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Scaling of average receiving time on weighted polymer networks with some topological properties

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In this paper, a family of the weighted polymer networks is introduced depending on the number of copies *f* and a weight factor *r*. The topological properties of weighted polymer networks can be completely analytically characterized in terms of the involved parameters and/or of the fractal dimension. Moreover, assuming that the walker, at each step, starting from its current node, moves to any of its neighbors with probability proportional to the weight of edge linking them, namely weight-dependent walk. Then, we calculate the average receiving time (ART) with weighted-dependent walks, which is the sum of mean first-passage times (MFPTs) for all nodes absorpt at the trap located at the central node as a recursive relation. The obtained remarkable results display that when $\frac{1}{f+1} < r < 1$, the ART grows sublinearly with the network size; when $r = \frac{1}{f+1}$, ART grows with increasing size N_g as $\ln^2 N_g$; when $0 < r < \frac{1}{f+1}$, ART grows with increasing size N_g as $\ln N_g$. In the treelike polymer networks, ART grows with linearly with the network size N_g when $r = 1$. Thus, the weighted polymer networks are more efficient than treelike polymer networks in receiving information.

Complex networks, as a powerful tool to describe and characterize the natural and man-made systems, have attracted considerable attention in many fields, such as mathematics, biology, life science and engineering disciplines[1–3]. Besides, as rapid developing discipline, polymer science has attracted much attention in the past few years, since it provides a powerful tool to study the macromolecules with various structures[4]. Flexible polymer structures are various, such as dendrimers[5], mesh-like polymers[6,7], fractals[8,9], dendritic[10,11], regular hyperbranched structures[12,13], scale-free and small-world networks[14,15], and so on.

Weighted networks represent the natural framework to describe natural, social, and technological systems, in which the intensity of a relation or the traffic between elements is an important parameters[16,17]. In general terms, weighted networks are extension of networks or graphs[18,19], in which each edge between nodes i and j is associated with a variable w_{ij}, called the weight. Much attention has been paid to the study of weighted networks because most real networks, which include airport networks[20], ecosystems[21], the Internet networks[22] and so on, often show weighted properties, so it is also meaningful to investigate the behavior on the weighted networks[23]. Motivated by complex networks and polymer structures, Zhang *et al.* defined a category of treelike polymer networks controlled by a parameter, which is built in an iterative way[24,25]. Combining the weighted networks[26] and polymer structures, a family of the weighted polymer networks is introduced depending on the number of copies f and a weight factor r.

In 2015, Dai *et al.* introduced comprehensively three kinds of walks: random walk, weigh-dependent walk and strength-dependent walk on the weighted networks[27]. On weighted networks, the walker will choose an edge according to its weight of the node connected by it, i.e. weight-dependent walk. A key quantity related to weighted networks is the mean first-passage time(MFPT), that is, the expected first time for the walker starting from a source node to a given target node. The average receiving time (ART) is the sum of mean first-passage times (MFPTs) for all nodes absorpt at the trap located at a given target node[28–31].

In this paper, we define a family of weighted polymer networks controlled by two parameters, which is built in an iterative way. According to the construction, we study some structural properties of the weighted polymer networks, showing that (1) in the limit of large network order g, the average degree of weighted polymer networks

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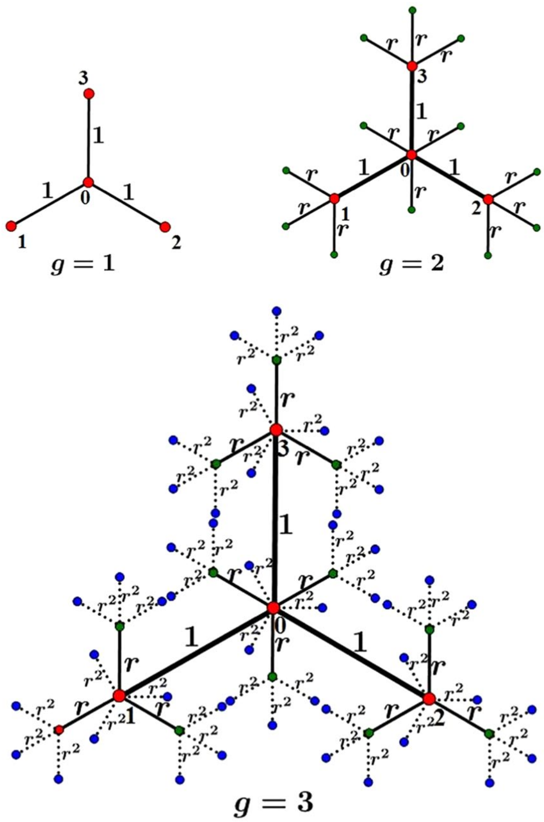

Figure 1. Iterative construction method for weighted polymer networks from $g = 1$ to $g = 3$ for the case of $f = 3$.

tends to 2; (2) when $0 < r < 1$, their average node strength goes to zero as g increases; (3) their node strength distribution follows a power-law distribution; (4) the weighted polymer networks networks have small-world property: in the infinite network, the average weighted shortest path (AWSP) tends to be a constant value which depends on two parameters f, r. However, the average shortest path (ASP) increases logarithmically with the network size. Then, by applying recursive relations of weighted polymer networks, we calculate the average receiving time (ART) with weighted-dependent walks, which is the sum of mean first-passage times (MFPTs) for all nodes absorpt at the trap located at the central node. So we derive exactly the ART formula, which displays that in large networks, the leading behaviors of ART for the weighted polymer networks follow distinct scalings, with the trapping efficiency associated with the network size N_g, the number of copies f and a weight factor r.

This paper is organized as follow. Based on weighted networks[26] and polymer structures, a family of the weighted polymer networks is introduced depending on the number of copies f and a weight factor r in the next section. In Section 3, some a priori prescribed topology is described in terms of average degree, average node strength, node strength distribution, and the average weighted shortest path, depending on the two main parameters: the number of copies f and the weight factor r. In Section 4, the average receiving time (ART) with weighted-dependent walk is obtained by recursive formulas for $F_1(g)$ and $T_{tot}(g)$. In the last section, we draw some conclusions that (1) the topology of weighted polymer networks can be completely analytically characterized in terms of the involved parameters and/or of the fractal dimension; (2) the weighted polymer networks are more efficient than treelike polymer networks in term of receiving information.

Weighted treelike networks

In this section, a family of weighted polymer networks are introduced. Intuited by polymer networks[24, 25] and Weighted Fractal Networks (WFN for short)[26], a family of weighted polymer networks are constructed in a deterministically iterative way.

Let r($0 < r < 1$) be a positive real numbers, and f($f \geq 1$) be a positive integer. Denote by G_g the weighted polymer networks after g iterations, and the following is the iterative algorithm to create weighted polymer networks:

(1) For $g = 0$, G_0 consists of an isolated node, called the central node. For $g = 1$, f new nodes are generated connecting the central node to form G_1. Let G_1 be our base graph, composed by $f + 1$ nodes and f edges with unit weight. The $f + 1$ nodes in G_1 are all the attaching nodes, labeled by $0, 1, 2, \cdots, f$.

(2) For $g = 2$, G_2 is obtained from G_1: Let $G_1^{(0)}, G_1^{(1)}, \cdots, G_1^{(f)}$ be $f + 1$ replicas of G_1, whose weighted edges have been scaled by the weight factor r. For $i = 0, 1, 2, \cdots, f$, let us denote by i' the central node in $G_1^{(i)}$. Then merge the central node i' in $G_1^{(i)}$ and Node i in G_1 into a single new node, still labelled by $i(i = 0, 1, \cdots, f)$. Figure 1 illustrates the iterative construction processes of a particular network from $g = 1$ to $g = 3$ for the

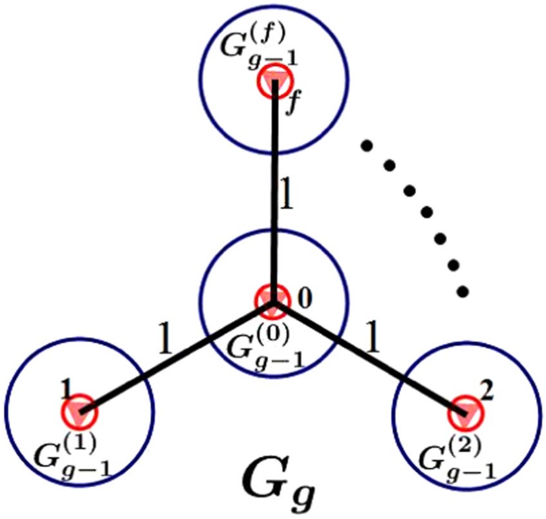

Figure 2. Construction method of weighted polymer networks. The open circles and triangles represent Node i' of $G_{g-1}^{(i)}$ and i of $G_1 (i = 0, 1, 2, \cdots, f)$, respectively.

case of $f = 3$.

(3) For $g \geq 1$, G_g is obtained from G_{g-1} (see Fig. 2): Let $G_{g-1}^{(0)}, G_{g-1}^{(1)}, \cdots, G_{g-1}^{(f)}$ be $f + 1$ replicas of G_{g-1}, whose weighted edges have been scaled by the weight factor r. For $i = 0, 1, 2, \cdots, f$, let us denote by i' the central node in $G_{g-1}^{(i)}$. Then merge the central node i' in $G_{g-1}^{(i)}$ and Node i in G_1 into a single new node, still labelled by $i (i = 0, 1, \cdots, f)$. The weighted polymer networks is set up.

The weighted polymer networks is one of type of WFN. According to Carletti and Righi[26], WFN are scale-free, the exponent being the fractal dimension. WFN exhibit the "small-world" property (i.e. slow (logarithmic) increase of the average shortest path with the network size) and large average clustering coefficient. Thus, the fractal dimension of weighted polymer networks is completely characterized by two main parameters: the number of copies $f \geq 1$ and the weight factor $0 < r < 1$. We have that the fractal dimension of the weighted polymer networks is $d_{fract} = -\frac{\log(f+1)}{\log r}$.

According to the construction approach, it is easy to derive that at each iterative step $g_i (g_i \geq 1)$, the number of newly generated nodes is $L(g_i) = f(f + 1)^{g_i - 1}$. Then the total number of nodes at each generation g is

$$N_g = 1 + \sum_{g_i=1}^{g} L(g_i) = (f + 1)^g, \tag{1}$$

and the total number of edges in G_g is $E_g = N_g - 1 = (f + 1)^g - 1$.

Topological properties of weighted polymer networks

The aim of this section is to characterize the topology of weighted polymer networks, by analytically studying their properties such as the average degree, the average node strength, the node strength distribution, and the average weighted shortest path.

Average degree and average node strength. The degree of a node i in a network, that is, the number of connections or edges the node i has to other nodes, is denoted by $deg(i)$. The average degree of the weighted polymer networks G_g, denoted by $ad(G_g)$, is defined as $ad(G_g) = \frac{2E_g}{N_g}$[32]. Hence in the limit of large g, the average degree $ad(G_g)$ is finite and it is asymptotically given by

$$ad(G_g) = \frac{2E_g}{N_g} = \frac{2(f + 1)^g - 2}{(f + 1)^g} \to 2, \quad g \to \infty.$$

In the weighted polymer networks G_g, a weight w_{ij} is assigned to the edge connecting the nodes i and j, and the strength of node i can be defined as

$$s_i = \sum_{j \in \nu(i)} w_{ij}, \tag{2}$$

where the sum index j runs over the set $\nu(i)$ of neighbors of i. The strength of a node integrates the information concerning its connectivity and the weights of its links[33,34]. Then using the recursive construction, we can explicitly compute the total node strength $S_g = \sum_{i \in G_g} s_i$, and, provided $r \neq \frac{1}{f+1}$, easily show that

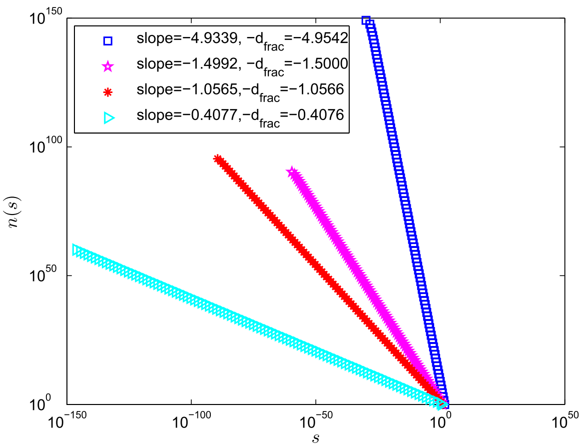

Figure 3. The log-log plot of $n(s)$ versus s for different weight factor r and different copy number f.

$$S_g = 2f\frac{(fr+r)^g - 1}{fr+r-1}.$$

Because $r < 1$, we trivially find that the average node strength goes to zero as g increases: $S_g/N_g = \frac{2fr^g}{fr+r-1} - \frac{2f}{(fr+r-1)(f+1)^g} \to 0(g \to \infty)$.

Especially, let us observe that the same is true if $r = \frac{1}{f+1}$; in this case, in fact $S_g = 2fg$ grows linearly with g, thus slower than N_g.

Node strength distribution. Let $n(s)$ denote the number of nodes in the weighted polymer networks G_g that have strength s. Let $s_i(g_i)$ be the strength of any one of newly generated nodes i at each iterative step $g_i(g_i > 0)$. Assume that node i entered the networks at generation $g(g > 0)$, then $s_i(g) = r^{g-1}$. By construction, the strength of node i entered the networks at generation $g_i(0 < g_i < g)$ is $s_i(g_i) = r^{g_i-1} + f(r^{g-1} + r^{g-2} + \cdots + r^{g_i})$. For $g_i = 0$, the strength of the initial central node labeled by 0, equals to $s_0(0) = f(r^{g-1} + r^{g-2} + \cdots + r + 1)$. Using the property of iterative construction method, we can conclude:

$$n(s_i(g_i)) = f(f+1)^{g_i-1}, \quad and \quad n(s_0(0)) = 1.$$

For $g = 100$, $n(s)$ versus s in the weighted polymer networks is on a log-log scale in Fig. 3. The slope of the weighted polymer networks G_{100} with 8^{100} nodes, $f = 7$ and $r = 1/4$ is -1.4992, which differs by 0.0008 from its the negative of the fractal dimension $-d_{frac} = \frac{\log 8}{\log 1/4} = -1.5000$. The slope of the weighted polymer networks G_{100} with 9^{100} nodes, $f = 8$ and $r = 1/8$ is -1.0565, which differs by 0.0001 from its the negative of the fractal dimension $-d_{frac} = \frac{\log 9}{\log 1/8} = -1.0566$. The slope of the weighted polymer networks G_{100} with 31^{100} nodes, $f = 30$ and $r = 1/2$ is -4.9339, which differs by 0.0203 from its the negative of the fractal dimension $-d_{frac} = \frac{\log 31}{\log 1/2} = -4.9542$. The slope of the weighted polymer networks G_{100} with 4^{100} nodes, $f = 3$ and $r = 1/30$ is -0.4077, which differs by -0.0001 from its the negative of the fractal dimension $-d_{frac} = \frac{\log 4}{\log 1/30} = -0.4076$. The results show that for different weight factor r and different copy number f, every line slope of $\log n(s)$ versus $\log s$ is nearly equal to the negative of the fractal dimension $-d_{fract} = \frac{\log(f+1)}{\log r}(0 < r < 1)$. This implies that $n(s)$ are distributed according to a power law with exponent $d_{fract} = -\frac{\log(f+1)}{\log r}$. And therefore, for large g, $n(s)$ can be obtained as

$$n(s) \sim cs^{-d_{fract}},$$

where c is constant. Thus, $n(s)$ also follows a power-law distribution.

Average weighted shortest path. By definition the average weighted shortest path(AWSP) of the weighted networks G_g[35] is given by

$$\overline{d}_g = \frac{2}{N_g(N_g-1)}S_{tot}(g), \tag{3}$$

where

$$S_{tot}(g) = \sum_{i,j \in G_g, i \neq j} d_{ij}(g), \tag{4}$$

$d_{ij}(g)$ being the weighted shortest path linking nodes i and j in G_g.

The modular recursive construction of G_g allows us to calculate the exact value of $S_{tot}(g)$. At step $g+1$, we incise G_g into $f+1$ branches, which we label as $G_g^{(i)}(i = 0, 1, \cdots, f)$. Each branch $G_g^{(i)}(i = 0, 1, \cdots, f)$ is a copy of G_g and has the same structure as G_g, while their edge weights have been scaled by a weight factor r. The central nodes i' of $G_g^{(i)}(i = 1, \cdots, f)$ are all connected to central node $0'$ of $G_g^{(0)}$ by f edges with unit weight. Thus, the total of shortest distances $S_{tot}(g)$ satisfies the following recursion:

$$S_{tot}(g+1) = (f+1)rS_{tot}(g) + \Omega_g, \tag{5}$$

where Ω_g is the sum over all weighted shortest paths whose nodes are not in the same copy of $G_g^{(i)}(i = 0, 1, \cdots, f)$. Note that the weighted paths that contribute to Ω_g must all go through central node $0'$ of $G_g^{(0)}$ at which the different $G_g^{(i)}(i = 0, 1, \cdots, f)$ branches are connected. This recursive relation can be elaborated as follows:

The first term on the rhs of (5) describes the sum of the weighted shortest path linking nodes i and j in $G_g^{(i)}(i = 0, 1, \cdots, f)$, respectively, i.e.,

$$\sum_{i,j\in G_g^{(0)}} + \sum_{i,j\in G_g^{(1)}} + \cdots + \sum_{i,j\in G_g^{(f)}}$$

Using the scaling mechanism for the edges, the above sum can be easily identified with

$$r\sum_{i,j\in G_g} + r\sum_{i,j\in G_g} + \cdots + r\sum_{i,j\in G_g} = (f+1)r\sum_{i,j\in G_g} = (f+1)rS_{tot}(g)$$

One can prove (see Method) that

$$\Omega_g = \begin{cases} \dfrac{f^2}{r-1}r^g(f+1)^{2g} + \dfrac{f^2(r-2)}{r-1}(f+1)^{2g}, & \textit{if} \quad 0 < r < 1, \\ f^2(g+1)(f+1)^{2g}, & \textit{if} \quad r = 1. \end{cases} \tag{6}$$

Considering $S_{tot}(1) = f^2$, we can solve Eq. (4) recursively to yield

$$S_{tot}(g) = \begin{cases} \dfrac{f^2(r-2)}{(f+1-r)(r-1)}(f+1)^{2g-1} + \dfrac{f(fr-1)}{(f+1)(r-1)}r^{g-1}(f+1)^g \\ +\left(\dfrac{f}{r-1} - \dfrac{f^2(r-2)}{(f+1-r)(r-1)}\right)r^{g-1}(f+1)^{2g-1}, & \textit{if} \quad 0 < r < 1, \\ (fg-1)(f+1)^{2g-1} + (f+1)^{g-1}, & \textit{if} \quad r = 1. \end{cases} \tag{7}$$

We find that if $r=1$ then $S_{tot}(g) = (fg-1)(f+1)^{2g-1} + (f+1)^{g-1}$, which coincides with the $S_{tot}(g)$ in ref. 24. Therefore

$$\overline{d}_g = \begin{cases} \dfrac{2f^2(r-2)}{(f+1-r)(r-1)}\dfrac{(f+1)^{g-1}}{(f+1)^g - 1} + \dfrac{2f(fr-1)}{(f+1)(r-1)}\dfrac{r^{g-1}}{(f+1)^g - 1} \\ +2\left(\dfrac{f}{r-1} - \dfrac{f^2(r-2)}{(f+1-r)(r-1)}\right)\dfrac{r^{g-1}(f+1)^{g-1}}{(f+1)^g - 1}, & \textit{if} \quad 0 < r < 1, \\ \dfrac{2(fg-1)(f+1)^g + 2}{(f+1)^{g+1} - (f+1)}, & \textit{if} \quad r = 1. \end{cases} \tag{8}$$

which provides the following asymptotic behavior in the limit of large g (see Fig. 4). When $g \to \infty$,

$$\overline{d}_g \to \frac{2f^2(r-2)}{(f+1-r)(r-1)}, \quad \textit{if} \quad 0 < r < 1 \tag{9}$$

Thus the network grows unbounded but with the logarithm of the network size, while the weighted shortest distances stay bounded.

Recalling $N_g = (f+1)^g$ as given in Eq. (1), we have $g = \ln N_g/\ln(f+1)$. We can also compute the average shortest path (ASP), $\overline{d}_g$, formally obtained by setting $r=1$. Hence, when the network size is large enough, we have

$$\overline{d}_g \cong \frac{2f}{(f+1)\ln(f+1)}\ln N_g, \quad \textit{if} \quad r = 1. \tag{10}$$

Average receiving time on weighted polymer networks

The purpose of this section is to determine explicitly the average receiving time $\langle T\rangle_g$ and show how it scales with network order. We aim at a particular case on G_g with the perfect trap being located at the central node, labelled by 0. The process of biased walks is that the particle (walker), at each time step, starting from its current Node i, jumps to its neighbor Node j with probability $p^w_{i\to j}$ (see Eq. (11)).

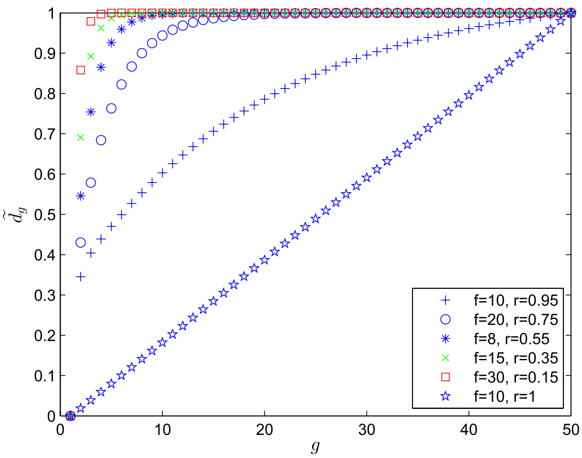

Figure 4. The average weighted shortest path. Plot of the renormalized average weighted shortest path $\widetilde{d}_g$ versus the iteration g, where $\widetilde{d}_g = \frac{\overline{d}_g - \min\{\overline{d}_g\}}{\max\{\overline{d}_g\} - \min\{\overline{d}_g\}}$.

For weight-dependent walk, a walker chooses one of its nearest neighbors with probability proportional to the weight of edge linking them[36, 37]. The transition probability from node i to its neighbor j is

$$p_{i\to j}^{w} = \frac{w_{ij}}{s_i} = \frac{w_{ij}}{\sum_{j\in\nu(i)} w_{ij}}, \tag{11}$$

where s_i denotes the strength of node i (see Eq. (2)).

For convenience of description, let us denote by $0, 1, 2, \cdots, f$ the $f+1$ attaching nodes in G_g, and by $4, 5, \cdots, N_g - 2$ and $N_g - 1$ all other nodes except for the $f+1$ attaching nodes. Let $F_{ij}(g)$ be the mean first-passage time (MFPT) for a walker starting from Node i to Node j. Let $F_i(g)$ be the MFPT from Node i to the trap. $\langle T\rangle_g$ is the average receiving time (ART), which is defined as the average of $F_i(g)$ over all starting nodes other than the trap. $\langle T\rangle_g$ is the key question considered in this section.

By definition, $\langle T\rangle_g$ is given by

$$\langle T\rangle_g = \frac{1}{N_g}\sum_{i=0}^{N_g-1} F_i(g).$$

Here we denote by $T_{tot}(g)$ the sum of MFPTs for all nodes to absorption at the trap located the central Node 0, i.e.,

$$T_{tot}(g) = \sum_{i=0}^{N_g-1} F_i(g).$$

Thus, the problem of determining $\langle T\rangle_g$ is reduced to finding $T_{tot}(g)$. We will compute $T_{tot}(g)$ by segmenting G_g.

From the iterative construction method of G_g, G_g can be regarded as merging $f+1$ groups, sequentially denoted by $G_{g-1}^{(i)}(i = 0, 1, \cdots, f)$. The $f+1$ groups are obtained as follows: $G_{g-1}^{(0)}, G_{g-1}^{(1)}, \cdots, G_{g-1}^{(f)}$ are $f+1$ replicas of G_{g-1}, whose weighted edges have been scaled by the weight factor r. For $i = 0, 1, 2, \cdots, f$, let us denote by i' the central node in $G_{g-1}^{(i)}$. Then merge the central node i' in $G_{g-1}^{(i)}$ and Node i in G_1 into a single new node, still labelled by $i(i = 0, 1, \cdots, f)$. The process is described in Fig. 2.

Through this division we could rewrite the sum $T_{tot}(g)$ as follows:

$$T_{tot}(g) = (f+1)T_{tot}(g-1) + N_{g-1}\sum_{i=1}^{f} F_i(g). \tag{12}$$

We now elaborate Eq. (12). The first term on the rhs of Eq. (12) describes the sum of MFPTs for all nodes in $G_{g-1}^{(i)}$ to reach its attaching nodes $i(i = 0, 1, \cdots, f)$. Recalling $G_{g-1}^{(i)}$ linked to Node $i(i = 0, 1, \cdots, f)$ is a copy of G_{g-1} and the scaling mechanism for edges, the first term in the rhs of Eq. (12) can be identified with $(f+1)T_{tot}(g-1)$; the second term describes the sum of MFPTs for all nodes in $G_{g-1}^{(j)}$ from Node $j(j = 1, 2, \cdots, f)$ to Central Node 0.

Because of the symmetry of nodes $1, 2, \cdots, f$, $F_1(g) = F_2(g) = \cdots = F_f(g)$. Eq. (12) can be simplified as

$$T_{tot}(g) = (f+1)T_{tot}(g-1) + fN_{g-1}F_1(g). \tag{13}$$

Thus, the problem of determining $T_{tot}(g)$ is reduced to finding $F_1(g)$. Using the construction of the weighted polymer networks G_g and the scaling mechanism for edges, we obtain

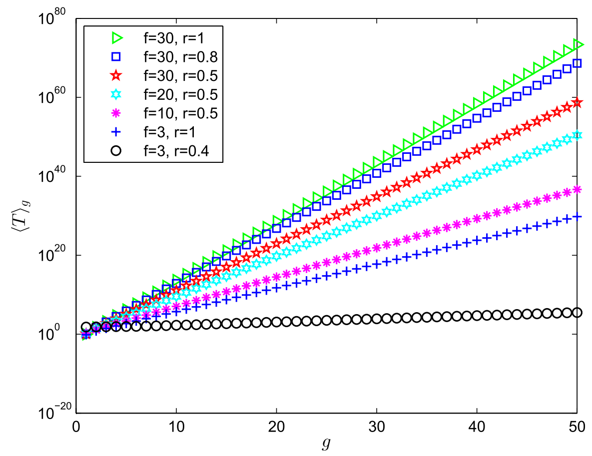

Figure 5. Average receiving time $\langle T\rangle_g$ versus g is on a semilogarithmic scale for the range of $\frac{1}{f+1} < r \le 1$.

$$\begin{aligned} F_1(g) &= (1 + fr + fr^2 + \cdots + fr^{g-1}) + f[r^{g-1}F_1(1) + r^{g-2}F_1(2) \\ &\quad + \cdots + r^2F_1(g-2) + rF_1(g-1)], \end{aligned} \tag{14}$$

and

$$\begin{aligned} F_1(g-1) &= (1 + fr + fr^2 + \cdots + fr^{g-2}) + f[r^{g-2}F_1(1) \\ &\quad + r^{g-3}F_1(2) + \cdots + r^2F_1(g-3) + rF_1(g-2)]. \end{aligned} \tag{15}$$

From Eqs (14) and (15), we can further have

$$F_1(g) - (fr + r)F_1(g-1) = 1 + fr - r. \tag{16}$$

Considering the initial condition $F_1(1) = 1$, we can solve recursively Eq. (16) to obtain

$$F_1(g) = \begin{cases} \dfrac{2fr(fr+r)^{g-1}}{fr+r-1} - \dfrac{fr+1-r}{fr+r-1}, & if \quad r \ne \dfrac{1}{f+1}, \\ (1+fr-r)g - (fr-r), & if \quad r = \dfrac{1}{f+1}. \end{cases} \tag{17}$$

Considering $T_{tot}(1) = f$ and inserting Eq. (17), we can solve Eq. (13) inductively to yield

$$\begin{aligned} T_{tot}(g) &= \frac{2f^2r^{g+1}}{(fr+r-1)^2}(f+1)^{2g-1} - \frac{f(fr+1-r)}{fr+r-1}g(f+1)^{g-1} \\ &\quad - \frac{2f^2r}{(fr+r-1)^2}(f+1)^{g-1}, \quad if \quad r \ne \frac{1}{f+1}, \\ T_{tot}(g) &= \frac{f(1+fr-r)}{2}g^2(f+1)^{g-1} - \frac{f(1+3fr-3r)}{2}g(f+1)^{g-1} \\ &\quad + f(1+fr-r)(f+1)^{g-1}, \quad if \quad r = \frac{1}{f+1}. \end{aligned} \tag{18}$$

Hence, $\langle T\rangle_g$, which we are concerned about, could be expressed as follows:

$$\begin{aligned} \langle T\rangle_g &= \frac{2f^2r}{(fr+r-1)^2(f+1)}(fr+r)^g - \frac{f(fr+1-r)}{(fr+r-1)(f+1)}g \\ &\quad - \frac{2f^2r}{(fr+r-1)^2(f+1)}, \quad if \quad r \ne \frac{1}{f+1}, \\ \langle T\rangle_g &= \frac{f(1+fr-r)}{2(f+1)}g^2 - \frac{f(1+3fr-3r)}{2(f+1)}g + \frac{f(1+fr-r)}{f+1}, \quad if \quad r = \frac{1}{f+1}. \end{aligned} \tag{19}$$

We find that if $r = 1$ then

$$\langle T\rangle_g = 2(f+1)^{g-1} - \frac{(fg+2)}{f+1}, \tag{20}$$

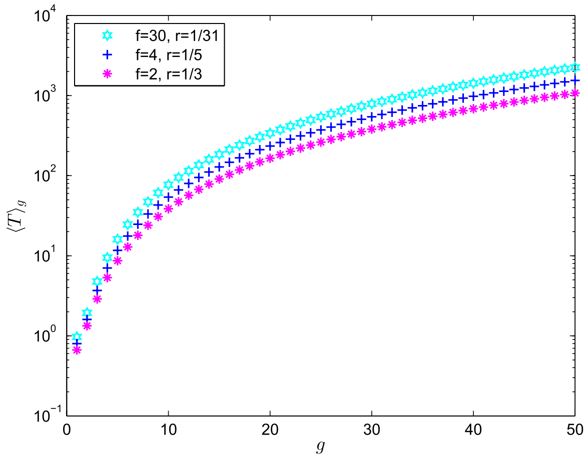

Figure 6. Average receiving time $\langle T\rangle_g$ versus g is on a semilogarithmic scale for the range of $r=\frac{1}{f+1}$.

which coincides with the $\langle T\rangle_g$ in ref. 24.

Recalling $N_g=(f+1)^g$ and $g=\ln N_g/\ln(f+1)$, we have

$$\langle T\rangle_g \approx \frac{2N_g}{f+1}-\frac{f\ \ln N_g}{(f+1)\ \ln(f+1)}\simeq\frac{2N_g}{f+1},\quad if\quad r=1, \tag{21}$$

which coincides with that in ref. 24.

For systems with large order, i.e. $N_g\to\infty$,

$$\begin{aligned}\langle T\rangle_g &\approx \frac{2N_g}{f+1},\quad if\quad r=1,\\ \langle T\rangle_g &\approx \frac{2f^2r}{(fr+r-1)^2(f+1)}N_g^{1+\log_{f+1}r}\\ &= \frac{2f^2r}{(fr+r-1)^2(f+1)}N_g^{1-\frac{1}{d_{fract}}},\quad if\quad \frac{1}{f+1}<r<1,\\ \langle T\rangle_g &\approx \frac{f^2}{(f+1)^2\ln^2(f+1)}\ln^2N_g,\quad if\quad r=\frac{1}{f+1},\\ \langle T\rangle_g &\approx \frac{f(fr+1-r)}{(1-fr-r)(f+1)\ln(f+1)}\ln N_g,\quad if\quad 0<r<\frac{1}{f+1}.\end{aligned} \tag{22}$$

According to Eqs (19) and (20), ART $\langle T\rangle_g$ versus g for the range of $g\le 50$ on a semilogarithmic scale is shown in Figs 5, 6 and 7. From Eq. (22), we can have draw the conclusions as follows:

Case 1: $\langle T\rangle_g \approx \frac{2N_g}{f+1}$.

When $r=1$, ART grows linearly with the network size N_g. Figure 5 shows that ART increases with the increase of the values of f. That is to say, the smaller the value of f is, the more efficient the trapping process is.

Case 2: $\langle T\rangle_g \approx \frac{2f^2r}{(fr+r-1)^2(f+1)}N_g^{1+\log_{f+1}r}=\frac{2f^2r}{(fr+r-1)^2(f+1)}N_g^{1-\frac{1}{d_{fract}}}$.

When $\frac{1}{f+1}<r<1$, in large network, the ART grows as a power-law function of the network size N_g with the exponent, represented by $\theta(f,r)=1+\log_{f+1}r=1-\frac{1}{d_{fract}}$ as follows:

(1) When f is kept fixed, the exponent $\theta(f, r)$ is an increasing function of $r(0<r<1)$ in Fig. 5. When r grows from 0 to 1, the exponent increases from 0 and approaches 1, indicating that ART grows sublinearly with the network size N_g. This also means that the efficiency of the trapping process depends on the parameter r: the smaller the value of r, the more efficient the trapping process is.
(2) When $r(r\neq 1)$ is kept fixed, ART grows sublinearly with the network size N_g and the exponent $\theta(f, r)$ is an increasing function of the values of f in Fig. 5. That is to say, the smaller the value of f is, the more efficient the trapping process is.
(3) The fractal dimension d_{fract} of the weighted polymer networks play a relevant role in calculating the ART. The exponent $1-\frac{1}{d_{fract}}$ is an increasing function of the values of d_{fract}, which means that the smaller the value of d_{fract} is, the more efficient the trapping process is.

Case 3: $\langle T\rangle_g \approx \frac{f(1+fr-r)}{2(f+1)\ln^2(f+1)}\ln^2N_g$.

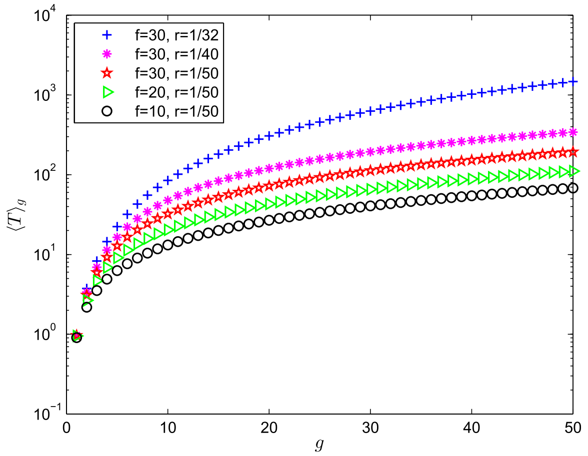

Figure 7. Average receiving time $\langle T\rangle_g$ versus g is on a semilogarithmic scale for the range of $0 < r < \frac{1}{f+1}$.

When $r = \frac{1}{f+1}$, ART grows with increasing size N_g as $\ln^2 N_g$ according to Eq. (22). Figure 6 shows the smaller the value of f is, the more efficient the trapping process is.

Case 4: $\langle T\rangle_g \approx \frac{f(fr+1-r)}{(1-fr-r)(f+1)\ln(f+1)}\ln N_g$.

When $0 < r < \frac{1}{f+1}$, ART grows with increasing size N_g as $\ln N_g$ in Fig. 7. When f is kept fixed, the smaller the value of r, the more efficient the trapping process is. When $r(r \neq 1)$ is kept fixed, the smaller the value of f is, the more efficient the trapping process is.

Method

The analytical expression for Ω_g is not difficult to find, we denote as Ω_g^{ab} the sum of all shortest paths with nodes in $\Omega_g^{(a)}$ and $\Omega_g^{(b)}$ $(a, b = 0, 1, 2, \cdots, f)$. Then the sum Ω_g is

$$\Omega_g = \Omega_g^{10} + \Omega_g^{20} + \cdots + \Omega_g^{f0} + \Omega_g^{12} + \Omega_g^{13} + \cdots + \Omega_g^{1f} + \Omega_g^{23} + \Omega_g^{24} + \cdots + \Omega_g^{25} + \cdots + \Omega_g^{(f-1)f}.$$

By symmetry,

$$\Omega_g^{10} = \Omega_g^{20} = \cdots = \Omega_g^{f0},$$

and

$$\Omega_g^{12} = \Omega_g^{13} = \cdots = \Omega_g^{1f} = \Omega_g^{23} = \Omega_g^{24} = \cdots = \Omega_g^{25} = \cdots = \Omega_g^{(f-1)f},$$

so that

$$\Omega_g = f\Omega_g^{10} + \frac{(f-1)f}{2}\Omega_g^{12}. \tag{23}$$

In order to find Ω_g^{10} and Ω_g^{12}, we define

$$\Delta_g = \sum_{i\in G_g, i\neq 0} d_{i0}.$$

We have

$$\Delta_1 = \sum_{i=1}^{f} d_{i0} = f.$$

Considering the self-similar network structure, we can easily know that at step g, the quantity Δ_g evolves recursively as

$$\Delta_g = r\Delta_{g-1} + f(r\Delta_{g-1} + N_{g-1}) = (fr + r)\Delta_{g-1} + fN_{g-1}.$$

Using $\Delta_1 = f$, we have

$$\Delta_g = (fr + r)^{g-1}\Delta_1 + f(f+1)^{g-1}(r^{g-2} + \cdots + r + 1).$$

If $0 < r < 1$, then

$$\Delta_g = \frac{fr}{r-1}(fr+r)^{g-1} - \frac{f}{r-1}(f+1)^{g-1}.$$

If $r = 1$, then

$$\Delta_g = fg(f+1)^{g-1}.$$

On the other hand, we have

$$\begin{aligned} \Omega_g^{10} &= \sum_{i \in G_g^{(1)} j \in G_g^{(0)}} d_{ij} \\ &= \sum_{i \in G_g^{(1)} j \in G_g^{(0)}} (d_{i1} + d_{10} + d_{j0}) \\ &= N_g \sum_{i \in G_g^{(1)}} d_{i1} + N_g^2 + N_g \sum_{i \in G_g^{(0)}} d_{j0} \\ &= N_g r \Delta_g + N_g^2 + N_g r \Delta_g = 2N_g r \Delta_g + N_g^2, \end{aligned} \tag{24}$$

$$\begin{aligned} \Omega_g^{12} &= \sum_{i \in G_g^{(1)} j \in G_g^{(2)}} d_{ij} \\ &= \sum_{i \in G_g^{(1)} j \in G_g^{(2)}} (d_{i1} + d_{12} + d_{j2}) \\ &= N_g \sum_{i \in G_g^{(1)}} d_{i1} + 2N_g^2 + N_g \sum_{i \in G_g^{(2)}} d_{j2} \\ &= N_g r \Delta_g + 2N_g^2 + N_g r \Delta_g = 2N_g r \Delta_g + 2N_g^2, \end{aligned} \tag{25}$$

where $d_{10} = 1$, $d_{12} = 2$ have been used.

Substituting Eqs (24) and (25) into Eq. (23), we have

$$\begin{aligned} \Omega_g &= f(2N_g r \Delta_g + N_g^2) + \frac{f(f-1)}{2}(2N_g r \Delta_g + 2N_g^2) \\ &= fr(f+1)N_g \Delta_g + f^2 N_g^2 \end{aligned}$$

If $0 < r < 1$, then

$$\Omega_g = \frac{f^2}{r-1} r^g (f+1)^{2g} + \frac{f^2(r-2)}{r-1}(f+1)^{2g}.$$

If $r = 1$, then

$$\Omega_g = f^2(g+1)(f+1)^{2g}.$$

Conclusions

In this paper, we have introduced a family of weighted polymer networks, and studied its topological structure:(1) in the limit of large g, the average degree of weighted polymer networks tends to 2; (2) when $0 < r < 1$, their average node strength goes to zero as g increases; (3) their node strength distribution follows a power-law distribution; (4) the weighted polymer networks networks have small-world property: in the infinite network, the AWSP tends to be a constant value which depends on two parameters f, r. However, the ASP increases logarithmically with the network size. Finally, we calculate the average receiving time (ART) with weighted-dependent walks on weighted polymer networks. Our analysis has indicated that (1) when $\frac{1}{f+1} < r < 1$, in large network, the ART grows as a power-law function of the network size N_g with the exponent, represented by $\theta(f, r) = 1 + \log_{f+1} r = 1 - \frac{1}{d_{fract}}$. ART grows sublinearly with the network size N_g and the exponent $1 - \frac{1}{d_{fract}}$ is an increasing function of the values of d_{fract}, which means that the smaller the value of d_{fract} is, the more efficient the trapping process is; (2) when $r = \frac{1}{f+1}$, ART grows with increasing size N_g as $\ln^2 N_g$; (3) when $0 < r < \frac{1}{f+1}$, ART grows with increasing size N_g as $\ln N_g$. In the treelike polymer networks, ART grows with linearly with the network size N_g when $r = 1$. Thus, the weighted polymer networks are more efficient than treelike polymer networks in receiving information.

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Author Contributions

Y.Z.Z., D.D.Y. designed the research. S.L. and J.L. collected the data. F.Z. and C.L.H. wrote the manuscript, and W.C. prepared Figures 1–7. All authors discussed the results and reviewed the manuscript.

Additional Information

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