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Research article

Bayesian analysis of meta-analytic models incorporating dependency: new approaches for the hierarchical Bayesian delta-splitting model

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ABSTRACT

Dependence between studies in meta-analysis is an assumption which is imposed on the structure of hierarchical Bayesian meta-analytic models. Dependence in meta-analysis can occur as a result of study reports using the same data or from the same authors. In this paper, the hierarchical Bayesian delta-splitting (HBDS) model (Steven and Taylor, 2009), which allows for dependence between studies and sub-studies by introducing dependency at the sampling and hierarchical levels, is developed using Bayesian approaches. Parameter estimation obtained from the joint posterior distributions of all parameters for the HBDS model was conducted using the Metropolis within Gibbs algorithm. The estimation of parameters for simulation studies using **R** code confirmed the consistency of the model parameters. These parameters were then tested successfully on studies to assess the effects of native-language vocabulary aids on second language reading as a case study.

1. Introduction

Meta-analytic models have been developed to incorporate heterogeneity within studies, between studies or between subgroups in order to obtain an overall conclusion (Kontopantelis and Reeves, 2012; Lunn et al., 2013; Newcombe et al., 2012; Böhning et al., 2014). Heterogeneity within and between studies due to differences in some aspects of the research, such as the statistical, methodological and clinical approaches, is a crucial issue which needs to be overcome when attempting to conduct a meta-analysis (Abrams et al., 2005; Dohoo et al., 2007). Frequentist and Bayesian statistical methods are the two techniques used to accommodate heterogeneity in meta-analysis (Lunn et al., 2013). The Frequentist framework is used when heterogeneity arises only based on the data, By contrast, Bayesian approaches consider the true underlying parameter value as a random variable when conducting meta-analysis.

An increasing variety of Bayesian approaches for estimating parameters have been developed in meta-analysis (Blackwood et al., 2012; Chen and Pei, 2009; Lunn et al., 2013; Robinson et al., 2009). In all three cases, a hierarchical Bayesian model was used because of its predictive nature. For example, meta-analysis using the Bayesian approach was employed by Chen and Pei (2009) to assess the effectiveness of a tumour necrosis factor (TNF) polymorphic marker in determining risk of hepatitis C virus (HCV) infection; providing a more definitive association between TNF polymorphism and the risk of HCV infection. Lunn et al. (2013) used the Metropolis-within-Gibbs algorithm to estimate the joint posterior distribution of all parameters for the hierarchical Bayesian model. A Markov chain was constructed to generate the parameters for the model using the formulation of the algorithm. The model was applied to data on the effect of diuretics on the risk of pre-eclampsia during pregnancy using the OpenBugs meta-analysis package.

The meta-analysis approaches that have been applied to gene expression studies provide examples in which dependency, originating both at the sampling level and at the hierarchical level, is accommodated. An example of this is found in the meta-analysis performed by Stevens and Nicholas (2009). In this meta-analysis, sampling dependence occurred since multiple measures of differential expression were produced for each gene using the same sample of data. At the hierarchical level, dependency occurred because some studies were conducted in the same laboratory or by the same research team. Gilbert-Norton et al. (2010) used the hierarchical Bayesian linear model to address some unresolvable questions about corridor effectiveness using meta-analysis. A corridor is defined as long, a narrow strip of land which helps in the movement of species between disconnected areas of their natural habitat. Gilbert-Norton et al. (2010) used a conservative hierarchical Bayesian

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model that accounted for sampling and hierarchical dependence to answer questions about the effectiveness of corridors in increasing movement of species, comparing their effectiveness for different species and investigating whether artificially created and maintained experimental corridors were more effective than naturally occurring ones. It was concluded that corridors existing in the landscape prior to the study had more movement than those artificially created for the study. The results suggested that, in general, corridors increase species movement between disconnected areas of habitat and that maintaining and creating corridors was worthwhile.

In this paper, we developed the hierarchical Bayesian delta-splitting (HBDS) model which allows dependency between studies of meta-analytic. The Metropolis within Gibbs algorithm is approach used to approximate the joint posterior distributions of all parameters of the model. Application of the model using the developed algorithm is given to the effects of native-language vocabulary aids on second language reading.

This paper is organised as follows. The HBDS model is introduced in Section 2. An approach used to formulate the joint posterior distribution of the model which was derived by the multiplication of the likelihood with the prior(s) is discussed in Section 3. The Metropolis within Gibbs algorithm which was developed to estimate the parameters of interest for the HBDS model is also given in Section 3. A simulation study for this model was conducted in which the dependence assumptions were imposed on the variance-covariance matrix. This simulation study is discussed in Section 4. The data obtained from the simulation study that was used to determine and evaluate the performance of known parameters for the HBDS model followed by application of the model is provided in Section 4.

2. The hierarchical Bayesian delta splitting (HBDS) model

Following Dumouchel and Normand (2000) and Dumouchel and Harris (1983), Stevens (2005) extended the hierarchical Bayesian approach to the non-independent case. This model can be used to obtain overall conclusions from a meta-analysis of several studies in which a dependence structure occurs due to the use of the same data at the sampling level and the same laboratory at the hierarchical level.

The Hierarchical Bayesian Linear Model (HBLM) framework incorporating the *l*-dependence group (Stevens and Taylor, 2009) is summarized as follows:

$$\begin{aligned}
&\tilde{\theta} = X\beta + \delta + \varepsilon \\
&\quad = \theta' + \varepsilon \\
&\delta \sim N\left(0, \tau^2 I\right) \\
&\varepsilon \sim N(0, V) \\
&\theta' | \beta, \tau \sim N\left(X\beta, \tau^2 I\right) \\
&\beta | \tau \sim N(b, D) \\
&D = \operatorname{diag}\left(d_1^2, \cdots d_p^2\right) \\
&\tau^2 \sim \text{Inverse Gamma}(q, r)
\end{aligned} \tag{1}$$

where $\tilde{\theta}_{n\text{x}1}$ is a vector of effect size estimates, $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_n)^T$,(*n* is the number of studies; $\theta'_{n\text{x}1}$ is the vector of the underlying effect sizes being estimated in each study, $\theta' = (\theta'_1, \theta'_2, \ldots, \theta'_n)^T$; *X*is the *nxp* design matrix representing known (covariate) differences between studies, $X = \begin{bmatrix} x_{1,1} & \ldots & x_{1,p} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \ldots & x_{n,p} \end{bmatrix}_{nxp}$; $\beta_{p\text{x}1}$is a vector of parameters representing the effects of the different covariates (or unknown parameters to be estimated), $\beta = (\beta_0, \beta_1, \ldots, \beta_{p-1})^T$(*p* is the number of different covariates); $\delta_{n\text{x}1}$ is the vector of random deviation of $X\beta$ from θ', $\theta' = X\beta + \delta$, $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^T$;and $\varepsilon_{n\text{x}1}$ is the vector of sampling errors for each study, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T$.

Stevens (2005) stated that dependence can occur at the level of groups of studies (or substudies). For example, in the case study performed by Joyce (1997), dependence between substudies occurred since the English test was given to the same group of French students in the first second and third semesters. In these cases it is not appropriate to assume that $\delta \sim N(0, \tau^2 I)$ is independent because of dependency between groups of substudies. Following Stevens (2005), this assumption for the dependence group can be written as follows $\delta \sim N(0, \Delta)$ where Δ is as defined in Eq. (2). This hierarchical dependence structure can be accommodated by essentially splitting the δ, into two components: a study (or experiment or researcher) component and a substudy-within-study component. For this reason, this approach may be referred to as "delta-splitting".

Equivalently, the variance-covariance matrix of the vector δ can be given a block diagonal structure Δ instead of the previous diagonal structure $\tau^2 I$, thus

$$\Delta = \begin{pmatrix} \begin{pmatrix} \tau^2 & \ddots & \varphi \\ \varphi & & \tau^2 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \begin{pmatrix} \tau^2 & \ddots & \varphi \\ \varphi & & \tau^2 \end{pmatrix} & \cdots & 0 \\ \cdots & 0 & \ddots & \cdots \\ 0 & 0 & \cdots & \begin{pmatrix} \tau^2 & \ddots & \varphi \\ \varphi & & \tau^2 \end{pmatrix} \end{pmatrix} \tag{2}$$

The blocks on the diagonal correspond to separate dependency groups, and it is possible that each block may represent several dependent studies. As before, τ^2 represents the level of variability between what each of the studies are measuring (θ'). φ represents the covariance between studies within dependency groups. Let Δ denote the resulting block-diagonal variance-covariance matrix for vector δ, with φ representing the hierarchical covariance between related sub-studies within the same study (or studies by the same researcher). Then the delta-splitting meta-analysis linear model assumes

$$\begin{aligned}
&\tilde{\theta} \sim N(X\beta, \psi), \\
&\psi = V + \Delta I + \varphi M,
\end{aligned} \tag{3}$$

where M is an appropriate 0–1 matrix (with 1s corresponding to the nonzero off-diagonal values in the diagonal blocks of Δ_i).

Stevens and Taylor (2009) developed the meta-analytic model to accommodate the variation in studies. This model was based on the modification of the HBLM to incorporate the covariance delta-splitting framework. The dependence structure was imposed on the model using the variance-covariance matrix. The model is namely the hierarchical Bayesian delta-splitting (HBDS) model and presented in Eq. (4).

The hierarchical Bayesian meta-analysis approach with a delta-splitting framework (Stevens and Taylor, 2009) can be summarized as follows:

$$\begin{aligned}
&\tilde{\theta} = X\beta + \delta + \varepsilon \\
&\quad = \theta' + \varepsilon \\
&\delta \sim N(0, \Delta) \\
&\Delta = \Delta(\tau, \varphi) = \tau^2 I + \varphi M \\
&\varepsilon \sim N(0, V) \\
&\theta' | \beta, \varphi, \tau \sim N(X\beta, \Delta) \\
&\beta | \varphi, \tau \sim N(b, D) \\
&D = \operatorname{diag}\left(d_1^2, \cdots d_p^2\right) \\
&\varphi | \tau \sim P(\varphi | \tau) \\
&\tau \sim P(\tau)
\end{aligned} \tag{4}$$

where $\tilde{\theta}_{n\text{x}1}$ is a vector of effect size estimates, $\tilde{\theta} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_n)^T$ (*n* is the number of studies); $\theta'_{n\text{x}1}$ is the vector of the underlying effect size being

estimated in each study, $\theta^{'} = (\theta_1^{'}, \theta_2^{'}, \ldots, \theta_n^{'})^T$; X is the *nxp* design matrix which represents known (covariate) differences between studies, $X = \begin{bmatrix} x_{1,1} & \ldots & x_{1,p} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \ldots & x_{n,p} \end{bmatrix}_{nxp}$; β_{px1} is a vector of parameters which represent the effects of the difference covariates (or unknown parameters to be estimated), $\beta = (\beta_0, \beta_1, \ldots, \beta_{p-1})^T$ (p is the number of different covariates); δ_{nx1} is the vector representing the random deviation of $X\beta$,$\theta^{'} = X\beta + \delta$, $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^T$; ε_{nx1} is the vector representing the sampling error within each study, $\varepsilon = (\delta_1, \delta_2, \ldots, \delta_n)^T$; Δ is a block-diagonal matrix of the form given in Eq. (2); I_{nxn} is the n x n identity matrix and M_{nxn} is a matrix with 0s on the diagonal and 1s in those off-diagonal entries which represent the correlation between studies having the same author. Note that $\Delta = \tau^2 I + \varphi M$.

3. Bayesian analysis

Formulation of the joint posterior distributions of all parameters for the model is derived in this section.

3.1. Posterior analysis of HBDS model using Metropolis within Gibbs

This section presents the hierarchical Bayesian delta-splitting (HBDS) model which can be used to obtain overall conclusions in meta-analysis. Heterogeneity between studies which are *dependent* as a result of the sharing data and authors or laboratories can be accommodated by this model. This model was developed by Stevens and Taylor (2009) by essentially splitting the δ in Eq. (1) into two components: a study (or experiment or researcher) component and a substudy-within-study component. The variance-covariance (Δ) component in Eq. (3) was split into two parameters to overcome the dependence structure in the meta-analysis.

Recall that the HBDS model in (4) can be expressed as follows:

$$\begin{aligned} &\tilde{\boldsymbol{\theta}} \sim MVN(\boldsymbol{\theta}^{'}, V) \\ &\varepsilon_{(l)} \sim MVN(0, V) \\ &\boldsymbol{\theta}^{'} \sim MVN(X\boldsymbol{\beta}, \tau^2 I + \varphi M) \\ &\delta \sim MVN(0, \tau^2 I + \varphi M) \\ &\boldsymbol{\beta} \sim MVN(b, D) \\ &\varphi|\tau \sim \text{Unif}\left(-\frac{\tau^2}{(K-1)}, \tau^2 \right) \\ &\tau \sim \log - \text{logistic}(c_0, 1) \end{aligned} \tag{5}$$

where $\tilde{\boldsymbol{\theta}}_{nx1}$ is the vector of effect size estimates given by $\tilde{\boldsymbol{\theta}} = (\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_n)^T$ (n is the number of studies). Let $\boldsymbol{\theta}^{'}_{nx1}$ be the vector of the underlying effect size whose components are estimated separately in each study ($i = 1, 2, \ldots, n$) given by $\boldsymbol{\theta}^{'} = (\theta_1^{'}, \theta_2^{'}, \ldots, \theta_n^{'})^T$. Let X be the *nxp* design matrix which represents known (covariate) differences between the studies:

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,p} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,p} \end{bmatrix}_{nxp} \tag{6}$$

Let $\boldsymbol{\beta}_{px1}$ be a vector of parameters which represent the effects of the different covariates (or unknown parameters to be estimated),

$$\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{p-1})^T, \tag{7}$$

where p is the number of different covariates. Let δ_{nx1} be the vector of random deviation of $X\boldsymbol{\beta}$ from $\boldsymbol{\theta}^{'}$, $\boldsymbol{\theta}^{'} = X\boldsymbol{\beta} + \delta$, and $\delta = (\delta_1, \delta_2, \ldots, \delta_n)^T$. Now let ε_{nx1} be the vector of sampling error within each study, $\varepsilon = (\delta_1, \delta_2, \ldots, \delta_n)^T$. The matrix $\tau^2 I + \varphi M$ is a block-diagonal matrix. Here I_{nxn} is the identity matrix and M_{nxn} is 0–1 matrix (with 1s corresponding to the nonzero off-diagonal entries in the diagonal block of $\tau^2 I + \varphi M$) and K is the size of the largest block on the diagonal of matrix $\tau^2 I + \varphi M$.

The log-logistic distribution used for $P(\tau)$ in Eq. (8) is defined by

$$P(\tau) = \frac{c_0}{(c_0 + \tau)^2}; \qquad \tau > 0 \tag{8}$$

This is a special case that belongs to an extended two-parameter family of prior distributions, a so-called location-scale family based on $\log(\tau)$.

For $\tau > 0$ and $\gamma > 0$, let

$$P(\tau; \delta, \gamma) = \frac{\gamma \delta \tau^{(\gamma-1)}}{(\delta + \tau^{\gamma})^2}, \tag{9}$$

where δ and γ are the median and shape of τ, respectively. The default prior distribution corresponds to choosing $\delta = c_0$ and $\gamma = 1$, such that Eq. (9) is equal to Eq. (8). This particular selection which was suggested by Dumouchel and Normand (2000) offers several advantages.

Firstly, the prior has a maximum at 0 and is a decreasing function of τ. This conforms to the belief that it is definitely possible for τ to be near 0. The second advantage is that the quartiles of the distribution of $P(\tau)$ are $c_0/3$, c_0 and $3c_0$. As a result, the distribution is automatically scaled to be in a sensible range and in the correct units. Moreover, this prior distribution is right-skewed, with the given quartiles and is highly dispersed (with infinite expected values for both τ and τ^{-1}). This is consistent with the fact that τ can be close to zero, but that τ is allowed to vary substantially from zero when the sampling variances are larger (Stevens and Taylor, 2009).

The analytical form of the joint posterior distribution of all parameters for the HBDS model was derived by multiplying the likelihood with prior(s). The Metropolis within Gibbs algorithm (Hoff, 2009; Millar and Meyer, 2000) was used to approximate the parameters in the model. This algorithm was selected since the conditional posterior distribution of φ given $\boldsymbol{\theta}^{'}, \boldsymbol{\beta}$ and τ, and the conditional posterior distribution of τ given $\boldsymbol{\theta}^{'}, \boldsymbol{\beta}$ and φ were not in standard form. The conditional posterior distributions of $\boldsymbol{\theta}^{'}$, given $\boldsymbol{\beta}$, φ and τ^2 and the conditional posterior distribution of $\boldsymbol{\beta}$, given $\boldsymbol{\theta}^{'}$, φ and τ^2 of the model were estimated using the Gibbs sampler algorithm.

3.1.1. Posterior Analysis

The joint posterior distribution of all parameters for the model is

$$\begin{aligned} P(\boldsymbol{\theta}^{'}, \boldsymbol{\beta}, \varphi, \tau|\tilde{\boldsymbol{\theta}}) &\propto P(\tilde{\boldsymbol{\theta}}|\boldsymbol{\theta}^{'}, \boldsymbol{\beta}, \varphi, \tau) \times P(\boldsymbol{\theta}^{'}, \boldsymbol{\beta}, \varphi, \tau) \\ &= P(\tilde{\boldsymbol{\theta}}|\boldsymbol{\theta}^{'}, \boldsymbol{\beta}, \varphi, \tau) \times P(\boldsymbol{\theta}^{'}|\boldsymbol{\beta}, \varphi, \tau) \times P(\boldsymbol{\beta}|\varphi, \tau) \times P(\varphi|\tau) \times P(\tau), \end{aligned} \tag{10}$$

where $P(\tilde{\boldsymbol{\theta}}|\boldsymbol{\theta}^{'}, \boldsymbol{\beta}, \varphi, \tau)$ is the joint likelihood, $P(\boldsymbol{\theta}^{'}|\boldsymbol{\beta}, \varphi, \tau)$ is the conditional prior of $\boldsymbol{\theta}^{'}$ given $\boldsymbol{\beta}, \varphi$ and τ, $P(\boldsymbol{\beta}|\varphi, \tau)$ is the conditional prior of $\boldsymbol{\beta}$ given φ and τ, $P(\varphi|\tau)$ is a conditional prior of φ given τ, and $P(\tau)$ is prior distribution of τ. Recall from Eq. (5) that the joint likelihood and priors of the HBDS model can be expressed as follows.

The joint likelihood is:

$$P(\tilde{\boldsymbol{\theta}}|\boldsymbol{\theta}^{'}, \boldsymbol{\beta}, \varphi, \tau) = (2\pi)^{-\frac{n}{2}} |V|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}^{'})^T V^{-1} (\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}^{'}) \right\} \tag{11}$$

where

$$\tilde{\boldsymbol{\theta}} = [\tilde{\theta}_1 \quad \tilde{\theta}_2 \quad \tilde{\theta}_3 \quad \tilde{\theta}_4 \quad \tilde{\theta}_5 \quad \ldots \quad \tilde{\theta}_n]^T,$$

$$\boldsymbol{\theta}^{'} = [\theta_1^{'} \quad \theta_2^{'} \quad \theta_3^{'} \quad \theta_4^{'} \quad \theta_5^{'} \quad \ldots \quad \theta_n^{'}]^T,$$

and the dependency is encoded by the variance-covariance matrix. An example of the form of the variance-covariance matrix is given below.

$$V=\begin{bmatrix} Var(V_1) & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & Var(V_2) & Cov(V_2,V_3) & 0 & 0 & \dots & 0 \\ 0 & Cov(V_3,V_2) & Var(V_3) & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & Var(V_4) & 0 & \dots & 0 \\ \vdots & 0 & 0 & 0 & Var(V_5) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & Var(V_n) \end{bmatrix}.$$

In the following, the forms of the conditional prior distributions of each parameter for the HBDS model are discussed.

3.1.2. Conditional prior distribution of $\theta^{'}$ given β, φ and τ

The derivation of this prior begins with the variance-covariance matrix $\tau^2 I + \varphi M$ where the matrix M has a similar form to the one shown below:

$$M=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{nxn}.$$

Using Eq. (5), the conditional prior probability density function of $\boldsymbol{\theta}^{'}$, given $\boldsymbol{\beta}, \varphi$ and, is written as

$$P(\boldsymbol{\theta}^{'}|\boldsymbol{\beta},\varphi,\tau)=(2\pi)^{-\frac{n}{2}}\left|\tau^2 I+\varphi M\right|^{-\frac{1}{2}}\ \exp\Bigg\{ -\frac{1}{2}(\boldsymbol{\theta}^{'}-X\boldsymbol{\beta})^T\left(\tau^2 I+\varphi M\right)^{-1}(\boldsymbol{\theta}^{'}-X\boldsymbol{\beta})\Bigg\}, \tag{12}$$

where X and $\boldsymbol{\beta}$ are as given in Eqs. (6) and (7), respectively.

3.1.3. Conditional prior distribution of β

Using Eq. (5), the conditional prior probability density function of $\boldsymbol{\beta}$, independent of φ and τ, can be written as

$$P(\boldsymbol{\beta})=(2\pi)^{-\frac{p}{2}}|D|^{-\frac{1}{2}}\exp\Bigg\{ -\frac{1}{2}(\boldsymbol{\beta}-b)^T D^{-1}(\boldsymbol{\beta}-b)\Bigg\}, \tag{13}$$

where

$$D=\begin{bmatrix} d_1^2 & 0 & \dots & 0 \\ 0 & d_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_p^2 \end{bmatrix}_{pxp},$$

and $d_1, \dots, d_p$ are arbitrary real numbers and b is a vector of arbitrary real numbers of size px1.

3.1.4. Conditional prior distribution of φ given τ

The next step is the selection of an appropriate prior distribution for φ, given τ. This distribution is conditional on τ because of the numerical constraint required to ensure positive definiteness of $\tau^2 I + \varphi M$ (Stevens and Taylor, 2009). Using Eq. (5), the conditional prior distribution of φ, given τ, is a uniform distribution of the form

$$P(\varphi|\tau)=\left\{\left[1+\frac{1}{(K-1)}\right](\tau)^2\right\}^{-1}. \tag{14}$$

3.1.5. Prior distribution of τ

The prior distribution of τ is given in Eq. (8).

Using Eqs. (11), (12), (13), (14), and (8), the joint posterior distribution of all parameters for the HBDS model is given by

$$\begin{aligned} P(\boldsymbol{\theta}^{'},\boldsymbol{\beta},\varphi,\tau|\tilde{\theta}) &= (2\pi)^{-\frac{n}{2}}|V|^{-\frac{1}{2}}\exp\Bigg\{ -\frac{1}{2}(\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}^{'})^T V^{-1}(\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}^{'})\Bigg\} \\ &\times (2\pi)^{-\frac{n}{2}}\left|\tau^2 I+\varphi M\right|^{-\frac{1}{2}}\exp\Bigg\{ -\frac{1}{2}(\boldsymbol{\theta}^{'}-X\boldsymbol{\beta})^T\left(\tau^2 I+\varphi M\right)^{-1}(\boldsymbol{\theta}^{'}-X\boldsymbol{\beta})\Bigg\} \\ &\times (2\pi)^{-\frac{p}{2}}|D|^{-\frac{1}{2}}\exp\Bigg\{ -\frac{1}{2}(\boldsymbol{\beta}-b)^T D^{-1}(\boldsymbol{\beta}-b)\Bigg\} \\ &\times \left\{\left[1+\frac{1}{(K-1)}\right](\tau)^2\right\}^{-1}\times\frac{c_0}{(c_0+\tau)^2}. \end{aligned} \tag{15}$$

The following describes the derivation of the conditional posterior distribution of $\theta^{'}$, given β, φ and τ and the conditional posterior distribution of β given $\theta^{'}, \varphi$ and τ , from Eq. (15).

3.1.6. Conditional posterior distribution of $\theta^{'}$ given β, φ and τ

Using Eq. (15) as detailed in Appendix, the conditional posterior distribution of $\boldsymbol{\theta}^{'}|\boldsymbol{\beta},\varphi,\tau$ is the multivariate normal distribution with mean $\mu_{\theta^{'}}$ and variance-covariance matrix, $\Lambda_{\theta^{'}}$ as given in equations (28) and (27) (see Appendix). Furthermore, the distribution $\boldsymbol{\theta}^{'}|\boldsymbol{\beta},\varphi,\tau$ can be rewritten in the form $\theta_1^{'},\theta_2^{'},\dots,\theta_n^{'}|\boldsymbol{\beta},\varphi,\tau \sim N_n(\mu_{\theta^{'}},\Lambda_{\theta^{'}})$ where n is the number of studies.

The conditional posterior distribution of

$$\theta_i^{'}|\theta_{(-i)}^{'},\boldsymbol{\beta},\varphi,\tau \tag{16}$$

where $\theta_{(-i)}^{'} = (\theta_1^{'}, \theta_2^{'}, \dots, \theta_{i-1}^{'}, \theta_{i+1}^{'}, \dots\theta_{n-1}^{'}, \theta_n^{'})$, the vector parameter excluding $\theta_i^{'}$, can be derived using Theorem 3.31 of Flury (1997). The parameter of $\boldsymbol{\theta}^{'}$ which is normally distributed, $\boldsymbol{\theta}^{'} \sim N_n(\mu_{\theta^{'}},\Lambda_{\theta^{'}})$, is partitioned into $m = 1$ and (n-m) components, $\boldsymbol{\theta}_i^{'}$ and $\boldsymbol{\theta}_{(-i)}^{'}$, where the size of these matrices are 1×1 and (n-1) x 1, respectively. The parameter of $\boldsymbol{\mu}$ is partitioned into matrices $(\mu_{\theta^{'}})_i$ and $(\mu_{\theta^{'}})_{-i}$ with the size 1×1 and (n-1) x 1, respectively. The variance-covariance is formed by $\Lambda_{\theta^{'}} = \begin{pmatrix} (\Lambda_{\theta^{'}})_{i,i} & (\Lambda_{\theta^{'}})_{i,-i} \\ (\Lambda_{\theta^{'}})_{-i,i} & (\Lambda_{\theta^{'}})_{-i,-i} \end{pmatrix}_{nxn}$.

As in (Flury, 1997, Theorem 3.31), assumed that $(\Lambda_{\theta^{'}})_{i,i}$ is positive definite, it then follows that the conditional posterior distribution of $\boldsymbol{\theta}_i^{'}$, given $\boldsymbol{\theta}_{(-i)}^{'}$ is a variate normal distribution with the parameters

$$\mu_{\boldsymbol{\theta}_i^{'}|\boldsymbol{\theta}_{(-i)}^{'}} = E\left[\boldsymbol{\theta}_i^{'}|\boldsymbol{\theta}_{(-i)}^{'}\right] = (\mu_{\theta^{'}})_i + (\Lambda_{\theta^{'}})_{i,-i}(\Lambda_{\theta^{'}})_{-i,-i}^{-1}(\boldsymbol{\theta}_{(-i)}^{'} - (\mu_{\theta^{'}})_{-i}) \tag{17}$$

and

$$\Lambda_{\boldsymbol{\theta}_i^{'}|\boldsymbol{\theta}_{(-i)}^{'}} = \mathrm{Cov}\left[\boldsymbol{\theta}_i^{'}|\boldsymbol{\theta}_{(-i)}^{'}\right] = (\Lambda_{\theta^{'}})_{i,i} - (\Lambda_{\theta^{'}})_{i,-i}(\Lambda_{\theta^{'}})_{-i,-i}^{-1}(\Lambda_{\theta^{'}})_{-i,i}. \tag{18}$$

In summary, the conditional posterior distribution given in (16) is the normal distribution with mean and variance as shown in Eqs. (17) and (18), respectively.

3.1.7. Conditional posterior distribution of β given $\theta^{'}, \varphi$ and τ

Using Eq. (15), the conditional posterior distribution of $\boldsymbol{\beta}$ given $\boldsymbol{\theta}^{'}$, φ and τ is derived by considering $\boldsymbol{\beta}$ to be a random variable and $\boldsymbol{\theta}^{'}$, φ, τ to be constants as detailed in Appendix. In summary, the conditional posterior distribution of $\boldsymbol{\beta}|\boldsymbol{\theta}^{'},\varphi,\ \tau$ is the multivariate normal distribution with mean $\mu_\beta$and corresponding variance-covariance matrix, Λ_β as given in equations (33) and (32), respectively (see Appendix). Furthermore, the distribution $\boldsymbol{\beta}|\boldsymbol{\theta}^{'},\varphi,\tau$ may now be rewritten in the form $\beta_0,\beta_1,\dots,\beta_{p-2},\beta_{p-1}|\theta_1^{'},\theta_2^{'},\dots,\theta_n^{'},\varphi,\tau \sim N_p(\mu_\beta,\Lambda_\beta)$ where p is the number of covariates.

The conditional posterior distribution of

$$\beta_k|\theta_1^{'},\dots,\theta_n^{'},\beta_{(-k)},\varphi,\tau \tag{19}$$

where $\beta_{(-k)} = (\beta_0, \beta_1, \dots, \beta_{k-1}, \beta_{k+1}, \dots, \beta_{p-1})$, the vector parameter excluding β_k, can be derived using Theorem 3.31 of Flury (1997).

The parameter of $\boldsymbol{\beta}$ which is normally distributed, $\boldsymbol{\beta} \sim N_p(\mu_\beta, \Lambda_\beta)$, is partitioned into $q = 1$ and (r-q) components, $\boldsymbol{\beta}_k$ and $\boldsymbol{\beta}_{(-k)}$, where the size of these matrices are 1×1 and (r-1) x 1, respectively. The parameter of $\boldsymbol{\mu}$ is partitioned into matrices $(\mu_\beta)_k$ and $(\mu_\beta)_{-k}$ with the size 1×1 and (r-1) x 1, respectively. The variance-covariance is formed by $\boldsymbol{\Lambda_\beta} = \begin{pmatrix} (\Lambda_\beta)_{k,k} & (\Lambda_\beta)_{k,-k} \\ (\Lambda_\beta)_{-k,k} & (\Lambda_\beta)_{-k,-k} \end{pmatrix}_{pxp}$.

As in (Flury, 1997, Theorem 3.31), assumed that $(\Lambda_\beta)_{k,k}$ is positive definite, it then follows that the conditional posterior distribution of $\boldsymbol{\beta}_k$, given $\boldsymbol{\beta}_{(-k)}$ is a variate normal distribution with the parameters

$$\mu_{\beta_k|\beta_{(-k)}} = E\left[\boldsymbol{\beta}_k|\boldsymbol{\beta}_{(-k)}\right] = (\mu_\beta)_k + (\Lambda_\beta)_{k,-k}(\Lambda_\beta)^{-1}_{-k,-k}(\boldsymbol{\beta}_{(-k)} - (\mu_\beta)_{-k}) \tag{20}$$

and

$$\Lambda_{\beta_k|\beta_{(-k)}} = \mathrm{Cov}\left[\boldsymbol{\beta}_k|\boldsymbol{\beta}_{(-k)}\right] = (\Lambda_\beta)_{k,k} - (\Lambda_\beta)_{k,-k}(\Lambda_\beta)^{-1}_{-k,-k}(\Lambda_\beta)_{-k,k}. \tag{21}$$

In summary, the conditional posterior distribution given in (19) is the normal distribution with mean and variance as shown in Eqs. (20) and (21), respectively.

Using Eq. (15), the conditional posterior distribution of φ given $\boldsymbol{\theta}^{'}, \boldsymbol{\beta}$ and τ was formed from the product of the prior distributions $P(\theta^{'}|\beta, \varphi, \tau)$ and $P(\varphi|\tau)$.

$$f(\varphi|\boldsymbol{\theta}^{'}, \boldsymbol{\beta}, \tau) \propto (2\pi)^{-\frac{n}{2}}\left|\tau^2 I + \varphi M\right|^{-\frac{1}{2}} \times \exp\left\{-\frac{1}{2}(\boldsymbol{\theta}^{'} - X\boldsymbol{\beta})^T\left(\tau^2 I + \varphi M\right)^{-1}(\boldsymbol{\theta}^{'} - X\boldsymbol{\beta})\right\} \times \left\{\left[1 + \frac{1}{(K-1)}\right](\tau)^2\right\}^{-1}. \tag{22}$$

Unfortunately, in the multiplication of priors in Eq. (22) was in non-standard form. To overcome this issue, the Metropolis-Hasting algorithm was used to generate the parameter estimates from $\varphi|\boldsymbol{\theta}^{'}$, $\boldsymbol{\beta}$, τ. This algorithm also used to generate the posterior distribution of τ given $\boldsymbol{\theta}^{'}$, $\boldsymbol{\beta}$ and φ given in Eq. (23) as the product of the prior distributions $P(\theta^{'}|\beta, \varphi, \tau), P(\varphi|\tau)$ and $P(\tau)$ was in non-standard form.

$$f(\tau|\boldsymbol{\theta}^{'}, \boldsymbol{\beta}, \varphi) = (2\pi)^{-\frac{n}{2}}\left|\tau^2 I + \varphi M\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{\theta}^{'} - X\boldsymbol{\beta})^T\left(\tau^2 I + \varphi M\right)^{-1}(\boldsymbol{\theta}^{'} - X\boldsymbol{\beta})\right\} \times \left\{\left(1 + \frac{1}{(K-1)}\right)(\tau)^2\right\}^{-1} \times \frac{c_0}{(c_0 + \tau)^2}. \tag{23}$$

In the following section, an approximation to the joint posterior distribution of all parameters for the HBDS model is obtained using the Metropolis within Gibbs algorithm.

3.2. Metropolis within gibbs algorithm for HBDS model

As the conditional posterior distributions for τ, given $\boldsymbol{\theta}^{'}, \boldsymbol{\beta}$ and φ is in non-standard form, the Metropolis within Gibbs provides an alternative MCMC algorithm to estimate the joint posterior distribution of the HBDS model. The steps that were followed to complete this process are listed below.

3.2.1. The algorithm

1. Let $\boldsymbol{\theta}^{'(0)}, \boldsymbol{\beta}^{(0)}, \varphi^{(0)}$ and $\tau^{2(0)}$ denote the starting point of a Markov chains. The value of these starting points can be randomly drawn from a starting distribution or simply chosen deterministically.

Let $j = 1, 2, \ldots, t$, where t is the number of iterations, $i = 1, 2, \ldots, n$, n is the number of studies and $k = 0, 1, \ldots, p$-1, for p is the number of covariates.

2. $\theta_i^{'(j)}$ given $\theta_{(-i)}^{'(j-1)}, \beta_0^{(j-1)}, \ldots, \beta_{p-1}^{(j-1)}, \varphi$ and $\tau^{(j-1)}$, was generated using

$$\theta_i^{'(j)}\left|\theta_{(-i)}^{'(j-1)}, \beta_0^{(j-1)}, \ldots, \beta_{p-1}^{(j-1)}, \varphi, \tau^{(j-1)} \sim N_n\left(\mu_{\theta_i^{'}}, \Lambda_{\theta_i^{'}}\right)\right.$$

where $\mu_{\theta_i^{'}}$ and $\Lambda_{\theta_i^{'}}$ are defined in Eqs. (17) and (18), respectively, and μ_θ and $\Lambda_{\theta^{'}}$ are defined in equation (28) and (27) (see Appendix).

3. $\beta_k^{(j)}$ given $\theta_1^{'(j)}, \ldots, \theta_n^{'(j)}, \beta_{(-k)}^{(j-1)}, \varphi$ and $\tau^{(j-1)}$ was generated:

$$\beta_k^{(j)}\left|\theta_1^{'(j)}, \ldots, \theta_n^{'(j)}, \beta_{(-k)}^{(j-1)}, \varphi, \tau^{(j-1)} \sim N_n(\mu_{\beta_k}, \Lambda_{\beta_k})\right.$$

where μ_{β_k} and Λ_{β_k} are defined in Eqs. (20) and (21), respectively and μ_β and Λ_β are defined in equation (33) and (32) (see Appendix).

Steps 2 and 3 used the Gibbs sampler algorithm. In order to complete the following steps, the Metropolis-Hasting algorithm was used to generate the parameters.

4. $\varphi^{(j)}$ given $\boldsymbol{\theta}^{'(j)}, \boldsymbol{\beta}^{(j)}$ and $\tau^{(j-1)}$ was generated by implementing the following steps:

   a. It was proposed that $\varphi^* \sim \mathrm{N}(\varphi^{(j-1)}, \omega_1^2)$.
   b. The acceptance ratio (probability) for the parameter φ was as follows.

$$r_\varphi = \frac{P(\boldsymbol{\theta}^{'(j)}, \boldsymbol{\beta}^{(j)}, \varphi^*, \tau^{(j-1)}|\tilde{\boldsymbol{\theta}})}{P(\boldsymbol{\theta}^{'(j)}, \boldsymbol{\beta}^{(j)}, \varphi^{(j-1)}, \tau^{(j-1)}|\tilde{\boldsymbol{\theta}})} \times \frac{J_\varphi(\varphi^{(j-1)})}{J_\varphi(\varphi^*)} \propto \frac{\left|\tau^{(j-1)} I + \varphi^* M\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\theta^{'(j)} - X\beta^{(j)})^T\left(\tau^{(j-1)} I + \varphi^* M\right)^{-1}(\theta^{'(j)} - X\beta^{(j)})\right\}}{\left|\tau^{(j-1)} I + \varphi^{(j-1)} M\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\theta^{'(j)} - X\beta^{(j)})^T\left(\tau^{(j-1)} I + \varphi^{(j-1)} M\right)^{-1}(\theta^{'(j)} - X\beta^{(j)})\right\}} \times \frac{1}{exp\left[-\frac{(\varphi^* - \varphi^{(j-1)})^2}{2\omega_1^2}\right]}.$$

   c. The parameter U was sampled from $U \sim$Uniform(0,1).
   If $r_\varphi > U$, then $\varphi^{(j)} = \varphi^*$ otherwise $\varphi^{(j)} = \varphi^{(j-1)}$.

5. $\tau^{(j)}$ given $\boldsymbol{\theta}^{'(j)}, \boldsymbol{\beta}^{(j)}$ and $\varphi^{(j)}$ was generated by implementing the following steps:

   a. It was proposed that $\tau^* \sim$ Gamma $(\delta + \tau^{(j-1)}, \omega + \tau^{(j-1)})$.
   The proposal distribution for τ^* was:

$$J_\tau(\tau^*) = \frac{1}{(\omega + \tau^{(j-1)})^{(\delta + \tau^{(j-1)})}\Gamma(\delta + \tau^{(j-1)})}(\tau^*)^{(\delta + \tau^{(j-1)} - 1)} exp\left(-\frac{\tau^*}{\omega + \tau^{(j-1)}}\right), \tau^* \in (0, \infty).$$

where $\delta + \tau^{(j-1)} > 0$ and $\omega + \tau^{(j-1)}$ are shape and scale, respectively.
   b. The acceptance ratio (probability) for the parameter τ was as follows.

$$r_\tau = \frac{P(\boldsymbol{\theta}^{'(j)}, \boldsymbol{\beta}^{(j)}, \varphi^{(j)}, \tau^*|\tilde{\boldsymbol{\theta}})}{P(\boldsymbol{\theta}^{'(j)}, \boldsymbol{\beta}^{(j)}, \varphi^{(j)}, \tau^{(j-1)}|\tilde{\boldsymbol{\theta}})} \mathrm{x} \frac{J_\tau(\tau^{(j-1)})}{J_\tau(\tau^*)} \propto \frac{\left|\tau^* I + \varphi^{(j)} M\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\theta^{'(j)} - X\beta^{(j)})^T\left(\tau^* I + \varphi^{(j)} M\right)^{-1}(\theta^{'(j)} - X\beta^{(j)})\right\}}{\left|\tau^{(j-1)} I + \varphi^{(j)} M\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\theta^{'(j)} - X\beta^{(j)})^T\left(\tau^{(j-1)} I + \varphi^{(j)} M\right)^{-1}(\theta^{'(j)} - X\beta^{(j)})\right\}} \mathrm{x} \frac{\left\{\left[1 + \frac{1}{(K-1)}\right](\tau^*)^2\right\}^{-1}}{\left\{\left[1 + \frac{1}{(K-1)}\right](\tau^{(j-1)})^2\right\}^{-1}} \mathrm{x} \frac{\frac{1}{(c_0 + \tau^*)^2}}{\frac{1}{(c_0 + \tau^{(j-1)})^2}} \mathrm{x} \frac{(\tau^{(j-1)})^{(\delta + \tau^{(j-1)} - 1)} exp\left(-\frac{\tau^{(j-1)}}{\omega + \tau^{(j-1)}}\right)}{(\tau^*)^{(\delta + \tau^{(j-1)} - 1)} exp\left(-\frac{\tau^*}{\omega + \tau^{(j-1)}}\right)}.$$

   c. The parameter U was sampled from $U \sim$Uniform(0,1).

If $r_\tau > U$, then $\tau^{(j)} = \tau^*$ otherwise $\tau^{(j)} = \tau^{(j-1)}$.

6. Steps 2, 3, 4and 5 were repeated until the chains reached convergence.

4. Empirical results

Simulated data and case study used to estimate the parameters are presented in this section.

4.1. Simulation study

A simulation study was performed for the HBDS model to confirm the validity of programming using **R**. Thirty studies ($n = 30$) with two dependent groups and eight covariates ($p = 8$) were simulated to obtain the so-called simulated effect sizes.

The steps involved to conduct the simulation study can be described as follows.

1. The value of a positive real number (τ) was fixed.
2. The value of φ given τ was calculated from the uniform distribution with the minimum and maximum values are $-\frac{\tau^2}{(K-1)}$ and τ^2, respectively.
3. Matrices b_{px1} and D_{pxp} were fixed, where p is the number of covariates. A vector of parameters $\boldsymbol{\beta}_{px1}$ was generated from the multivariate normal distribution with mean b_{px1} and variance-covariance matrix D_{pxp}.
4. The matrix (X_{nxp}) and identity matrix (I_{nxn}) were constructed. Parameters $\boldsymbol{\theta}'_{nx1}$ were then generated from the multivariate normal distribution, with mean $X_{nxp}\boldsymbol{\beta}_{px1}$ and variance-covariance matrix $\tau^2 I + \varphi M$.
5. The variance-covariance matrix $(V_{(l)nxn})$ was fixed where l is the number of dependent groups. The effects size vector $(\tilde{\boldsymbol{\theta}})$ was generated from the multivariate normal distribution, with mean $\boldsymbol{\theta}'_{nx1}$ and variance-covariance matrix (V_{nxn}).

10,000 independent samples were simulated following the given steps to obtain the values of parameters $\tau, \varphi, \beta_0, ..., \beta_7, \theta'_1, ..., \theta'_{30}$ and $\tilde{\theta}_1$, $....\tilde{\theta}_{30}$. The resulting simulated parameters are given in Table 1. Furthermore, the values of $\tau, \varphi, \beta_0, ..., \beta_7, \theta'_1, ..., \theta'_{30}$ were considered to be the true values of the parameters and $\tilde{\theta}_1, ...\tilde{\theta}_{30}$ were considered to be the simulated effects sizes for the study.

4.1.1. Estimation of parameters

The parameters for the HBDS model were estimated using the simulated data $(\tilde{\theta}_1, \ldots, \tilde{\theta}_{30})$ given in Table 1. It was expected that the estimated values would be close to the true values. The Metropolis within Gibbs formulation was used to approximate the joint posterior distribution of parameters for the HBDS model. A cycle of 50,000 iterations was executed, but only the last 10,000 iterations were of use in determining the convergence of the chains of parameters.

The Geweke, Heidelberger & Welch (H-W), and Raftery & Lewis (R-L) tests were the diagnostic tests used to determine whether the chains of parameters in the HBDS model had converged. The results of the MCMC convergence diagnostics using CODA and the values of the estimated parameters are presented in Tables 2 and 3, respectively.

The z-scores for $\beta_0, \ \ldots, \ \beta_7$ were all between -2 and 2 for the Geweke diagnostic test, confirming convergence at a 5% significance level. The p-value of $\beta_3(0.021)$ was lower than 0.05, indicating that the null hypothesis was rejected. However, it was not consistent with other β's. The stationarity tests for $\beta_0, \ \ldots, \ \beta_7$ were passed. The dependence factors (I) for the R-L diagnostic test were all below 5.0, which suggested that the

Table 1. The true values of $\tau^2, \varphi, \beta_0, ..., \beta_7, \theta'_1, ..., \theta'_{30}$ and results of the simulated effects sizes $(\tilde{\theta}_1, ...\tilde{\theta}_{30})$ for the HBDS model.

True value of τ^2			
$\tau^2 = 1.2$			
True value of φ			
$\varphi = 0.347$			
True value of $\beta_0, ...\beta_7$			
$\beta_0 = 0.9753$	$\beta_1 = 0.9571$	$\beta_2 = 1.0353$	$\beta_3 = 0.9799$
$\beta_4 = 1.0491$	$\beta_5 = 1.0149$	$\beta_6 = 1.0180$	$\beta_7 = 0.9454$
True value of $\theta'_1, \ ..., \ \theta'_{30}$			
$\theta'_1 = 4.3092$	$\theta'_2 = 5.8636$	$\theta'_3 = 4.2561$	$\theta'_4 = 4.3704$
$\theta'_5 = 8.0226$	$\theta'_6 = 8.0265$	$\theta'_7 = 7.1088$	$\theta'_8 = 4.4449$
$\theta'_9 = 6.5337$	$\theta'_{10} = 4.9341$	$\theta'_{11} = 4.9774$	$\theta'_{12} = 6.6576$
$\theta'_{13} = 4.2838$	$\theta'_{14} = 4.7533$	$\theta'_{15} = 7.4797$	$\theta'_{16} = 7.3650$
$\theta'_{17} = 7.1009$	$\theta'_{18} = 4.2994$	$\theta'_{19} = 4.9647$	$\theta'_{20} = 4.9702$
$\theta'_{21} = 4.2967$	$\theta'_{22} = 4.2911$	$\theta'_{23} = 4.3091$	$\theta'_{24} = 4.7636$
$\theta'_{25} = 8.0322$	$\theta'_{26} = 8.0403$	$\theta'_{27} = 7.1030$	$\theta'_{28} = 4.3720$
$\theta'_{29} = 4.9734$	$\theta'_{30} = 4.9614$		
Simulated effects sizes of $\theta'_1, \ ..., \ \theta'_{30}$			
$\tilde{\theta}_1 = 4.2964$	$\tilde{\theta}_2 = 5.8364$	$\tilde{\theta}_3 = 4.2582$	$\tilde{\theta}_4 = 4.7369$
$\tilde{\theta}_5 = 8.0108$	$\tilde{\theta}_6 = 8.0179$	$\tilde{\theta}_7 = 7.1082$	$\tilde{\theta}_8 = 4.3724$
$\tilde{\theta}_9 = 6.4914$	$\tilde{\theta}_{10} = 4.9184$	$\tilde{\theta}_{11} = 4.9896$	$\tilde{\theta}_{12} = 6.6690$
$\tilde{\theta}_{13} = 4.2183$	$\tilde{\theta}_{14} = 4.7967$	$\tilde{\theta}_{15} = 7.4988$	$\tilde{\theta}_{16} = 7.3506$
$\tilde{\theta}_{17} = 7.1052$	$\tilde{\theta}_{18} = 4.2995$	$\tilde{\theta}_{19} = 4.9587$	$\tilde{\theta}_{20} = 4.9738$
$\tilde{\theta}_{21} = 4.2982$	$\tilde{\theta}_{22} = 4.2811$	$\tilde{\theta}_{23} = 4.2944$	$\tilde{\theta}_{24} = 4.7237$
$\tilde{\theta}_{25} = 8.0303$	$\tilde{\theta}_{26} = 8.0418$	$\tilde{\theta}_{27} = 7.0955$	$\tilde{\theta}_{28} = 4.3435$
$\tilde{\theta}_{29} = 4.9857$	$\tilde{\theta}_{30} = 4.9446$		

Table 2. The MCMC convergence diagnostics for $\tau, \varphi, \beta_0, \ldots, \beta_7$ using Geweke, H-W and R-L tests (the simulated effect sizes for HBDS model).

Test Variable	Geweke	H-W	R-L
τ	z-score -0.3947	Stationarity test: passed *p*-value: 0.195	Dependence factor (I) 2.5
φ	z-score -0.5551	Stationarity test: passed *p*-value: 0.867	Dependence factor (I) 36.8
β_0	z-score -0.4393	Stationarity test: passed *p*-value: 0.4372	Dependence factor (I) 1.28
β_1	z-score -1.3295	Stationarity test: passed *p*-value: 0.3100	Dependence factor (I) 3.04
β_2	z-score -1.2546	Stationarity test: passed *p*-value: 0.471	Dependence factor (I) 1.11
β_3	z-score 1.5083	Stationarity test: passed *p*-value: 0.0213	Dependence factor (I) 3.37
β_4	z-score -1.6836	Stationarity test: passed *p*-value: 0.2609	Dependence factor (I) 3.28
β_5	z-score 0.8170	Stationarity test: passed *p*-value: 0.6736	Dependence factor (I) 2.70
β_6	z-score -1.5592	Stationarity test: passed *p*-value: 0.0512	Dependence factor (I) 1.21
β_7	z-score 2.3017	Stationarity test: passed *p*-value: 0.1069	Dependence factor (I) 3.20

The *z*-score for τ was -0.3947 for the Geweke test. As this value lay between -2 and 2, it could be concluded that the chains of parameters had reached convergence at a 5% significance level. The stationarity test for τ was passed with a *p*-value of 0.195 for the H-W diagnostic test, under the null hypothesis that the MCMC chain was stationary. The R-L test showed that the dependence factor (I) for τ was lower than 5.0, indicating that the sample was less correlated confirming the convergence.

sample was less correlated. All of these results together suggested that the chains of parameters had converged.

The *z*-score for φ was -0.5551 for the Geweke test. As this value was between -2 and 2, it could be concluded that the chains of parameters had reached convergence at a 5% significance level. However, the *p*-value of φ was 0.013 for the H-W diagnostic test, indicating the null hypothesis was rejected. The R-L test showed that the dependence factor (I) for φ was more than 5.0, indicating that the sample was highly correlated. This showed that the chains of parameters were not convergence.

4.1.2. Estimation results

The estimated values of τ, φ, β_0, ..., β_7 and $\theta_1^{'}$, ..., $\theta_{30}^{'}$, together with corresponding 95% credible intervals (CrIs) and standard deviations (SD) are presented below. This data will be used to draw conclusions about the parameters for the model. Table 3 shows the results of parameter estimates obtained using the Metropolis within Gibbs algorithm under the assumption of a dependence structure on the HBDS model.

As can be seen from Table 3, the estimated value of τ was close to the true value. Moreover, from the first and second columns of Table 3, the estimated values of some parameters β's were not very close to the true values. Some of standard deviations are quite large compare to their point estimates. This indicated that the data was spread out over a large range of values. However, all of the estimated values of β's and their corresponding true values lay within of their credible interval. This indicated that a 95% the true value will lie within the range.

For example, the estimated value of τ was 1.12, associated with the 95% CrI (0.7133, 2.6653). This was close to the true value of τ^2 (1.2). The estimated value of the intercept $(\widehat{\beta}_0)$ was 1.0508, associated with its 95% CrI (-3.0193, 5.273). This shows that the true value of the intercept, β_0 (0.9753) lay within the 95% credible interval of $\widehat{\beta}_0$.

The estimated value of φ (7.57) was not close to the true value (0.347) and the true value was not lay within the credible interval of the estimated parameter of φ (density plot of φ can be seen in Appendix). This issue might likely happen as when the parameters were generated, the $\tau^2 I + \varphi M$ matrix became semi-positive definite. Chivers (2013) developed the "MHadaptive" package (http://cran.r-project.org/web/packages/MHadaptive/MHadaptive.pdf) in order to overcome this problem. This package was used by forcing semi-positive definite matrix, $\tau^2 I + \varphi M$ to be positive definite. However, the elements of matrix $\tau^2 I + \varphi M$ were changed. In consequences, the matrix was not similar with the original matrix. Even though the estimator of φ showed that the statistics performance were not really good as expected, it was not consistent with other parameters.

Figure 1 displays the trace plots of $\beta_0, \ldots, \beta_7$. These figures show that the final 10,000 iterations for the chains of parameters were relatively stable with very small fluctuations only, confirming convergence. The marginal posterior densities of $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ and β_7 which are shown in Figure 2 and labelled V1, V2, V3, V4, V5, V6, V7 and V8, respectively, are unimodal and symmetric.

Figure 3 shows that the density of τ was right-skew with a mean value of 1.12 as τ is an log-logistic distribution. The trace plot of τ is displayed in Figure 4. This figure shows that the chains of τ mixed relatively well with some fluctuations, confirming the convergence. Moreover, the acceptance rate for τ was 71.9%. This indicated that the chain moves rapidly across the whole distribution, without getting stuck in any one place.

4.2. Case study: application the HBDS to the native language vocabulary data

The Metropolis within Gibbs algorithm for the HBDS model was applied to the data presented by Stevens and Taylor (2009). The results obtained from this approach were compared to the results found by Stevens and Taylor (2009) which used a numerical approach. The purpose was to determine whether the Metropolis within Gibbs algorithm can be used to approximate the joint posterior of all parameters for the HBDS model by the implementation to the data.

4.2.1. Estimations of parameters

A total of 50,000 iterations were executed, but only the last 10,000 iterations were of use in determining the convergence of the parameters. The Geweke, Heidelberger & Welch (H-W), and Raftery & Lewis (R-L) tests were the diagnostic tests used to determine whether the chains of parameters had converged.

Table 3. Estimated parameters for the HBDS model using the Metropolis within Gibbs (simulated data).

True value	Estimated value of τ and φ with the 95% CrI and SD		
$\tau = 1.2$	$\widehat{\tau} = 1.12$ with (0.7133, 2.6653) and 0.4229		
$\varphi = 0.347$	$\widehat{\varphi} = 7.57$ with (7.006, 8.373) and 0.4583		
	Estimated value of $\beta_0, \ldots, \beta_7$ with the 95% CrI and SD		
	Parameter estimates	95 % CrI	SD
$\beta_0 = 0.9753$	$\widehat{\beta}_0 = 1.0508$	(-3.0193, 5.273)	2.1134
$\beta_1 = 0.9571$	$\widehat{\beta}_1 = 1.0190$	(-1.3613, 3.336)	1.2122
$\beta_2 = 1.0353$	$\widehat{\beta}_2 = 1.0326$	(-4.5219, 6.590)	2.8610
$\beta_3 = 1.0503$	$\widehat{\beta}_3 = 0.9888$	(-1.9723, 3.956)	1.5014
$\beta_4 = 0.9799$	$\widehat{\beta}_4 = 0.9882$	(-0.9972, 2.908)	0.9810
$\beta_5 = 1.0491$	$\widehat{\beta}_5 = 1.0237$	(-2.4082, 4.369)	1.7205
$\beta_6 = 1.0180$	$\widehat{\beta}_6 = 0.9442$	(-4.5225, 6.414)	2.7815
$\beta_7 = 0.9454$	$\widehat{\beta}_7 = 0.9241$	(-0.7259, 2.578)	0.8498
$\boldsymbol{\theta_1', \ldots, \theta_{30}'}$	Estimated value of $\theta_1', \ldots, \theta_{30}'$ with the 95% CrI and SD		
$\theta_1' = 4.3092$	$\widehat{\theta}_1' = 4.276$	(2.252, 6.276)	1.0349
$\theta_2' = 5.8636$	$\widehat{\theta}_2' = 5.842$	(3.023, 8.638)	1.4272
$\theta_3' = 4.2561$	$\widehat{\theta}_3' = 4.\ 261$	(1.554, 6.964)	1.3765
$\theta_4' = 4.7519$	$\widehat{\theta}_4' = 4.\ 727$	(2.969, 6.437)	0.8985
$\theta_5' = 8.0226$	$\widehat{\theta}_5' = 8.037$	(5.446, 10.615)	1.3260
$\theta_6' = 8.0265$	$\widehat{\theta}_6' = 8.021$	(6.229, 9.815)	0.9146
$\theta_7' = 7.1088$	$\widehat{\theta}_7' = 7.101$	(5.299, 8.883)	0.9141
$\theta_8' = 4.3704$	$\widehat{\theta}_8' = 4.349$	(1.986, 6.716)	1.2236
$\theta_9' = 6.5337$	$\widehat{\theta}_9' = 6.462$	(3.378, 9.552)	1.5997
$\theta_{10}' = 4.9341$	$\widehat{\theta}_{10}' = 4.918$	(2.832, 7.037)	1.0903
$\theta_{11}' = 4.9774$	$\widehat{\theta}_{11}' = 5.002$	(3.440, 6.565)	0.7926
$\theta_{12}' = 6.6576$	$\widehat{\theta}_{12}' = 6.646$	(4.878, 8.492)	0.9336
$\theta_{13}' = 4.2838$	$\widehat{\theta}_{13}' = 4.258$	(2.333, 6.281)	1.0038
$\theta_{14}' = 4.7533$	$\widehat{\theta}_{14}' = 4.737$	(2.588, 6.888)	1.1091
$\theta_{15}' = 7.4797$	$\widehat{\theta}_{15}' = 7.438$	(5.021, 9.886)	1.2498
$\theta_{16}' = 7.3650$	$\widehat{\theta}_{16}' = 7.319$	(4.847, 9.835)	1.2827
$\theta_{17}' = 7.1009$	$\widehat{\theta}_{17}' = 7.086$	(5.000, 9.141)	1.0607
$\theta_{18}' = 4.2994$	$\widehat{\theta}_{18}' = 4.264$	(2.326, 6.201)	0.9906
$\theta_{19}' = 4.9647$	$\widehat{\theta}_{19}' = 4.910$	(3.290, 6.519)	0.8162
$\theta_{20}' = 4.9702$	$\widehat{\theta}_{20}' = 4.924$	(3.166, 6.686)	0.8845
$\theta_{21}' = 4.2967$	$\widehat{\theta}_{21}' = 4.294$	(3.063, 5.582)	0.6383
$\theta_{22}' = 4.2911$	$\widehat{\theta}_{22}' = 4.279$	(2.675, 5.872)	0.8072
$\theta_{23}' = 4.3091$	$\widehat{\theta}_{23}' = 4.299$	(2.274, 6.324)	1.0325
$\theta_{24}' = 4.7636$	$\widehat{\theta}_{24}' = 4.719$	(2.550, 6.936)	1.1381
$\theta_{25}' = 8.0322$	$\widehat{\theta}_{25}' = 8.036$	(6.417,9.629)	0.8150
$\theta_{26}' = 8.0403$	$\widehat{\theta}_{26}' = 8.034$	(6.686, 9.356)	0.6830
$\theta_{27}' = 7.1030$	$\widehat{\theta}_{27}' = 7.097$	(5.610, 8.606)	0.7662
$\theta_{28}' = 4.3720$	$\widehat{\theta}_{28}' = 4.350$	(1.913, 6.870)	1.2654
$\theta_{29}' = 4.9734$	$\widehat{\theta}_{29}' = 4.934$	(3.016, 6.843)	0.9789
$\theta_{30}' = 4.9614$	$\widehat{\theta}_{30}' = 4.923$	(3.541, 6.322)	0.7062

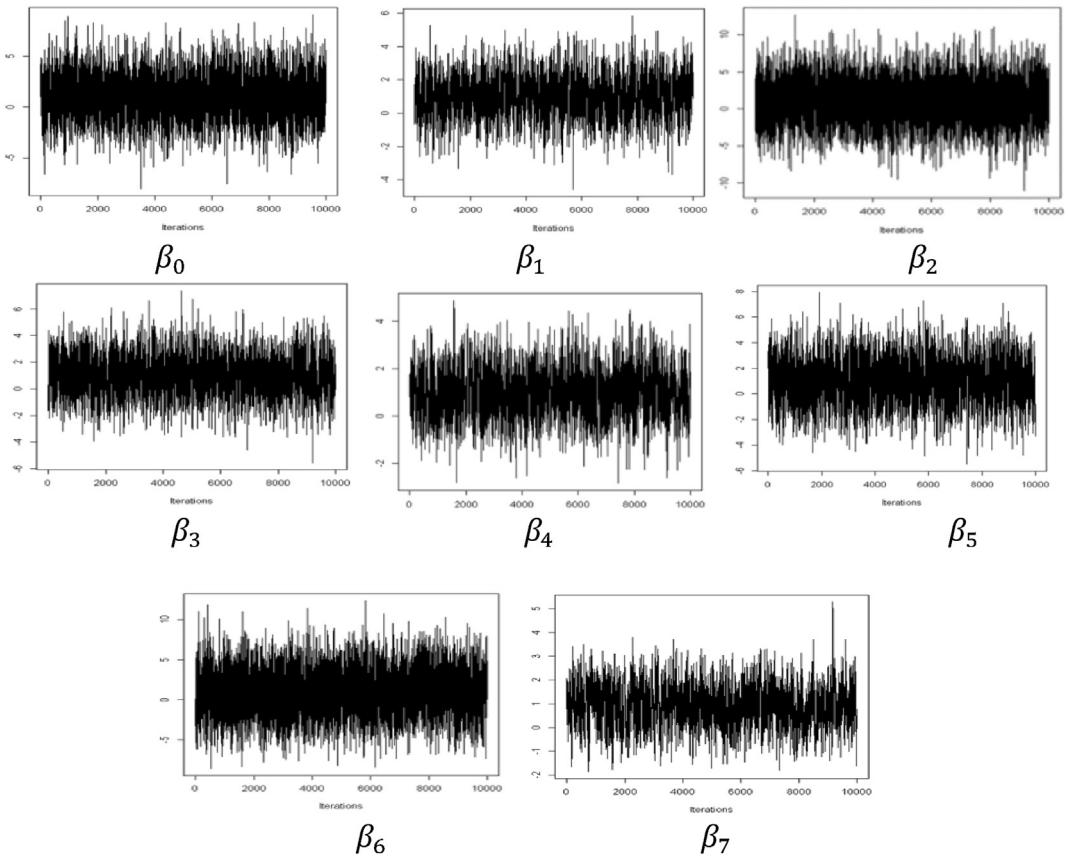

Figure 1. Trace plots of $\beta_0, \ldots, \beta_7$ for the HBDS model.

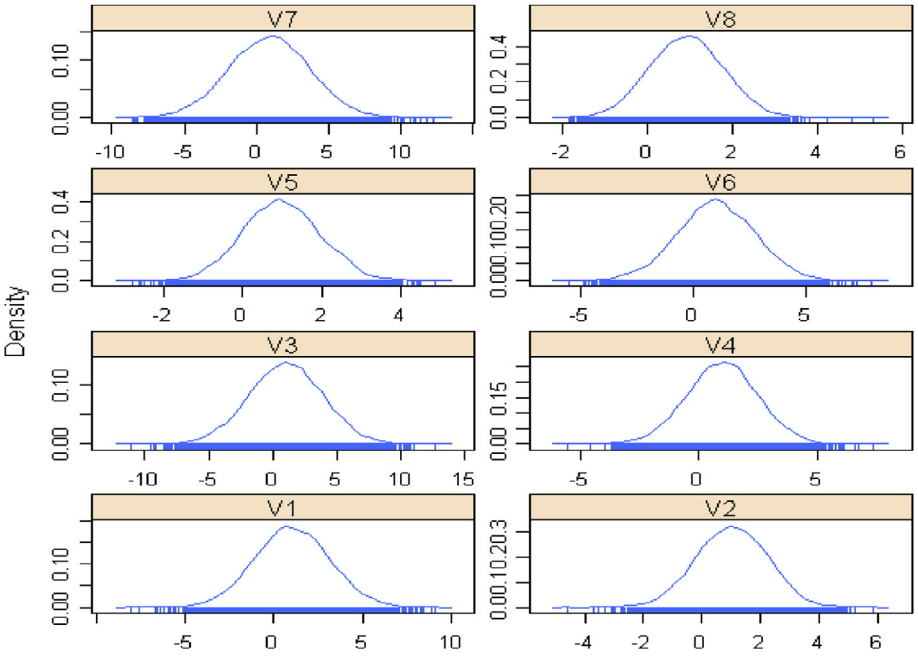

Figure 2. Density plots of $\beta_0, \ldots, \beta_7$ for the HBDS model.

Table 4 shows the results of the convergence diagnostic tests of parameters for HBDS model. The z-score of τ^2 was 0.3726. As this value lay between -2 and 2, it could be concluded that the τ^2 convergence at a 5% significance level. The p-value of τ^2 was 0.525. This confirmed that the null hypothesis of τ^2 was not rejected. The stationarity and half-width tests were passed for the H-W diagnostic. The dependence factor (I) for the R-L was lower than 5, indicating a less correlated samples, it was likely that the convergence of the chains for τ^2 had been achieved.

The z-scores of parameters $(\beta_0, \ldots, \beta_5)$ were consistent between -2 and 2 for the Geweke diagnostic tests, confirming the chains of parameters

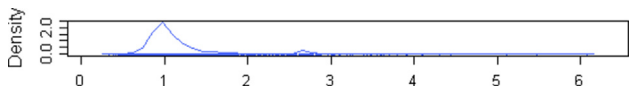

Figure 3. Density plot of τ for the HBDS model.

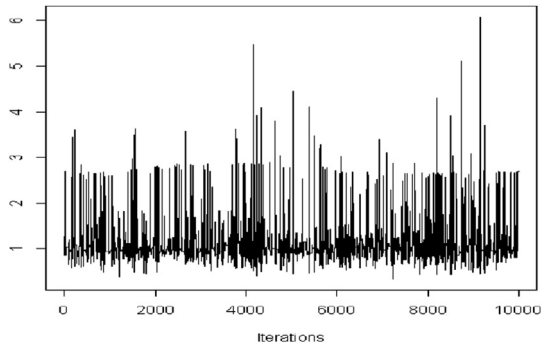

Figure 4. Trace plot of τ for the HBDS model.

reached convergence at a 5% significance level. Even though the *p*-value of β_2 (0.99) was larger than 0.95, indicating the null hypothesis was rejected, this was not consistent with the *p*-value of other β's. Moreover, the stationarity tests for all parameters were passed after discarding 50% of the chains. The half-width tests were also passed. Even though the dependence factor for β_2 greater than 5 the dependence factors (I) other parameters were lower than 5 for the R-L diagnostic test, suggesting that the sample was less correlated. Together all of these diagnostic tests indicated that the chains for $\beta_0, \beta_1, \beta_3, \beta_4, \beta_5$ had converged.

The dependence factor (I) of φ for the R-L test was higher than 5, indicating a highly correlated samples. However, it did not indicate by other diagnostic tests for φ. The z-score of φ was 0.7225. This value lay between -2 and 2, confirming the convergence at a 5% significance level. The *p*-value of φ was 0.768. This confirmed that the null hypothesis of φ was not rejected. The stationarity and half-width tests were passed for the H-W diagnostic. It was likely to conclude that the convergence of the chains for φ had been achieved.

4.2.2. Estimation results

The parameter estimates for φ, τ^2 $\beta_0, \ldots, \beta_5$, and their associated credible intervals and standard deviations are presented in Table 5. The point estimate of τ^2 was 0.7942, associated with a 95% credible interval of (0.2735, 1.9929). This was far from the value found by Stevens and Taylor (2009) (0.2573), and its 95% credible interval (0, 0.6241).

The point estimate of φ and its credible interval obtained using the Metropolis within Gibbs was also far from the result obtained by Stevens and Taylor (2009). The point estimate of the intercepts (β_0) obtained using this approach was 0.5066, and its credible interval (0.4073, 0.744), was tighter than the result obtained by Stevens and Taylor (2009) indicating more precise for the population mean effect size (intercepts). From the application of the model to the data, it shown only 50.6% the native-language vocabulary aids were effective as second language reading comprehension aids.

The results obtained by the use of the Metropolis within Gibbs to approximate the parameters in the HBDS model by applying the Stevens and Taylor's (2009) data were not really good as expected. This issue might likely happen as when the parameters were generated, the $\tau^2 I + \varphi M$ matrix became semi-positive definite, although the restricted condition for parameter φ given τ (Stevens and Taylor, 2009) had been implemented. "MHadaptive" package (http://cran.r-project.org/web/packages/MHadaptive/MHadaptive.pdf) in order to overcome this problem was also implemented by forcing the positive semidefinite matrix, $\tau^2 I + \varphi M$ to be positive definite. However, the elements of matrix were changed. Forcing the semi-positive definite matrix to be a positive definite matrix however, could create new matrix which was not similar to the original matrix.

The density plot displayed in panel (a) of Figure 5 shows that the marginal posterior density of β_0 (intercept) was symmetric. This indicates that β_0 was normally distributed. Moreover, the trace plot displayed in panel (b) of Figure 5 shows that the last 10,000 iterations in the estimation of β_0 had relatively good mixing, suggesting that the chains had converged.

Table 4. MCMC convergence diagnostics for values of φ, τ^2, $\beta_0, \ldots, \beta_5$ for the HBDS model (case study).

Test Variable	Geweke	H-W	R-L
τ^2	z-score 0.3726	Stationarity test: passed *p*-value: 0.525 Half-width test: passed Half-width: 0.0131	Dependence factor (I) 1.35
φ	z-score 0.7225	Stationarity test: passed *p*-value: 0.768 Half-width test: passed Half-width: 2.96	Dependence factor (I) 52.1
β_0	z-score -1.0855	Stationarity test: passed *p*-value: 0.755 Half-width test: passed Half-width: 0.0077	Dependence factor (I) 1.03
β_1	z-score -1.8605	Stationarity test: passed *p*-value: 0.359 Half-width test: passed Half-width: 0.0237	Dependence factor (I) 0.99
β_2	z-score -0.1602	Stationarity test: passed *p*-value: 0.990 Half-width test: passed Half-width: 0.0256	Dependence factor (I) 6.36
β_3	z-score -1.1828	Stationarity test: passed *p*-value: 0.466 Half-width test: passed Half-width: 0.0234	Dependence factor (I) 1.07
β_4	z-score -0.1625	Stationarity test: passed *p*-value: 0.769 Half-width test: passed Half-width: 0.0144	Dependence factor (I) 1.04
β_5	z-score -0.9618	Stationarity test: passed *p*-value: 0.940 Half-width test: passed Half-width: 0.0229	Dependence factor (I) 1.01

Table 5. Parameter estimates for the HBDS model using the Metropolis within Gibbs (case study).

Estimate result of τ^2 with the 95% CrI and SD		
$\widehat{\tau}^2 = 0.7942$, CrI (0.2735, 1.9929), SD = 0.489 $\widehat{\tau}^2 = 0.2573$, CrI (0, 0.6241), SD = 0.1834 (Stevens and Taylor, 2009)		
Estimate result of φ with the 95% CrI and SD		
$\widehat{\varphi} = 3.3671$, CrI (2.676, 4.053), SD = 3.836 $\widehat{\varphi} = 0.1117$, CrI (0, 0.4472), SD = 0.1678 (Stevens and Taylor, 2009)		
Estimates results of $\beta_0, \ldots, \beta_5$ with the 95% CrI and SD		
Parameter Estimates	95 % CrI	SD
$\widehat{\beta}_0 = 0.5066$ (0.6280) (Stevens)	(0.4037, 0.7444) (0.2979, 0.9581) (Stevens)	0.2274 0.1650 (Stevens)
$\widehat{\beta}_1 = -0.6322$ (-1.0699) (Stevens)	(-1.9459, 0.1328)	1.1513
$\widehat{\beta}_2 = -0.2953$ (-0.2184) (Stevens)	(-0.7733, 0.2608)	0.3423
$\widehat{\beta}_3 = 0.2985$ (0.1368) (Stevens)	(-0.3606, 0.9554)	0.7968
$\widehat{\beta}_4 = 1.1143$ (0.7116) (Stevens)	(-0.1175, 0.8474)	0.5146
$\widehat{\beta}_5 = 0.6968$ (0.4562) (Stevens)	(-0.0827, 1.2247)	0.6946

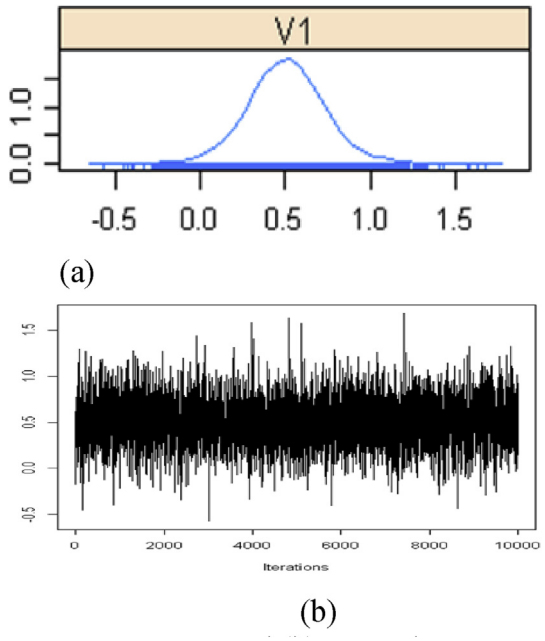

Figure 5. (a) Density plot of β_0 and (b) trace plot of β_0 (case study).

5. Conclusion

This paper discussed the hierarchical Bayesian delta splitting (HBDS) model. This model was used to obtain overall conclusions in meta-analysis by combining results from several studies. These models could accommodate heterogeneity that arose in the meta-analysis due to the different outcomes or treatments occurring in each study under consideration. The existence of correlations within studies and between studies arising due to the dependence structure was assumed in the model.

The validity of the programming to estimate parameters for the HBDS model was confirmed using the simulated data. The joint posterior distributions of all parameters for the models were derived using the Metropolis within Gibbs algorithm. The formula for the posterior distribution was implemented in **R** and the resulting code was executed in order to estimate the parameters for the model.

The MCMC convergence diagnostics using CODA were applied to determine whether the chains of parameters had converged. The Geweke, Heidelberger & Welch, Raftery & Lewis diagnostic tests showed that the chains of estimated parameters for the model had converged. The estimation of parameters using **R** code confirmed the consistency of the parameters for the model. Although several of the point estimates were not really close to their corresponding target values, they were still inside their corresponding credible intervals. The true values of the parameters also lay inside the credible intervals, indicating that the parameters were consistent. Furthermore, the trace and density plots showed that the parameters were stable and symmetric.

Even though, the results of parameter estimates obtained by the use of the HBDS model not exactly with what we expect is probably due to forcing the semi-positive definite matrix becomes positive, it is still likely to conclude that the Metropolis within Gibbs algorithm is a useful approach to approximate the joint posterior distribution of all parameters for the hierarchical Bayesian models in meta-analysis.

Declarations

Author contribution statement

Junaidi: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Wrote the paper.

D. Nur: Performed the experiments; Contributed reagents, materials, analysis tools or data.

I. Hudson: Contributed reagents, materials, analysis tools or data.

E. Stojanovski: Conceived and designed the experiments.

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Competing interest statement

The authors declare no conflict of interest.

Additional information

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