Chapter 5
Switching Control Strategies

This chapter is motivated by the application of control strategies to eradicate epidemics. The previous switched epidemic models are reintroduced with continuous control (e.g., vaccination of newborns continuously in time) or switching control (i.e., piecewise continuous application of vaccination or treatment schemes) for evaluation and optimization. As discussed earlier, infectious disease models are a crucial component in designing and implementing detection, prevention, and control programs (e.g., WHO's program against smallpox, leading to its global eradication by 1977). The switched SIR model is first returned to analyze vaccination of the susceptible group (e.g., newborns or the entire cohort). Subsequently, the developed theoretical methods are applied to the switched SIR model with a treatment program in effect. Common Lyapunov functions are used to provide controlled eradication of diseases modeled by the so-called SEIR (Susceptible-Exposed-Infected-Recovered) model with seasonal variations captured by switching. A screening process, where traveling individuals are examined for infection, is proposed and studied for the switched multi-city model of the previous chapter. Switching control of diseases such as dengue and chikungunya, which are spread via mosquito–human interactions, is also investigated.

5.1 Vaccination of the Susceptible Group

The majority of developed countries have in place cohort immunization programs (also called time-constant immunization or vaccination programs here) for a number of diseases with varying degrees of success [1]. For example, measles immunization in many areas of the Western world recommends vaccinations at 15 months of age and 6 years of age [139]. Studies analyzing this type of program mathematically can be found in, for example, [4, 69, 75, 83, 92, 100–102, 110, 138, 147, 173].

The mathematical formulation of a newborn continuous vaccination strategy takes the following form [69, 138, 173]: assume that a fraction $\rho \in [0, 1]$ of susceptible newborns are vaccinated, moving them to the recovered class R, continuously in time. In this model, natural and vaccine-acquired immunity are viewed as the same. Applied to the classical endemic model SIR model (3.9) gives

$$\dot{S}(t) = (1 - \rho)\mu - \beta S(t)I(t) - \mu S(t),$$

$$\dot{I}(t) = \beta S(t)I(t) - (g + \mu)I(t), \tag{5.1}$$

$$\dot{R}(t) = \mu\rho + gI(t) - \mu R(t).$$

Newborn vaccinations reduce the birth rate μ of the susceptible population to $(1 - \rho)\mu$. Equation (5.1) admits the following equilibria: a disease-free solution $(1 - \rho, 0, \rho) \equiv Q_{\text{DFS}}^{(5.1)}$ and an endemic solution

$$Q_{\text{ES}}^{(5.1)} \equiv \left(\frac{\mu + g}{\beta}, \frac{\mu}{\beta}(R_0^{(5.1)} - 1), \frac{g}{\beta}(R_0^{(5.1)} - 1) + \rho \right), \tag{5.2}$$

where

$$R_0^{(5.1)} \equiv \frac{\beta(1 - \rho)}{\mu + g}$$

is the basic reproduction number of (5.1). The underlying mechanics of the newborn vaccination can be translated into something more familiar by the following change of variables [69]: let $S \equiv \widehat{S}(1 - \rho)$, $I \equiv \widehat{I}(1 - \rho)$, and $R \equiv \widehat{R}(1 - \rho) + \rho$. Then (5.1) is equivalently written as

$$\frac{d\widehat{S}}{dt}(t) = \mu - \beta(1 - \rho)\widehat{S}(t)\widehat{I}(t) - \mu\widehat{S}(t),$$

$$\frac{d\widehat{I}}{dt}(t) = \beta(1 - \rho)\widehat{S}(t)\widehat{I}(t) - (g + \mu)\widehat{I}(t), \tag{5.3}$$

$$\frac{d\widehat{R}}{dt}(t) = g\widehat{I}(t) - \mu\widehat{R}(t).$$

This control strategy therefore has the effect of transforming the contact rate from β to $\beta(1 - \rho)$. This is most clearly reflected in the basic reproduction number $R_0^{(5.1)}$, which dictates the usual threshold for long-term behavior (i.e., disease eradication versus endemicity). The condition $R_0^{(5.1)} < 1$ yields a critical vaccination rate to achieve herd immunity [63]:

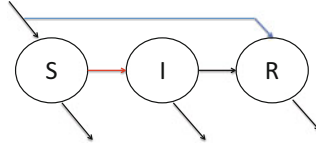$$\rho_{\text{crit}} \equiv 1 - 1/R_0^{(3.8)} \in [0, 1).$$

Fig. 5.1 Flow of the switched SIR system with newborn vaccinations (5.4). The *red line* represents the horizontal transmission and the *blue line* represents the vaccination scheme

With seasonality modeled by a switched contact rate β_σ (where $\sigma \in \mathscr{S}_{\text{dwell}}$), the model is given by

$$
\begin{aligned}
\dot{S}(t) &= \mu(1 - \rho) - \beta_\sigma S(t)I(t) - \mu S(t), \\
\dot{I}(t) &= \beta_\sigma S(t)I(t) - (g + \mu)I(t), \\
\dot{R}(t) &= gI(t) - \mu R(t) + \rho\mu, \\
(S(0), I(0), R(0)) &= (S_0, I_0, R_0),
\end{aligned}
\tag{5.4}
$$

with physical domain

$$
D_{(5.4)} \equiv \{(S, I, R) \in \mathbb{R}^3_+ : S + I + R = 1\} = D_{(3.8)}
$$

(which is positively invariant to (5.4)). See Fig. 5.1 for the flow diagram associated with (5.4).

Although the present focus is on disease eradication by control, we mention that each mode admits an endemic equilibria of the form

$$
Q_{\text{ES}}^{(5.4),i} \equiv \left(\frac{\mu + g}{\beta_i}, \frac{\mu}{\beta_i}(R_0^{(5.4),i} - 1), \frac{g}{\beta_i}(R_0^{(5.4),i} - 1) + \rho \right), \quad \forall i \in \mathscr{M},
\tag{5.5}
$$

with mode basic reproduction numbers

$$
R_0^{(5.4),i} \equiv \frac{\beta_i}{\mu + g}(1 - \rho) = (1 - \rho)R_0^{(3.8),i}, \quad \forall i \in \mathscr{M}.
\tag{5.6}
$$

Recall Theorem 3.1, in which the switched SIR model (3.8) was shown to achieve eradication if

$$
R_0^{(3.8)} = \frac{1}{\omega} \sum_{i=1}^m R_0^{(3.8),i} \tau_i = \frac{\sum_{i=1}^m \beta_i \tau_i}{\omega(\mu + g)} < 1
$$

whenever $\sigma \in \mathscr{S}_{\text{periodic}}(\omega)$ (in such a way that the disease-free solution is globally asymptotically I-stable). Moreover, $R_0^{(3.8)} > 1$ implies persistence of the disease (see Theorem 3.4). In contrast, consider the following theorem.

Theorem 5.1 *If $\sigma \in \mathscr{S}_{\text{periodic}}(\omega)$ and*

$$R_0^{(5.4)} \equiv \frac{1}{\omega} \sum_{i=1}^{m} R_0^{(5.4),i} \tau_i = \frac{\sum_{i=1}^{m} \beta_i (1-\rho) \tau_i}{\omega(\mu + g)} < 1,$$

then the disease-free solution $Q_{\text{DFS}}^{(5.4)} \equiv (1 - \rho, 0, \rho)$ of the switched SIR system with newborn vaccination (5.4) is globally attractive and globally asymptotically I-stable in the meaningful domain.

Proof By (5.4),

$$\dot{S}(t) = \mu(1 - \rho) - \beta_\sigma S(t) I(t) - \mu S(t),$$
$$\leq \mu(1 - \rho) - \mu S(t).$$

Consider the comparison system

$$\dot{x}(t) = \mu(1 - \rho) - \mu x(t),$$
$$x(0) = S_0, \tag{5.7}$$

which has unique solution $x(t) \equiv (S_0 - (1 - \rho)) \exp(-\mu t) + (1 - \rho)$ that satisfies

$$\lim_{t \to \infty} x(t) = 1 - \rho.$$

By the comparison theorem, for any $\epsilon > 0$ there exists a time $t^* > 0$ such that $S(t) \leq x(t) \leq 1 - \rho + \epsilon$ for $t \geq t^*$, and so

$$\dot{I}(t) = \beta_\sigma S(t) I(t) - (\mu + g) I(t),$$
$$\leq (\beta_\sigma [1 - \rho + \epsilon] - \mu - g) I(t),$$
$$\equiv \lambda_{\sigma,\epsilon} I(t),$$

where $\lambda_{i,\epsilon} \equiv \beta_i(1 - \rho) - g - \mu + \epsilon \beta_i$ for each $i \in \mathcal{M}$. Choose N to be the smallest integer such that $N\omega > t^*$. Then, as in the proof of Theorem 3.1,

$$I((N + 1)\omega) \leq I(N\omega) \exp\left(\sum_{i=1}^{m} \lambda_{i,\epsilon} \tau_i\right),$$
$$= \eta(\epsilon) I(N\omega),$$

where

$$\eta(\epsilon) \equiv \exp\left(\sum_{i=1}^{m} \lambda_{i,\epsilon} \tau_i\right).$$

Now, $R_0^{(5.4)} < 1$ gives that $\sum_{i=1}^{m} \lambda_i \tau_i < 0$. Then it holds that $\sum_{i=1}^{m} \lambda_i \tau_i < -\delta$ for some $\delta > 0$, and

$$\sum_{i=1}^{m} \lambda_{i,\epsilon} \tau_i = \sum_{i=1}^{m} \lambda_i \tau_i + \epsilon \sum_{i=1}^{m} \beta_i \tau_i < -\delta + \epsilon \sum_{i=1}^{m} \beta_i \tau_i.$$

Choosing

$$\epsilon = \frac{\delta}{2 \sum_{i=1}^{m} \beta_i \tau_i}$$

implies that $\eta(\epsilon) < 1$. It can be similarly shown that $I((N+h+1)\omega) \leq \eta I((N+h)\omega)$ for any integer $h \in \mathbb{N}$ and the rest of the proof of Theorem 3.1 may be applied to produce the result.

The threshold condition $R_0^{(5.4)} < 1$ defines a critical newborn vaccination rate:

$$R_0^{(5.4)} = \frac{\sum_{i=1}^{m} \beta_i (1 - \rho) \tau_i}{\omega(\mu + g)} < 1$$

implies that

$$(1 - \rho) \frac{\sum_{i=1}^{m} \beta_i \tau_i}{\omega(\mu + g)} = (1 - \rho) R_0^{(3.8)} < 1,$$

and hence the critical rate is given as

$$\rho_{\text{crit}} \equiv 1 - 1/R_0^{(3.8)} = 1 - \frac{\omega(\mu + g)}{\sum_{i=1}^{m} \beta_i \tau_i} \in [0, 1),$$

which guarantees disease eradication. That is, if the disease persists in the switched SIR model ($R_0^{(3.8)} > 1$), then disease eradication can be achieved by newborn vaccinations as long as $\rho \geq \rho_{\text{crit}}$. If $R_0^{(3.8)} = 1$ then $\rho_{\text{crit}} = 0$ and as $R_0^{(3.8)} \to \infty$ then $\rho_{\text{crit}} \to 1$. Other controlled eradication results can also be shown under different classes of switching rules, as in Sect. 3.4 (i.e., if $\sigma \in \mathscr{S}_{\text{dwell}}$ according to Theorems 3.2 and 3.3).

Example 5.1 Consider (5.4) with $\mathscr{M} = \{1, 2\}$, σ defined as in (3.37), and initial conditions $(S_0, I_0, R_0) = (0.75, 0.25, 0)$. Motivated by the measles parameters of [138], let $\beta_1 = 18$, $\beta_2 = 3$, $g = 1$, $\mu = 0.1$, and $\rho = 0$ which give that $R_0^{(5.4)} = 6.136$ and persistence of the disease, i.e., by Theorem 3.4 (see Fig. 5.2 for an illustration; the solution I oscillates approximately between the endemic minimum and maximum, $I_{\min} = 0.0576$ and $I_{\max} = 0.0854$). With $\rho = 0.85$ ($\rho_{\text{crit}} = 0.84$), $R_0^{(5.4)} = 0.920$ and the disease is eradicated according to Theorem 5.1 (see Fig. 5.3).
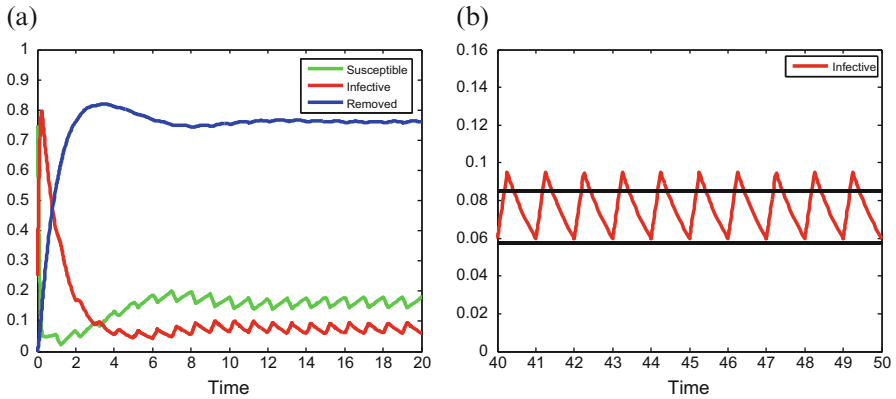
(a)

(b)

Fig. 5.2 Simulation of Example 5.1. (**a**) $\rho = 0$. (**b**) The *black lines* represent $I_{\min} = 0.0576$ and $I_{\max} = 0.0854$
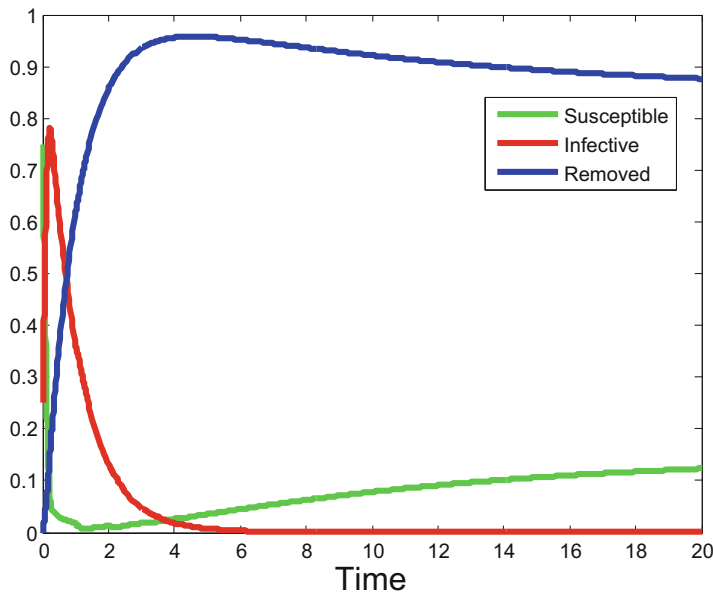

Fig. 5.3 Simulation of Example 5.1 with $\rho = 0.85$

Instead of a newborn vaccination strategy, consider an immunization strategy applied to the entire susceptible cohort in an SIR model (3.8). Mathematically, suppose that the susceptible population is vaccinated at a rate $v \geq 0$ per unit time and again assume permanent immunity is acquired through vaccination (which is
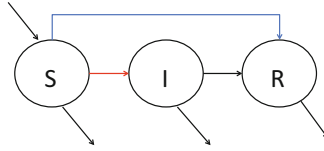
Fig. 5.4 Flow of the switched SIR system with susceptible vaccinations (5.8). The *red line* represents the horizontal transmission and the *blue line* represents the vaccination scheme

indistinguishable from naturally acquired immunity). Thus, the dynamics of the model

$$
\begin{aligned}
\dot{S}(t) &= \mu - \beta_\sigma S(t)I(t) - \mu S(t) - vS(t), \\
\dot{I}(t) &= \beta_\sigma S(t)I(t) - (g + \mu)I(t), \\
\dot{R}(t) &= gI(t) + vS(t) - \mu R(t), \\
(S(0), I(0), R(0)) &= (S_0, I_0, R_0),
\end{aligned}
\tag{5.8}
$$

are investigated. The flow diagram for (5.8) is shown in Fig. 5.4.

As before, the meaningful domain is unchanged ($D_{(5.8)} = D_{(3.8)}$) and positively invariant (thus giving a global unique solution for appropriate initial conditions). However, the set of mode basic reproduction numbers of (5.8) is changed from the uncontrolled switched SIR model (as expected):

$$
R_0^{(5.8),i} \equiv \frac{\beta_i}{\mu + g}\frac{\mu}{\mu + v}, \quad \forall i \in \mathcal{M},
\tag{5.9}
$$

while the disease-free solution is calculated as

$$
Q_{\mathrm{DFS}}^{(5.8)} \equiv \left(\frac{\mu}{\mu + v}, 0, 1 - \frac{\mu}{\mu + v} \right).
\tag{5.10}
$$

Each mode $i \in \mathcal{M}$ admits an endemic equilibrium:

$$
Q_{\mathrm{ES}}^{(5.8),i} \equiv \left(\frac{\mu + g}{\beta_i}, \frac{\mu}{\mu + g}\left(1 - \frac{1}{R_0^{(5.8),i}} \right), \frac{\mu}{\mu + g}\left(1 - \frac{1}{R_0^{(5.8),i}} \right) + \frac{v}{\mu}\frac{\mu + g}{\beta_i} \right).
\tag{5.11}
$$

Different from the newborn vaccination scheme, (5.8) gives that

$$
\begin{aligned}
\dot{S}(t) &= \mu - \beta_\sigma S(t)I(t) - \mu S(t) - vS(t), \\
&\le \mu - (\mu + v)S(t),
\end{aligned}
$$

which motivates analyzing the comparison system

$$\dot{x}(t) = \mu - (\mu + v)x(t),$$
$$x(0) = S_0,$$

(5.12)

that has unique solution

$$x(t) \equiv \left(S_0 - \frac{\mu}{\mu + v} \right) \exp(-(\mu + v)t) + \frac{\mu}{\mu + v},$$

satisfying

$$\lim_{t \to \infty} x(t) = \frac{\mu}{\mu + v}$$

(the first component of the disease-free solution). Thus, the same analysis as in Theorem 5.1 yields that the solution of (5.8) converges to the disease-free solution $Q_{\mathrm{DFS}}^{(5.8)}$ (which is globally asymptotically I-stable in the meaningful domain) if

$$\mathrm{R}_0^{(5.8)} \equiv \frac{1}{\omega} \sum_{i=1}^m \mathrm{R}_0^{(5.8),i} \tau_i = \frac{\sum_{i=1}^m \beta_i \tau_i}{\omega(\mu + g)} \frac{\mu}{\mu + v};$$

the critical cohort immunization rate as

$$v_{\mathrm{crit}} \equiv \mu(\mathrm{R}_0^{(3.8)} - 1) = \mu \left(\frac{\sum_{i=1}^m \beta_i \tau_i}{\omega(\mu + g)} - 1 \right) \in \mathbb{R}_+$$

(assuming that $\mathrm{R}_0^{(3.8)} \geq 1$).

Example 5.2 Consider (5.8) with $\mathscr{M} = \{1, 2\}$, σ defined as in (3.37), and initial conditions $(S_0, I_0, R_0) = (0.75, 0.25, 0)$. Motivated by the measles parameters of [138], let $\beta_1 = 18$, $\beta_2 = 3$, $g = 1$, $\mu = 0.1$, and $v = 0.57$ ($v_{\mathrm{crit}} = 0.51$). Then $\mathrm{R}_0^{(5.8)} = 0.92$ (see Fig. 5.5 for an illustration).

The vaccination models thus far assume immediate movement from susceptible to vaccinated. This ignores the time it takes to obtain immunity by completing a vaccination program. The following assumptions are made [101]:

1. The mean period of vaccine-induced immunity is $1/\gamma$ for some $\gamma > 0$.
2. Individuals in the vaccinated class contract the disease at a reduced rate β_σ^V (i.e., $\beta_i^V < \beta_i$ for each $i \in \mathscr{M}$ since individuals may have partial immunity during the vaccination process).
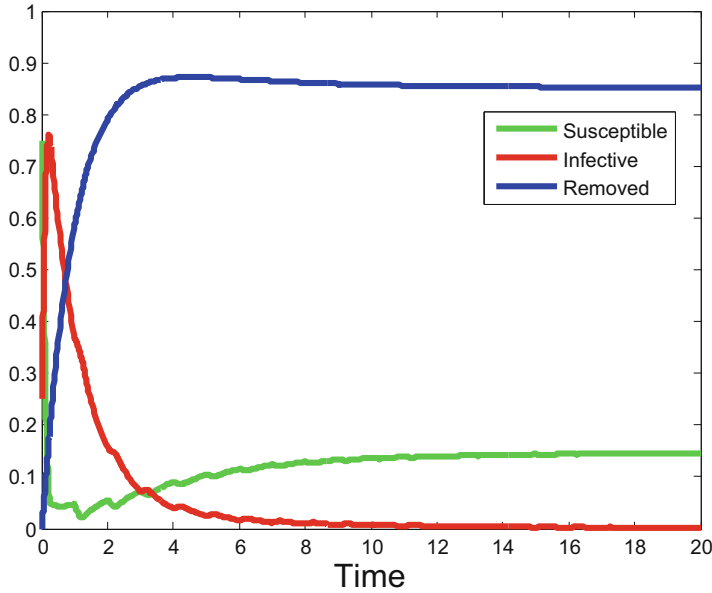
Fig. 5.5 Simulation of Example 5.2 with $v = 0.57$

Under these assumptions, the SVIR model with switching is written as

$$\dot{S}(t) = \mu - \beta_\sigma S(t)I(t) - \mu S(t) - vS(t),$$
$$\dot{V}(t) = vS(t) - \beta_\sigma^V V(t)I(t) - \gamma V(t) - \mu V(t),$$
$$\dot{I}(t) = \beta_\sigma S(t)I(t) + \beta_\sigma^V V(t)I(t) - gI(t) - \mu I(t), \qquad (5.13)$$
$$\dot{R}(t) = gI(t) + \gamma V(t) - \mu R(t),$$
$$(S(0), V(0), I(0), R(0)) = (S_0, V_0, I_0, R_0).$$

For (5.13), the set of mode basic reproduction numbers can be calculated as follows:

$$R_0^{(5.13),i} \equiv \left(\frac{\beta_i}{\mu + g} + \frac{\beta_i^V}{\mu + g} \frac{v}{\mu + \gamma} \right) \frac{\mu}{\mu + v}, \quad \forall i \in \mathcal{M}. \qquad (5.14)$$

The flow diagram of (5.13) is illustrated in Fig. 5.6.

Observe that as the efficacy of the vaccine is increased (i.e., β_i^V decreases or γ increases), the mode reproduction numbers reduce to those of the SIR model (3.8) (and are equal in the limit $\beta_i^V \to 0$ for each i or $\gamma \to \infty$). However, as noted in [101], increasing the efficacy of the vaccine is usually more difficult than controlling the vaccination rate v. There is a single disease-free equilibrium point [101]:
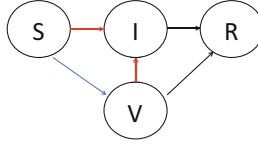
Fig. 5.6 Flow diagram of the switched SVIR system (5.13). The *red line* represents the transmission of the disease and the *blue line* represents the vaccination

$$Q_{\text{DFS}}^{(5.13)} \equiv (\bar{S}, \bar{V}, \bar{I}, \bar{R}) \equiv \left(\frac{\mu}{\mu + v}, \frac{v\mu}{(\mu + \gamma)(\mu + v)}, 0, \frac{v\gamma}{(\mu + \gamma)(\mu + v)} \right)$$
(5.15)

and, as per usual, any mode for which $R_0^{(5.8),i} \geq 1$ admits an endemic equilibrium

$$Q_{\text{ES}}^{(5.13),i} \equiv (S_i^*, V_i^*, I_i^*, R_i^*), \quad \forall i \in \mathcal{M},$$

where I_i^* is the positive root of the function $I \mapsto A_1 I^2 + A_2 I + A_3(1 - R_0^{(5.8),i})$ where

$$A_1 \equiv (\mu + g)\beta_i \beta_i^V > 0, \quad \forall i \in \mathcal{M},$$

$$A_2 \equiv (\mu + g)[(\mu + v)\beta_i^V + (\mu + \gamma)\beta_i] - \beta_i^V \beta_i \mu, \quad \forall i \in \mathcal{M},$$

$$A_3 \equiv (\mu + g)(\mu + v)(\mu + \gamma) > 0, \quad \forall i \in \mathcal{M},$$

and

$$S_i^* \equiv \frac{\mu}{\mu + v + \beta_i I_i^*}, \quad \forall i \in \mathcal{M},$$

$$V_i^* \equiv \frac{v\mu}{(\mu + v + \beta_i I_i^*)(\mu + \gamma + \beta_i^V I_i^*)}, \quad \forall i \in \mathcal{M},$$

$$R_i^* \equiv 1 - S_i^* - I_i^* - V_i^*, \quad \forall i \in \mathcal{M}.$$

Theorem 5.2 *If* $\sigma \in \mathcal{S}_{\text{periodic}}(\omega)$ *and*

$$R_0^{(5.13)} \equiv \frac{1}{\omega} \sum_{i=1}^{m} R_0^{(5.13),i} \tau_i < 1,$$

then the disease-free solution $Q_{\text{DFS}}^{(5.13)}$ *of the switched SIR system with progressive immunity* (5.13) *is globally attractive and globally asymptotically I-stable in the meaningful domain.*

Proof Observe from (5.8) that

$$\dot{S}(t) = \mu - \beta_\sigma S(t)I(t) - \mu S(t) - v S(t),$$

$$\leq \mu - (\mu + v)S(t).$$

Similarly,

$$\dot{V}(t) = vS(t) - \beta_\sigma^V V(t)I(t) - \gamma V(t) - \mu V(t),$$
$$\leq vS(t) - \gamma V(t) - \mu V(t).$$

The comparison system

$$\dot{x}(t) = \mu - (\mu + v)x(t),$$
$$\dot{y}(t) = vx(t) - (\gamma + \mu)y(t), \qquad\qquad (5.16)$$
$$(x(0), y(0)) = (S_0, V_0),$$

gives the appropriately needed result; for any $\epsilon > 0$, there exists $t^* > 0$ such that $S(t) \leq x(t) \leq \bar{S} + \epsilon$ and $V(t) \leq y(t) \leq \bar{V} + \epsilon$ for $t \geq t^*$. Returning to the differential equation for I,

$$\dot{I}(t) = \beta_\sigma S(t)I(t) + \beta_\sigma^V V(t)I(t) - gI(t) - \mu I(t),$$
$$\leq (\beta_\sigma[\bar{S} + \epsilon] + \beta_\sigma^V[\bar{V} + \epsilon] - \mu - g)I(t),$$
$$= \lambda_{\sigma,\epsilon} I(t),$$

where $\lambda_{i,\epsilon} \equiv \beta_\sigma[\bar{S} + \epsilon] + \beta_\sigma^V[\bar{V} + \epsilon] - \mu - g$ for each $i \in \mathscr{M}$. The condition $\mathrm{R}_0^{(5.4)} < 1$ gives that

$$\sum_{i=1}^m (\beta_i \bar{S} + \beta_\sigma^V \bar{V} - \mu - g)\tau_i < 0.$$

As in the proof of Theorem 5.1, $\epsilon > 0$ can be chosen sufficiently small so that $\sum_{i=1}^m \lambda_{i,\epsilon}\tau_i < 0$ and it follows that $\lim_{t\to\infty} I(t) = 0$. The limiting equation for S is $\dot{S}(t) = \mu - \mu S(t) - vS(t)$; S converges to $\bar{S} = \mu/(\mu + v)$, and the limiting equation for V is $\dot{V}(t) = v\mu/(\mu+v) - \gamma V(t) - \mu V(t)$, from which it follows that V converges to $\bar{V}$. Finally, the limiting equation for R is $\dot{R}(t) = \gamma v\mu/[(\mu + v)(\gamma + \mu)] - \mu R(t)$, from which convergence of R to $\bar{R}$ follows. Therefore, the solution of system (5.13) converges to the disease-free equilibrium $Q_{\mathrm{DFS}}^{(5.13)}$. Asymptotic I-stability follows as usual.

The critical vaccination rate in the case of progressive immunity is calculated by setting

$$\mathrm{R}_0^{(5.13)} = \frac{1}{\omega}\left(\frac{\sum_{i=1}^m \beta_i\tau_i}{\mu + g} + \frac{\sum_{i=1}^m \beta_i^V \tau_i}{\mu + g}\frac{v}{\mu + \gamma}\right)\frac{\mu}{\mu + v} = 1.$$

Namely,

$$v_{\mathrm{crit}} \equiv \mu\left(\frac{\sum_{i=1}^m \beta_i\tau_i}{\omega(\mu + g)} - 1\right)\left(1 - \mu\frac{\sum_{i=1}^m \beta_i^V \tau_i}{\omega(\mu + g)}\right)^{-1}.$$
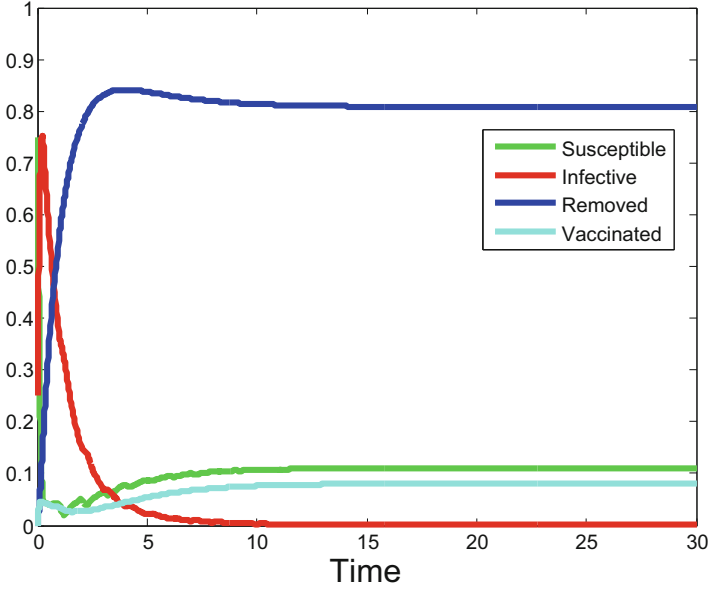
Fig. 5.7 Simulation of Example 5.3 with $v = 0.8$

Example 5.3 Consider (5.13) with $\mathcal{M} = \{1, 2\}$, σ defined as in (3.37), and initial conditions $(S_0, V_0, I_0, R_0) = (0.75, 0, 0.25, 0)$. Given $\beta_1 = 18$, $\beta_2 = 3$, $g = 1$, $\mu = 0.1$, $\gamma = 1$ and vaccine-reduced contact rates $\beta_1^V = 1$ and $\beta_2^V = 0.17$. Then $v = 0.8$ ($v_{\text{crit}} = 0.51$) implies that $\mathrm{R}_0^{(5.13)} = 0.580$ (see Fig. 5.7 for an illustration).

5.2 Treatment Schedules for Classes of Infected

The control strategy of treating infections is investigated. More specifically, a piecewise constant switching control is presented. Assume that $p_i \geq 0$, $i \in \mathcal{M}$, are treatment rates, per unit time, of the infected population which may be applied to the infected population. The value p_i can be broken down as $p_i = v_i/q$ where $1/q > 0$ is average treatment period and $v_i > 0$ is the treatment success rate. Assuming movement to the recovered class from the treatment process, the switched system is written as follows:

$$\dot{S}(t) = \mu - \beta_\sigma S(t)I(t) - \mu S(t),$$
$$\dot{I}(t) = \beta_\sigma S(t)I(t) - gI(t) - \mu I(t) - p_\sigma I(t), \qquad (5.17)$$
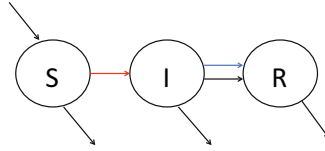$$\dot{R}(t) = gI(t) - \mu R(t) + p_\sigma I(t).$$

Fig. 5.8 Flow of the switched SIR system with treatment (5.17). The *red line* represents the horizontal transmission and the *blue line* represents the treatment strategy

The variables here have been normalized by the total population (since $S+I+R = 1$ is an invariant of (5.17)). Indeed, the physically meaningful domain of (5.17) is equal to

$$D_{(5.17)} \equiv \{(S, I, R) \in \mathbb{R}^3_+ : S + I + R = 1\} = D_{(3.8)}.$$

See Fig. 5.8 for an illustration of the flow diagram for (5.17).

   The treatment rate acts to reduce the average infectious period (from an average of $1/(\mu + g)$ to $1/(\mu + g + p_i)$); the set of mode basic reproduction numbers are reduced as

$$R_0^{(5.17),i} \equiv \frac{\beta_i}{\mu + g + p_i} \leq \frac{\beta_i}{\mu + g} = R_0^{(3.8),i}, \quad \forall i \in \mathcal{M}. \qquad (5.18)$$

Disease eradication by switching treatment can immediately be proved from the techniques of Sect. 3.4 by making the following observation:

$$\dot{I}(t) = \beta_\sigma S(t)I(t) - gI(t) - \mu I(t) - p_\sigma I(t) \leq \lambda_i I(t),$$

where $\lambda_i \equiv \beta_i - g - \mu - p_i$ for each i. By repeating the standard attractivity and partial stability switched systems methods already used, the following result is provided.

Theorem 5.3 *Consider the switched SIR model with switching treatment (5.17). Global attractivity of the disease-free solution $Q_{\mathrm{DFS}}^{(5.17)} \equiv (1, 0, 0)$ holds under any of the following conditions:*

(i) $\sigma \in \mathscr{S}_{\mathrm{periodic}}(\omega)$ and $R_0^{(5.17)} \equiv \frac{1}{\omega} \sum_{i=1}^{m} R_0^{(5.17),i} \tau_i < 1$;
(ii) $\sigma \in \mathscr{S}_{\mathrm{dwell}}$ and there exists $h > 0$ such that

$$\left\langle R_0^{(5.17)} \right\rangle \equiv \sup_{t \geq h} \frac{\sum_{i=1}^{m} \beta_i T_i(t)}{t(\mu + g) + \sum_{i=1}^{m} p_i T_i(t)} < 1; \qquad (5.19)$$

(iii) $\sigma \in \mathscr{S}_{\mathrm{dwell}}$ satisfies $T^+ \leq N_0 + qT^-(t)$ for some $q \in (0, 1)$ and $N_0 \geq 0$ such that

$$\max\{R_0^{(5.17),i} : i \in \mathcal{M}^-\} - 1 < q(\max\{R_0^{(5.17),i} : i \in \mathcal{M}^+\} - 1),$$

where $\mathcal{M}^- \equiv \{i \in \mathcal{M} : R_0^{(5.17),i} < 1\}$ and $\mathcal{M}^+ \equiv \{i \in \mathcal{M} : R_0^{(5.17),i} \geq 1\}$.

On the other hand, if $\sigma \in \mathscr{S}_{\text{periodic}}(\omega)$ and $R_0^{(5.17)} > 1$, then the disease persists uniformly in (5.17).

In the setting of Theorem 5.3, case (i) also implies asymptotic I-stability of $Q_{\text{DFS}}^{(5.17)}$ in the meaningful domain. Cases (ii)-(iii) give exponential I-stability.

Example 5.4 Consider (5.17) with $\mathscr{M} = \{1, 2\}$, σ defined as in (3.37), initial conditions $(S_0, I_0, R_0) = (0.75, 0.25, 0)$, and model parameters $\beta_1 = 18$, $\beta_2 = 3$, $g = 1$, $\mu = 0.1$. Given $p = 1$ (recall $v = 0.57$ ensured disease eradication in the cohort immunization scheme (5.8)), then $R_0^{(5.17)} = 3.21$ and the scheme is ineffective (see Fig. 5.9 for an illustration).

This treatment strategy can be extended to generalized forces of infections (recall the formulation in the switched SIR model with general switched incidence rates (3.29)): suppose that the incidence rate takes the form $(t, S, I) \mapsto h_\sigma(I)S$ to give the system

$$
\begin{aligned}
\dot{S}(t) &= \mu - h_\sigma(I(t))S(t) - \mu S(t), \\
\dot{I}(t) &= h_\sigma(I(t))S(t) - (g + \mu + p_\sigma)I(t), \\
\dot{R}(t) &= (g + p_\sigma)I(t) - \mu R(t), \\
(S(0), I(0), R(0)) &= (S_0, I_0, R_0),
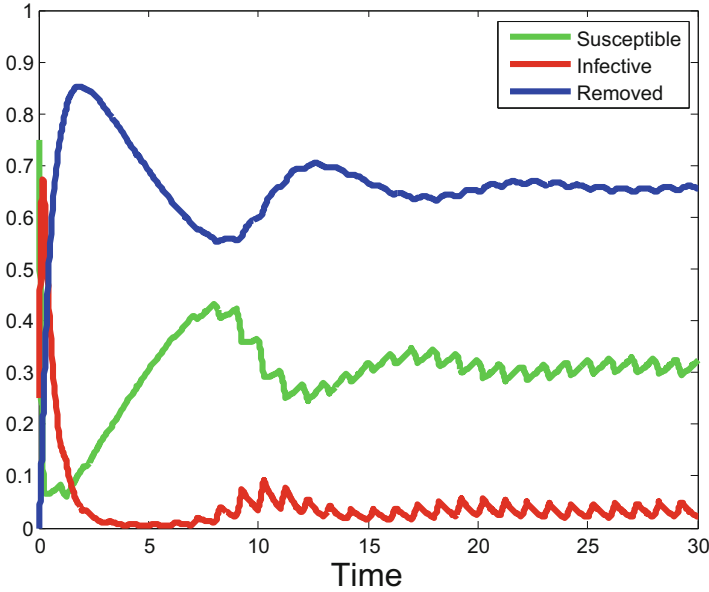\end{aligned}
\tag{5.20}
$$

Fig. 5.9 Simulation of Example 5.4 with $p = 1$

where the forces of infection h_i are assumed to satisfy necessary physical conditions (i.e., so that (a)–(d) in Sect. 3.5 are satisfied by $f_i(S, I) \equiv h_i(I)S$). The treatment rate can be used to control the disease to eradication, via the set of mode reproduction numbers

$$\mathrm{R}_0^{(5.20),i} \equiv \frac{1}{\mu + g + p_i} \frac{dh_i}{dI}(0), \quad \forall i \in \mathcal{M},$$

as follows.

Theorem 5.4 *Assume that $h_i \in C^2([0, 1], \mathbb{R}_+)$ satisfies $\frac{d^2 h_i}{dI^2}(I) \leq 0$ for all $I \in [0, 1]$, $i \in \mathcal{M}$. If $\sigma \in \mathscr{S}_{\mathrm{dwell}}$ and*

$$p_i > \frac{dh_i}{dI}(0) - (\mu + g), \quad \forall i \in \mathcal{M}, \tag{5.21}$$

then $Q_{\mathrm{DFS}}^{(5.20)} \equiv (1, 0, 0)$ is globally asymptotically stable in the meaningful domain

$$D_{(5.20)} \equiv \{(S, I, R) \in \mathbb{R}_+^3 : S + I + R = 1\};$$

the disease is eradicated by the switching treatment control.

Proof Define the mapping

$$V(S, I) \equiv S - \ln(S) + I - 1$$

which is continuously differentiable on

$$\Omega_{SI}^\epsilon \equiv \{(S, I) \in \mathbb{R}_+^2 : S + I \leq 1\} \cap \{(S, I) : S \geq \epsilon\} = \{(S, I) : S \geq \epsilon, S + I \leq 1\},$$

for $\epsilon > 0$ [73]. Observe that $V(1, 0) = 0$, $V > 0$ for $(S, I) \in \Omega_{SI}^\epsilon \setminus \{(1, 0)\}$,

$$\frac{\partial V}{\partial S}(S, I) = 1 - 1/S, \quad \frac{\partial^2 V}{\partial S^2}(S, I) = 1/S^2, \quad \frac{\partial V}{\partial I}(S, I) = 1, \quad \frac{\partial^2 V}{\partial I^2}(S, I) = 0,$$

implying that $(S, I) = (1, 0)$ is the unique (global) minimum of the Lyapunov function in Ω_{SI}^ϵ. The time-derivative of V along trajectories of (5.20) yields that

$$\dot{V}_{(5.20)}(t, S, I, R) = (1 - 1/S)(\mu - h_\sigma(I)S - \mu S) + h_\sigma(I)S - (\mu + g + p_\sigma)I,$$

$$= \mu[(1 - 1/S)(1 - S)] + (\mu + g + p_\sigma)I\left(\frac{h_\sigma(I)}{(\mu + g + p_\sigma)I} - 1\right).$$

Proceed by arguments in [73]: observe that $(1 - 1/S)(1 - S) < 0$ for $\epsilon \leq S < 1$; $(1 - 1/S)(1 - S) = 0$ if $S = 1$. The concavity condition on the set of functions h_i

implies that $h_i(I)/I \leq \frac{dh_i}{dI}(0)$ for all $I > 0$. It follows that

$$\frac{h_i(I)}{(\mu + g + p_i)I} \leq \frac{1}{\mu + g + p_i} \frac{dh_i}{dI}(0) \leq R_0^{(5.20),i}, \quad \forall i \in \mathcal{M}.$$

The condition (5.21) implies that

$$R_0^{(5.20),i} < 1, \quad \forall i \in \mathcal{M}.$$

Moreover, $h_i(I)/[(\mu_i + g_i)I] - 1 < 0$ for each $i \in \mathcal{M}$, so that $\dot{V}_{(5.20)}(t, S, I, R) < 0$ holds unless $(S, I) = (1, 0)$ and the arbitrary choice of ϵ yields global asymptotic stability of $(1, 0)$ in $\Omega_{SI}^{\epsilon \to 0}$. The equation $R = 1 - I - S$ implies the conclusion holds in $D_{(5.20)}$.

Example 5.5 Consider (5.20) with $\mathcal{M} = \{1, 2\}$, σ defined as in (3.37), initial conditions $(S_0, I_0, R_0) = (0.75, 0.25, 0)$, and $h_i(I) = \beta_i \sin(\pi I/2)$ for $i = 1, 2$. Let $\beta_1 = 4$, $\beta_2 = 1.6$, $g = 1.9$, $\mu = 0.1$. Observe that $f_i(S, I) \equiv h_i(I)S$ satisfies $f_i(t, S, I) > 0$ for $S, I \neq 0$, $f_i(t, S, 0) = f_i(t, 0, I) = 0$, $\frac{\partial f_i(t, S, I)}{\partial I} > 0$ for $0 \leq I < 1$. Then $p_1 = 5$ and $p_2 = 1$ imply that (5.21) holds and global asymptotic stability of the disease-free solution by Theorem 5.4. On the other hand, if $p_1 = p_2 = 0$ then, since $\beta_i \sin(\pi I/2)S \geq \beta_i SI$ for each i, the disease persists by a straightforward calculation of

$$\overline{R_0}^{(3.29)} = 1.1$$

(where $\overline{R_0}^{(3.29)}$ is outlined in Sect. 3.5). The situation is illustrated in Fig. 5.10.
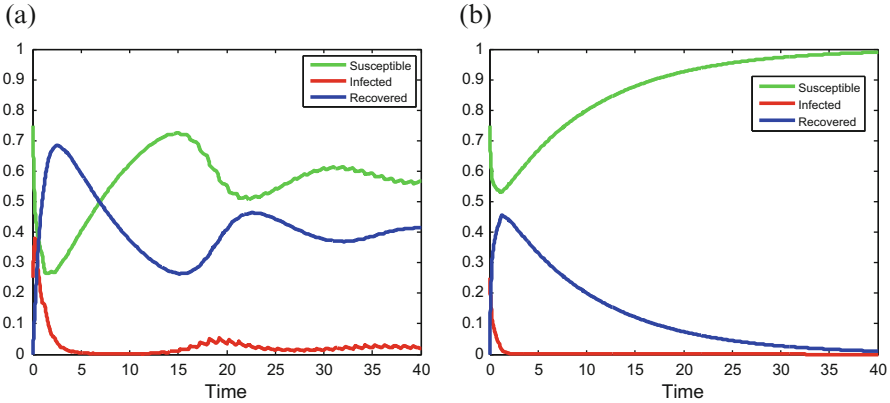

Fig. 5.10 Simulations of Example 5.5. (**a**) $p_1 = p_2 = 0$. (**b**) $p_1 = 5, p_2 = 1$

5.3 Introduction of the Exposed: A Controlled SEIR Model

A number of diseases exhibit a period of latency where individuals have been infected but are not yet infectious. (An incubation period is the time between infection and clinical onset of the disease; i.e., appearance of symptoms.) Motivated by this fact, we re-examine the assumption made earlier of a negligible latency period; assume that once a susceptible individual makes an adequate contact with an infected individual they enter a latent period before becoming infectious. Let E denote the class of individuals who have been exposed but are not yet infectious. Assume that individuals who have been exposed become infectious at a rate $a > 0$ (i.e., average incubating period of $1/a$). With other physiological and epidemiological assumptions matching those of the classical endemic model (i.e., (3.9)), the model is given by the following dynamic system:

$$\dot{S}(t) = \mu - \beta S(t)I(t) - \mu S(t),$$
$$\dot{E}(t) = \beta S(t)I(t) - aE(t) - \mu E(t),$$
$$\dot{I}(t) = aE(t) - gI(t) - \mu I(t), \tag{5.22}$$
$$\dot{R}(t) = gI(t) - \mu R(t),$$
$$(S(0), E(0), I(0), R(0)) = (S_0, E_0, I_0, R_0),$$

where

$$(S_0, I_0, E_0, R_0) \in D_{(5.22)} \equiv \{(S, E, I, R) \in \mathbb{R}^4_+ : S + E + I + R = 1\},$$

which is invariant to (5.22); $\{\dot{S} + \dot{E} + \dot{I} + \dot{R}\}|_{S+I+E+R=1} = 0$, $\dot{S}|_{S=0} = \mu > 0$, $\dot{E}|_{E=0} = \beta SI \geq 0$, $\dot{I}|_{I=0} = 0$, and $\dot{R}|_{R=0} = gI \geq 0$. The basic reproduction number of (5.22) is calculated as

$$R_0^{(5.22)} \equiv \frac{\beta a}{(\mu + g)(\mu + a)}; \tag{5.23}$$

the average number of new cases is the product of the contact rate, β, the average fraction surviving the latent period, $a/(a + \mu)$, and the average infectious period $1/(\mu + g)$ [65]. There is a single disease-free equilibrium

$$Q_{\text{DFS}}^{(5.22)} \equiv (1, 0, 0, 0)$$

and an endemic equilibrium:

$$Q_{\text{ES}}^{(5.22)} \equiv \left(\frac{1}{R_0^{(5.22)}}, \frac{\mu(\mu + g)}{\beta a}(R_0^{(5.22)} - 1), \frac{\mu}{\beta}(R_0^{(5.22)} - 1), \frac{g}{\beta}(R_0^{(5.22)} - 1) \right).$$

The invariant $S + E + I + R = 1$ implies that the equation for R can be omitted (i.e., (5.22) is intrinsically three-dimensional).

Recall the basic reproduction number of the classical endemic SIR model (see Eq. (3.11)) and note that for the SEIR model, the reproduction number

$$\mathrm{R}_0^{(5.22)} = \mathrm{R}_0^{(3.9)} \frac{a}{\mu + a},$$

which implies that $\mathrm{R}_0^{(5.22)} \leq \mathrm{R}_0^{(3.9)}$. Moreover, since the mean lifetime of an individual is much greater than the average latency period (i.e., $1/\mu \gg 1/a$) then $a \gg \mu$ so that $a/(a + \mu) \approx 1$ [69]. Thus, $\mathrm{R}_0^{(5.22)} \approx \mathrm{R}_0^{(3.9)}$ in most cases. If the latent period is small compared to the infectious period (i.e., $a/g \gg 1$), which is often the case, the latent period can be ignored [103], which justifies the assumption made for the classical endemic model. The dynamics of (5.22) are again dictated by the basic reproduction number:

$$\mathrm{R}_0^{(5.22)} \leq 1$$

implies asymptotic stability of the disease-free equilibrium $Q_{\mathrm{DFS}}^{(5.22)}$ in $D_{(5.22)}$;

$$\mathrm{R}_0^{(5.22)} > 1$$

implies asymptotic stability of the endemic equilibrium $Q_{\mathrm{ES}}^{(5.22)}$ in $D_{(5.22)}$ (see, e.g., [81]) and is approached in a damped oscillatory fashion [69]. In fact, the period of oscillations is approximately equal to

$$2\pi \sqrt{\frac{1}{\mu(\mathrm{R}_0^{(5.22)} - 1)} \left(\frac{1}{\mu + g} + \frac{1}{\mu + a} \right)}$$

where [69]:

1. The term $1/(\mu(\mathrm{R}_0^{(5.22)} - 1))$ is the average age of infection.
2. The term $1/(\mu + g) + 1/(\mu + a)$ is the average period of host's infectivity.

In effect, the SEIR model (5.22) admits a slower rate of growth of the disease after its introduction because the latent period delays an exposed person from becoming infectious [69].

With seasonal variations in (5.22), and a treatment of infected by the switching rate p_σ $(p_1, \ldots, p_m \geq 0)$, the model is given by
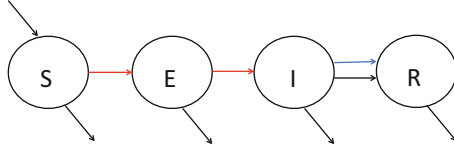
Fig. 5.11 Flow of the switched SEIR system with treatment (5.24). The *red line* represents the horizontal transmission and the *blue line* represents the treatment strategy

$$\dot{S}(t) = \mu - \beta_\sigma S(t)I(t) - \mu S(t),$$
$$\dot{E}(t) = \beta_\sigma S(t)I(t) - aE(t) - \mu E(t),$$
$$\dot{I}(t) = aE(t) - gI(t) - \mu I(t) - p_\sigma I(t), \qquad (5.24)$$
$$\dot{R}(t) = gI(t) + p_\sigma I(t) - \mu R(t),$$
$$(S(0), E(0), I(0), R(0)) = (S_0, E_0, I_0, R_0).$$

Here, it is assumed that infected individuals seek treatment but those who have been exposed and are in the latent period (possibly asymptomatic) do not seek treatment. See Fig. 5.11 for the flow of individuals in the population. The mode basic reproduction numbers are thus

$$R_0^{(5.24),i} \equiv \frac{\beta_i a}{(\mu + g + p_i)(\mu + a)}, \quad \forall i \in \mathcal{M}. \qquad (5.25)$$

Intuitively, the basic reproduction number in each isolated mode equal to $\beta_i/(\mu + g + p_i)$ (average contact rate times average period of infection) multiplied by $1/(\mu + a)$ (average latent period). Equation (5.24) still admits a common disease-free equilibrium

$$Q_{\text{DFS}}^{(5.24)} \equiv (1, 0, 0, 0) = Q_{\text{DFS}}^{(5.22)},$$

while each mode admits an endemic equilibrium:

$$Q_{\text{ES}}^{(5.24),i} \equiv \left(\frac{1}{R_0^{(5.24),i}}, \frac{\mu(\mu + g + p_i)}{\beta_i a}(R_0^{(5.24),i} - 1), \frac{\mu}{\beta_i}(R_0^{(5.24),i} - 1), \frac{g + p_i}{\beta_i}(R_0^{(5.24),i} - 1) \right),$$

for each $i \in \mathcal{M}$. The long-term behavior of (5.24) is characterized via common Lyapunov function techniques and the switching invariance principle.

Theorem 5.5 *If $\sigma \in \mathcal{S}_{\text{dwell}}$ and*

$$p_i > \frac{\beta_i a}{\mu + a} - (\mu + g), \quad \forall i \in \mathcal{M}, \qquad (5.26)$$

then $Q_{\text{DFS}}^{(5.24)}$ is globally attractive in the meaningful domain; the disease is eradicated by the switching treatment control.

Proof Define the mapping $V(E, I) \equiv aE + (a + \mu)I$ (similar to the one from [140]) and define the following set:

$$\Omega_{EI} = \{(E, I) \in \mathbb{R}_+^2 : E + I \leq 1\}.$$

Observe that $V(0, 0) = 0$ and $V(E, I) > 0$ for $(E, I) \in \Omega_{EI} \setminus \{(0, 0)\}$. The time-derivative of V along trajectories of (5.24) is given by:

$$\begin{aligned}
\dot{V}_{(5.24)}(t, S, E, I, R) &= a(\beta_\sigma SI - aE - \mu E) + (a + \mu)(aE - gI - \mu I - p_\sigma I), \\
&= \beta_\sigma aSI - (\mu + g + p_\sigma)(\mu + a)I, \\
&= (\text{R}_0^{(5.24)}S - 1)(\mu + g + p_\sigma)(\mu + a)I.
\end{aligned}$$

Condition (5.26) precisely implies that $\text{R}_0^{(5.24),i} < 1$ for all $i \in \mathcal{M}$. From this it follows that $\dot{V}_{(5.24)}(t, S, E, I, R) \leq 0$; $V(E, I)$ is a common weak Lyapunov function of (5.24). The set

$$\{(E, I) \in \Omega_{EI} : \dot{V}_{(5.24)} = 0\} = \{(E, I) \in \Omega_{EI} : (E, I) = (c, 0), \quad \forall 0 \leq c \leq 1\}$$

and, by inspection of the limiting equations of (5.24) with $I = 0$, the solution converges to the disease-free equilibrium $Q_{\text{DFS}}^{(5.24)}$ by Theorem 2.2.

Example 5.6 Consider (5.24) with $\mathcal{M} = \{1, 2\}$, σ defined as in (3.37), initial conditions $(S_0, E_0, I_0, R_0) = (0.9, 0, 0.1, 0)$, and $\beta_1 = 8$, $\beta_2 = 1.6$, $g = 0.9$, $\mu = 0.1$. Let the latent period equal $1/a = 1/0.3$ from [103]. Given $p_1 = 6$ and $p_2 = 1$, then

$$6 = p_1 > \frac{\beta_1 a}{\mu + a} - (\mu + g) = 5,$$

and

$$1 = p_2 > \frac{\beta_2 a}{\mu + a} - (\mu + g) = 0.2;$$

the conditions of Theorem 5.5 are satisfied and $Q_{\text{DFS}}^{(5.24)}$ is globally attractive in the meaningful domain. See Fig. 5.12 for an illustration.

Motivated by the number of infectious diseases transmitted by both horizontal and vertical modes (e.g., rubella, herpes simplex, hepatitis B, Chagas' disease [140]), consider (5.24) with the additional assumption of vertical transmission:
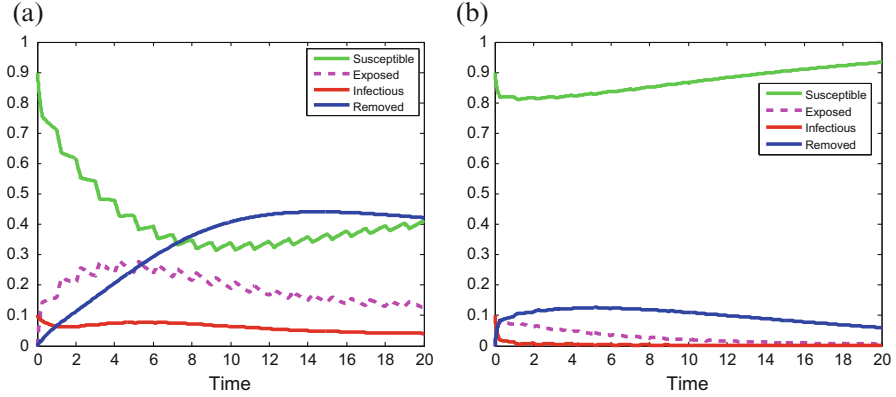
Fig. 5.12 Simulations of Example 5.5. (**a**) $p_1 = p_2 = 0$. (**b**) $p_1 = 6, p_2 = 1$

$$\dot{S}(t) = \mu(1 - \rho E(t) - q I(t)) - \beta_\sigma S(t) I(t) - \mu S(t),$$

$$\dot{E}(t) = \mu(\rho E(t) + q I(t)) + \beta_\sigma S(t) I(t) - a E(t) - \mu E(t),$$

$$\dot{I}(t) = a E(t) - g I(t) - \mu I(t) - p_\sigma I(t),$$

$$\dot{R}(t) = g I(t) + p_\sigma I(t) - \mu R(t),$$

$$(S(0), E(0), I(0), R(0)) = (S_0, E_0, I_0, R_0),$$

$$(5.27)$$

where $\rho \in [0, 1]$ and $q \in [0, 1]$ represent vertical transmission via exposed and infected individuals, respectively. The set

$$D_{(5.27)} \equiv \{(S, E, I, R) \in \mathbb{R}_+^4 : S + E + I + R = 1\}$$

is the meaningful domain (which is positively invariant). The mode basic reproduction numbers are (from the case [140]):

$$R_0^{(5.27),i} \equiv \frac{\beta_i a}{(\mu + g + p_i)(\mu(1 - \rho) + a) - \mu q a}, \quad \forall i \in \mathcal{M}, \qquad (5.28)$$

which can be interpreted via a Taylor expansion of the transmission of diseases through the generations of offspring in each mode [140] (where the authors also present the endemic equilibria and stability results for the time-invariant and uncontrolled version of (5.27)). Equation (5.27) admits a disease-free equilibrium $Q_{\text{DFS}}^{(5.27)} \equiv (1, 0, 0, 0)$ and mode-dependent endemic equilibria $Q_{\text{ES},i}^{(5.27)} \equiv (S_i^*, E_i^*, I_i^*, R_i^*)$ where, for each $i \in \mathcal{M}$,
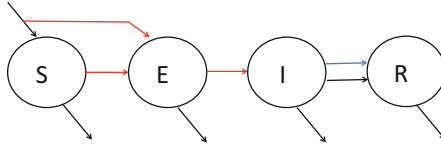
Fig. 5.13 Flow of the switched SEIR system with vertical transmission and treatment (5.27). The *red lines* represent the horizontal and vertical transmission and the *blue line* represents the treatment strategy

$$S_i^* \equiv \frac{1}{R_0^{(5.27),i}},$$

$$E_i^* \equiv 1 - S_i^* - I_i^* - R_i^*,$$

$$I_i^* \equiv \frac{a\mu R_0^{(5.27),i}}{\beta_i a + \rho\mu(g + \mu + p_i)R_0^{(5.27),i} + q\mu a R_0^{(5.27),i}}(1 - 1/R_0^{(5.27),i}),$$

$$R_i^* \equiv \frac{a(g + p_i)R_0^{(5.27),i}}{\beta_i a + \rho\mu(g + \mu + p_i)R_0^{(5.27),i} + q\mu a R_0^{(5.27),i}}(1 - 1/R_0^{(5.27),i}).$$

The flow of (5.27) is illustrated in Fig. 5.13. Eradication is established as follows.

Theorem 5.6 *If $\sigma \in \mathscr{S}_{\text{dwell}}$ and*

$$p_i > \frac{(\beta_i + \mu q)a}{\mu(1 - \rho) + a} - (\mu + g), \quad \forall i \in \mathscr{M}, \tag{5.29}$$

then $Q_{\text{DFS}}^{(5.27)}$ is globally attractive in the meaningful domain; the disease is eradicated by the switching treatment control.

Proof The proof proceeds similar to the proof of Theorem 5.5 by replacing the Lyapunov candidate function with

$$V(E, I) \equiv aE + (a + \mu - \rho\mu)I$$

(adopted from one in [140]). Observe that $V(0, 0) = 0$ and $V(E, I) > 0$ when $(E, I) \in \Omega_{EI} \setminus \{(0, 0)\}$ (where Ω_{EI} is defined in the proof of Theorem 5.5). The time-derivative of V along trajectories of (5.27) is given by

$$\dot{V}_{(5.27)}(t, S, E, I, R) = a(\beta_\sigma SI + \rho\mu E + q\mu I - aE - \mu E)$$
$$+ (a + \mu - \rho\mu)(aE - gI - p_\sigma I - \mu I),$$
$$= \beta_\sigma aSI - [(\mu + g + p_\sigma)(\mu + a - \rho\mu) - \mu qa]I,$$
$$= (R_0^{(5.27)}S - 1)(\mu + g + p_\sigma)(\mu + a - \rho\mu - \mu qa)I.$$

Hence, if $R_0^{(5.27),i} < 1$ for all i, then $\dot{V}_{(5.27)}(t, S, E, I, R) \leq 0$. The remaining part follows by similar arguments to the proof of Theorem 5.5.

Appropriate for a disease like AIDS [140], suppose that the assumptions on the population dynamics and disease-induced mortality are relaxed; assume that the birth rate is $b > 0$, the natural death rate is $d > 0$ and the disease-induced death rate is $\alpha > 0$. Applied to (5.24) yields the following dynamic system:

$$\dot{S}_c(t) = b - \beta_\sigma \frac{S_c(t)I_c(t)}{N(t)} - dS_c, (t)$$

$$\dot{E}_c(t) = \beta_\sigma \frac{S_c(t)I_c(t)}{N(t)} - aE_c(t) - dE_c(t), \tag{5.30}$$

$$\dot{I}_c(t) = aE_c(t) - gI_c(t) - dI_c(t) - \alpha I_c(t) - p_\sigma I_c(t),$$

$$\dot{R}_c(t) = gI_c(t) + p_\sigma I_c(t) - dR_c(t),$$

where S_c, E_c, I_c, R_c represent the number of individuals in the susceptible, exposed, infectious, and removed classes, respectively. The total population $N \equiv S_c + E_c + I_c + R_c$ satisfies the differential equation

$$\dot{N}(t) = (b - d)N(t) - \alpha I_c(t).$$

Normalizing the equations via $S \equiv S_c/N, E \equiv E_c/N, I \equiv I_c/N, R \equiv R_c/N$ yields the following switching system:

$$\dot{S}(t) = b - \beta_\sigma S(t)I(t) - bS(t) + \alpha S(t)I(t),$$

$$\dot{E}(t) = \beta_\sigma S(t)I(t) - aE(t) - bE(t) + \alpha E(t)I(t),$$

$$\dot{I}(t) = aE(t) - gI(t) - bI(t) - \alpha I(t) - p_\sigma I(t) + \alpha I^2(t),$$

$$\dot{R}(t) = gI(t) + p_\sigma I(t) - bR(t) + \alpha R(t)I(t),$$

$$(S(0), E(0), I(0), R(0)) = (S_0, E_0, I_0, R_0), \tag{5.31}$$

where the normalized variables satisfy $S(t) + E(t) + I(t) + R(t) = 1$. The initial conditions satisfy $(S_0, E_0, I_0, R_0) \in D_{(5.31)} = D_{(5.24)}$, which is invariant to (5.31); $\{\dot{S} + \dot{E} + \dot{I} + \dot{R}\}|_{S+I+E+R=1} = 0$, $\dot{S}|_{S=0} = b > 0$, $\dot{E}|_{E=0} = \beta_i SI \geq 0$, $\dot{I}|_{I=0} = aE \geq 0$ and $\dot{R}|_{R=0} = gI \geq 0$. A consequence of the disease-induced mortality, the terms $\alpha SI, \alpha EI, \alpha IR,$ and αI^2 act as positive feedback in the dynamic system. The mode basic reproduction numbers are

$$R_0^{(5.31),i} \equiv \frac{\beta_i a}{(b + g + \alpha + p_i)(b + a)}, \quad \forall i \in \mathcal{M}, \tag{5.32}$$

which reflect the time-invariant case [81]. Comparing (5.32) to (5.25) reveals a reduction in the mode-dependent basic reproduction numbers; the disease-induced mortality reduces the infectious period and therefore the rate of transmission. As in the previously studied SEIR models, (5.31) again admits a disease-free equilibrium

$Q_{\text{DFS}}^{(5.31)} \equiv (1, 0, 0, 0)$ and m endemic equilibria $Q_{\text{ES}}^{(5.31),i} \equiv (S_i^*, E_i^*, I_i^*, R_i^*)$. Here, I_i^* satisfies the following cubic equation [81]:

$$\left(1 - \frac{\alpha}{a+b}I_i^*\right)\left(1 - \frac{\alpha}{\alpha + g + b + p_i}I_i^*\right)\left(1 + \frac{\beta_i - \alpha}{b}I_i^*\right) = \text{R}_0^{(5.31),i}, \quad (5.33)$$

for each $i \in \mathcal{M}$. If $\text{R}_0^{(5.31),i} > 1$, then (5.33) admits a unique positive solution [81]. The other endemic equilibria states satisfy

$$S_i^* \equiv \frac{b}{b + \beta_i I_i^* - \alpha I_i^*},$$

$$E_i^* \equiv \frac{g + \alpha + b + p_i - \alpha I_i^*}{a}I_i^*,$$

$$R_i^* \equiv 1 - S_i^* - E_i^* - I_i^*,$$

for each $i \in \mathcal{M}$. Stability of the disease-free solution can be established using the following lemma from [81].

Lemma 5.1 *Let $\Omega \equiv \{(x, y) \in \mathbb{R}_+^2 : x + y \leq 1\}$ and $h(x, y) \equiv (a-b)x + (c-b)y + b$, where $a, b, c > 0$ are constants. Then it follows that*

$$\max\{h(x, y) : (x, y) \in \Omega\} = \max\{a, b, c\}.$$

Theorem 5.7 *If $\sigma \in \mathscr{S}_{\text{dwell}}$ and*

$$p_i > \frac{\beta_i a}{b + a} - (\mu + g + \alpha), \quad \forall i \in \mathcal{M}, \quad (5.34)$$

then $Q_{\text{DFS}}^{(5.31)}$ is globally attractive in the meaningful domain. Moreover, the total number of infected individuals approaches zero (i.e., $\lim_{t \to \infty} I_c(t) = 0$).

Proof Define the mapping $V(E, I) \equiv aE + (a+b)I$ [81], which satisfies $V(0, 0) = 0$ and $V(E, I) > 0$ for $(E, I) \in \Omega_{EI} \setminus \{(0, 0)\}$ (where Ω_{EI} is outlined in the proof of Theorem 5.5). The time-derivative of V trajectories of (5.31) is given by

$$\dot{V}_{(5.27)}(t, S, E, I, R) = a(\beta_\sigma SI - aE - bE + \alpha EI)$$

$$+ (a + b)(aE - gI - bI - \alpha I - p_\sigma I + \alpha I^2),$$

$$= [\beta_\sigma aS - (a + b)(g + \alpha + b + p_\sigma) + \alpha aE + \alpha(a + b)I]I,$$

$$\leq [\beta_\sigma a(1 - E - I) - (a + b)(g + \alpha + b + p_\sigma)$$

$$+ \alpha aE + \alpha(a + b)I]I,$$

$$= [h_\sigma(E, I) - (a + b)(g + \alpha + b + p_\sigma)]I,$$

where

$$h_i(E, I) \equiv (\alpha a - \beta_i a)E + (\alpha(a + b) - \beta_i a)I + \beta_i a, \quad \forall i \in \mathcal{M}.$$

Applying Lemma 5.1 with the function h_i and set Ω_{EI} gives that

$$\dot{V}_{(5.27)} \leq [\max\{\alpha a, \beta_\sigma a, \alpha(a + b)\} - (a + b)(g + \alpha + b + p_\sigma)]I.$$

Then, since $R_0^{(5.31),i} < 1$ for each $i \in \mathcal{M}$ by Eq. (5.34), it follows that $\dot{V}_{(5.27)} \leq 0$. Note that $\dot{V} = 0$ if $(E, I) = (c, 0)$ or if $\max\{\alpha a, \beta_i a, \alpha(a + b)\} = (a + b)(g + \alpha + b + p_i)$, which implies $R_0^{(5.31),i} = 1$ (and is therefore not possible). It then follows by similar arguments to the proof of Theorem 5.5 that $\lim_{t \to \infty} I(t) = 0$. Recall that $I_c \equiv IN$ and

$$\dot{N}(t) = (b - d)N(t) - \alpha I_c(t) = (b - d - \alpha I(t))N(t).$$

The case $b < d$ is straightforward. The case $b = d$ yields that $\dot{N}(t) = -\alpha I(t)N(t) \leq 0$, from which it follows that N is bounded for all t since $I \to 0$. The case $b > d$ gives that N grows without bound since $I \to 0$. Then, $S_c \equiv SN$ and $S \to 1$ implies that $S_c \to N$. The fact that $N \equiv S_c + E_c + I_c + R_c$ yields the result.

Equations (5.26), (5.29), and (5.34) define mode-dependent critical control rates for the SEIR model (5.24), SEIR model with vertical transmission (5.27), and SEIR model with disease-induced mortality (5.31), respectively:

$$p_i^{(5.26),\text{crit}} \equiv \frac{\beta_i a}{\mu + a} - (\mu + g), \quad \forall i \in \mathcal{M},$$

$$p_i^{(5.29),\text{crit}} \equiv \frac{(\beta_i + \mu q)a}{\mu(1 - \rho) + a} - (\mu + g), \quad \forall i \in \mathcal{M},$$

$$p_i^{(5.34),\text{crit}} \equiv \frac{\beta_i a}{b + a} - (\mu + g + \alpha), \quad \forall i \in \mathcal{M}.$$

Observe that

$$p_i^{(5.34),\text{crit}} \leq p_i^{(5.26),\text{crit}} \leq p_i^{(5.29),\text{crit}}, \quad \forall i \in \mathcal{M},$$

as expected; the disease-induced mortality effectively reduces the average infectious period (making the disease easier to control and thus a decrease in the critical treatment rates) while the vertical transmission has the effect of increasing the basic reproduction number of each mode (hence an increase in the critical treatment rates).

5.4   Screening of Traveling Individuals

In this part, we return to (4.21) and consider restricting the travel of infected individuals as a control method for preventing epidemics. We consider the following control strategy:

1. Assume that $\theta^{(j)} \in [0, 1]$ is the probability of successfully detecting an infected individual entering city $j \in \mathcal{N} \equiv \{1, \ldots, n\}$ by travel.
2. Assume that susceptible individuals are not falsely identified as being infected. Denote infected individuals traveling to city j who are properly screened by $Q^{(j)}$.
3. Assume that individuals who are being screened do not die or give birth.
4. When the infected individuals are identified, assume that they will be isolated for treatment.
5. Assume that individuals in the screened classes recover at a switched rate $q_\sigma^{(j)} > 0$ in city j and enter the recovered population.

With the additional assumptions that the immigration rate is $m^{(j)}$ in city j and the functions $f_i^{(j)}$ and $h_i^{(j)}$ having standard incidence rate structures, the controlled version of (4.21) becomes the following switching system:

$$\dot{S}^{(j)}(t) = m^{(j)} - \beta_\sigma^{(j)} \frac{S^{(j)}(t) I^{(j)}(t)}{N^{(j)}(t)} - \mu^{(j)} S^{(j)}(t) + \sum_{l \in \mathcal{N}} \alpha^{(l,j)} S^{(j)}(t)$$

$$- \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} \gamma_\sigma^{(j)} \frac{S^{(l)}(t) I^{(l)}(t)}{N^{(l)}(t)},$$

$$\dot{I}^{(j)}(t) = \beta_\sigma^{(j)} \frac{S^{(j)}(t) I^{(j)}(t)}{N^{(j)}(t)} - g^{(j)} I^{(j)}(t) - \mu^{(j)} I^{(j)}(t) + \alpha^{(j,j)} I^{(j)}(t)$$

$$+ (1 - \theta^{(j)}) \left[\sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} I^{(l)}(t) + \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} \gamma_\sigma^{(j)} \frac{S^{(l)}(t) I^{(l)}(t)}{N^{(l)}(t)} \right],$$

$$\dot{Q}^{(j)}(t) = -q_\sigma^{(j)} Q^{(j)} + \theta^{(j)} \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} I^{(l)}(t)$$

$$+ \theta^{(j)} \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} \gamma_\sigma^{(j)} \frac{S^{(l)}(t) I^{(l)}(t)}{N^{(l)}(t)},$$

$$\dot{R}^{(j)}(t) = g^{(j)} I^{(j)}(t) + q_\sigma^{(j)} Q^{(j)}(t) - \mu^{(j)} R^{(j)}(t) + \sum_{l \in \mathcal{N}} \alpha^{(l,j)} R^{(l)}(t),$$

$$(S^{(j)}(0), I^{(j)}(0), Q^{(j)}(0), R^{(j)}(0)) = (S_0^{(j)}, I_0^{(j)}, Q_0^{(j)}, R_0^{(j)}),$$

$$\tag{5.35}$$

for all $j \in \mathcal{N}$. The flow diagram for the model with screening is given in Fig. 5.14.
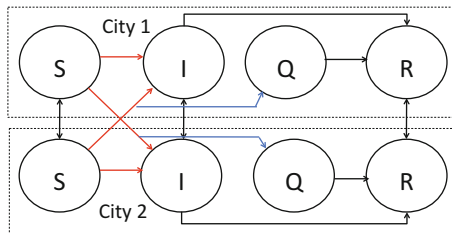
Fig. 5.14 Flow of the multi-city model with screening (5.35) with $n = 2$. The *red lines* represent new infections (including from traveling individuals) and the *blue lines* represent the screening process. The population dynamics in each city have been omitted here

The meaningful physical domain for this system is

$$D_{(5.35)} \equiv \{(S, I, Q, R) \in \mathbb{R}_+^{4n} : \sum_{j \in \mathcal{N}} S^{(j)} + I^{(j)} + Q^{(j)} + R^{(j)} \le N^*\},$$

where

$$N^* = \frac{\sum_{j=1}^{n} m^{(j)}}{\min\{\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(n)}\}} > 0.$$

For a given initial condition, the solution of (5.35) enters $D_{(5.35)}$ (in finite or infinite time); $D_{(5.35)}$ is positively invariant to (5.35) (see Proposition 2.1 in [167]).

Let us next consider the existence of a disease-free solution of (5.35) with $I^{(j)}(t) \equiv 0$ for all $j \in \mathcal{N}$: It is apparent that the screening class converges to zero, and thus the recovered class (since $\mu^{(j)} > 0$ for each $j \in \mathcal{N}$). The limiting system is given by

$$\dot{S}^{(j)}(t) = m^{(j)} - \mu^{(j)} S^{(j)}(t) + \sum_{l \in \mathcal{N}} \alpha^{(l,j)} S^{(j)}(t), \quad \forall j \in \mathcal{N}. \tag{5.36}$$

Using the notation and methodology in [167], define the irreducible matrices $A \equiv (\alpha^{(l,j)})_{1 \le l,j \le n}$ and $U \equiv \text{diag}\{\mu^{(1)}, \dots, \mu^{(n)}\}$ and the vector $m = (m^{(1)}, \dots, m^{(n)})$. Then (5.36) is in vector form as

$$\dot{S}(t) = m + (A - U)S(t),$$

whose unique solution satisfies

$$S(t) \equiv (S_0 + (A - U)^{-1}m) \exp((A - U)t) - (A - U)^{-1}m,$$

and

$$\lim_{t \to \infty} S(t) = -(A - U)^{-1}m \equiv S^*,$$

where $A - U$ is nonsingular and m has nonnegative components (i.e., $(A - U)^{-1}m$ has nonnegative entries). Hence, Eq. (5.35) admits the disease-free solution

$$Q_{\text{DFS}}^{(5.35)} \equiv (S^*, 0, 0, 0).$$

The basic reproduction number of (5.35) is, in general, the spectral radius of an integral operator. It is possible to provide a threshold theorem involving an approximation of the basic reproduction number, as follows.

Theorem 5.8 *Let* $\alpha_{\min} \equiv \min\{\alpha^{(l,j)} : l, j \in \mathcal{N}, l \neq j\}$, $\alpha_{\max} \equiv \max\{\alpha^{(l,j)} : l, j \in \mathcal{N}, l \neq j\}$, $\theta_{\min} \equiv \min\{\theta^{(j)} : j \in \mathcal{N}\}$, $g_{\min} \equiv \min\{g^{(j)} : j \in \mathcal{N}\}$ *and* $\mu_{\min} \equiv \min\{\mu^{(j)} : j \in \mathcal{N}\}$. *For each* $i \in \mathcal{M}$, *let* $\beta_i \equiv \max\{\beta_i^{(j)} : j \in \mathcal{N}\}$, *and* $\gamma_i \equiv \max\{\gamma_i^{(j)} : j \in \mathcal{N}\}$. *If* $\sigma \in \mathcal{S}_{\text{periodic}}(\omega)$ *and*

$$\widehat{\mathrm{R}}_0^{(5.35)} \equiv \frac{\sum_{i=1}^m (\beta_i S^* + (n-1)(1 - \theta_{\min})\alpha_{\max}\gamma_i S^*)\tau_i}{\omega(g_{\min} + \mu_{\min} + (n-1)\theta_{\min}\alpha_{\min})} < 1, \tag{5.37}$$

then $Q_{\text{DFS}}^{(5.35)}$ *is globally attractive in the meaningful domain; the disease is eradicated by the screening control.*

Proof From Eq. (5.35) note that

$$\dot{S}^{(j)}(t) \leq m^{(j)} - \mu_j S^{(j)}(t) + \sum_{l \in \mathcal{N}} \alpha^{(l,j)} S^{(j)}(t).$$

Consider the comparison system

$$\dot{x}^{(j)}(t) = m^{(j)} - \mu^{(j)}x^{(j)}(t) + \sum_{l \in \mathcal{N}} \alpha^{(l,j)}x^{(j)}(t),$$
$$x(0) = (S_0^{(1)}, \ldots, S_0^{(n)}) = S_0. \tag{5.38}$$

As remarked above, the solution of this system converges to $-(A - U)^{-1}m = S^*$. Choose

$$0 < \epsilon = \frac{(1 - \widehat{\mathrm{R}}_0^{(5.35)})(\omega(g_{\min} + \mu_{\min} + (n-1)\theta_{\min}\alpha_{\min})}{2\sum_{i=1}^m (\beta_i S^* + (n-1)(1 - \theta_{\min})\alpha_{\max}\gamma_i)\tau_i}.$$

By comparison theorem, there exists a time $t^* > 0$ such that $S^{(j)}(t) \leq x^{(j)}(t) \leq S^* + \epsilon$ for $t \geq t^*$ and each $j \in \mathcal{N}$ (i.e., by Theorem 1.10). Choose $N \in \mathbb{N}$ as the smallest integer such that $N\omega > t^*$. The following inequalities follow from (5.35) for each $j \in \mathcal{N}$:

$$\sum_{j \in \mathcal{N}} \dot{I}_j(t) = \sum_{j \in \mathcal{N}} [\beta_\sigma^{(j)} \frac{S^{(j)}(t)I^{(j)}(t)}{N^{(j)}(t)} - g^{(j)}I^{(j)}(t) - \mu^{(j)}I^{(j)}(t)$$

$$- \theta^{(j)} \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} I^{(l)}(t) + (1 - \theta^{(j)}) \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} \gamma_\sigma^{(j)} \frac{S^{(l)}(t) I^{(l)}(t)}{N^{(l)}(t)}],$$

$$\leq \sum_{j \in \mathcal{N}} [\beta_\sigma \frac{S^{(j)}(t) I^{(j)}(t)}{N^{(j)}(t)} - g_{\min} I^{(j)}(t) - \mu_{\min} I^{(j)}(t) - \theta^{(j)} \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} I^{(j)}(t)$$

$$+ (1 - \theta^{(j)}) \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} \gamma_\sigma \frac{S^{(l)}(t) I^{(l)}(t)}{N^{(l)}(t)}].$$

Since

$$\sum_{j \in \mathcal{N}} \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} S^{(l)} I^{(l)} = \sum_{j \in \mathcal{N}} \sum_{l \in \mathcal{N} \setminus \{j\}} \alpha^{(l,j)} S^{(j)} I^{(j)} \leq \sum_{j \in \mathcal{N}} (n-1) \alpha_{\max} S^{(j)} I^{(j)},$$

then

$$\sum_{j \in \mathcal{N}} \dot{I}_j(t) \leq \sum_{j \in \mathcal{N}} [\beta_\sigma (S^* + \epsilon) - g_{\min} - \mu_{\min}$$

$$- (n-1) \theta_{\min} \alpha_{\min} + (n-1)(1 - \theta_{\min}) \alpha_{\max} \gamma_\sigma (S^* + \epsilon)] I^{(j)}(t),$$

$$\leq \sum_{j \in \mathcal{N}} [\beta_\sigma S^* + (n-1)(1 - \theta_{\min}) \alpha_{\max} \gamma_\sigma S^*$$

$$+ \epsilon (\beta_\sigma + (n-1)(1 - \theta_{\min}) \alpha_{\max} \gamma_\sigma) - g_{\min} - \mu_{\min} - (n-1) \theta_{\min} \alpha_{\min}] I^{(j)}(t),$$

$$= \lambda_{k,\epsilon} \sum_{j \in \mathcal{N}} I^{(j)}(t), \tag{5.39}$$

where

$$\lambda_{i,\epsilon} \equiv \beta_i S^* + (n-1)(1 - \theta_{\min}) \alpha_{\max} \gamma_i S^*$$

$$+ \epsilon (\beta_i + (n-1)(1 - \theta_{\min}) \alpha_{\max} \gamma_i) - g_{\min} - \mu_{\min} - (n-1) \theta_{\min} \alpha_{\min}. \tag{5.40}$$

Equation (5.39) implies

$$\sum_{j \in \mathcal{N}} I^{(j)}((N+1)\omega) \leq \sum_{j \in \mathcal{N}} I^{(j)}(N\omega) \exp \left(\sum_{i=1}^m \lambda_{i,\epsilon} \tau_i \right)$$

$$= \exp \left(\sum_{i=1}^m \lambda_{i,\epsilon} \tau_i \right) \sum_{j \in \mathcal{N}} I^{(j)}(N\omega),$$

from which it follows that $\sum_{j \in \mathcal{N}} I^{(j)}((N+1)\omega) \leq \eta \sum_{j \in \mathcal{N}} I^{(j)}(N\omega)$, where

$$\eta \equiv \exp\left(\sum_{i=1}^{m} \lambda_{i,\epsilon} \tau_i\right).$$

Note that $\eta \in (0,1)$ since $\widehat{R_0}^{(5.35)} < 1$ and by the choice of ϵ above. Thus,

$$\sum_{j \in \mathcal{N}} I^{(j)}((N+1)\omega) \leq \eta \sum_{j \in \mathcal{N}} I^{(j)}(N\omega) < \sum_{j \in \mathcal{N}} I^{(j)}(N\omega).$$

Similarly, it can be shown that

$$\sum_{j \in \mathcal{N}} I^{(j)}((N+h+1)\omega) \leq \eta \sum_{j \in \mathcal{N}} I^{(j)}((N+h)\omega), \quad \forall h \in \mathbb{N},$$

and so

$$\sum_{j \in \mathcal{N}} I^{(j)}((N+h+1)\omega) \leq \eta \sum_{j \in \mathcal{N}} I^{(j)}((N+h)\omega),$$

$$\leq \eta(\eta \sum_{j \in \mathcal{N}} I^{(j)}((N+h-1)\omega),$$

$$\vdots$$

$$\leq \eta^{h+1} \sum_{j \in \mathcal{N}} I^{(j)}(N\omega).$$

Therefore the sequence $\{\sum_{j \in \mathcal{N}} I^{(j)}((N+h)\omega)\}_{h=0}^{\infty}$ converges to zero as $h \to \infty$. Since $I^{(j)}(t)$ is bounded on $t \in [0, N\omega]$ for each j and since $\sum_{j \in \mathcal{N}} I^{(j)}$ is bounded on each interval $[t_0 + (N+h)\omega, (N+h+1)\omega]$ for $h \in \mathbb{N} \cup \{0\}$, then it follows that $I^{(j)}$ converges to zero as $h \to \infty$ for each j. The limiting system is given by Eq. (5.36), which converges to the disease-free solution $Q_{\text{DFS}}^{(5.35)}$.

Requiring that $\widehat{R_0}^{(5.35)} < 1$ in Eq. (5.37) defines a critical screening rate θ_{crit} that guarantees disease eradication. More precisely,

$$\theta_{\text{crit}} \equiv \frac{\sum_{i=1}^{m}(\beta_i S^* m - g_{\min} - \mu_{\min} + (n-1)\alpha_{\max}\gamma_i S^*)\tau_i}{\sum_{i=1}^{m}(\alpha_{\max}\gamma_i)\tau_i + \omega\alpha_{\min}}. \tag{5.41}$$

Example 5.7 Consider (5.35) with $\mathcal{N} = \{1,2\}$ and $\mathcal{M} = \{1,2\}$. Suppose that σ follows the seasonal switching rule outlined in (3.37) and the initial conditions are given by $(S^{(1)}, I^{(1)}, Q^{(1)}, R^{(1)}, S^{(2)}, I^{(2)}, Q^{(2)}, R^{(2)}) = (0.5, 0.1, 0, 0, 0.4, 0, 0, 0)$ (i.e., the disease begins in city 1). The following model parameters are used: $\beta_1 = 4.5$,
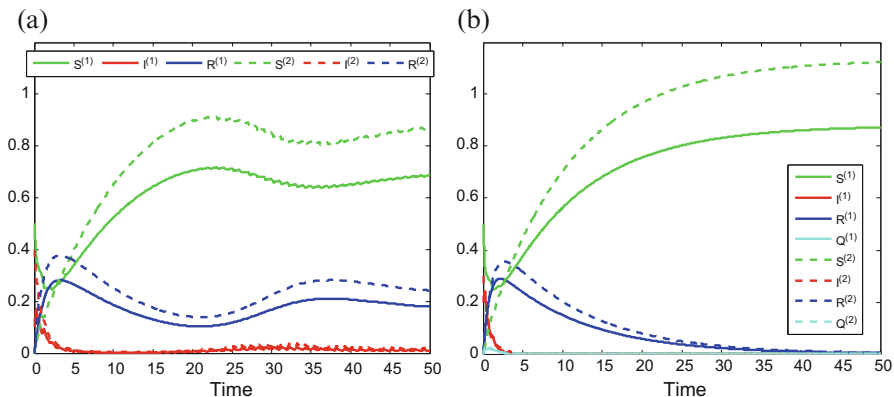
Fig. 5.15 Simulations of Example 5.7. (**a**) $\theta^{(1)} = \theta^{(2)} = 0$. (**b**) $\theta^{(1)} = 0.95$, $\theta^{(2)} = 0.9$

$\beta_2 = 0.5$, $g^{(1)} = 1.5$, $g^{(2)} = 1.2$, $m^{(1)} = 0.1$, $m^{(2)} = 0.09$, $\mu^{(1)} = 0.1$, $\mu^{(2)} = 0.09$, $\gamma = 1$, $\alpha^{(1,1)} = -0.4$, $\alpha^{(1,2)} = 0.4$, $\alpha^{(2,2)} = -0.3$, $\alpha^{(2,1)} = 0.3$. That is,

$$A = \begin{bmatrix} -0.4 & 0.3 \\ 0.4 & -0.3 \end{bmatrix}, \quad U = \mathrm{diag}\{0.1, 0.09\}, \quad m = (0.1, 0.09),$$

giving that $S^* = (0.880, 1.13)$. If $\theta^{(1)} = \theta^{(2)} = 0$ then $\widehat{R_0}^{(5.35)} = 1.47$ (see Fig. 5.15a). If $\theta^{(1)} = 0.95$ and $\theta^{(2)} = 0.9$ then $\widehat{R_0}^{(5.35)} = 0.987$ (see Fig. 5.15b); in this case, the critical screening rate is given by $\theta_{\mathrm{crit}} = 0.871$.

5.5 Switching Control for Vector-borne Diseases

In this part we return to the vector-borne model (4.35) for control strategy analysis. Switching cohort immunization is considered here: assume that a switching vaccination control is applied at a rate $v_\sigma > 0$ to the susceptible population (where immunization immediately moves an individual to the vaccinated class, V). Assume also that a switching treatment control is applied at a rate $p_\sigma > 0$. Motivated by realistic difficulties and failures of a vaccine program, the probability that a vaccinated individual can still become infected through transmission is assumed to be nonzero (but reduced when compared to the susceptible individuals); let

$$\xi \beta_\sigma V(t) \int_0^d f(u) I(t - u) \mathrm{d}u,$$

where $\xi \in [0, 1]$, correspond to such a reduced transmission between vaccinated and infected (i.e., ξ is a measure of the vaccine efficacy). Applied to (4.35), the control model is given by

$$\dot{S}(t) = \mu(1 - S(t)) - \beta_\sigma S(t) \int_0^d f(u)I(t-u)\mathrm{d}u - v_\sigma S(t) + \theta V(t),$$

$$\dot{I}(t) = \beta_\sigma (S(t) + \xi V(t)) \int_0^d f(u)I(t-u)\mathrm{d}u - (g + \mu + p_\sigma)I(t),$$

$$\dot{R}(t) = gI(t) + p_\sigma I(t) - \mu R(t),$$

$$\dot{V}(t) = v_\sigma S(t) - \xi\beta_\sigma V(t) \int_0^d f(u)I(t-u)\mathrm{d}u - (\theta + \mu)V(t),$$

$$(S(s), I(s), R(s), V(s)) = (S_0, I_0(s), R_0, V_0), \quad \forall s \in [-d, 0].$$

$$(5.42)$$

The physical domain of interest for (5.42) is given by

$$D_{(5.42)} \equiv \{(S, I, R, V) \in \mathbb{R}_+^4 : S + I + R + V = 1\},$$

and it is assumed that $(S_0, I_0(0), R_0, V_0) \in D_{(5.42)}$. Equation (5.42) admits m disease-free equilibria due to the time-varying vaccination rates:

$$Q^{(5.42)} \equiv (S_i^*, I_i^*, R_i^*, V_i^*) \equiv \left(\frac{\mu(\theta + \mu)}{\mu + v_i(1 - \theta)}, 0, 0, \frac{v_i S_i^*}{\theta + \mu} \right),$$

for all $i \in \mathcal{M}$. The movement of the population between compartments is illustrated in Fig. 5.16.

In the absence of infection, the solution of (5.42) traverses between the disease-free equilibria as the vaccination rates vary with respect to time. This observation motivates studying convergence to a disease-free set: when $I(t) \equiv 0$, the number of individuals in the recovered class approaches zero exponentially and the reduced model is given by

$$\dot{S}(t) = \mu(1 - S(t)) - v_\sigma S(t) + \theta V(t),$$
$$\dot{V}(t) = v_\sigma S(t) - (\mu + \theta)V(t).$$

$$(5.43)$$

Fig. 5.16 Flow of the vector-borne model with treatment and vaccination (5.42). The *red lines* represent the horizontal transmission and the *blue lines* represent the treatment and vaccination strategies. Births/deaths are omitted here for illustrative purposes
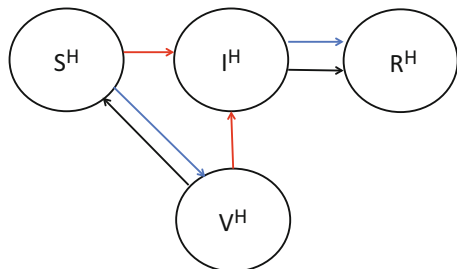
Define $v_{\min} \equiv \min\{v_i : i \in \mathcal{M}\}$ and $v_{\max} \equiv \max\{v_i : i \in \mathcal{M}\}$. Since $S + V = 1$ is invariant to (5.43),

$$\dot{S}(t) \leq \mu - (\mu + v_{\min})S(t) + \theta(1 - S(t)),$$

$$= (\mu + v_{\min} + \theta)\left(\frac{\overline{S}_{\max}}{S(t)} - 1\right)S(t),$$

so that $\dot{S}(t) \leq 0$ if $1 \geq S(t) \geq \overline{S}_{\max}$ where

$$\overline{S}_{\max} \equiv \frac{\mu + \theta}{\mu + v_{\min} + \theta}.$$

Similarly, if $0 < S(t) \leq \overline{S}_{\min} \equiv \mu(1 + \theta)/(\mu + v_{\max} + \theta)$, then $\dot{S}(t) \geq 0$ since

$$\dot{S}(t) \geq \mu - (\mu + v_{\max})S(t) + \theta(1 - S(t)),$$

$$= (\mu + v_{\max} + \theta)\left(\frac{\overline{S}_{\min}}{S(t)} - 1\right)S(t).$$

Further, $\dot{V}(t) \leq 0$ whenever $1 \geq V(t) \geq \overline{V}_{\max} \equiv v_{\max}/(\mu + v_{\max} + \theta)$ since

$$\dot{V}(t) \leq v_{\max}(1 - V(t)) - (\mu + \theta)V(t),$$

$$= (\mu + v_{\max} + \theta)\left(\frac{\overline{V}_{\max}}{V(t)} - 1\right)V(t).$$

Finally, if $0 < V(t) \leq \overline{V}_{\min} \equiv v_{\min}/(\mu + v_{\min} + \theta)$, then $\dot{V}(t) \geq 0$ since

$$\dot{V}(t) \geq v_{\min}(1 - V(t)) - (\mu + \theta)V(t),$$

$$= (\mu + v_{\min} + \theta)\left(\frac{\overline{V}_{\min}}{V(t)} - 1\right)V(t).$$

The solution of (5.43) converges to the set

$$\{(S, V) \in \mathbb{R}_+^2 : \overline{S}_{\min} \leq S \leq \overline{S}_{\max}, \overline{V}_{\min} \leq V \leq \overline{V}_{\max}\},$$

which can be shown as follows: Consider the comparison system

$$\dot{x}(t) = \begin{cases} (\mu + v_{\min} + \theta)\left(\frac{\overline{S}_{\max}}{x(t)} - 1\right)x(t), & \text{if } x(t) \neq 0, \\ \mu + \theta, & \text{if } x(t) = 0, \end{cases} \qquad (5.44)$$

$$x(0) = S_0.$$

The solution of (5.44) converges to $\overline{S}_{\max}$. By the comparison theorem,, for any $\epsilon > 0$ there exists $t^* > 0$ such that $S(t) \leq x(t) \leq \overline{S}_{\max} + \epsilon$ for all $t \geq t^*$. Similarly, the comparison system

$$\dot{x}(t) = \begin{cases} (\mu + v_{\max} + \theta)\left(\frac{\overline{S}_{\min}}{x(t)} - 1\right) x(t), & \text{if } x(t) \neq 0, \\ \mu - (\mu + v_{\max})x(t) + \theta(1 - x(t)), & \text{if } x(t) = 0, \end{cases} \tag{5.45}$$

$$x(0) = S_0,$$

yields that $\lim_{t \to \infty} S(t) \geq \overline{S}_{\min}$, with similar arguments with respect to V giving the desired result. Therefore, under the assumption that $I(t) \equiv 0$, the solution of (5.42) converges to the disease-free convex set

$$\Psi_{\text{cohort}} \equiv \{(S, I, R, V) \in \mathbb{R}_+^4 : \overline{S}_{\min} \leq S \leq \overline{S}_{\max}, I = 0, R = 0, \overline{V}_{\min} \leq V \leq \overline{V}_{\max}\}.$$

Define the following constants:

$$\lambda_i \equiv \beta_i(\overline{S}_{\max} + \xi\overline{V}_{\max}) - (\mu + g + p_i), \quad \forall i \in \mathcal{M}.$$

$\mathcal{M}^- \equiv \{i \in \mathcal{M} : \lambda_i < 0\}$ and $\mathcal{M}^+ \equiv \{i \in \mathcal{M} : \lambda_i \geq 0\}$. The idea here is that the switched system is composed of a mixture of stable and unstable modes, where $\mathcal{M}^-$ and $\mathcal{M}^+$ denote the stable and unstable modes, respectively. To prove threshold conditions for disease eradication, we focus on the set Ψ_{cohort}. Before proceeding, we remind the reader of some switched systems notions: let $\sigma \in \mathcal{S}_{\text{dwell}}$, $t^2 > t^1 \geq 0$, and let

$$T_i(t^1, t^2) \equiv |\{t \in [t^1, t^2] : \sigma(t) = i\}|,$$

$$T^+(t^1, t^2) \equiv |\{t \in [t^1, t^2] : \sigma(t) \in \mathcal{M}^+\}|,$$

$$T^-(t^1, t^2) \equiv |\{t \in [t^1, t^2] : \sigma(t) \in \mathcal{M}^-\}|,$$

$$N_i(t^1, t^2) \equiv |\{t_k \in [t^1, t^2) : \sigma(t_k) = i\}|,$$

$$N(t^1, t^2) \equiv |\{t_k \in [t^1, t^2)\}|,$$

$$N^-(t^1, t^2) \equiv |\{t_k \in [t^1, t^2) : \sigma(t_k) \in \mathcal{M}^-\}|.$$

Roughly, these are the activation time in the ith mode, set $\mathcal{M}^+$, set $\mathcal{M}^-$, and the number of switches activating the ith mode, the total number of switches, and the number of switches activating modes in the set $\mathcal{M}^+$, respectively. Note that $\bigcup_{i=1}^m T_i(t_0, t) = [t_0, t]$. As an illustration, consider the switching rule in Fig. 5.17, which gives that

$$T_1(0, 5) = 2, \quad T_1(0, 4) = 1, \quad T_2(3, 3.5) = 0.5, \quad T_3(0, 5) = 2$$

$$N_1(0, 5) = 2, \quad N_1(0, 4) = 1, \quad N_2(3, 3.5) = 1, \quad N_3(0, 5) = 1.$$

Fig. 5.17 Example of a
switching rule $\sigma \in \mathscr{S}_{\text{dwell}}$
with switch times $t_k = 1, 3, 4$
and $\mathscr{M} = \{1, 2, 3\}$
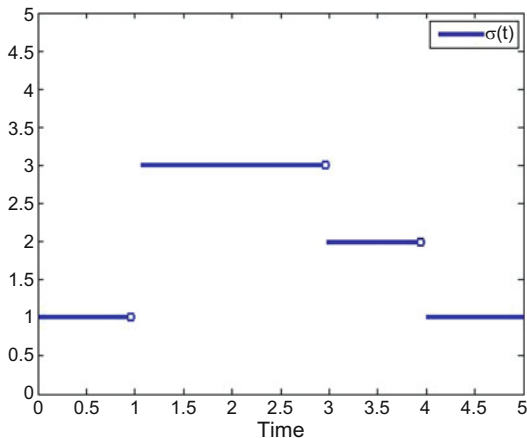
Some necessary Halanay-like results are needed for the disease eradication
proofs and are reviewed here. In [174], Zhu used the following Halanay-like lemma
to study switched system stability.

Lemma 5.2 *Assume that $\beta, \alpha > 0$ and the function $u : [t_0 - d, \infty) \to \mathbb{R}_+$ satisfies
the following delay differential inequality:*

$$\dot{u}(t) \leq \beta \|u_t\|_d - \alpha u(t), \quad \forall t \geq t_0.$$

If $\beta - \alpha \geq 0$, then

$$u(t) \leq \|u_{t_0}\|_d \exp((\beta - \alpha)(t - t_0)), \quad \forall t \geq t_0.$$

*If $\beta - \alpha < 0$, then there exists a positive constant η satisfying $\eta + \beta \exp(\eta d) - \alpha < 0$
such that*

$$u(t) \leq \|u_{t_0}\|_d \exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where $\|u_t\|_d \equiv \sup_{-d \leq s \leq 0} u(t + s)$.

For completeness, an impulsive delayed version of a switching Halany-like result
is presented here without proof (see Proposition 1 in [142]).

Proposition 5.1 *For $i \in \mathscr{M}$, let $a_i, b_i, g_i, h_i \geq 0$ be constants and assume that a
function $u : [t_0 - d, +\infty) \to \mathbb{R}_+$ satisfies*

$$
\begin{aligned}
\dot{u}(t) &\leq b_\sigma \|u_t\|_d - a_\sigma u(t), & t \neq t_k, \quad t \geq t_0, \\
u(t) &\leq g_\sigma u(t^-) + h_\sigma \|u_t\|_d, & t = t_k, \quad k \in \mathbb{N},
\end{aligned}
\tag{5.46}
$$

for some $\sigma \in \mathscr{S}_{\text{dwell}}(d)$ (i.e., $t_k - t_{k-1} \geq d$ for all $k \in \mathbb{N}$). Then, for $t \geq t_0$,

$$u(t) \leq \|u_{t_0}\|_d \left(\prod_{j=1}^{N(t_0,t)} \delta_{i_j} \right) \exp \left(\sum_{i \in \mathscr{M}^+} \lambda_i T_i(t_0, t) - \sum_{i \in \mathscr{M}^-} \eta_i \tilde{T}_i(t_0, t) \right), \qquad (5.47)$$

where $\tilde{T}_i(t_0, t) \equiv T_i(t_0, t) - N_i(t_0, t)d$, $\lambda_i \equiv b_i \max_{i \in \mathscr{M}}\{1/\delta_i, 1\} - a_i$, $\delta_i \equiv g_i + h_i \exp(\xi d)$, $\xi \equiv \max\{\xi_i : i \in \mathscr{M}^-\}$, $\xi_i > 0$ is chosen for $i \in \mathscr{M}^-$ so that $\xi_i + b_i \exp(\xi_i d) - a_i < 0$, $\mathscr{M}^+ \equiv \{i \in \mathscr{M} : \lambda_i \geq 0\}$, $\mathscr{M}^- \equiv \{i \in \mathscr{M} : \lambda_i < 0\}$, and $\eta_i > 0$ is chosen for $i \in \mathscr{M}^-$ so that $\eta_i + b_i \max_{i \in \mathscr{M}}\{1/\delta_i, 1\} \exp(\eta_i d) - a_i < 0$.

Proposition 5.1 is placed into the following useful form for this section (set $g_i = 1$, $h_i = 0$, and $\delta_i = 1$ for each $i \in \mathscr{M}$).

Proposition 5.2 *For $i \in \mathscr{M}$, let $\beta_i \geq 0$ and $\alpha_i \geq 0$. Assume that a function $u : [t_0 - d, \infty) \to \mathbb{R}_+$ satisfies the following switching delay differential inequality:*

$$\dot{u}(t) \leq \beta_\sigma \|u_t\|_d - \alpha_\sigma u(t),$$

and $\sigma \in \mathscr{S}_{\text{dwell}}(d)$. Let $\mathscr{M}^+ \equiv \{i \in \mathscr{M} : \lambda_i \geq 0\}$ and $\mathscr{M}^- \equiv \{i \in \mathscr{M} : \lambda_i < 0\}$ where $\lambda_i \equiv \beta_i - \alpha_i$ for each $i \in \mathscr{M}$. For each $i \in \mathscr{M}^-$, choose $\eta_i > 0$ such that

$$\eta_i + \beta_i \exp(\eta_i d) - \alpha_i < 0.$$

Then,

$$u(t) \leq \|u_{t_0}\|_d \exp \left(\sum_{i \in \mathscr{M}^+} \lambda_i T_i(t_0, t) - \sum_{i \in \mathscr{M}^-} \eta_i (T_i(t_0, t) - N_i(t_0, t)d) \right), \quad \forall t \geq t_0.$$

$$(5.48)$$

Note that if $\lambda_i = \beta_i - \alpha_i < 0$ for $i \in \mathscr{M}$ then it is always possible to choose $\eta_i > 0$ satisfying $\eta_i + \beta_i \exp(\eta_i d) - \alpha_i < 0$. Letting $F_i(x) \equiv x + \beta_i \exp(xd) - \alpha_i$, $F(0) = \beta_i - \alpha_i < 0$ and $F'(\eta_i) = 1 + \beta_i d \exp(\eta_i d) > 0$. By continuity of F_i, there exists $\eta_i^* > 0$ such that $F(\eta_i^*) = 0$ and η_i can be chosen as $0 < \eta_i < \eta_i^*$.

Proposition 5.3 *Assume that $\beta_i \geq 0$ and $\alpha_i \geq 0$ for $i \in \mathscr{M}$. Assume that a function $u : [t_0 - d, \infty) \to \mathbb{R}_+$ satisfies the following switching delay differential inequality:*

$$\dot{u}(t) \leq \beta_\sigma \|u_t\|_d - \alpha_\sigma u(t),$$

and $\sigma \in \mathscr{S}_{\text{periodic}}(\omega)$. Let $\mathscr{M}^+ \equiv \{i \in \mathscr{M} : \lambda_i \geq 0\}$ and $\mathscr{M}^- \equiv \{i \in \mathscr{M} : \lambda_i < 0\}$ where $\lambda_i \equiv \beta_i - \alpha_i$ for each $i \in \mathscr{M}$. For each $i \in \mathscr{M}^-$, choose $\eta_i > 0$ such that

$$\eta_i + \beta_i \exp(\eta_i d) - \alpha_i < 0.$$

Then, u is bounded on any compact interval and satisfies

$$u(t_0 + j\omega) \leq \|u_{t_0}\|_d \chi^j, \quad \forall j \in \mathbb{N}, \qquad (5.49)$$

where

$$\chi = \exp\left[\sum_{i \in \mathcal{M}^+} \lambda_i \tau_i - \sum_{i \in \mathcal{M}^-} \eta_i(\tau_i - d)\right].$$

Proof The boundedness of u on any compact interval follows immediately from Theorem 5.2. From Eq. (5.48), for $j \in \mathbb{N}$

$u(t_0 + j\omega)$

$$\leq \|u_{t_0}\|_d \exp\left[\sum_{i \in \mathcal{M}^+} \lambda_i T_i(t_0, t_0 + j\omega) - \sum_{i \in \mathcal{M}^-} \eta_i(T_i(t_0, t_0 + j\omega) - N_i(t_0, t_0 + j\omega)d)\right],$$

$$= \|u_{t_0}\|_d \exp\left[\sum_{i \in \mathcal{M}^+} \lambda_i j T_i(t_0, t_0 + \omega) - \sum_{i \in \mathcal{M}^-} \eta_i j(T_i(t_0, t_0 + \omega) - N_i(t_0, t_0 + \omega)d)\right],$$

$$= \|u_{t_0}\|_d \exp\left[j\sum_{i \in \mathcal{M}^+} \lambda_i \tau_i - j\sum_{i \in \mathcal{M}^-} \eta_i(\tau_i - d)\right],$$

$$= \|u_{t_0}\|_d \chi^j,$$

since $\sigma \in \mathscr{S}_{\text{periodic}}(\omega)$ implies that $T_i(t_0, t_0 + j\omega) = jT_i(t_0, t_0 + \omega)$ and $N_i(t_0, t_0 + j\omega) = jN_i(t_0, t_0 + \omega)$. $\qquad\blacksquare$

We are now in a position to prove some eradication results.

Theorem 5.9 *For each $i \in \mathcal{M}^-$, let $\eta_i > 0$ satisfy*

$$\eta_i + \beta_i(\overline{S}_{\max} + \xi\overline{V}_{\max})\exp(\eta_i d) - (\mu + g + p_i) < 0.$$

Let $\lambda^+ \equiv \max\{\lambda_i : i \in \mathcal{M}^+\}$ and $\lambda^- \equiv \min\{\eta_i : i \in \mathcal{M}^-\}$. Let $\sigma \in \mathscr{S}_{\text{dwell}}(d)$ such that there exists $M > 0$ and $\widetilde{t} > 0$ satisfying

$$\sup_{t \geq \widetilde{t}} \frac{t - \widetilde{t}}{T^-(\widetilde{t}, t) - N^-(\widetilde{t}, t)d} \leq M. \tag{5.50}$$

If there exists $q \geq 0$ such that

$$T^+(\widetilde{t}, t) \leq q(T^-(\widetilde{t}, t) - N^-(\widetilde{t}, t)d)), \tag{5.51}$$

$$q\lambda^+ < \lambda^-, \tag{5.52}$$

then the solution of (5.42) satisfies $\lim_{t \to \infty}(S(t), I(t), R(t), V(t)) \in \Psi_{\text{cohort}}$; the solution converges to the disease-free set and is therefore eradicated.

Proof From the switched model of a vector-borne disease (5.42),

$$\dot{S}(t) = \mu - \beta_\sigma S(t) \int_0^d f(u)I(t-u)\mathrm{d}u - v_\sigma S(t) + \theta V(t),$$

$$\leq \mu(1 - S(t)) - v_\sigma S(t) + \theta V(t),$$

$$\leq \mu(1 - S(t)) - v_{\min}S(t) + \theta V(t),$$

$$\leq \mu + \theta - (\mu + \theta + v_{\min})S(t),$$

since $V(t) = 1 - S(t) - I(t) - R(t) \leq 1 - S(t)$. Similarly,

$$\dot{V}(t) = v_\sigma S(t) - \xi\beta_\sigma V(t) \int_0^d f(u)I(t-u)\mathrm{d}u - (\mu + \theta)V(t),$$

$$\leq v_\sigma S(t) - (\mu + \theta)V(t),$$

$$\leq v_{\max}S(t) - (\mu + \theta)V(t),$$

$$\leq v_{\max}(1 - V(t)) - (\mu + \theta)V(t),$$

$$\leq v_{\max} - (v_{\max} + \mu + \theta)V(t).$$

For any $\epsilon > 0$, there exists a time $t^* > 0$ for which $S(t) \leq \overline{S}_{\max} + \epsilon$ and $V(t) \leq \overline{V}_{\max} + \epsilon$ for all $t \geq t^*$. Let l be the smallest positive integer such that $t_l > \max\{\widetilde{t}, t^*\}$. Then,

$$\dot{I}(t) = \beta_\sigma(S(t) + \xi V(t)) \int_0^d f(u)I(t-u)\mathrm{d}u - (\mu + g + p_\sigma)I(t),$$

$$\leq \beta_{\max}(1 + \xi) \sup_{t-d \leq s \leq t} I(s) - (\mu + g + p_{\min})I(t), \quad \forall t \in [0, t_l).$$

By inspection, $I(t) \leq \|I_0\|_d \exp(\eta t)$ for all $t \in [0, t_l)$ where $\eta > 0$ satisfies

$$\eta + \beta_{\max}(1 + \xi)\exp(\eta d) - (\mu + g + p_{\min}) > 0$$

by Lemma 5.2. In general,

$$\dot{I}(t) \leq \beta_\sigma[(\overline{S}_{\max} + \epsilon) + \xi(\overline{V}_{\max} + \epsilon)] \sup_{t-d \leq s \leq t} I(s) - (\mu + g + p_\sigma)I(t), \qquad (5.53)$$

for all $t \in [t_{k-1}, t_k)$ and $k - 1 \geq l$, where $I_{t_l} \in \mathrm{PC}([-d, 0], \mathbb{R}_+)$. Define the constants

$$\lambda_{i,\epsilon} \equiv \beta_i[(\overline{S}_{\max} + \epsilon) + \xi(\overline{V}_{\max} + \epsilon)] - (\mu + g + p_i), \quad \forall i \in \mathcal{M}.$$

For each $i \in \mathcal{M}^-$, let $\eta_{i,\epsilon} > 0$ satisfy

$$\eta_{i,\epsilon} + \beta_i[(\overline{S}_{\max} + \epsilon) + \xi(\overline{V}_{\max} + \epsilon)]\exp(\eta_{i,\epsilon}d) - (\mu + g + p_i) < 0.$$

Proposition 5.2 thus implies that

$$I(t) \leq I_0^* \exp \left[\sum_{i \in \mathcal{M}^+} \lambda_{i,\epsilon} T_i(t_l, t) - \sum_{i \in \mathcal{M}^-} \eta_{i,\epsilon}(T_i(t_l, t) - N_i(t_l, t)d) \right], \qquad (5.54)$$

for all $t \in [t_{k-1}, t_k)$, $k - 1 \geq l$, where $I_0^* \equiv \|I_0\|_d \exp(\eta t_l)$.

Define $\lambda_\epsilon^+ \equiv \max\{\lambda_{i,\epsilon} : i \in \mathcal{M}^+\}$ and $\lambda_\epsilon^- \equiv \{\eta_{i,\epsilon} : i \in \mathcal{M}^-\}$. Then, by definition,

$$\beta_i[(\bar{S}_{\max} + \epsilon) + \xi(\bar{V}_{\max} + \epsilon)] \exp(\eta_{i,\epsilon} d) - (\mu + g + p_i) < -\eta_{i,\epsilon} \leq -\lambda_\epsilon^-, \quad \forall i \in \mathcal{M},$$

which can be rewritten as

$$\beta_i(\bar{S}_{\max} + \xi\bar{V}_{\max}) \exp(\eta_{i,\epsilon} d) - (\mu + g + p_i) + G_i\epsilon < -\eta_{i,\epsilon} \leq -\lambda_\epsilon^-, \quad \forall i \in \mathcal{M},$$

where

$$G_i \equiv \beta_i(1 + \xi) \exp(\eta_{i,\epsilon} d), \quad \forall i \in \mathcal{M}.$$

Also,

$$\beta_i(\bar{S}_{\max} + \xi\bar{V}_{\max}) \exp(\eta_i d) - (\mu + g + p_i) < -\eta_i \leq -\lambda^-, \quad \forall i \in \mathcal{M}.$$

Therefore, there exists a constant F_1 such that $-\lambda_\epsilon^+ \leq -\lambda^- + F_1\epsilon$. Letting $\nu \in \arg\max\{\lambda_i : i \in \mathcal{M}^+\}$,

$$q\lambda_\epsilon^+ \leq q\lambda^+ + F_2\epsilon,$$

where

$$F_2 \equiv q\beta_\nu(\bar{S}_{\max} + \xi\bar{V}_{\max}) - (\mu + g + c_\nu).$$

Hence,

$$q\lambda_\epsilon^+ - \lambda_\epsilon^- \leq q\lambda^+ - \lambda^- + (F_1 + F_2)\epsilon.$$

Since $q\lambda^+ - \lambda^- < 0$, there exists a positive constant δ such that $q\lambda_\epsilon^+ - \lambda_\epsilon^- \leq -0.5\delta$. Choose

$$\epsilon = \frac{\delta(F_1 + F_2)}{2},$$

then $q\lambda_\epsilon^+ - \lambda_\epsilon^- \leq -0.5\delta$.

It thus follows from Eqs. (5.50), (5.51), and (5.54) that

$$I(t) \leq I_0^* \exp\left[\lambda_\epsilon^+ \sum_{i \in \mathcal{M}^+} T_i(t_l, t) - \lambda_\epsilon^- \sum_{i \in \mathcal{M}^-} (T_i(t_l, t) - N_i(t_l, t)d)\right],$$

$$= I_0^* \exp[\lambda_\epsilon^+ T^+(t_l, t) - \lambda_\epsilon^- (T^-(t_l, t) - N^-(t_l, t)d)],$$

$$\leq I_0^* \exp[q\lambda_\epsilon^+ (T^-(t_l, t) - N^-(t_l, t)d) - \lambda_\epsilon^- (T^-(t_l, t) - N^-(t_l, t)d)],$$

$$= I_0^* \exp[(q\lambda_\epsilon^+ - \lambda_\epsilon^-)(T^-(t_l, t) - N^-(t_l, t)d)],$$

$$\leq I_0^* \exp\left[(q\lambda_\epsilon^+ - \lambda_\epsilon^-)\frac{(t - t_l)}{M}\right], \quad \forall t \geq t_l.$$

Equation (5.51) guarantees that $T^-(t_l, t) - N^-(t_l, t)d \geq 0$. Therefore,

$$I(t) \leq I_0^* \exp[-0.5\delta(t - t_l)], \quad \forall t \geq t_l.$$

It follows that $\lim_{t \to \infty} R(t) = 0$ and (5.42) reduces to system (5.43), from which the result follows.

Intuitively, Eqs. (5.51) and (5.52) describe the time spent in the unstable mode $\mathcal{M}^+$, with corresponding worst-case growth rate λ^+, versus the time spent in the stable modes $\mathcal{M}^-$, with corresponding most conservative decay rate λ^-. The constant q characterizes said relationships. If Eq. (5.52) holds, then

$$q \max_{i \in \mathcal{M}^+} \{\beta_i(\overline{S}_{\max} + \xi \overline{V}_{\max}) - (\mu + g + p_i)]\}$$

$$+ \min_{i \in \mathcal{M}^-} \{\beta_i(\overline{S}_{\max} + \xi \overline{V}_{\max}) \exp(\eta_i d) - (\mu + g + p_i)\} < 0.$$

Let $\nu \in \arg\max\{\lambda_i : i \in \mathcal{M}^+\}$ and $\zeta \in \arg\min\{\eta_i : i \in \mathcal{M}^+\}$. Then

$$\lambda^+ = \beta_\nu(\overline{S}_{\max} + \xi \overline{V}_{\max}) - (\mu + g + p_\nu)$$

and

$$\lambda^- = \beta_\zeta(\overline{S}_{\max} + \xi \overline{V}_{\max}) \exp(\eta_\zeta d) - (\mu + g + p_\zeta).$$

Hence, (5.52) implies that

$$\overline{R}_0^{(5.42)} \equiv q\frac{(\beta_\nu + \beta_\zeta \exp(\eta_\zeta d))(\overline{S}_{\max} + \xi \overline{V}_{\max})}{2\mu + 2g + p_\nu + p_\zeta} < 1, \tag{5.55}$$

an approximation of the disease's basic reproduction number. In fact, (5.52) implies (5.55); (5.52) is a stricter requirement on the model parameters. Controlled eradication under periodic variations can be established as follows.

Theorem 5.10 *For each $i \in \mathcal{M}^-$, let $\eta_i > 0$ satisfy*

$$\eta_i + \beta_i(\overline{S}_{\max} + \xi \overline{V}_{\max}) \exp(\eta_i d) - (\mu + g + p_i) < 0.$$

If $\sigma \in \mathcal{S}_{\text{periodic}}(\omega)$ and

$$\Lambda_{\text{cohort}} \equiv \sum_{i \in \mathcal{M}^+} \lambda_i \tau_i - \sum_{i \in \mathcal{M}^-} \eta_i(\tau_i - d) < 0, \tag{5.56}$$

then the solution of (5.42) satisfies $\lim_{t \to \infty}(S(t), I(t), R(t), V(t)) \in \Psi_{\text{cohort}}$; the solution converges to the disease-free set and is therefore eradicated.

Proof Beginning from Eq. (5.53) in the proof of Theorem 5.9, choose the smallest positive integer B such that $B\omega > \max\{\tilde{t}, t^*\}$. By Proposition 5.3, $I((B+j)\omega) \leq \|I_{B\omega}\|_d \delta^j$ for each $j \in \mathbb{N}$, where

$$\delta \equiv \exp\left[\sum_{i \in \mathcal{M}^+} \lambda_{i,\epsilon} \tau_i - \sum_{i \in \mathcal{M}^-} \eta_{i,\epsilon}(\tau_i - d) \right]$$

and $\|I_{B\omega}\|_d \leq K$ for some constant $K > 0$. It follows from (5.56) and the arguments in the proof of Theorem 5.9 that $\epsilon > 0$ can be chosen sufficiently small to guarantee that $0 < \delta < 1$. Thus, $\lim_{t \to \infty} I(t) = 0$, from which $\lim_{t \to \infty} R(t) = 0$ follows. Equation (5.42) reduces to (5.43) and the result holds.

Equation (5.56) implies that

$$\sum_{i \in \mathcal{M}^+} [\beta_i(\overline{S}_{\max} + \xi \overline{V}_{\max}) - (\mu + g + p_i)]\tau_i$$

$$+ \sum_{i \in \mathcal{M}^-} [\beta_i(\overline{S}_{\max} + \xi \overline{V}_{\max}) \exp(\eta_i d) - (\mu + g + p_i)](\tau_i - d),$$

$$< \sum_{i \in \mathcal{M}^+} [\beta_i(\overline{S}_{\max} + \xi \overline{V}_{\max}) - (\mu + g + p_i)]\tau_i + \sum_{i \in \mathcal{M}^-} (-\eta_i)(\tau_i - d),$$

$$= \sum_{i \in \mathcal{M}^+} \lambda_i \tau_i - \sum_{i \in \mathcal{M}^-} \eta_i(\tau_i - d),$$

$$< 0.$$

That is, (5.56) implies that $\overline{R_0}^{(5.42)} < 1$ where

$$\overline{R_0}^{(5.42)} \equiv \frac{\sum_{i \in \mathcal{M}^+} \beta_i(\overline{S}_{\max} + \xi \overline{V}_{\max})\tau_i + \sum_{i \in \mathcal{M}^-} \beta_i \exp(\eta_i d)(\overline{S}_{\max} + \xi \overline{V}_{\max})(\tau_i - d)}{\sum_{i \in \mathcal{M}^+} (\mu + g + p_i)\tau_i + \sum_{i \in \mathcal{M}^-} (\mu + g + p_i)(\tau_i - d)}. \tag{5.57}$$

Table 5.1 Epidemiological parameters

Parameter	Description	Value
β_σ	Average number of contacts per unit time	[8, 1.6]
μ	Natural birth/death rate	1
g	Recovery rate	1.5
d	Upper bound on the incubation time	0.1

The parameter values given in brackets represent the switching value associated with $\sigma = 1$ and $\sigma = 2$, respectively

$\overline{R}_0^{(5.42)}$ may be viewed as an approximate basic reproduction number and it should be noted that the theorem condition is stricter than requiring $\overline{R}_0^{(5.42)} < 1$. A comparison of these switching control strategies (i.e., switching vaccination and switching treatment) is reserved for Sect. 6.2.1.

The results of this section are illustrated with simulation. Consider the initial conditions $(S_0, I_0, R_0, V_0) = (0.9, 0.1, 0, 0)$, baseline model parameters in Table 5.1, and dwell-time satisfying periodic switching rule

$$\sigma = \begin{cases} 1, & \text{if } t \in [k, k + \frac{3}{12}), k \in \mathbb{N} \cup \{0\}, \\ 2, & \text{if } t \in [k + \frac{3}{12}, k + 1), \end{cases} \tag{5.58}$$

which is motivated from seasonal variations in the model parameters. The period of the switching rule is $\omega = 1$ with $\tau_1 = 3/12$ (modeling a winter season or rainy season, depending on climate) and $\tau_2 = 9/12$ (summer seasons or dry season). As in [104], let

$$f(u) \equiv \frac{\exp(-u)}{1 - \exp(-d)}.$$

Let $v_1 = 3$, $v = 2$, $p = 2$, $p = 0.5$. For the susceptible cohort immunization program, the model parameters give $\overline{S}_{\max} = 0.3548$, $\overline{V}_{\max} = 0.7317$, and $\lambda_1 = 0.8948$ (i.e., $\mathscr{M}^+ = \{1\}$). Letting $\eta_2 = 1$ (i.e., $\mathscr{M}^- = \{2\}$) implies that

$$\Lambda_{\text{cohort}} = -0.4263,$$

and convergence to the disease-free set Ψ_{cohort} by Theorem 5.10 (Fig. 5.18).

5.6   Discussions

Cohort immunization (i.e., time-constant vaccination) has been implemented by a number of countries, as discussed earlier. As mentioned, the predominant strategy for measles immunization follow a recommendation of doses at 15 months of

Fig. 5.18 Simulations of the switching control scheme (5.42) with parameters in Table 5.1
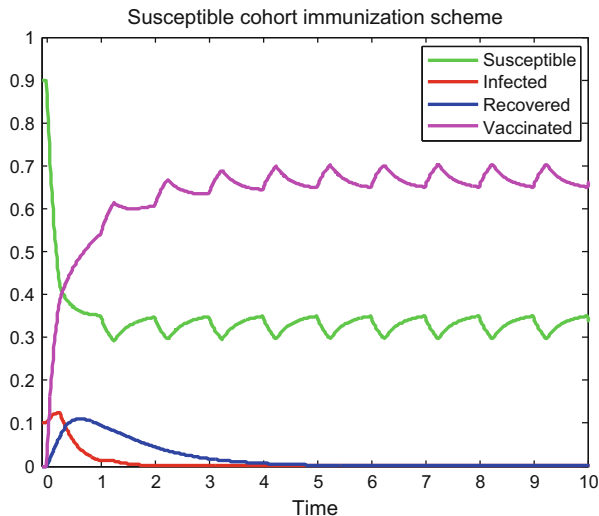
age and approximately 6 years of age in many parts of the Western world [139]. For background studies in the literature on epidemic models with such a control program, the reader is referred to [4, 69, 102, 138] and the references therein. The control strategies considered in Sect. 5.1 assume immediate movement from susceptible classes to the recovered class. This ignores the time involved to obtain immunity by completing a vaccination program. Motivated by this, consider the usual vaccination schedule for hepatitis B where individuals are given three vaccinations separated by 1 month and 6 months [101]. The authors further note that approximately 30–50% of individuals will gain anti-HB antibodies after the first dose, 80–90% after the second dose, and virtually all 1 month after the final dose. Based on their work on hepatitis B and measles in [101], the model (5.13) was analyzed. The application of vaccination and treatment schemes to switched SIR models in Sects. 5.1 and 5.2 are largely based on, and extend, the works in [94, 96].

Hepatitis B, Chagas' disease, HIV/AIDS, and tuberculosis are examples of diseases displaying latency periods [103, 140] and therefore appropriately modeled by the SEIR model (5.22), which has been extensively studied in the literature (e.g., see [69, 72, 80, 81, 103, 134]). The SEIR model with vertical transmission (5.27) was analyzed with switching because of the number of infectious diseases with latency period that are transmitted by both horizontal and vertical modes (e.g., rubella, herpes simplex, hepatitis B, Chagas' disease [140]). The SEIR model with disease-induced deaths (i.e., Eq. (5.31)) is an appropriate modeling choice for disease like AIDS [140]. A summary of the critical control rates guaranteeing eradication in the various theorems provided in Sects. 5.1, 5.2, and 5.3 is given in Table 5.2.

In the mathematical epidemic modeling literature, a time-constant entry/exit screening strategy was studied by Liu and Takeuchi [100] consisting of a two-city SIS model with transport-related infections and a screening process. Entry and

Table 5.2 Critical control rates of the epidemic models with periodic switching

Control strategy	Disease model	Critical control rate
Newborn vaccinations	(5.4)	$\rho_{\text{crit}} \equiv 1 - \omega(\mu + g)/(\sum_{i=1}^{m} \beta_i \tau_i)$
Susceptible vaccinations	(5.8)	$v_{\text{crit}} \equiv \mu\left(\sum_{i=1}^{m} \beta_i \tau_i/(\omega(\mu + g)) - 1\right)$
Vaccinations with progressive immunity	(5.13)	$v_{\text{crit}} \equiv \dfrac{\mu\left(\sum_{i=1}^{m} \beta_i \tau_i/(\omega(\mu+g)) - 1\right)}{1 - \mu \sum_{i=1}^{m} \beta_i^V \tau_i/(\omega(\mu+g))}$
Treatment of infected	(5.17)	$p_{\text{average-crit}} \equiv \sum_{i=1}^{m} \beta_i \tau_i/\omega - (\mu + g)$
SIR with general FOI and treatments	(5.20)	$p_i^{\text{crit}} \equiv h_i'(0) - (\mu + g)$
SEIR with treatments	(5.24)	$p_i^{\text{crit}} \equiv \beta_i a/(\mu + a) - (\mu + g)$
Vertical SEIR with treatments	(5.27)	$p_i^{\text{crit}} \equiv \dfrac{(\beta_i + \mu q)a}{\mu(1-\rho) + a} - (\mu + g)$
Disease-induced mortality SEIR with treatments	(5.31)	$p_i^{\text{crit}} \equiv \beta_i a/(b + a) - (\mu + g + \alpha)$

Note that $p_{\text{average-crit}} \equiv \frac{\sum_{i=1}^{m} p_i^{\text{crit}} \tau_i}{\omega}$. The critical rates indexed by i are mode-dependent

exit screening were performed during the spread of SARS in 2003; temperature screening using thermal scanning and questionnaires were given to assess symptoms for possible exposure at mass transit centers [100]. More recently, global travel has been a major factor in the spread of the H1N1 strain of influenza in 2009, Ebola virus in 2015, and Zika virus in 2016. Motivated by this and the time-invariant entry screening models investigated in [100, 147], screening strategies were considered in Sect. 5.4. The formulation and analysis of the screening strategy for a switched multi-city model in Sect. 5.4. In Sect. 5.5, Halanay-like switching results were used to prove convergence to disease-free sets (and thus disease eradication). Halanay-like inequalities have been generalized to include switching (for example, [164]), time-varying parameters (for example, [120, 172]), and impulsive effects (for example, [160, 163]). The works [142, 143] form the basis for the derivations and results found in Sect. 5.5. Other switching control strategies (e.g., reduced contact rates) are detailed later in this monograph, while other possibilities (e.g., purposeful shifts in population behavior) are theoretically unlocked by the findings here.