,\sigma'\rangle \right\}_{a_{\sigma,\sigma'}, b_\sigma, c_\sigma, d_{\sigma,\sigma'} \in \mathbb{C}} \tag{35}$$

$$\widetilde{\text{SEP}}_{\text{SSR}} = \left\{ \sum_{\sigma,\sigma'=0}^1 a_{\sigma,\sigma'} a_{L,\sigma}^\dagger a_{L,\sigma'}^\dagger |\text{vac}\rangle + \sum_{\sigma=0}^1 b_\sigma a_{L,\sigma}^\dagger \sum_{\sigma=0}^1 c_\sigma a_{R,\sigma}^\dagger |\text{vac}\rangle + \sum_{\sigma,\sigma'=0}^1 d_{\sigma,\sigma'} a_{R,\sigma}^\dagger a_{R,\sigma'}^\dagger |\text{vac}\rangle \right\}_{a_{\sigma,\sigma'}, b_\sigma, c_\sigma, d_{\sigma,\sigma'} \in \mathbb{C}} \tag{36}$$

All other pure states are SSR-entangled.

SSR-entanglement depends on the choice of the spatial basis $\{|L\rangle, |R\rangle\}$, as entanglement-III does, and, recalling Eqs. (25) and (26), $P_{LR}\text{SEP}_{\text{SSR}} = P_{LR}\text{SEP}_{\text{III}}$. Moreover, SSR-separable states in $P_{LR}\text{SEP}_{\text{III}}$ are isomorphic to $|\sigma\rangle \otimes |\sigma'\rangle$ and indeed represent states of particles effectively distinguished by their spatial localisations. The name superselection rule entanglement is due to the fact that Definition 5 is derived within the supports of the projectors P_{LL}, P_{LR}, P_{RR} (and their many-particle generalisations) and ignoring their superpositions^{17–20}. The physical meaning of this superselection rule is that SSR-entanglement recovers standard entanglement when groups of particles can be effectively distinguished. SSR-entanglement is a constrained version of mode-entanglement since $|\Psi\rangle \in \text{SEP}_{\text{SSR}}$ if and only if $P_X|\Psi\rangle \in \text{SEP}_{\text{mode}}$, with $X \in \{LL, LR, RR\}$. Specializing the general argument in Ref.¹⁷ to the two-particle system described here, the compatibility with locality can be verified when \mathcal{A} is made of all functions of $a_{L,\sigma}$ and $a_{L,\sigma}^\dagger$ and \mathcal{B} of all functions of $a_{R,\sigma}$ and $a_{R,\sigma}^\dagger$, as for mode-entanglement but with the further constraints $P_X\mathcal{A}P_Y = P_X\mathcal{B}P_Y = 0$ for all $X \neq Y$, $X, Y \in \{LL, LR, RR\}$. Operators $A \in \mathcal{A}$, $B \in \mathcal{B}$ and AB do not generate SSR-entanglement and are the local operators of this theory.

We analysed different definitions of entanglement for indistinguishable particles in the light of locality. In particular, entanglement is a form of quantum correlations between subsystems, and therefore entanglement is fully specified only after identifying subsystems. In many of the existing approaches, subsystems are vaguely considered to be particles. We have looked for operators whose correlations correspond to each notion of entanglement. These operators, that define the subsystems by identifying their measurable quantities, form commuting subalgebras, where commutativity entails independence of the subsystems. Nevertheless, we stress that we used neither the commutativity nor the algebra structure in our main proofs. Therefore, our results are more general than what we need for the analysis of entanglement.

The results of our investigation is that three of the five existing entanglement definitions are incompatible with any locality notion formalised as above, because they do not correspond to correlations either between particles or between more general and abstract subsystems. Indeed, for any couple of non-trivial operators there are non-entangled pure states that show correlations. Therefore, these definitions do not generalise the Werner's formulation of entanglement⁴¹, i.e. the requirement (2) for pure separable states, to the domain of indistinguishable particles for any partitioning of the system. Their practical usefulness may be shown in the framework of different resource theories⁴⁰ that do not share some of their properties with entanglement theory. Thus, our results open the way to a deeper investigation of connections between indistinguishable particles entanglement and other resource theories.

Our results are relevant when particle indistinguishability cannot be neglected (see Fig. 1). When, on the other hand, particles can be distinguished by means of unambiguous properties, i.e. orthogonal states of certain degrees of freedom like different position eigenstates^{41,49–52}, then the standard theory of entanglement applies. Since entanglement of distinguishable particles is a resource for quantum technologies, our analysis shed light into the possibility to identify individually addressable subsystems when particle distinguishability cannot be implemented, as in miniaturised quantum devices with all degrees of freedom employed in the device functioning.

Methods

Lemma 1 *The states $|\Psi\rangle$ that solve Eq. (21) span the whole symmetrised Hilbert space $\mathbf{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.*

Proof If Eq. (21) is a tautology, its solutions correspond to all states $|\Psi\rangle \in \text{SEP}_I$ which form an overcomplete basis of the symmetrised Hilbert space $\mathbf{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Therefore the statement is proved.

If Eq. (21) is not a tautology, it has at least one and at most four solutions $c_1 = f(c_0)$ for each fixed c_0 and fixed A , from the fundamental theorem of algebra. Denote by $|\Psi(c_0)\rangle \in \text{SEP}_I$ the state corresponding to one of these solutions with $c_0 \neq 0$. Consider the rescaling $c'_0 = \lambda c_0$, and assume that the new state $|\Psi(c'_0)\rangle$ is proportional to $|\Psi(c_0)\rangle$. Therefore $|\Psi(c'_0)\rangle = \lambda^2 |\Psi(c_0)\rangle$ and $c'_1 = \lambda c_1$. We implicitly assumed that $c_1 = f(c_0)$ and $c'_1 = f(c'_0)$ are the same function, which is always possible because the polynomial P_A depends only on the matrix A that is not changed.

Equations $f(\lambda c_0) = f(c'_0) = c'_1 = \lambda c_1 = \lambda f(c_0)$ say that the function $c_1 = f(c_0)$ is homogeneous of degree one and therefore linear. This linear behaviour implies that $P_A(c_0, c_1) \propto c_0^4$, because P_A is homogeneous in c_0 and c_1 . Therefore $P_A = 0$ implies $c_0 = 0$, but $c_0 \neq 0$ by construction. Therefore, $|\Psi(c_0)\rangle$ and $|\Psi(c'_0)\rangle$ are linearly independent. We can iterate this argument with a second rescaling $c''_0 = \eta c'_0 = \eta \lambda c_0$ to find a third state linearly independent from the previous two. This proves the Theorem, since the symmetrised Hilbert space $\mathbf{S}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ is three-dimensional. \square

Note that the linear span of solutions $|\Psi\rangle$ of Eq. (21) is not the space of solutions, because P_A is not linear in $|\Psi\rangle$.

Lemma 2 *Given two rational functions $f(x)$ and $g(x)$ of several variables $x = (x_1, x_2, \dots, x_n)$ such that $f(x)g(x) = 0$ for all x in the domain of the two functions, then either $f(x) = 0$ or $g(x) = 0$ for all x .*

Proof Fix $x_j \geq 2$ so that the two functions $f(x) =: \tilde{f}_{x_2, \dots, x_n}(x_1)$ and $g(x) =: \tilde{g}_{x_2, \dots, x_n}(x_1)$ either are identically zero or have a finite number of zeros $x_1 = x_1(x_2, \dots, x_n)$. If both \tilde{f} and \tilde{g} have a finite number of zeros, after fixing $x_j \geq 2$, then there are infinitely many n -tuples x such that $f(x)g(x) \neq 0$, contradicting the hypothesis of the Lemma. \square

Lemma 3 *Given an operator A , if for any separable-III states $|\Psi_X\rangle \in P_X \cdot \text{SEP}_{III}$ and $|\Psi_Y\rangle \in P_Y \cdot \text{SEP}_{III}$*

$$\frac{\langle \Psi_X | A_{X,X} | \Psi_X \rangle}{\langle \Psi_X | \Psi_X \rangle} - \frac{\langle \Psi_Y | A_{Y,Y} | \Psi_Y \rangle}{\langle \Psi_Y | \Psi_Y \rangle} = 0, \quad (37)$$

then $A_X = \alpha P_X$ and $A_Y = \alpha P_Y$.

Proof Equation (37) implies that

$$\begin{aligned} \langle \Psi_X | A_{X,X} | \Psi_X \rangle &= \alpha \langle \Psi_X | \Psi_X \rangle, \\ \langle \Psi_Y | A_{Y,Y} | \Psi_Y \rangle &= \alpha \langle \Psi_Y | \Psi_Y \rangle. \end{aligned} \quad (38)$$

with some α independent of the supports of both P_X and P_Y . Without loss of generality, we now focus on the first of equations (38), and consider the three cases $X = LL, LR, RR$ with states $|\Psi_X\rangle$ represented as discussed before Theorem 5.

If $X = LL$, separable-III states $|\Psi_X\rangle$ can be represented as in Eq. (5). Since the state norm is $\langle\Psi_X|\Psi_X\rangle = |c_0|^4 + |c_1|^4 + |c_0|^2|c_1|^2$, and from the arbitrariness of the coefficients $c_{0,1}$, all monomials that appear only on the left-hand-side of Eq. (38), e.g. $\bar{c}_0^2 c_1^2$, must be multiplied by a vanishing coefficient. Moreover, the coefficients of the remaining monomials must match between the left and the right hand sides of Eq. (38). These requirements imply $A_{LL,LL} = \alpha P_{LL}$.

The case $X = LR$ is similar, with the difference that separable-III states $|\Psi_{LR}\rangle$ can be represented as $(c_0|0\rangle + c_1|1\rangle) \otimes (d_0|0\rangle + d_1|1\rangle)$ and $\langle\Psi_{LR}|\Psi_{LR}\rangle = (|c_0|^2 + |c_1|^2)(|d_0|^2 + |d_1|^2)$. Now, we must compare coefficients of monomials in $c_{0,1}$, $d_{0,1}$, and their complex conjugates. The result is $A_{LR,LR} = \alpha P_{LR}$.

The case $X = RR$ is straightforward because all states $|\Psi_{RR}\rangle = P_{RR} |\Psi_{RR}\rangle$ are separable-III. Therefore, Eq. (38) must be fulfilled for all bases of the support of P_{RR} that is possible only if $A_{RR,RR} = \alpha P_{RR}$. \square

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Author contributions

U.M. conceived and supervised the project. T.J.F.J. and U.M. performed the analytic work, discussed the results, and wrote the paper.

Competing interests

The authors declare no competing interests.

Additional information

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