The Reduct of a Fuzzy β-Covering

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Abstract. This paper points some mistakes of three algorithms of updating the reduct in fuzzy β-covering via matrix approaches while adding and deleting some objects of the universe, and gives corrections of these mistakes. Moreover, we study the reduct of a fuzzy β-covering while adding and deleting objects further.

Keywords: Covering-based rough sets · Fuzzy sets · Matrix · Reduct

1 Introduction

Recently, fuzzy covering approximation spaces [1–3] were generalized to fuzzy β-covering approximation spaces by Ma [4] by replacing 1 with a parameter β, where 1 is a condition in fuzzy covering approximation spaces. Inspired by Ma's work, many researches were done. For example, some fuzzy covering-based rough set models were constructed by Yang and Hu [5–7], D'eer et al. [8,9] studied fuzzy neighborhood operators, and Huang et al. [10] presented a matrix approach for computing the reduct of a fuzzy β-covering.

The research idea of Ref. [10] is very good, but we find that Algorithms 1, 2 and 3 are incorrect after checking the paper carefully. Moreover, the result of a fuzzy β-covering can be studied further while adding and deleting objects. Hence, a further study about Ref. [10] can be done in this paper. Firstly, we explain the mistakes about Algorithms 1, 2 and 3 in Huang et al. (2020) [10]. Then, we give corresponding corrections of them. Finally, we present some new definitions and properties for updating the reduct while adding and deleting objects of a universe. The concepts about a fuzzy β-covering approximation space after adding and deleting objects are presented, respectively. Some new properties about the fuzzy β-covering approximation space and the new fuzzy β-covering approximation space after adding and deleting objects are given.

The rest of this paper is organized as follows. Section 2 reviews some fundamental definitions about fuzzy covering-based rough sets. In Sect. 3, we show some mistakes in [7]. Moreover, we give corresponding corrections of them. In Sect. 4, we present some new definitions and properties for updating the reduct while adding and deleting objects. This paper is concluded and further work is indicated in Sect. 5.

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2 Basic Definitions

This section recalls some fundamental definitions related to fuzzy covering-based rough sets. Supposing U is a nonempty and finite set called universe.

For any family $\gamma_i \in [0,1], i \in I$, $I \subseteq \mathbb{N}^+$ ($\mathbb{N}^+$ is the set of all positive integers), we write $\vee_{i \in I}\gamma_i$ for the supremum of $\{\gamma_i : i \in I\}$, and $\wedge_{i \in I}\gamma_i$ for the infimum of $\{\gamma_i : i \in I\}$. Some basic operations on $F(U)$ are shown as follows [11]: $A, B \in F(U)$,

(1) $A \subseteq B$ iff $A(x) \leq B(x)$ for all $x \in U$;

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(3) $A \cup B = \{\langle x, A(x) \vee B(x)\rangle : x \in U\}$;

(4) $A \cap B = \{\langle x, A(x) \wedge B(x)\rangle : x \in U\}$;

(5) $A' = \{\langle x, 1 - A(x) : x \in U\}$.

Ma [4] presented the notion of fuzzy β-covering approximation space.

Definition 1. *([4]) Let U be an arbitrary universal set and $F(U)$ be the fuzzy power set of U. For each $\beta \in (0,1]$, if $(\bigcup_{i=1}^{m} C_i)(x) \geq \beta$ for each $x \in U$, then we call $\widehat{C} = \{C_1, C_2, ..., C_m\}$ a fuzzy β-covering of U with $C_i \in F(U)$ $(i = 1, 2, ..., m)$. We also call $(U, \widehat{C})$ a fuzzy β-covering approximation space.*

The concept of reducible elements is important for us to deal with some problems in fuzzy covering-based rough sets [5]. Let $\widehat{C}$ be a fuzzy β-covering of U and $C \in \widehat{C}$. If C can be expressed as a union of some elements in $\widehat{C} - \{C\}$, then C is called a reducible element in $\widehat{C}$; otherwise C is called an irreducible element in $\widehat{C}$.

As shown in [5], if all reducible elements are deleted from a fuzzy β-covering $\widehat{C}$, then the remainder is still a fuzzy β-covering and this new fuzzy β-covering does not have any reducible element. We call this new fuzzy β-covering the reduct of the original fuzzy β-covering $\widehat{C}$. The following definition presents its concept.

Definition 2. *([5]) Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space. Then the family of all irreducible elements of $\widehat{C}$ is called the reduct of $\widehat{C}$, denoted as $\Gamma(\widehat{C})$.*

To calculate the result of a fuzzy β-covering by matrix, Huang et al. gave the following definition.

Definition 3. *([10]) Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space. The containing relation character matrix on U is denoted by $Q^U = (q_{ij}^U)_{m \times m}$, where*

$$q_{ij}^U = \begin{cases} 1, & C_i \subseteq C_j \wedge i \neq j; \\ 0, & \textit{otherwise}; \end{cases} \quad i, j \in \{1, 2, \cdots, m\}$$

3 Some Corrections on the Reduct of a Fuzzy β-covering

In [10], we find that Algorithms 1, 2 and 3 have mistakes after checking the paper carefully. Then we give corresponding corrections of the paper in this section.

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Algorithm 1: Algorithm to compute the reduct of a fuzzy β-covering.
   Input: (1)Ĉ = {C_1, C_2, ··· , C_m,} (2)U = {x_1, x_2, ··· , x_n}. (3)β ∈ (0, 1].
   Output: Γ(Ĉ), Q^U.
1  for i = 1, 2, ··· , m do
2  |  C(x_i) ← 0;
3  |  for j = 1, 2, ··· , m do
4  |  |  C(x_i) ← C(x_i) ∨ C_j(x_i)
5  |  if C(x_i) < β then
6  |  |  return Ĉ is not a fuzzy β-covering
7  Γ(Ĉ) ← Ĉ;
8  for k = 1, 2, ··· , m do
9  |  T ← ∅;
10 |  for l = 1, 2, ··· , m do
11 |  |  if C_k ⊆ C_l then
12 |  |  |  q^U_kl ← 1;
13 |  |  if q^U_kl = 1 then
14 |  |  |  T ← T ∪ C_k;
15 |  if T = C_l then
16 |  |  Γ(Ĉ) ← Γ(Ĉ) − {C_l};
17 return Γ(Ĉ)
```

Fig. 1. Algorithm 1 (In [10])

By Algorithm 1 (In [10]), we know that $\Gamma(\widehat{C}) = \emptyset$ for any fuzzy β-covering, which is incorrect. To explain the incorrect results in Algorithm 1, we show the Algorithm 1 (In [10]) in Fig. 1:

In Algorithm 1 (In [10]), $U = \{x_1, x_2, \cdots, x_n\}$. By Step 2, $C(x_i) \leftarrow 0$. Hence,

- Step 1: "$i = 1, 2, \cdots, m$" should be changed as "$i = 1, 2, \cdots, n$".

According to Definition 3 (Definition 5 in [10]) and Step 12, we find Step 11 of Algorithm 1 (In [10]) is incorrect. By Steps 11 and 12, if $C_k \subseteq C_l$ then $q^U_{kl} \leftarrow 1$. But according to Definition 3 (Definition 5 in [10]), if $C_k \subseteq C_l$ and $k \neq l$ then $q^U_{kl} \leftarrow 1$. Hence,

- Step 11: "**if** $C_k \subseteq C_l$ **then**" should be changed as "**if** $C_k \subseteq C_l$ and $k \neq l$ **then**".

From Steps 11 to 12 of Algorithm 1 (In [10]), it is to find all $C_k \in \widehat{C} - \{C_l\}$ which satisfy $C_k \subseteq C_l$ for any $C_l \in \widehat{C}$. From Steps 13 to 16 of Algorithm 1 (In [10]), if $\bigcup_{C_k \in \widehat{C} - \{C_l\}} = C_l$ then C_l is a reducible element in $\widehat{C}$. Hence, Steps 8 and 10 should be swaped places. That is to say,

- Step 8: "**for** $k = 1, 2, \cdots, m$ **do**" should be changed as "**for** $l = 1, 2, \cdots, m$ **do**".
- Step 10: "**for** $l = 1, 2, \cdots, m$ **do**" should be changed as "**for** $k = 1, 2, \cdots, m$ **do**".

The result of Algorithm 2 (In [10]) will be $\widehat{\mathcal{G}}$ all the time, which is incorrect. To explain the incorrect results in Algorithm 2, we show the Algorithm 2 (In [10]):

By Algorithm 2, we find:

– Step 15: "$T \leftarrow T \cup C_k$;" should be changed as "$T \leftarrow T \cup \mathcal{G}_k$".

From Steps 14 to 17 of Algorithm 2 (In [10]), if $\bigcup_{\mathcal{G}_k \in \widehat{\mathcal{G}} - \{\mathcal{G}_l\}} = \mathcal{G}_l$ then $\mathcal{G}_l$ is a reducible element in $\widehat{\mathcal{G}}$. Hence, Steps 11 and 13 should be swaped places. That is to say,

– Step 11: "**for** $k = 1, 2, \cdots, m$ **do**" should be changed as "**for** $l = 1, 2, \cdots, m$ **do**".
– Step 13: "**for** $l = 1, 2, \cdots, m$ **do**" should be changed as "**for** $k = 1, 2, \cdots, m$ **do**".

The result of Algorithm 3 (In [10]) will be $\widehat{\mathcal{G}}$ all the time, which is incorrect. To explain the incorrect results in Algorithm 3, we show the Algorithm 3 (In [10]):

By Algorithm 3, we find:

– Step 15: "$T \leftarrow T \cup C_k$;" should be changed as "$T \leftarrow T \cup \mathcal{G}_k$".

From Steps 14 to 17 of Algorithm 3 (In [10]), if $\bigcup_{\mathcal{G}_k \in \widehat{\mathcal{G}} - \{\mathcal{G}_l\}} = \mathcal{G}_l$ then $\mathcal{G}_l$ is a reducible element in $\widehat{\mathcal{G}}$. Hence, Steps 11 and 13 should be swaped places. That is to say,

– Step 11: "**for** $k = 1, 2, \cdots, m$ **do**" should be changed as "**for** $l = 1, 2, \cdots, m$ **do**".
– Step 13: "**for** $l = 1, 2, \cdots, m$ **do**" should be changed as "**for** $k = 1, 2, \cdots, m$ **do**".

4 New Properties of Reducts of Fuzzy β-Coverings While Adding and Deleting Some Objects

This section presents some new properties of reducts in fuzzy β-coverings while adding and deleting some objects, respectively. In this section, t denotes an integer which is more than 1.

Firstly, we give some new properties on reducts of fuzzy β-coverings while adding some objects of a universe. The concept of increasing fuzzy β-covering approximation space is presented in the following definition.

Definition 4. *Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U, where $U = \{x_1, x_2, \cdots, x_n\}$ and $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$. We call $(U^+, \widehat{C}^+)$ an increasing fuzzy β-covering approximation space from $(U, \widehat{C})$, where $U^+ = \{x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_{n+t}\}$, $\widehat{C}^+ = \{C_1^+, C_2^+, \cdots, C_m^+\}$, and for any $1 \leq j \leq m$,*

$$\begin{cases} C_j^+(x_i) = C_j(x_i), & 1 \leq i \leq n; \\ (\bigcup_{j=1}^m C_j^+)(x_i) \geq \beta, & n+1 \leq i \leq n+t. \end{cases}$$

The following proposition shows that an increasing fuzzy β-covering approximation space from a fuzzy β-covering approximation space is also a fuzzy β-covering approximation space.

Algorithm 2: Algorithm to update the reduct of a fuzzy β-covering while adding some objects into the universe.

Input: (1)$\widehat{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_m\}$ (2)$\mathcal{U} = \{x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_{n+t}\}$. (3)$\beta \in (0, 1]$,(4)$Q^U$.
Output: $\Gamma(\widehat{\mathcal{G}})$.

$Q^{\mathcal{U}} \leftarrow Q^U$;
for $i = 1, 2, \cdots, m$ **do**
  for $j = 1, 2, \cdots, m$ **do**
    if $q_{ij}^{\mathcal{U}} = 1$ **then**
      $s \leftarrow 1$;
      for $k = n+1, n+2, \cdots, n+t$ **do**
        if $\mathcal{G}_i(x_k) > \mathcal{G}_j(x_k)$ **then**
          $s \leftarrow 0$;
      $q_{ij}^{\mathcal{U}} \leftarrow s$;
$\Gamma(\widehat{\mathcal{G}}) \leftarrow \widehat{\mathcal{G}}$;
for $k = 1, 2, \cdots, m$ **do**
  $T \leftarrow \emptyset$;
  for $l = 1, 2, \cdots, m$ **do**
    if $q_{kl}^{\mathcal{U}} = 1$ **then**
      $T \leftarrow T \cup C_k$;
  if $T = \mathcal{G}_l$ **then**
    $\Gamma(\widehat{\mathcal{G}}) \leftarrow \Gamma(\widehat{\mathcal{G}}) - \{\mathcal{G}_l\}$;
return $\Gamma(\widehat{\mathcal{G}})$

Fig. 2. Algorithm 2 (In [10])

Proposition 1. *Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U, where $U = \{x_1, x_2, \cdots, x_n\}$ and $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$. Then $(U^+, \widehat{C}^+)$ is also a fuzzy β-covering approximation space of U^+.*

Proof. By Definition 4, $(\bigcup_{j=1}^m C_j^+)(x_i) = (\bigcup_{j=1}^m C_j)(x_i) \geq \beta$ for any $i \in \{1, 2, \cdots, n\}$, and $(\bigcup_{j=1}^m C_j^+)(x_i) \geq \beta$ for each $i \in \{n+1, \cdots, n+t\}$. Hence, $(U^+, \widehat{C}^+)$ is also a fuzzy β-covering approximation space of U^+ by Definition 1.

Example 1. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $\widehat{C} = \{C_1, C_2, C_3, C_4\}$, where

$$C_1 = \frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.7}{x_5},$$
$$C_2 = \frac{0.3}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5},$$
$$C_3 = \frac{0.7}{x_1} + \frac{0.6}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.7}{x_5},$$
$$C_4 = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5}.$$

According to Definition 1, we know $\widehat{C}$ is a fuzzy β-covering of U $(0 < \beta \leq 0.6)$. Suppose $\beta = 0.5$. Let $U^+ = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\widehat{\mathbf{C}}^+ = \{C_1^+, C_2^+, C_3^+, C_4^+\}$, where

$$C_1^+ = \frac{0.7}{x_1} + \frac{0.8}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.7}{x_5} + \frac{0.6}{x_6},$$
$$C_2^+ = \frac{0.3}{x_1} + \frac{0.8}{x_2} + \frac{0.3}{x_3} + \frac{0.5}{x_4} + \frac{0.6}{x_5} + \frac{0.5}{x_6},$$
$$C_3^+ = \frac{0.7}{x_1} + \frac{0.6}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.7}{x_5} + \frac{0.5}{x_6},$$
$$C_4^+ = \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.3}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}.$$

Algorithm 3: Algorithm to update the reduct of a fuzzy β-covering while deleting objects from the universe

Input: (1)$\widehat{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_m\}$ (2)$\mathcal{U} = \{x_1, x_2, \cdots, \cdots, x_{n-t}\}$. (3)$\beta \in (0,1]$,(4)$Q^{\mathcal{U}}$.
Output: $\Gamma(\widehat{\mathcal{G}})$.

$Q^{\mathcal{U}} \leftarrow Q^{U}$;
for $i = 1, 2, \cdots, m$ **do**
  for $j = 1, 2, \cdots, m$ **do**
    if $q_{ij}^{\mathcal{U}} = 0$ **then**
      $s \leftarrow 1$;
      for $k = 1, 2, \cdots, n-t$ **do**
        if $\mathcal{G}_i(x_k) > \mathcal{G}_j(x_k)$ **then**
          $s \leftarrow 0$;
      $q_{ij}^{\mathcal{U}} \leftarrow s$;
$\Gamma(\widehat{\mathcal{G}}) \leftarrow \widehat{\mathcal{G}}$;
for $k = 1, 2, \cdots, m$ **do**
  $T \leftarrow \emptyset$;
  for $l = 1, 2, \cdots, m$ **do**
    if $q_{kl}^{\mathcal{U}} = 1$ **then**
      $T \leftarrow T \cup C_k$;
  if $T = \mathcal{G}_l$ **then**
    $\Gamma(\widehat{\mathcal{G}}) \leftarrow \Gamma(\widehat{\mathcal{G}}) - \{\mathcal{G}_l\}$;
return $\Gamma(\widehat{\mathcal{G}})$

Fig. 3. Algorithm 3 (In [10])

According to Definitions 1 and 4, we know $\widehat{C}^{+}$ is a fuzzy 0.5-covering of U.

We give a relationship about the relation character matrices between a fuzzy β-covering approximation space and it's increasing fuzzy β-covering approximation space in the following proposition.

Proposition 2. *Let* $(U, \widehat{C})$ *and* $(U^{+}, \widehat{C}^{+})$ *be two fuzzy* β*-covering approximation spaces, where* $U = \{x_1, x_2, \cdots, x_n\}$ *and* $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$*. If* $q_{ij}^{U} = 0$*, then* $q_{ij}^{U^{+}} = 0$ *for any* $i, j \in \{1, 2, \cdots, m\}$*.*

Proof. For any $i, j \in \{1, 2, \cdots, m\}$, we have the following two conditions:
For $i = j$: if $i = j$, then $q_{ij}^{U} = 0$ and $q_{ij}^{U^{+}} = 0$;
For $i \neq j$: by Definition 3, if $q_{ij}^{U} = 0$, then there exists $k \in \{1, 2, \cdots, n\}$ such that $C_i(x_k) > C_j(x_k)$. Hence, there exists $k \in \{1, 2, \cdots, n\}$ such that $C_i^{+}(x_k) > C_j^{+}(x_k)$ according to Definition 4. Therefore, C_i^{+} is not contained in C_j^{+}. That is to say, $q_{ij}^{U^{+}} = 0$.

Example 2. (Continued from Example 1)

$$Q^U = (q_{ij}^U)_{4\times 4} = \begin{array}{c c} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array},$$

$$Q^{U^+} = (q_{ij}^{U^+})_{4\times 4} = \begin{array}{c c} & \begin{array}{cccc} C_1^+ & C_2^+ & C_3^+ & C_4^+ \end{array} \\ \begin{array}{c} C_1^+ \\ C_2^+ \\ C_3^+ \\ C_4^+ \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Hence, if $q_{ij}^U = 0$, then $q_{ij}^{U^+} = 0$ for any $i, j \in \{1, 2, \cdots, 4\}$.

We give a relationship about reducible elements between a fuzzy β-covering approximation space and it's increasing fuzzy β-covering approximation space in the following proposition.

Proposition 3. *Let $(U, \widehat{C})$ and $(U^+, \widehat{C}^+)$ be two fuzzy β-covering approximation spaces, where $U = \{x_1, x_2, \cdots, x_n\}$ and $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$. If C_i^+ is a reducible element in $\widehat{C}^+$, then C_i is a reducible element in $\widehat{C}$ for any $i \in \{1, 2, \cdots, m\}$.*

Proof. It is immediate by Definition 4 and the concept of reducible element.

The converse of Proposition 3 is not true, i.e., "If C_i is a reducible element in $\widehat{C}$, then C_i^+ is a reducible element in $\widehat{C}^+$ for any $i \in \{1, 2, \cdots, m\}$." is not true. Example 1 can explain this. In Example 1, since $C_1 = C_2 \bigcup C_3$, C_1 is a reducible element in $\widehat{C}$. However, C_1^+ is not a reducible element in $\widehat{C}^+$. Based on Proposition 3, we give the following corollary.

Corollary 1. *Let $(U, \widehat{C})$ and $(U^+, \widehat{C}^+)$ be two fuzzy β-covering approximation spaces, where $U = \{x_1, x_2, \cdots, x_n\}$ and $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$. If C_i is a irreducible element in $\widehat{C}$, then C_i^+ is a irreducible element in $\widehat{C}^+$ for any $i \in \{1, 2, \cdots, m\}$.*

Proof. By Proposition 3, it is immediate.

Example 3. (Continued from Example 1) C_2, C_3 and C_4 are irreducible elements in $\widehat{C}$. C_2^+, C_3^+ and C_4^+ are irreducible elements in $\widehat{C}^+$.

The converse of Corollary 1 is not true, i.e., "If C_i^+ is a irreducible element in $\widehat{C}^+$, then C_i is a irreducible element in $\widehat{C}$ for any $i \in \{1, 2, \cdots, m\}$." is not true. Example 1 can explain this. In Example 1, C_1^+ is a irreducible element in $\widehat{C}^+$. But C_1 is not a irreducible element in $\widehat{C}$. Inspired by Corollary 1, we give the following theorem.

Theorem 1. *Let $(U, \widehat{C})$ and $(U^+, \widehat{C}^+)$ be two fuzzy β-covering approximation spaces. Then $|\Gamma(\widehat{C})| \leq |\Gamma(\widehat{C}^+)|$.*

Proof. By Definition 2, $\Gamma(\widehat{C})$ and $\Gamma(\widehat{C}^{+})$ are families of all irreducible elements of $\widehat{C}$ and $\widehat{C}^{+}$, respectively. Hence, it is immediate by Corollary 1.

Note that $|\Gamma(\widehat{C})|$ and $|\Gamma(\widehat{C}^{+})|$ denote the cardinality of $\Gamma(\widehat{C})$ and $\Gamma(\widehat{C}^{+})$, respectively.

Example 4. (Continued from Example 1) $\Gamma(\widehat{C}) = \{C_2, C_3, C_4\}$, $\Gamma(\widehat{C}^{+}) = \{C_1^{+}, C_2^{+}, C_3^{+}, C_4^{+}\}$. Hence, $|\Gamma(\widehat{C})| = 3$ and $|\Gamma(\widehat{C}^{+})| = 4$. That is to say, $|\Gamma(\widehat{C})| \leq |\Gamma(\widehat{C}^{+})|$.

Then, we give some new properties on reducts of fuzzy β-coverings while deleting some objects of a universe. The concept of declining fuzzy β-covering approximation space is presented in the following definition.

Definition 5. *Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U, where $U = \{x_1, x_2, \cdots, x_n\}$ and $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$. We call $(U^{-}, \widehat{C}^{-})$ a declining fuzzy β-covering approximation space from $(U, \widehat{C})$, where $U^{+} = \{x_1, x_2, \cdots, x_{n-t}\}$, $\widehat{C}^{-} = \{C_1^{-}, C_2^{-}, \cdots, C_m^{-}\}$ and $C_j^{-}(x_i) = C_j(x_i)$ for any $1 \leq i \leq n-t$, $1 \leq j \leq m$.*

The following proposition shows that a declining fuzzy β-covering approximation space from a fuzzy β-covering approximation space is also a fuzzy β-covering approximation space.

Proposition 4. *Let $(U, \widehat{C})$ be a fuzzy β-covering approximation space of U, where $U = \{x_1, x_2, \cdots, x_n\}$ and $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$. Then $(U^{-}, \widehat{C}^{-})$ is also a fuzzy β-covering approximation space of U^{-}.*

Proof. By Definition 5, $(\bigcup_{j=1}^{m} C_j^{-})(x_i) = (\bigcup_{j=1}^{m} C_j)(x_i) \geq \beta$ for any $i \in \{1, 2, \cdots, n-t\}$. Hence, $(U^{-}, \widehat{C}^{-})$ is also a fuzzy β-covering approximation space of U^{-} by Definition 1.

Example 5. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $\widehat{C} = \{C_1, C_2, C_3, C_4\}$, where

$$\begin{aligned}
C_1 &= \tfrac{0.7}{x_1} + \tfrac{0.8}{x_2} + \tfrac{0.6}{x_3} + \tfrac{0.6}{x_4} + \tfrac{0.7}{x_5},\\
C_2 &= \tfrac{0.3}{x_1} + \tfrac{0.8}{x_2} + \tfrac{0.3}{x_3} + \tfrac{0.8}{x_4} + \tfrac{0.6}{x_5},\\
C_3 &= \tfrac{0.7}{x_1} + \tfrac{0.6}{x_2} + \tfrac{0.6}{x_3} + \tfrac{0.6}{x_4} + \tfrac{0.7}{x_5},\\
C_4 &= \tfrac{0.4}{x_1} + \tfrac{0.6}{x_2} + \tfrac{0.3}{x_3} + \tfrac{0.2}{x_4} + \tfrac{0.5}{x_5}.
\end{aligned}$$

According to Definition 1, we know $\widehat{C}$ is a fuzzy β-covering of U $(0 < \beta \leq 0.6)$. Suppose $\beta = 0.5$. Let $U^{-} = \{x_1, x_2, x_3\}$ and $\widehat{\mathbf{C}}^{-} = \{C_1^{-}, C_2^{-}, C_3^{-}, C_4^{-}\}$, where

$$\begin{aligned}
C_1^{-} &= \tfrac{0.7}{x_1} + \tfrac{0.8}{x_2} + \tfrac{0.6}{x_3},\\
C_2^{-} &= \tfrac{0.3}{x_1} + \tfrac{0.8}{x_2} + \tfrac{0.3}{x_3},\\
C_3^{-} &= \tfrac{0.7}{x_1} + \tfrac{0.6}{x_2} + \tfrac{0.6}{x_3},\\
C_4^{-} &= \tfrac{0.4}{x_1} + \tfrac{0.6}{x_2} + \tfrac{0.3}{x_3}.
\end{aligned}$$

According to Definitions 1 and 5, we know $\widehat{C}^{-}$ is a fuzzy 0.5-covering of U.

We give a relationship about the relation character matrices between a fuzzy β-covering approximation space and it's declining fuzzy β-covering approximation space in the following proposition.

Proposition 5. *Let* $(U, \widehat{C})$ *and* $(U^-, \widehat{C}^-)$ *be two fuzzy* β*-covering approximation spaces, where* $U = \{x_1, x_2, \cdots, x_n\}$ *and* $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$*. If* $q_{ij}^{U^-} = 0$*, then* $q_{ij}^{U} = 0$ *for any* $i, j \in \{1, 2, \cdots, m\}$*.*

Proof. For any $i, j \in \{1, 2, \cdots, m\}$, we have the following two conditions:

For $i = j$: if $i = j$, then $q_{ij}^{U} = 0$ and $q_{ij}^{U^-} = 0$;

For $i \neq j$: by Definition 3, if $q_{ij}^{U^-} = 0$, then there exists $k \in \{1, 2, \cdots, n-t\}$ such that $C_i(x_k) > C_j(x_k)$. Hence, there exists $k \in \{1, 2, \cdots, n-t\}$ such that $C_i(x_k) > C_j(x_k)$ according to Definition 5, i.e., there exists $k \in \{1, 2, \cdots, n\}$ such that $C_i(x_k) > C_j(x_k)$. Therefore, C_i is not contained in C_j. That is to say, $q_{ij}^{U} = 0$.

Example 6. (Continued from Example 5)

$$Q^U = (q_{ij}^U)_{4\times4} = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array},$$

$$Q^{U^-} = (q_{ij}^{U^-})_{4\times4} = \begin{array}{c} \\ C_1^- \\ C_2^- \\ C_3^- \\ C_4^- \end{array} \begin{array}{c} \begin{array}{cccc} C_1^- & C_2^- & C_3^- & C_4^- \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

Hence, if $q_{ij}^{U^-} = 0$, then $q_{ij}^{U} = 0$ for any $i, j \in \{1, 2, \cdots, 4\}$.

Huang et al. [10] gave a relationship about reducible elements between a fuzzy β-covering approximation space and it's declining fuzzy β-covering approximation space in the following proposition.

Lemma 1. *([10]) Let* $(U, \widehat{C})$ *and* $(U^-, \widehat{C}^-)$ *be two fuzzy* β*-covering approximation spaces, where* $U = \{x_1, x_2, \cdots, x_n\}$ *and* $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$*. If* C_i *is a reducible element in* $\widehat{C}$*, then* C_i^- *is a reducible element in* $\widehat{C}^-$ *for any* $i \in \{1, 2, \cdots, m\}$*.*

The converse of Lemma 1 is not true, i.e., "If C_i^- is a reducible element in $\widehat{C}$, then C_i is a reducible element in $\widehat{C}$ for any $i \in \{1, 2, \cdots, m\}$." is not true. Example 5 can explain this. In Example 5, since $C_1^- = C_2^- \bigcup C_3^-$, C_1^- is a reducible element in $\widehat{C}^-$. But C_1 is not a reducible element in $\widehat{C}$. Based on Lemma 1, we give the following corollary.

Corollary 2. *Let* $(U, \widehat{C})$ *and* $(U^-, \widehat{C}^-)$ *be two fuzzy* β*-covering approximation spaces, where* $U = \{x_1, x_2, \cdots, x_n\}$ *and* $\widehat{C} = \{C_1, C_2, \cdots, C_m\}$*. If* C_i^- *is a irreducible element in* $\widehat{C}^-$*, then* C_i *is a irreducible element in* $\widehat{C}$ *for any* $i \in \{1, 2, \cdots, m\}$*.*

Proof. By Lemma 1, it is immediate.

Example 7. (Continued from Example 5) C_2^-, C_3^- and C_4^- are irreducible elements in $\widehat{C}^-$. C_2, C_3 and C_4 are irreducible elements in $\widehat{C}$.

Based on Corollary 2, we give the following theorem.

Theorem 2. *Let $(U, \widehat{C})$ and $(U^-, \widehat{C}^-)$ be two fuzzy β-covering approximation spaces. Then $|\Gamma(\widehat{C})| \geq |\Gamma(\widehat{C}^-)|$.*

Proof. By Definition 2, $\Gamma(\widehat{C})$ and $\Gamma(\widehat{C}^-)$ are families of all irreducible elements of $\widehat{C}$ and $\widehat{C}^-$, respectively. Hence, it is immediate by Corollary 2.

Example 8. (Continued from Example 5) $\Gamma(\widehat{C}) = \{C_1, C_2, C_3, C_4\}$, $\Gamma(\widehat{C}^-) = \{C_2^-, C_3^-, C_4^-\}$. Hence, $|\Gamma(\widehat{C})| = 4$ and $|\Gamma(\widehat{C}^-)| = 3$. That is to say, $|\Gamma(\widehat{C})| \geq |\Gamma(\widehat{C}^-)|$.

5 Conclusions

In this paper, we explain the mistakes about Algorithms 1, 2 and 3 in Huang et al. (2020) [10]. Moreover, we present some new definitions and properties for updating the reduct while adding and deleting objects of a universe. It is helpful for others to investigate the work further. In future, updating the reduct while adding and deleting objects at the same time will be done. Neutrosophic sets and related algebraic structures [12–15] will be connected with the research content of this paper in further research.

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