



Research article

Generalization of minimax and maximin criteria in a game against nature for the case of a partial a priori uncertainty

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ABSTRACT

This study proposes a new criterion for choosing the optimal decision in a game against nature under a partial a priori uncertainty. The paper's main novelty consists in examining the situation when a part of the a priori probabilities of states of nature is known, and the other part is unknown. We prove the theorems for choosing the optimal decision as for the payoff and risk matrix, as well as for the profit matrix in the situation of a partial a priori uncertainty. The proposed approach also generalizes the Bayes, Wald, Savage, Hurwicz, and Laplace criteria since the minimum average payoff (or risk) for each of these criteria we can quickly obtain from the article's derived formulas. A practical example of a game against nature under a partial a priori uncertainty illustrates the proposed approach and shows its effectiveness compared to well-known criteria. We show that the introduced criterion provides the choice of a decision that is also optimal in conditions of risk, which indicates the effective use of the vector of known a priori probabilities.

1. Introduction

Game theory (GT) is increasingly penetrating the practice of economic, technical, and military decisions. It is a tool to increase the effectiveness of planned and managerial decisions. Game theory is of great importance in solving industry, transport, agriculture, and trade problems. So, GT helps to determine scientifically substantiated levels of lowering retail prices [1] and the optimal level of inventories [2]; solve the problems of tourist services [3], traffic control of urban transport [4], and artificial intelligence [5]; find emerging signal processing applications [6] and plan the military operations [7], etc. Usually, GT is defined as a branch of mathematics for the study of conflict situations. In general, we can divide the GT into the theory of strategic games [8] and statistical games [9], i.e., games against nature. In strategic games, some players, when choosing their decision, consider what impact this decision, according to their assumption, will have on the choice of policy by other players. Statistical game is a specific type of matrix game, in which one of the players is neutral, i.e., does not actively oppose the other participant in the game but keeps his decision secret. Usually, we call such a player "nature."

In this article, we consider only games against nature. As is well known [9], there are two players in a game against nature, only one of

which acts consciously. We will call this player the decision-maker and designate it as DM. Being the second participant in the game, nature is neither the opponent nor the ally of player DM. The player nature, being in one of the possible states, does not act against the player DM or in favor of him; it does not pursue a specific goal and is indifferent to the game's outcome. Moreover, player DM cannot exert any influence on the state of nature. One of the crucial assumptions in the GT against nature is the assumption that at any given time, the player nature can be in only one of n possible states S_1, S_2, \dots, S_n that are collectively exhaustive and mutually exclusive [10]. The set of states S_1, S_2, \dots, S_n is formed either based on real experience or assumptions of experts. To describe the game against nature, we also need a set of decisions of player DM: $\{\varphi_1, \varphi_2, \dots, \varphi_m\}$. When the probabilities of the states of nature $P(S_1), P(S_2), \dots, P(S_n)$ are known, such a situation is called the decision-making under risk [11]. When the probabilities $P(S_1), P(S_2), \dots, P(S_n)$ are unknown, the corresponding situation is called the decision-making under ignorance [11].

From the literature review (see Section 2) follows that over the past 75 years, in the GT against nature, depending on the availability of a priori information about the probabilities $P(S_1), P(S_2), \dots, P(S_n)$, the following criteria have been proposed to select the optimal decision of DM:

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- Bayes criterion
- Wald criterion
- Savage criterion
- Hurwicz criterion
- Laplace criterion

The Bayes criterion is used under the condition of risk, i.e., when the a priori probabilities $P(S_1), P(S_2), \dots, P(S_n)$ are known. In the situation of ignorance, i.e., when the probabilities $P(S_1), P(S_2), \dots, P(S_n)$ are unknown, the criteria of Wald and Savage are usually used. Thus, the Bayes criterion and the criteria of Wald and Savage correspond to two extreme cases of risk (a priori probabilities are known) and ignorance (a priori probabilities are unknown). The Hurwicz criterion recommends that when choosing a decision not to be guided by either extreme optimism or extreme pessimism. According to the Hurwicz criterion, the weighted average between the losses of extreme optimism and extreme pessimism is minimized, with the “weight” being the pessimism-optimism index α between 0 and 1. The Laplace criterion is used when we can assume that any of the states of nature is no more likely than the other. It means that all states of nature are equally probable. Thus, Bayes, Wald, and Savage created optimality criteria for complete a priori certainty (probabilities of nature states are known) or complete a priori uncertainty (probabilities of nature states are unknown). The Hurwicz and Laplace criteria are subjective because either they require choosing the value of the pessimism-optimism index or the equality of all probabilities of the states of nature, which in most cases cannot be justified.

In practice, however, there are cases of partial uncertainty when there is a priori information about some probabilities of states of nature. For example, such a problem appears in artificial intelligence and machine learning at an increase of groups of observed objects provided that the a priori probabilities of dividing the source objects into groups (classes) are known. The well-known optimality criteria do not cover such cases. Therefore, there is a certain gap not covered by classical criteria. The application of the Wald, Savage, Hurwicz and Laplace criteria under conditions of partial a priori uncertainty may turn out to be ineffective due to ignoring information about a part of the probabilities of the states of nature. The purpose of this study is to develop a new criterion for choosing the optimal decision of DM in a game against nature under a partial a priori uncertainty. Contrary to previous studies, the proposed approach assumes that a part of a priori probabilities is known. Thus, the proposed criterion is a kind of bridge between the two extreme cases that meet the criteria of Bayes and Wald. Besides, the developed criterion generalizes the Wald, Savage, Hurwicz, Laplace, and Bayes criteria since the minimum average payoff (or risk) for each of the listed we can easily obtain from the derived formulas in this article. A practical example of a game against nature under a partial a priori uncertainty shows that the proposed criterion provides the choice of the decision that is also optimal in the condition of risk, which indicates the effective use of the vector of known a priori probabilities.

The article's organization is as follows: Section 2 provides a literature review on the existing optimality criteria in a game against nature. Section 3 considers the decision-making under a partial a priori uncertainty. Section 4 presents the results and discussion. The conclusions are formulated in Section 5. We place nomenclature and references at the end of the article.

2. Review

Suppose that the payoff matrix $C_L = \| C_{ij} \|$ ($i = \overline{1, m}, j = \overline{1, n}$) and the risk matrix $R = \| R_{ij} \|$ are known, where C_{ij} is the payoff associated with the decision φ_i and state of the nature S_j , and R_{ij} is the difference between the payoff that the DM could get with the i -th decision and the effect that the DM would get if he knew the actual state of nature, i.e., $R_{ij} = C_{ij} - \min_{i=\overline{1, m}} C_{ij}$. At first, let's formulate the existing criteria when matrix C_L is considered in terms of the cost.

Then, the Bayes decision φ_v is the optimal because it minimizes the average risk [9, 10].

$$\varphi_v^{opt} \Rightarrow \sum_{j=1}^n R_{vj}P(S_j) = \min_{i=\overline{1, m}} \left[\sum_{j=1}^n R_{ij}P(S_j) \right]. \tag{1}$$

Criterion (1) is formulated similarly for the case of the payoff matrix $\| C_{ij} \|$.

The uncertainty associated with the complete lack of information about the probabilities of nature is the worst or hopeless. In this situation, the criteria of Wald [12] and Savage [13] are used to determine the best decision of DM. From the standpoint of the Wald criterion, the DM considers nature as an aggressively disposed and consciously acting adversary. If, according to the game condition, in the original matrix C_L , component C_{ij} represents the DM's payoff, then he selects a decision that satisfies the minimax value.

$$\varphi_v^{opt} \Rightarrow \min_{i=\overline{1, m}} \max_{j=\overline{1, n}} C_{ij}. \tag{2}$$

According to the Wald criterion, we select the best decision from all the most unlucky results; this is the reinsurance position of extreme pessimism designed for the worst case.

When using the Savage minimax risk criterion, the choice of decision is similar to that according to the Wald criterion with the difference that the DM is guided not by the payoff matrix C_L but by the risk matrix R :

$$\varphi_v^{opt} \Rightarrow \min_{i=\overline{1, m}} \max_{j=\overline{1, n}} R_{ij}. \tag{3}$$

Application of the Savage criterion allows the DM by any means to avoid a significant risk when choosing a decision.

The Hurwicz criterion of pessimism-optimism [14, 15] recommends that when choosing a decision to be guided by some average result characterizing the state between extreme pessimism and unbridled optimism. It uses the following two assumptions: nature can be in the most favorable state with probability α and in the most unfavorable state with probability $(1 - \alpha)$, where α is the pessimism-optimism index, $\alpha \in [0, 1]$. The best decision by the Hurwicz criterion satisfies the following condition:

$$\varphi_v^{opt} \Rightarrow \min_{i=\overline{1, m}} \left[\alpha \min_{j=\overline{1, n}} C_{ij} + (1 - \alpha) \max_{j=\overline{1, n}} C_{ij} \right] \tag{4}$$

At $\alpha = 0$, Hurwicz's criterion coincides with the Wald criterion. For $\alpha = 1$, the choice of DM decision is subject to the least of all possible payoffs ($\min C_{ij}$).

For the risk matrix R , the Hurwicz criterion of pessimism-optimism has the following form:

$$\varphi_v^{opt} \Rightarrow \min_{i=\overline{1, m}} \left[\alpha \min_{j=\overline{1, n}} R_{ij} + (1 - \alpha) \max_{j=\overline{1, n}} R_{ij} \right] \tag{5}$$

For $\alpha = 0$, the choice of decision is carried out according to the Savage minimax risk criterion; when $\alpha = 1$ - by the condition of the least of all possible risks ($\min R_{ij}$).

The DM chooses the value of α from 0 to 1, depending on his (or her) tendency to pessimism or optimism.

Giang [16] proposed a further extension of the Hurwicz criterion (τ -anchor criterion), developing a decision-making model under uncertainty that combines Hurwicz–Arrow's decision-making theory under ignorance with the theory of subjective probability [17].

Fargier and Guillaume [18] proposed new qualitative criteria (R_* and R^*) in the decision-making problem under ignorance that arbitrates between pure pessimism and pure optimism Hurwicz criterion.

Krug et al. [19] compared the Hurwicz, τ -anchor, and R_* and R^* criteria and concluded that these criteria could be classified into two groups: τ -anchor and R^* , and R_* and Hurwicz. The first group criteria

always lead to extreme solutions. The second group criteria may lead to a compromised solution.

The Laplace criterion [20] postulates that if no information is available about the a priori probabilities $P(S_1), P(S_2), \dots, P(S_n)$, it is reasonable to assume that they are equal. Therefore, if there are n nature states, the probability of each is $1/n$.

The best by the Laplace criterion decision satisfies the following condition:

$$\varphi_v^{opt} \Rightarrow \frac{1}{n} \sum_{j=1}^n R_{v,j} = \min_{i=1,m} \left[\frac{1}{n} \sum_{j=1}^n R_{i,j} \right]. \tag{6}$$

If in the original matrix C_L by the condition of the problem, the result $C_{i,j}$ represents the profit of the DM, the Bayes, Wald, Hurwicz, and Laplace criteria have the following well-known formulations:

$$\varphi_v^{opt} \Rightarrow \sum_{j=1}^n C_{v,j} P(S_j) = \max_{i=1,m} \left[\sum_{j=1}^n C_{i,j} P(S_j) \right], \tag{7}$$

$$\varphi_v^{opt} \Rightarrow \max_{i=1,m} \min_{j=1,n} C_{i,j}, \tag{8}$$

$$\varphi_v^{opt} \Rightarrow \max_{i=1,m} \left[\alpha \max_{j=1,n} C_{i,j} + (1 - \alpha) \min_{j=1,n} C_{i,j} \right], \tag{9}$$

$$\varphi_v^{opt} \Rightarrow \max_{i=1,m} \left[\frac{1}{n} \sum_{j=1}^n C_{i,j} \right]. \tag{10}$$

In recent decades, the classic criteria for decision-making under conditions of uncertainty have been extended based on the belief functions introduced by Dempster [21, 22] instead of the probabilities of nature states. Schafer [23] applied the theory of the belief functions to the decision-making process under uncertainty. For each action, a belief function is constructed considering possible consequences. Then, a combination of beliefs and goals is determined to estimate the secured value and the rejected value for each action. Denoeux [24] conducted a detailed review of using the belief functions in the game theory against nature. The author considered extensions of classical criteria in the case of using belief functions. He analyzes the lower and upper expected utilities [25, 26, 27] to construct generalized maximin and maximax criteria. Further, the author shows how the Hurwicz criterion can be generalized by introducing the expected utility and some other improvements proposed in [28, 29, 30]. The author also analyses the pignistic criterion developed in [31, 32], which generalizes the Laplace criterion of insufficient reason when belief functions evaluate uncertainty. Finally, the author analyses the extension of the minimax regret criterion to belief functions considered in [33].

From the analysis of published studies on the criteria for choosing the optimal decision in a game against nature, we can draw the following conclusions:

1. The classical criteria of optimality, Bayes, Wald, and Savage, are applicable either for conditions of complete a priori certainty or complete a priori uncertainty. The Hurwitz and Laplace criteria are subjective because they require either to choose the value of the pessimistic-optimistic index or the equality of all probabilities of nature states.
2. In recent years, new criteria have been developed. The τ -anchor criterion introduces the tolerance for ignorance (τ) defined in the interval $[-\infty, +\infty]$. The selection of the optimal decision is based on the representation theorem [16]. The R^* and R^τ criteria generalize the maximin and maximax criteria introducing the optimism threshold defined in the interval $[-\infty, +\infty]$.

3. The classical criteria of optimality under conditions of uncertainty have been extended based on the belief functions, which replaced the probabilities of states of nature. However, the choice of these functions is mainly subjective [34].

4. The classical optimality criteria do not consider the case of a partial a priori uncertainty in which we know a part of the a priori probabilities of the states of nature, and the other part is unknown. We can find an example of such a classification problem in trade with the classification of customers and goods. As shown in Section 4, the solution to such a problem based on the proposed approach allows optimization of marketing decisions by reducing costs compared to classical criteria under ignorance.

3. Decision making under a partial a priori uncertainty

Cases of complete a priori certainty and ignorance are extreme. The intermediate cases correspond to a situation with partial uncertainty. Consider when $r < n$ a priori probabilities are known, and the values of $n - r$ probabilities are unknown. We introduce the following designations: $\mathbf{P}_0 = \overline{P(S_1), P(S_n)}$ and $\mathbf{P}_{n-r} = \overline{P(S_{r+1}), P(S_n)}$.

Let us represent the average payoff for φ_i decision of DM as follows:

$$C_{payoff}(\varphi_i) = \sum_{j=1}^r C_{i,j} P(S_j) + \sum_{\omega=r+1}^n C_{i,\omega} P(S_\omega). \tag{11}$$

The second term in (11) includes unknown probabilities $\overline{P(S_{r+1}), P(S_n)}$.

Definition 1. The decision $\varphi_v = \varphi_v^{pmx}$ is called partially-minimax if

$$C_{payoff}[\varphi_v, \mathbf{P}_{n-r}^*(\varphi_v)] = \min_{i=1,m} \left[\sum_{j=1}^r C_{i,j} P(S_j) + \max_{\mathbf{P}_{n-r} \in \Psi} \sum_{\omega=r+1}^n C_{i,\omega} P(S_\omega) \right], \tag{12}$$

where $\mathbf{P}_{n-r}^*(\varphi_v)$ is the least favorable distribution of vector \mathbf{P}_{n-r} at decision φ_v and Ψ is the set of feasible solutions.

Theorem 1. If

$$\begin{aligned} & \sum_{j=1}^r C_{v,j} P(S_j) + \left[1 - \sum_{j=1}^r P(S_j) \right] \max_{\omega=r+1,n} C_{v,\omega} \\ & = \min_{i=1,m} \left\{ \sum_{j=1}^r C_{i,j} P(S_j) + \left[1 - \sum_{j=1}^r P(S_j) \right] \max_{\omega=r+1,n} C_{i,\omega} \right\} \end{aligned} \tag{13}$$

then φ_v is the partially-minimax decision, i.e., $\varphi_v = \varphi_v^{pmx}$.

Proof. With the decision of φ_v and the least favorable distribution of the vector of unknown probabilities $\mathbf{P}_{n-r}^*(\varphi_v)$, the average payoff is of the following form:

$$C_{payoff}[\varphi_v, \mathbf{P}_{n-r}^*(\varphi_v)] = \sum_{j=1}^r C_{v,j} P(S_j) + \max_{\mathbf{P}_{n-r} \in \Psi} \sum_{\omega=r+1}^n C_{v,\omega} P(S_\omega). \tag{14}$$

Let us write the second term of (14) separately as

$$\max_{\mathbf{P}_{n-r} \in \Psi} \sum_{\omega=r+1}^n C_{v,\omega} P(S_\omega). \tag{15}$$

Optimization problem (15) is a linear programming formulation [35]. As is well-known from the linear programming theory, the solution to the problem (15) exists if the set Ψ is convex polyhedron and the maximized objective function is bounded above. Let us show that these conditions are satisfied. Let $\mathbf{P}_{n-r}^{(1)}$ and $\mathbf{P}_{n-r}^{(2)}$ be arbitrary points of the set Ψ and $0 \leq \lambda \leq 1$. Then the set Ψ is a convex if

$$\mathbf{P}_{n-r}^{(0)} = \lambda \mathbf{P}_{n-r}^{(1)} + (1 - \lambda) \mathbf{P}_{n-r}^{(2)}, \tag{16}$$

which is equivalent to the condition

$$\sum_{\omega=r+1}^n P^{(0)}(S_\omega) = 1 - \sum_{j=1}^r P(S_j). \tag{17}$$

Indeed,

$$\begin{aligned} \sum_{\omega=r+1}^n P^{(0)}(S_\omega) &= \sum_{\omega=r+1}^n [\lambda P^{(1)}(S_\omega) + (1-\lambda)P^{(2)}(S_\omega)] = \lambda \sum_{\omega=r+1}^n P^{(1)}(S_\omega) + (1-\lambda) \sum_{\omega=r+1}^n P^{(2)}(S_\omega) \\ &= \lambda \left[1 - \sum_{j=1}^r P(S_j) \right] + (1-\lambda) \left[1 - \sum_{j=1}^r P(S_j) \right] = 1 - \sum_{j=1}^r P(S_j). \end{aligned} \tag{18}$$

The set Ψ is a convex polyhedron since it is formed by the intersection of a finite number of half-spaces $P(S_\omega) \geq 0$ and the hyperplane

$$\sum_{\omega=r+1}^n P(S_\omega) = 1 - \sum_{j=1}^r P(S_j). \tag{19}$$

The second term of function (11) is bounded above since the following inequality always holds:

$$\sum_{\omega=r+1}^n C_{i,\omega} P(S_\omega) \leq \max_{\omega=r+1, \dots, n} C_{i,\omega}. \tag{20}$$

Thus, the solution to the optimization problem (15) exists. As is well known, the solution to the linear programming problem (15) is located at one of the vertices of the set Ψ . Let us show that the points

$$\mathbf{P}_{n-r}^{(r+1)} = \left[1 - \sum_{j=1}^r P(S_j), \overbrace{0, 0, \dots, 0}^{n-r-1} \right], \mathbf{P}_{n-r}^{(r+2)} = \left[0, 1 - \sum_{j=1}^r P(S_j), \overbrace{0, 0, \dots, 0}^{n-r-2} \right], \dots, \mathbf{P}_{n-r}^{(n)} = \left[\overbrace{0, 0, \dots, 0}^{n-r-1}, 1 - \sum_{j=1}^r P(S_j) \right] \tag{21}$$

are the vertices of the set Ψ . Indeed, each point $\mathbf{P}_{n-r}^{(k)}$ ($k = \overline{r+1, n}$) satisfies a system of $n-r$ linearly independent constraints

$$P(S_k) = 1 - \sum_{j=1}^r P(S_j), P(S_\omega) = 0, \omega = \overline{r+1, n}, \omega \neq k, \tag{22}$$

as exact equalities and, therefore, is the vertex of the set Ψ . At an arbitrary vertex $\mathbf{P}_{n-r}^{(k)}$, the average payoff is defined as

$$C_{payoff}(\varphi_i, \mathbf{P}_{n-r}^{(k)}) = \sum_{j=1}^r C_{ij} P(S_j) + \left[1 - \sum_{j=1}^r P(S_j) \right] C_{ik}. \tag{23}$$

Since the number of vertices is finite, in one of them, the second term of function (23) reaches the maximum, which corresponds to the least favorable distribution of the vector of unknown probabilities at decision φ_i . Therefore,

$$C_{paooff}[\varphi_i, \mathbf{P}_{n-r}^*(\varphi_i)] = \sum_{j=1}^r C_{ij} P(S_j) + \left[1 - \sum_{j=1}^r P(S_j) \right] \max_{\omega=r+1, \dots, n} C_{i,\omega}, \tag{24}$$

where $\mathbf{P}_{n-r}^*(\varphi_i)$ is the least favorable distribution of vector \mathbf{P}_{n-r} at decision φ_i . It follows from condition (13) that the minimum of average payoff with the least favorable distribution of unknown probabilities will correspond to the decision φ_ν of DM, i.e.,

$$C_{payoff}[\varphi_\nu, \mathbf{P}_{n-r}^*(\varphi_\nu)] = \min_{i=1, \dots, m} \left\{ \sum_{j=1}^r C_{ij} P(S_j) + \left[1 - \sum_{j=1}^r P(S_j) \right] \max_{\omega=r+1, \dots, n} C_{i,\omega} \right\}. \tag{25}$$

Thus, φ_ν is the partially-minimax decision, i.e., $\varphi_\nu = \varphi_\nu^{pmax}$. The theorem is proved. The partially-minimax decision guarantees that for any a priori distribution of the vector \mathbf{P}_{n-r} other than $\mathbf{P}_{n-r}^*(\varphi_\nu)$, the average payoff will not exceed the value determined by (25). If we use the risk matrix $\| R_{ij} \|$, then (13) takes the following form:

$$\begin{aligned} &\sum_{j=1}^r R_{\nu j} P(S_j) + \left[1 - \sum_{j=1}^r P(S_j) \right] \max_{\omega=r+1, \dots, n} R_{\nu, \omega} \\ &= \min_{i=1, \dots, m} \left\{ \sum_{j=1}^r R_{ij} P(S_j) + \left[1 - \sum_{j=1}^r P(S_j) \right] \max_{\omega=r+1, \dots, n} R_{i, \omega} \right\}. \end{aligned} \tag{26}$$

As it is easy to see, we can obtain all the known criteria from (13) and (26). Indeed, for $r = n$, from (26) the Bayes criterion (1) follows; for $r = 0$, from (13) and (26), we obtain the Wald criterion (2) and the Savage criterion (3); if $r = 1$, from (13) and (26) we get a criterion that is close to

the Hurwitz criteria (4) and (5), and finally, from (26) when $r = n$ and $P(S_j) = 1/n$ we obtain the Laplace criterion (6).

Let us now consider when the original matrix $\| C_{ij} \|$ represents the profit of the DM.

Definition 2. The decision $\varphi_\nu = \varphi_\nu^{pmin}$ is called partially-maximin if

$$C_{prof}[\varphi_\nu, \mathbf{P}_{n-r}^*(\varphi_\nu)] = \max_{i=1, \dots, m} \left\{ \sum_{j=1}^r C_{ij} P(S_j) + \min_{\mathbf{P}_{n-r} \in \Psi} \sum_{\omega=r+1}^n C_{i,\omega} P(S_\omega) \right\}, \tag{27}$$

where C_{prof} is the average profit.

Theorem 2. If

$$\begin{aligned} &\sum_{j=1}^r C_{\nu j} P(S_j) + \left[1 - \sum_{j=1}^r P(S_j) \right] \min_{\omega=r+1, \dots, n} C_{\nu, \omega} \\ &= \max_{i=1, \dots, m} \left\{ \sum_{j=1}^r C_{ij} P(S_j) + \left[1 - \sum_{j=1}^r P(S_j) \right] \min_{\omega=r+1, \dots, n} C_{i, \omega} \right\} \end{aligned} \tag{28}$$

then φ_ν is the partially-maximin decision, i.e., $\varphi_\nu = \varphi_\nu^{pmin}$. The Proof of statement (28) is similar to the proof of Theorem 1. We can see from (28) that the proposed criterion generalizes criteria (7)–(10). Indeed, for $r = n$, from (28), the criterion of maximum average profit (7) follows; for $r = 0$, from (28), we obtain the maximin criterion (8); for $r = 1$, from (28), we get a criterion that is close to the Hurwicz criterion (9), and, finally, when $r = n$ and $P(S_j) = 1/n$ we obtain the Laplace criterion (10). Thus,

Table 1. Matrix of decisions and payoffs.

Decision φ_i /Payoff in the state S_i	S_1	S_2	S_3	S_4	S_5
φ_1	11,000	11,100	11,500	13,200	13,400
φ_2	11,900	12,700	13,000	13,500	14,000
φ_3	10,900	11,800	12,000	12,100	13,300
φ_4	12,200	12,500	13,100	13,400	13,900
φ_5	12,600	13,100	12,200	11,900	11,600

Table 2. Results of calculations.

Decisions	Expected gift costs (\$)			
	Proposed criterion	Wald criterion	Hurwicz criterion	Bayes criterion
φ_1	11,935	13,400	12,200	11,925
φ_2	12,960	14,000	12,950	12,935
φ_3	12,085	13,300	12,100	12,025
φ_4	12,990	13,900	13,050	12,965
φ_5	12,395	13,100	12,350	12,305

the proven mathematical relations (13), (26), and (28) generalize the relations for the minimum average payoff (or Bayes risk) and the maximum average income that meet the criteria of Bayes, Wald, Savage, Hurwitz, and Laplace.

4. Results and discussion

Let us consider the features of the application of the proposed criterion in the following example. A car dealer decided to run an advertising campaign “gift for buying a car.” Five possible gift options were chosen: φ_1 – a bicycle, φ_2 – a TV, φ_3 – a video camera, φ_4 – a refrigerator, and φ_5 – a paid tourist trip. The costs of this advertising company depend on the activity of car purchases during the campaign. The marketing department predicts five possible options for consumer activity: S_1, S_2, \dots, S_5 . Table 1 presents the total costs for each gift option depending on the consumer activity option.

Let vector of a priori probabilities of states S_1, S_2, \dots, S_5 be $P_0 = 0.3, 0.2, 0.15, X, X$ and $P_{n-r} = P_{5-3} = X, X$, i.e., the probabilities $P(S_4)$ and $P(S_5)$ are unknown. Using the proposed criterion, the DM should make the best decision on choosing a gift for this car dealer.

Using (25), we calculate the average payoff in choosing each of the gifts. Table 2 shows the calculation results for the proposed partially-minimax decision and the decisions of Wald, Hurwicz, and Bayes criteria. When calculating average payoff by the Hurwicz criterion, we, as usual, set $\alpha = 0.5$. Since the sum of the known probabilities is 0.65, then just for comparison we assumed that for the Bayes criterion $P(S_4) = 0.05$ and $P(S_5) = 0.3$.

As can be seen in Table 2, according to the proposed criterion, the optimal decision is to choose φ_1 , i.e., to choose a bicycle as a gift. According to the Wald criterion, optimal is the decision φ_5 , i.e., choosing a tourist trip as a gift. Optimal by the Hurwicz criterion is the φ_3 decision, i.e., the choice of a video camera. Finally, according to the Bayes criterion, the best decision is also φ_1 . The minimum expected gift costs for each criterion are bolded in Table 2. Thus, the proposed approach in this example provides the choice of a decision that is also optimal under risk, which indicates the effectiveness of the proposed approach in the condition of partial a priori uncertainty.

Figure 1 shows a graphical presentation of the minimum average payoff calculated according to the criteria of Wald (curve 1), Hurwicz (curve 2), Bayes (curve 3), and proposed (curve 4). As we can see in Figure 1, the proposed criterion gives the best approximation to the value of the average payoff calculated by the Bayes criterion under the condition that all a priori probabilities are known.

We should also note that the minimum average payoff by the Hurwicz criterion leads to the same payoff as by the Bayes criterion at $\alpha = 0.575$, but this value of α is not known a priori. Moreover, the optimal by the Bayes criterion is the decision φ_1 , and not φ_3 as according to the Hurwicz. Thus, the proposed criterion gives, firstly, the coincidence of the optimal decision with the Bayes decision under the assumption that all a priori probabilities are known and, secondly, a close value of the minimum payoff.

We should note that the proposed approach to decision-making covers only those cases of partial a priori uncertainty in which at least one or more probabilities of states of nature are known. However, there are other cases of partial a priori uncertainty in practice, in which the existing information can significantly increase the effectiveness of the made decisions. For

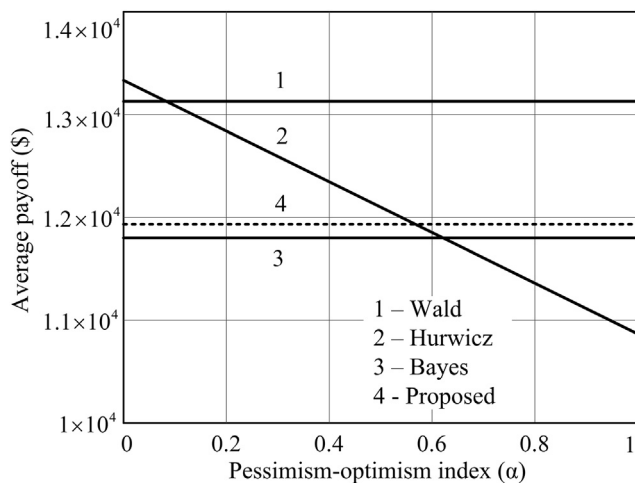


Figure 1. A graphical presentation of the average payoff calculated according to the criteria of Wald (curve 1), Hurwicz (curve 2), Bayes (curve 3), and proposed (curve 4).

example, sometimes, the probabilities of states of nature are functions of the distribution densities of random variables, and some of these densities are unknown. For this and other partial a priori uncertainty situations, one can prove theorems similar to [Theorems 1](#) and [2](#).

5. Conclusion

The article has proposed a new criterion for choosing the optimal decision in a game against nature under a partial a priori uncertainty. We have proved the theorems allowing us to choose the optimal decision in a situation when only a part of the a priori probabilities of the states of nature is known. The proposed criterion generalizes the Wald, Savage, Hurwicz, Bayes, and Laplace criteria since the minimum average payoff (or risk) for each of the listed we can easily obtain from the derived formulas in this article. Indeed, if all the a priori probabilities of the states of nature are unknown, then the Wald and Savage criteria follow from the proved theorems. If only one a priori probability is known or the pessimism-optimism index is introduced, we obtain the very close criterion to the Hurwicz. With all the a priori probabilities of the states of nature known, we obtain the Bayes criterion. And, finally, if there is no information about the prior probabilities of the states of nature, then assuming them to be equal, we obtain the Laplace criterion. A practical example of a game against nature under a partial a priori uncertainty shows that the proposed approach provides the choice of the decision that is also optimal under risk, i.e., in conditions of complete a priori certainty, which indicates the effective use of the vector of known a priori probabilities. Besides, it has been shown that the proposed criterion also provides a value of minimum average payoff very close to the amount of payoff by the Bayes criterion; the latter we calculated under the assumption that all a priori probabilities are known. Finally, we should note that the proposed criterion for choosing the optimal decision in a game against nature under conditions of a partial a priori uncertainty is a kind of bridge between the two extreme cases that meet the criteria of Savage and Bayes. On this bridge, we can smoothly move from the Savage criterion to the Bayes criterion as statistical information on the probabilities of nature states has been accumulating. Such a movement cannot be made using the Hurwicz criterion since the choice of the pessimism-optimism index is very subjective.

Our future work will be related to considering a situation in which the a priori probabilities of the states of nature are functions of the distribution densities of random variables, and some of these densities are unknown. For this situation of a partial a priori uncertainty, we plan to prove theorems that allow us to choose the optimal decision of the decision-maker.

Declarations

Author contribution statement

V. Ulansky: Conceived and designed the experiments; Analyzed and interpreted the data; Wrote the paper.

A. Raza: Performed the experiments; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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Additional information

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References

- [1] C.H. Chiu, T.M. Choi, D. Li, Price wall or war: the pricing strategies for retailers, *IEEE Trans. Syst., Man, Cybern. A, Syst., Humans* 39 (2) (2009) 331–343.
- [2] A. Chinchuluun, A. Karakitsiou, A. Mavrommati, Game theory models and their applications in inventory management and supply chain, in: Altannar Chinchuluun, Athanasia Karakitsiou, Athanasia Mavrommati (Eds.), *Pareto Optimality, Game Theory and Equilibria in, Springer Optimization and its Applications*, 2008, pp. 833–865.
- [3] F. Ding, T. Ma, Dynamic relationship between tourism and homogeneity of tourist destinations, *IEEE Access* 6 (2018) 51470–51476.
- [4] I.A. Villalobos, A.S. Poznyak, A.M. Tamayo, Urban traffic control problem: a game theory approach, *IFAC Proc. Volumes* 41 (2) (2008) 7154–7159.
- [5] G.F. Luger, *Artificial Intelligence: Structures and Strategies for Complex Problem Solving*, Pearson Education, Inc., Boston, 2009.
- [6] G. Bacci, S. Lasaulce, W. Saad, L. Sanguinetti, Game theory for networks: a tutorial on game-theoretic tools for emerging signal processing applications, *IEEE Signal Process. Mag.* 33 (1) (2016) 94–119.
- [7] J.B. Cruz, M.A. Simaan, A. Gacic, H. Jiang, B. Letellier, M. Li, et al., Game-theoretic modeling and control of a military air operation, *IEEE Trans. Aero. Electron. Syst.* 37 (4) (2001) 1393–1405.
- [8] P. Dutta, *Strategies and Games: Theory and Practice*, The MIT Press, Cambridge MA, 1999.
- [9] R.D. Luce, H. Raiffa, *Games and Decisions. Introduction and Critical Survey*, Dover Publications, New York, 1989.
- [10] D. Blackwell, M. Girshick, *Theory of Games and Statistical Decisions*, Dover Publications, Mineola, 1980.
- [11] R.A. Aliev, Brief review of theories of decision making, in: *Fundamentals of the fuzzy logic-based generalized theory of decisions*, *Stud. Fuzziness Soft Comput.* 293 (2013) 65–88.
- [12] A. Wald, Statistical decision functions which minimize the maximum risk, *Ann. Math.* 46 (2) (1945) 265–280.
- [13] L. Savage, The theory of statistical decision, *J. Am. Stat. Assoc.* 46 (1951) 55–67.
- [14] L. Hurwicz, The generalized Bayes minimax principle: a criterion for decision making under uncertainty Cowles Commission Discussion Paper, *Statistics* 335 (1951) 1–7.
- [15] K. Arrow, L. Hurwicz, Appendix: an optimality criterion for decision-making under ignorance, in: Kenneth J. Arrow, Leonid Hurwicz (Eds.), *Studies in Resource Allocation Processes*, 1977, pp. 461–472.
- [16] P.H. Giang, Decision making under uncertainty comprising complete ignorance and probability, *Int. J. Approx. Reason.* 62 (2015) 27–45.
- [17] F.J. Anscombe, R.J. Aumann, A definition of subjective probability, *Ann. Math. Stat.* 34 (1963) 199–205.
- [18] H. Fargier, R. Guillaume, Sequential decision making under ordinal uncertainty: a qualitative alternative to the Hurwicz criterion, *Int. J. Approx. Reason.* 116 (2020) 1–18.
- [19] Z. Krug, R. Guillaume, O. Battaia, Decision under ignorance: a comparison of existing criteria, in: M.J. Lesot, et al. (Eds.), *Information Processing and Management of Uncertainty in Knowledge-Based Systems in: IPMU 2020, Communications in Computer and Information Science*, 1237, 2020, pp. 1–14.
- [20] T. Biswas, Games against nature and the role of information in decision-making under uncertainty. *Decision-making under Uncertainty*, 1997, pp. 185–196.
- [21] A.P. Dempster, Upper and lower probabilities generated by a random closed interval, *Ann. Math. Stat.* 39 (3) (1968) 957–966.
- [22] A.P. Dempster, A generalization of Bayesian inference (with discussion), *J. Roy. Stat. Soc.* 30 (2) (1968) 205–247.
- [23] G. Shafer, Constructive decision theory, *Int. J. Approx. Reason.* 79 (2016) 45–62.
- [24] T. Denoeux, Decision-making with belief functions: a review, *Int. J. Approx. Reason.* 109 (2019) 87–110.
- [25] A.P. Dempster, Upper and lower probabilities induced by a multivalued mapping, *Ann. Math. Stat.* 38 (1967) 325–339.
- [26] A.P. Dempster, A. Kong, *Comment, Stat. Sci.* 2 (1) (1987) 32–36.
- [27] G. Shafer, Constructive probability, *Synthese* 48 (1) (1981) 1–60.
- [28] J.Y. Jaffray, Linear utility theory for belief functions, *Oper. Res. Lett.* 8 (2) (1989) 107–112.
- [29] T.M. Strat, Decision analysis using belief functions, *Int. J. Approx. Reason.* 4 (5-6) (1990) 391–417.
- [30] W. Ma, X. Luo, Y. Jiang, An Ambiguity Aversion Model for Decision Making under Ambiguity, *Thirty-First AAAI Conference on Artificial Intelligence*, San Francisco, 2017, pp. 614–621.
- [31] P. Smets, R. Kennes, The transferable belief model, *Artif. Intell.* 66 (1994) 191–243.
- [32] P. Smets, Decision making in a context where uncertainty is represented by belief functions, in: Rajendra P. Srivastava, J. Theodore, Mock (Eds.), *Belief Functions in Business Decisions in: Studies in Fuzziness and Soft Computing*, 2002, pp. 17–61.
- [33] R.R. Yager, Decision making using minimization of regret, *Int. J. Approx. Reason.* 36 (2) (2004) 109–128.
- [34] G. Shafer, Perspectives on the theory and practice of belief functions, *Int. J. Approx. Reason.* 4 (1990) 323–362.
- [35] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, New York, 1998.