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Research article

Development of Aczel-Alsina t-norm based linear Diophantine fuzzy aggregation operators and their applications in multi-criteria decision-making with unknown weight information

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ABSTRACT

Aczel-Alsina t-norm and t-conorm are intrinsically flexible and endow Aczel-Alsina aggregation operators with greater versatility and robustness in the aggregation process than operators rooted in other t-norms and t-conorm families. Moreover, the linear Diophantine fuzzy set (LD-FS) is one of the resilient extensions of the fuzzy sets (FSs), intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (PyFSs), and q-rung orthopair fuzzy sets (q-ROFSs), which has acquired prominence in decision analysis due to its exceptional efficacy in resolving ambiguous data. Keeping in view the advantages of both LD-FSs and Aczel-Alsina aggregation operators, this article aims to establish Aczel-Alsina operation rules for LD-FSs, such as Aczel-Alsina sum, Aczel-Alsina product, Aczel-Alsina scalar multiplication, and Aczel-Alsina exponentiation. Based on these operation rules, we expose the linear Diophantine fuzzy Aczel-Alsina weighted average (LDFAAWA) operator, and linear Diophantine fuzzy Aczel-Alsina weighted geometric (LDFAAWG) operator and scrutinize their distinctive characteristics and results. Additionally, based on these aggregation operators (AOs), a multi-criteria decision-making (MCDM) approach is designed and tested with a practical case study related to forecasting weather under an LD-FS setting. The developed model undergoes a comparative analysis with several prevailing approaches to demonstrate the superiority and accuracy of the proposed model. Besides, the influence of the parameter Λ on the ranking order is successfully highlighted.

1. Introduction and literature review

MCDM is an essential component of decision analysis that can rank the finite objects w.r.t. the attribute values. In MCDM, a decision-maker (DMr) evaluates given alternatives under various criteria, and their assessments are communicated as verbal or crisp values. Nonetheless, in practical scenarios, uncertain information is often exhibited in MCDM across domains like DM, clustering, supplier selection, pattern recognition, medical diagnosis, etc. Uncertainty plays a vital factor in the MCDM procedure, and it can be

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challenging for DMrs to accurately obtain results with imprecise, vague, and uncertain information. DMrs adopted myriad approaches to confronting MCDM issues, including FSs [1], interval-valued FS [2], IFS [3], PyFS [4], q-ROFS [5], and LD-FS [6], (2,1)-FS [7], (m,n)-FS [8], (m,n)-Fuzzy soft set [9], n^{th} power root FS [10,11], SR-FS [12], k_m^n-Rung Picture FS [13].

Zadeh invented the FS theory [1], revolutionizing science, technology, and mathematics. In FSs, a gradual evaluation of objects is specified with a membership degree (MD) range in $[0,1]$. Nonetheless, in several practical scenarios, only the MD is incapable of interpreting the ambiguous data. There is a requirement for a non-membership degree (NMD), and they may or may not be related to one another. This issue was resolved by Attanassov [3] by inventing the concept of IFS. IFSs significantly expand FS theory and deliberate an appropriate mechanism for handling uncertainties.

In IFSs, there is a constraint that the total of MD and NMD belong to $[0,1]$. It becomes challenging to obey such requirements when MD and NMD are allocated to specific items by distinct experts. Eventually, Yager integrated the conception of PyFS [4] and q-ROFS [5]. In PyFSs, the total of squares of the MD and the NMD fall in $[0,1]$ for all items. In q-ROFSs, the total of the q^{th} powers for MD and NMD lies in $[0,1]$. The substantial benefit of q-ROFSs compared to other variants is that these offer more excellent room for experts to articulate their opinions. Using q-ROFSs, DMrs can convey a larger spectrum of fuzzy information. As the parameter q grows, more orthopairs satisfy the bounding criterion.

While tackling real-world dilemmas, PyFSs [4] and q-ROFSs [5] provide more adaptable methods for specifying MD and NMD for the components of a set. These theories have gained much curiosity in current eras. Peng [14] evaluated the similarity and distance measures for PyFSs. Yager [15] devised an MCDM using PyFSs. Zhang et al. [16] examined new similarity measures of PyFSs. Ali [17] projected novel strategies in the framework of q-ROFSs. In [18], it is revealed that IFSs failed to yield satisfactory outcomes in many real-world scenarios. Ultimately, they provided justification for the demand for q-ROFSs. In [19], Wei et al. pioneered an MCDM mechanism in the context of Heronian mean operators of q-ROFSs.

Sometimes, uncertainty in the data was also not fully captured by PyFS [4] and q-ROFS [5]. To address this drawback, Riaz and Hashmi [6] generated a unique FS variant called LD-FS that includes reference parameters (RPs) of the MD and NMD, which fosters a fresh perspective for decision analysis research. Comparing the LD-FS model to IFSs, PyFSs, and q-ROFSs, the former is more fascinating and valuable. The existence of RPs is the primary benefit of the LD-FSs. These parameters give the MD and NMD additional room than IFSs, PyFSs, and q-ROFSs. LD-FSs offer several benefits over conventional FS variants, specifically in decision analysis. DMrs independently select the MD and NMD with no constraints.

Because LD-FSs give DMrs greater flexibility, many scholars dedicated themselves to integrating LD-FSs with other set theories. Hashmi et al. [20] interpreted the hybrid models of spherical LD-FSs, rough sets and soft sets to solve the MCDM problem. Ayub et al. [21,22] integrated some hybrid LD-FSs and rough sets models. In addition, Ayub et al. [23] interpreted the linear Diophantine fuzzy relation and their algebraic characteristics with applications in decision analysis. Moreover, Ayub et al. [24] integrated some hybridized models of LD-FSs and rough sets with applications in decision analysis. Some algebraic structures of LD-FSs were launched by Kamacı [25]. Some similarity measures for LD-FSs were incorporated by Mohammad et al. [26] for use in DM issues. Furthermore, Parimala et al. [27] integrated the idea of optimality conditions in networks for the solution algorithm's design using LD-FSs. Aydougdu [28] developed a novel linear Diophantine fuzzy information measures-based decision-making approach using the extended VIKOR method. In 2022, Kamacı [29] projected the notions of complex LD-FSs (CLD-FSs) and studied their fundamental characteristics. Riaz et al. [30] defined spherical LD-FSs.

Triangular norm (TN) and triangular conorm (TCN) have become essential for accomplishing FSs in decision analysis. Numerous aggregation operators (AOs) have been invented based on different TNs and TCNs. AOs are essential mechanisms that constitute a crucial role in information unification and receive more and more attention. Due to a broad spectrum of uncertain real-life problems, numerous AOs have been developed for multiple extensions of FSs [31–33]. Some AOs related to IFSs were integrated in [34–36]. Mahmood and Ali [37] invented Schweizer-Sklar Muirhead mean AOs in the framework of PyFSs with application in MCDM. Garg [38] formulated generalized Pythagorean fuzzy AOs using Einstein operations with applications in real-life dilemmas. Several interesting AOs for PyFSs were evaluated in [39–41]. A few AOs were devised by Liu and Wang [42] for q-ROFSs with MCDM applications. Riaz et al. [43,44] created the perception of some AOs in the environment of q-ROFSs with DM applications. Farid and Riaz [45] constructed Einstein's interactive geometric AOs for q-ROFSs with MCDM applications. Hayat et al. [46] offered a new group-based generalized interval-valued q-rung orthopair fuzzy soft AOs and their applications in sports decision-making problems. Raja et al. [47] introduced AOs based on group-based generalized q-rung orthopair fuzzy N-soft sets and applications in solar panel evaluation. Riaz et al. [48] developed an MCDM approach using linear Diophantine fuzzy weighted average (LDFWA) and linear Diophantine fuzzy weighted geometric (LDFWG) operators. Iampan et al. [49] generated Einstien AOs for LD-FS with MCDM applications. In [50], Farid et al. invented Einstein prioritized AOs for LD-FSs along with applications in decision analysis. Garg et al. [51] explored prioritized AOs for LD-FSs and their application in green sustainable chains. Riaz et al. [52] investigated generalized linear Diophantine fuzzy Choquet integral AOs with application to project management and risk analysis. Besides, Petchimuthu et al. [53] invented AOs of interval-valued LD-FSs (IVLD-FSs) with MCDM applications. In [54], Mahmood et al. interpreted power Muirhead mean operators for IVLD-FSs and their application in DM strategies. Riaz et al. [55] described Frank AOs for IVLD-FSs with MCDM applications. Almagrabi et al. [56] investigated q-LD-FSs and their associated AOs. Panpho and Yiarayong [57] devised (p, q)-Rung LD-FSs and their associated AOs with MCDM application.

In 1982, Aczel and Alsina [58] invented novel operations termed Aczel-Alsina TN (AA TN) and Aczel-Alsina TCN (AA TCN), which concentrate a strong emphasis on parameter variability. Recently, Aczel-Alsina AOs (AA-AOs) attained a lot of attention from numerous research scientists and played an effective role in decision-making dilemmas. Senapati et al. [59] devised AA-AOs and demonstrated their application in the MCDM method in the mechanism of IFSs. In recent times, Senapati et al. [60,61] have integrated some AA-AOs for interval-valued IFSs. Likewise, Senapati et al. [62] investigated Pythagorean fuzzy AA-AOs. Hussain et al. [63] developed AA-AOs for picture FSs based on an unknown degree of weight. The idea of q-ROF AA-AOs was offered by Khan et al. [64]. Senapati et al. [65] studied the selection of appropriate global partners for companies using q-rung orthopair fuzzy AA-AOs. Gayen et al. [66] established an innovative decision scheme under a dual hesitant q-ROF context using AA-AOs. Hussain et al. [67] studied AA-AOs for complex spherical FSs. Zeb et al. [68] provides a decision analytics approach for sustainable urbanization using q-rung orthopair fuzzy soft set-based AA- AOs. Ali and Naeem [69] integrated AA-AOs in p,q-QOFSs setting and their applicability in MCDM. Zhang et al. [70] proposed AA-AOs using generalized orthopair fuzzy aggregation information. Ali et al. [71] undertook an investigation of AA-AOs within the framework of q-rung orthopair hesitant FSs with their usage in MCDM dilemmas. Feng et al. [72] invented the WASPAS technique, integrating AA-AOs to handle complex interval-valued intuitionistic fuzzy data with applicability in decision analysis. Latif et al. [73] proposed a decision support system for single- valued neutrosophic AA-AOs. Mahmood and Ali [74] introduced an MCDM method based on Aczel-Alsina power AOs for complex IFSs.

1.1. Research gaps and motivations

In the light of earlier literature survey, our key motives, knowledge gaps, and the uniqueness of this script are summed up as follows:

1. The preexisting variants of FSs have been porously applied across various real-world dilemmas. Yet, all strategies have their drawbacks connected to the MD and NMD. To eradicate these shortcomings, Riaz and Hashmi [6] put forward a new concept of LD-FS by integrating RPs associated with the MD and NMD. Due to the addition of RPs, the LD-FS model is a rigorous approach with a broader region of MD and NMD compared to the preexisting FS variants. By modifying the physical meaning of RPs, LD-FS often classifies the data into MCDM problems. To expand the application spectrum of LD-FSs, there is a need to explore some other operational rules and their characteristics.
2. Aggregation is an essential phase in fuzzy information decision-making techniques that employ AOs in the final algorithmic stage. Numerous AOs play a vital role in the aggregation procedure throughout MCDM applications. In MCDM applications, many AOs are crucial to the aggregation process. Out of these, AA TN and TCN are prominent because they offer AA-AOs with a higher degree of adaptability and robustness in the aggregation process than operators rooted in other TN and TCN families. Although several MCDM schemes have been developed within the framework of LD-FSs. The prevailing studies [48–50] in LD-FSs and associated AOs lack flexibility in the information fusion process, and no prior investigation has been conducted into the creation of AOs based on AA TN and TCN. Therefore, designing AOs based on AA TN and TCN within the context of LD-FSs is indispensable. Thus, in this script, we consider how to extend the Aczel-Alsina operators to aggregate the information based on LD-FSs by introducing the conception of DFAAWA and LDFAAWG operators.
3. The primary benefit of these AOs stems from their parametric structure. These AOs have an inherent parameter Λ, which yields a robust and flexible aggregation procedure. As a result, the suggested AOs are felicitous and practical to address real-life MCDM issues more precisely than the existing schemes.
4. In MCDM situations, the significance of weights allocated to criteria cannot be overstated. In the evaluation of the MCDM process, criteria weights are often derived from a singular viewpoint or assigned directly, which lacks reliability and yields an imbalance of decision outcomes. As per the authors' knowledge, no such method exists in the framework of LD-FSs, where an aggregation operators-based model determines the completely unknown criteria weights information to avoid the loss of information. Therefore, in this study, we introduce a novel approach to deciding criteria weights using entropy measure.

1.2. Main objectives

Under the contributions of the above investigations, the key objectives of this research article are as follows:

1. To formulate several novel operations for LD-FNs in the context of the AA TN and TCN.
2. In light of the benefits of the Aczel-Alsina operational laws, efforts are made to discover the LDFAAWA operator and LDFAAWG operator.
3. To establish an innovative MCDM scheme based on developed AOs.
4. To demonstrate the realistic usage of the invented MCDM approach through concrete examples in diverse DM dilemmas.
5. To demonstrate the developed work's superiority, performance, and validity through a comparative analysis between the developed approach and some prevailing studies.
6. To systematically analyze the impact of the critical parameter Λ, upon ranking outcomes in the constructed AOs. This analysis sheds light on the interaction between parameters and decision outcomes.

1.3. Organization of this article

The layout of the script is systematized as follows: Section 2 includes a brief review of some cardinal terminologies related to the suggested study. In Section 3, we invented the novel operational laws using Aczel-Alsina TN and TCN in the context of LD-FNs. Section 4 outlines a range of Aczel-Alsina AOs of LD-FNs and investigates some cardinal characteristics of these operators. Section 5 harnesses the established Aczel-Alsina AOs of LD-FNs to construct an MCDM approach. Furthermore, we examine a practical example to disclose the applicability of the anticipated methodology. Section 6 includes a comparative analysis to exhibit the validity and

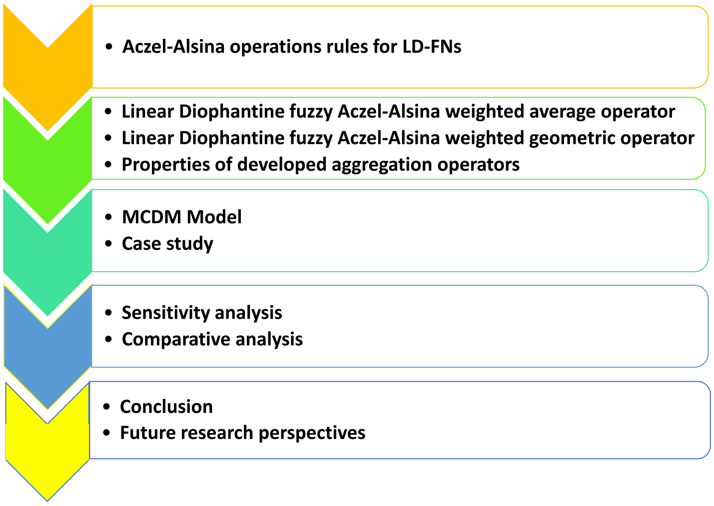


Fig. 1. Flowchart illustration for our proposed work.

Table 1
List of notations used in this study.

Abbreviations	Description
FS	Fuzzy set
IFS	Intuitionistic fuzzy set
PyFS	Pythagorean fuzzy set
q-ROFS	q-rung orthopair fuzzy set
LD-FS	linear Diophantine fuzzy set
LD-FN	linear Diophantine fuzzy number
AO	Aggregation operator
AA-AOs	Aczel-Alsina aggregation operators
MCDM	Multi-criteria decision-making
DMr	Decision-maker
MD	Membership degree
NMD	Non-membership degree
RPs	Reference parameters
TN	Triangular norm
TCN	Triangular conorm
AA TN	Aczel-Alsina triangular norm
AA TCN	Aczel-Alsina triangular conorm

efficacy of the designed approach. The conclusions and future study plans are provided in Section 8. This organization is outlined in Fig. 1.

2. Background knowledge

This segment discusses a few rudimentary ideas that will be utilized in the subsequent segments. Table 1 describes the notations included in the entire article.

Definition 2.1. [3] By an IFS over a fixed set $\wp$, we mean a structure of the form:

$$\mathfrak{I}=\left\{\left\langle \flat,\mu_{\mathfrak{I}}(\flat),\nu_{\mathfrak{I}}(\flat)\right\rangle:\flat\in\wp\right\}, \tag{2.1}$$

where $\mu_{\mathfrak{I}}:\wp\longrightarrow[0,1]$ and $\eta_{\mathfrak{I}}:\wp\longrightarrow[0,1]$ signify the MD and NMD, respectively, such that $0\leq\mu_{\mathfrak{I}}(\flat)+\nu_{\mathfrak{I}}(\flat)\leq 1$ for all $\flat\in\wp$.

Definition 2.2. [4,15] A PyFS over $\wp$ is a structure of the form:

$$\mathcal{P}=\left\{\left\langle \flat,\mu_{\mathcal{P}}(\flat),\nu_{\mathcal{P}}(\flat)\right\rangle:\flat\in\wp\right\}, \tag{2.2}$$

where $\mu_{\mathcal{P}}:\wp\longrightarrow[0,1]$ and $\nu_{\mathcal{P}}:\wp\longrightarrow[0,1]$ are the MD and NMD such that $0\leq\mu^2_{\mathcal{P}}(\flat)+\nu^2_{\mathcal{P}}(\flat)\leq 1$ for all $\flat\in\wp$.

Definition 2.3. [5] A q-ROFS over $\wp$ is a mathematical structures having the form:

$$\mathcal{Q} = \left\{ \left\langle \flat, \mu_{\mathcal{Q}}(\flat), \nu_{\mathcal{Q}}(\flat) \right\rangle : \flat \in \wp \right\}, \tag{2.3}$$

where $\mu_{\mathcal{Q}} : \wp \longrightarrow [0,1]$ and $\nu_{\mathcal{Q}} : \wp \longrightarrow [0,1]$ denote the MD and NMD such that $0 \leq \mu_{\mathcal{Q}}^q(\flat) + \nu_{\mathcal{Q}}^q(\flat) \leq 1$ for each $\flat \in \wp$ and $q \geq 1$. The hesitation part is formulated as:

$$\Xi_{\mathcal{Q}}(\flat) = \sqrt[q]{1 - \left(\mu_{\mathcal{Q}}^q(\flat) + \nu_{\mathcal{Q}}^q(\flat)\right)}. \tag{2.4}$$

Definition 2.4. [6] An LD-FS over $\wp$ is a structure of the form:

$$\Upsilon = \left\{ \left(\flat, \left\langle f(\flat), g(\flat) \right\rangle, \left\langle \alpha, \beta \right\rangle \right) : \flat \in \wp \right\}, \tag{2.5}$$

where $f : \wp \longrightarrow [0,1]$ and $g : \wp \longrightarrow [0,1]$ denote MD and NMD, respectively, and $\alpha, \beta \in [0,1]$ are RPs with $0 \leq \alpha + \beta \leq 1$ and $0 \leq \alpha f(\flat) + \beta g(\flat) \leq 1$ for all $\flat \in \wp$.

The degree of hesitation of any $\flat \in \wp$ is denoted and defined as:

$$\mathcal{H}_{\pounds}(\flat) = 1 - \left(\alpha f(\flat) + \beta g(\flat)\right). \tag{2.6}$$

Let $\Upsilon = \left\{ \left(\flat, \left\langle f(\flat), g(\flat) \right\rangle, \left\langle \alpha, \beta \right\rangle \right) : \flat \in \wp \right\}$ be an LD-FS over $\wp$. If Υ has a single element, Υ is termed a linear Diophantine fuzzy number (LD-FN). For the sake of simplicity, an LD-FN is signified by $\Upsilon = \left(\left\langle f, g \right\rangle, \left\langle \alpha, \beta \right\rangle\right)$.

The collection of all LD-FNs over $\wp$ will be denoted by $\mathfrak{LDFN}(\wp)$.

Definition 2.5. [6] A LD-FN of the form $\mathcal{A} = \left(\left\langle 1, 0 \right\rangle, \left\langle 1, 0 \right\rangle\right)$ is termed an absolute LD-FN over $\wp$. Moreover, a LD-FN of the form $\mathcal{N} = \left(\left\langle 0, 1 \right\rangle, \left\langle 0, 1 \right\rangle\right)$ is called a null LD-FN over $\wp$.

2.1. Elementary operations of LD-FNs

Some cardinal operations for LD-FNs are characterized as follows.

Definition 2.6. [6] Assume that $\Upsilon_1 = \left(\left\langle f_1, g_1 \right\rangle, \left\langle \alpha_1, \beta_1 \right\rangle\right)$ and $\Upsilon_2 = \left(\left\langle f_2, g_2 \right\rangle, \left\langle \alpha_2, \beta_2 \right\rangle\right)$ be two LD-FNs over $\wp$. Then, for each $\xi > 0$, we have

(1) $\Upsilon_1 \oplus \Upsilon_2 = \left(\left\langle f_1 + f_2 - f_1 f_2, g_1 g_2 \right\rangle, \left\langle \alpha_1 + \alpha_2 - \alpha_1 \alpha_2, \beta_1 \beta_2 \right\rangle\right);$
(2) $\Upsilon_1 \otimes \Upsilon_2 = \left(\left\langle f_1 f_2, g_1 + g_2 - g_1 g_2 \right\rangle, \left\langle \alpha_1 \alpha_2, \beta_1 + \beta_2 - \beta_1 \beta_2 \right\rangle\right);$
(3) $\xi \Upsilon_1 = \left(\left\langle 1 - \left(1 - f_1\right)^{\xi}, g_1^{\xi} \right\rangle, \left\langle 1 - \left(1 - \alpha_1\right)^{\xi}, \beta_1^{\xi} \right\rangle\right);$
(4) $\Upsilon_1^{\xi} = \left(\left\langle f_1^{\xi}, 1 - \left(1 - g_1\right)^{\xi} \right\rangle, \left\langle \alpha_1^{\xi}, 1 - \left(1 - \beta_1\right)^{\xi} \right\rangle\right).$

Definition 2.7. [6] Let $\Upsilon = \left(\left\langle f, g \right\rangle, \left\langle \alpha, \beta \right\rangle\right)$ be a LD-FN over $\wp$. Then, the score value (SV), quadratic score function (QSF), and expected score function (ESF) of Υ can be calculated as, respectively:

$$\widetilde{\mathfrak{S}}(\Upsilon) = \frac{1}{2}\left[\left(f - g\right) + \left(\alpha - \beta\right)\right], \tag{2.7}$$

$$\mathfrak{Q}(\Upsilon) = \frac{1}{2}\left[\left(f^2 - g^2\right) + \left(\alpha^2 - \beta^2\right)\right], \tag{2.8}$$

$$\mathfrak{E}(\Upsilon) = \frac{1}{2}\left[\frac{f - g + 1}{2} + \frac{\alpha - \beta + 1}{2}\right]. \tag{2.9}$$

2.2. Triangular norm and triangular conorm

In FS theory, TN and TCN are fundamental notions applied to construct generic operational laws of FSs and find widespread employment across various disciplines, particularly in decision analysis. In the following, we examine the critical core ideas for moving this discussion further.

Definition 2.8. [75] A function $\top : [0,1]^2 \longrightarrow [0,1]$ is said to be a TN if for all $\theta_1, \theta_2, \theta_3 \in [0,1]$, the subsequent axioms hold:

1. $\top(\theta_1, \theta_2) = \top(\theta_2, \theta_1)$;
2. $\top\left(\theta_1, \top(\theta_2, \theta_3)\right) = \top\left(\top(\theta_1, \theta_2), \theta_3\right)$;
3. $\top(\theta_1, \theta_2) \leq \top(\theta_1, \theta_3)$ if $\theta_2 \leq \theta_3$;

4. $\top(\theta_1, 1) = \theta_1$.

Following are a few examples of TNs:

- Minimum TN: $\top_\wedge(\theta_1, \theta_2) = min(\theta_1, \theta_2)$;
- Product TN: $\top_P(\theta_1, \theta_2) = \theta_1\theta_2$;
- Lukasiewicz TN: $\top_\pounds(\theta_1, \theta_2) = max(\theta_1 + \theta_2 - 1, 0)$;
- Drastic TN: $\top_D(\theta_1, \theta_2) = \begin{cases} \theta_1, \text{ if } \theta_2 = 1 \\ \theta_2, \text{ if } \theta_1 = 1 \\ 0, \text{ else.} \end{cases}$

Definition 2.9. A TCN is a map $\perp : [0,1]^2 \longrightarrow [0,1]$ satisfying the subsequent features:

1. $\perp(\theta_1, \theta_2) = \perp(\theta_2, \theta_1)$;
2. $\perp\big(\theta_1, \perp(\theta_2, \theta_3)\big) = \perp\big(\perp(\theta_1, \theta_2), \theta_3\big)$;
3. $\perp(\theta_1, \theta_2) \leq \perp(\theta_1, \theta_3)$ if $\theta_2 \leq \theta_3$;
4. $\perp(\theta_1, 0) = \theta_1$,

for all $\theta_1, \theta_2, \theta_3 \in [0,1]$.

Following are few examples of TCNs:

- Maximum TCN: $\perp_\vee(\theta_1, \theta_2) = max(\theta_1, \theta_2)$;
- Probabilistic TCN: $\perp_P(\theta_1, \theta_2) = \theta_1 + \theta_2 - \theta_1\theta_2$;
- Lukasiewicz TCN: $\perp_\pounds(\theta_1, \theta_2) = min(\theta_1 + \theta_2, 1)$;
- Drastic TCN: $\perp_D(\theta_1, \theta_2) = \begin{cases} \theta_1, \text{ if } \theta_2 = 0 \\ \theta_2, \text{ if } \theta_1 = 0 \\ 1, \text{ else.} \end{cases}$

2.3. Aczel-Alsina operations

In 1982, Aczel and Alsina [58] invented a new category of TN and TCN known as Aczel-Alsina TN (AA TN) and Aczel-Alsina (AA TCN), which emphasized a high premium on parameter variability.

Definition 2.10. [58] The AA TN $\left(\top_A^\Lambda\right)_{\Lambda\in[0,\infty]}$ is interpreted as:

$$\top_A^\Lambda(u_1, u_2) = \begin{cases} \top_D(u_1, u_2), & \text{if } \Lambda = 0 \\ min(u_1, u_2), & \text{if } \Lambda = \infty \\ e^{-\left(\left(-\ln u_1\right)^\Lambda + \left(-\ln u_2\right)^\Lambda\right)^{1/\Lambda}}, & \text{else.} \end{cases} \tag{2.10}$$

Some particular cases: $\top_A^\infty = \top_\wedge$, $\top_A^0 = \top_D$, $\top_A^1 = \top_P$.

Definition 2.11. [76] The AA TCN $\left(\perp_A^\Lambda\right)_{\Lambda\in[0,\infty]}$ is stated as:

$$\perp_A^\Lambda(u_1, u_2) = \begin{cases} \perp_D(u_1, u_2), & \text{if } \Lambda = 0 \\ max(u_1, u_2), & \text{if } \Lambda = \infty \\ 1 - e^{-\left(\left(-\ln(1-u_1)\right)^\Lambda + \left(-\ln(1-u_2)\right)^\Lambda\right)^{1/\Lambda}}, & \text{else.} \end{cases} \tag{2.11}$$

Some particular cases: $\perp_A^\infty = \perp_\vee$, $\perp_A^0 = \perp_D$, $\perp_A^1 = \perp_P$.

3. Proposed Aczel-Alsina operational laws for LD-FNs

Within this segment, we shall provide a set of Aczel-Alsina operational rules by employing Definitions 2.10 and 2.11 for LD-FNs and studied some fundamental features.

Definition 3.1. Let $\Upsilon_1 = \left(\left\langle f_1, g_1 \right\rangle, \left\langle \alpha_1, \beta_1 \right\rangle\right)$ and $\Upsilon_2 = \left(\left\langle f_2, g_2 \right\rangle, \left\langle \alpha_2, \beta_2 \right\rangle\right)$ be two LD-FNs and $\Lambda \geq 1$ be any real number, then the AA TN and AA TCN operations for LD-FNs are postulated as:

(1)

$$\Upsilon_1 \oplus \Upsilon_2 = \begin{pmatrix} \left\langle 1 - e^{-\left(\left(-\ln\left(1-f_1\right)^{\Lambda}\right)+\left(-\ln\left(1-f_2\right)^{\Lambda}\right)\right)^{1/\Lambda}}, e^{-\left(\left(-\ln g_1\right)^{\Lambda}+\left(-\ln g_2\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\left(-\ln\left(1-\alpha_1\right)^{\Lambda}\right)+\left(-\ln\left(1-\alpha_2\right)^{\Lambda}\right)\right)^{1/\Lambda}}, e^{-\left(\left(-\ln \beta_1\right)^{\Lambda}+\left(-\ln \beta_2\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{pmatrix}; \tag{3.1}$$

(2)

$$\Upsilon_1 \otimes \Upsilon_2 = \begin{pmatrix} \left\langle e^{-\left(\left(-\ln f_1\right)^{\Lambda}+\left(-\ln f_2\right)^{\Lambda}\right)^{1/\Lambda}}, 1 - e^{-\left(\left(-\ln\left(1-g_1\right)^{\Lambda}\right)+\left(-\ln\left(1-g_2\right)^{\Lambda}\right)\right)^{1/\Lambda}} \right\rangle, \\ \left\langle e^{-\left(\left(-\ln \alpha_1\right)^{\Lambda}+\left(-\ln \alpha_2\right)^{\Lambda}\right)^{1/\Lambda}}, 1 - e^{-\left(\left(-\ln\left(1-\beta_1\right)^{\Lambda}\right)+\left(-\ln\left(1-\beta_2\right)^{\Lambda}\right)\right)^{1/\Lambda}} \right\rangle \end{pmatrix}; \tag{3.2}$$

(3)

$$\xi \Upsilon_1 = \begin{pmatrix} \left\langle 1 - e^{-\left(\xi\left(-\ln\left(1-f_1\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\xi\left(-\ln g_1\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\xi\left(-\ln\left(1-\alpha_1\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\xi\left(-\ln \beta_1\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{pmatrix}, \ \xi > 0; \tag{3.3}$$

(4)

$$\Upsilon_1^{\xi} = \begin{pmatrix} \left\langle e^{-\left(\xi\left(-\ln f_1\right)^{\Lambda}\right)^{1/\Lambda}}, 1 - e^{-\left(\xi\left(-\ln\left(1-g_1\right)\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle e^{-\left(\xi\left(-\ln \alpha_1\right)^{\Lambda}\right)^{1/\Lambda}}, 1 - e^{-\left(\xi\left(-\ln\left(1-\beta_1\right)\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{pmatrix}, \ \xi > 0. \tag{3.4}$$

Theorem 3.2. *Let $\Upsilon_1 = \left(\left\langle f_1, g_1 \right\rangle, \left\langle \alpha_1, \beta_1 \right\rangle\right)$ and $\Upsilon_2 = \left(\left\langle f_2, g_2 \right\rangle, \left\langle \alpha_2, \beta_2 \right\rangle\right)$ be any two LD-FNs and $\xi, \xi_1, \xi_2 > 0$ be any real numbers. Then, the subsequent properties holds:*

1. $\Upsilon_1 \oplus \Upsilon_2 = \Upsilon_2 \oplus \Upsilon_1$;
2. $\Upsilon_1 \otimes \Upsilon_2 = \Upsilon_2 \otimes \Upsilon_1$;
3. $\xi(\Upsilon_1 \oplus \Upsilon_2) = \xi\Upsilon_1 \oplus \xi\Upsilon_2$;
4. $(\Upsilon_1 \otimes \Upsilon_2)^{\xi} = \Upsilon_1^{\xi} \otimes \Upsilon_2^{\xi}$;
5. $\xi_1\Upsilon_1 \oplus \xi_2\Upsilon_1 = (\xi_1 + \xi_2)\Upsilon_1$;
6. $\Upsilon_1^{\xi_1} \otimes \Upsilon_1^{\xi_2} = \Upsilon_1^{\xi_1+\xi_2}$;
7. $\left(\Upsilon_1^{\xi_1}\right)^{\xi_2} = \Upsilon_1^{\xi_1\xi_2}$;
8. $\xi_1(\xi_2\Upsilon_1) = (\xi_1\xi_2)\Upsilon_1$.

Proof. Straightforward. □

4. Linear Diophantine fuzzy Aczel-Alsina aggregation operators

In the current section, we introduce two types of AA-AOs for LD-FNs. We also highlight the desirable characteristics of theses AOs, based on Aczel-Alsina operational laws.

4.1. Linear Diophantine fuzzy Aczel-Alsina weighted average (LDFAAWA) operator

This subsection presents the LDFAAWA operator and discusses its cardinal features.

Definition 4.1. Let $\Upsilon_j = \left(\left\langle f_j, g_j \right\rangle, \left\langle \alpha_j, \beta_j \right\rangle : j = 1, 2, \cdots, k\right)$ be a family of LD-FNs over $\wp$ and $\mathcal{W} = (\varpi_1, \varpi_2, \cdots, \varpi_k)^T$ is a weight vector (WV) such that $\varpi_j > 0$ and $\sum_{j=1}^{k} \varpi_j = 1$. Then, the map

$$LDFAAWA : \mathfrak{LDFN}(\wp)^k \longrightarrow \mathfrak{LDFN}(\wp)$$

is called LDFAAWA operator postulated by:

$$LDFAAWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) = \bigoplus_{j=1}^{k} \left(\varpi_j \Upsilon_j\right). \tag{4.1}$$

Theorem 4.2. *Let $\Upsilon_j = \left(\langle f_j, g_j \rangle, \langle \alpha_j, \beta_j \rangle : j = 1, 2, \cdots, k\right)$ be a family of LD-FNs over $\wp$. Then, the aggregated result attained by the LDFAAWA operator is still an LD-FN. Moreover,*

$$LDFAAWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) = \bigoplus_{j=1}^{k} \left(\varpi_j \Upsilon_j\right)$$

$$= \left(\begin{array}{l} \left\langle 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array} \right) \tag{4.2}$$

Proof. This can be deduced by employing the principle of Mathematical induction on k as follows:

Step 1. If $k = 2$, then according to Eq. (4.2) it becomes

$$LDFAAWA(\Upsilon_1, \Upsilon_2) = \varpi_1 \Upsilon_1 \oplus \varpi_2 \Upsilon_2$$

$$= \left(\begin{array}{l} \left\langle 1 - e^{-\left(\varpi_1 \left(-\ln\left(1-f_1\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\varpi_1 \left(-\ln g_1\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\varpi_1 \left(-\ln\left(1-\alpha_1\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\varpi_1 \left(-\ln \beta_1\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array} \right)$$

$$\oplus \left(\begin{array}{l} \left\langle 1 - e^{-\left(\varpi_2 \left(-\ln\left(1-f_2\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\varpi_2 \left(-\ln g_2\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\varpi_2 \left(-\ln\left(1-\alpha_2\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\varpi_2 \left(-\ln \beta_2\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{l} \left\langle 1 - e^{-\left(\varpi_1 \left(-\ln\left(1-f_1\right)^{\Lambda}\right) + \varpi_2 \left(-\ln\left(1-f_2\right)^{\Lambda}\right)\right)^{1/\Lambda}}, e^{-\left(\varpi_1 \left(-\ln g_1\right)^{\Lambda} + \varpi_2 \left(-\ln g_2\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\varpi_1 \left(-\ln\left(1-\alpha_1\right)^{\Lambda}\right) + \varpi_2 \left(-\ln\left(1-\alpha_2\right)^{\Lambda}\right)\right)^{1/\Lambda}}, e^{-\left(\varpi_1 \left(-\ln \beta_1\right)^{\Lambda} + \varpi_2 \left(-\ln \beta_2\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{l} \left\langle 1 - e^{-\left(\sum_{j=1}^{2} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{2} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\sum_{j=1}^{2} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{2} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array} \right).$$

Hence, Eq. (4.2) is valid for $k = 2$.

Step 2. Next, assume that Eq. (4.2) valid for $k = r$, we have

$$LDFAAWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_r) = \bigoplus_{j=1}^{r} \left(\varpi_j \Upsilon_j\right)$$

$$= \left(\begin{array}{l} \left\langle 1 - e^{-\left(\sum_{j=1}^{r} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{r} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\sum_{j=1}^{r} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{r} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array} \right).$$

Step 3. Now, for $k = r + 1$, it follows that

$$LDFAAWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_r, \Upsilon_{r+1}) = \bigoplus_{j=1}^{r} \left(\varpi_j \Upsilon_j\right) \bigoplus \left(\varpi_{r+1} \Upsilon_{r+1}\right)$$

$$= \left(\begin{array}{l} \left\langle 1 - e^{-\left(\sum_{j=1}^{r} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{r} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\sum_{j=1}^{r} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{r} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array} \right)$$

$$\oplus \left(\begin{array}{l} \left\langle 1 - e^{-\left(\varpi_{r+1} \left(-\ln\left(1-f_{r+1}\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\varpi_{r+1} \left(-\ln g_{r+1}\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\varpi_{r+1} \left(-\ln\left(1-\alpha_{r+1}\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\varpi_{r+1} \left(-\ln \beta_{r+1}\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{l} \left\langle 1 - e^{-\left(\sum_{j=1}^{r+1} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{r+1} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\sum_{j=1}^{r+1} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{r+1} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array} \right).$$

Therefore, for $k = r + 1$, Eq. (4.2) holds.

In the light of steps (1), (2) and (3), it become evident that Eq. (4.2) holds valid for any positive integers k.

In the following, we prove that $LDFAAWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k)$ is still an LD-FN.

Since, $0 \le \alpha_j + \beta_j \le 1$, we have

$$\Rightarrow \beta_j \le 1 - \alpha_j$$

$$\Rightarrow \ln\left(\beta_j\right) \le \ln\left(1 - \alpha_j\right)$$

$$\Rightarrow -\ln\left(\beta_j\right) \ge -\ln\left(1 - \alpha_j\right)$$

$$\Rightarrow \left(-\ln\left(\beta_j\right)\right)^{\Lambda} \ge \left(-\ln\left(1 - \alpha_j\right)\right)^{\Lambda}$$

$$\Rightarrow \sum_{j=1}^{k} \varpi_j \left(-\ln\left(\beta_j\right)\right)^{\Lambda} \ge \sum_{j=1}^{k} \varpi_j \left(-\ln\left(1 - \alpha_j\right)\right)^{\Lambda}$$

$$\Rightarrow \left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(\beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda} \ge \left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1 - \alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}$$

$$\Rightarrow -\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(\beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda} \le -\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1 - \alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}$$

$$\Rightarrow e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(\beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} \le e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}$$

$$\Rightarrow -e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} + e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(\beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} \le 0$$

$$\Rightarrow 0 \le 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} + e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(\beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} \le 1.$$

Hence, we get

$$0 \leq \left(1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}\right)\left(1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}\right) + \left(e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(\beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}\right)\left(e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}}\right) \leq 1$$

since

$$0 \leq 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \leq 1$$

and

$$0 \leq 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} + e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(\beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} \leq 1. \quad \square$$

Example 4.3. Consider three LD-FNs: $\Upsilon_1 = (\langle 0.8, 0.7\rangle, \langle 0.5, 0.3\rangle)$, $\Upsilon_2 = (\langle 0.5, 0.9\rangle, \langle 0.1, 0.8\rangle)$ and $\Upsilon_3 = (\langle 0.8, 0.3\rangle, \langle 0.2, 0.7\rangle)$. Let $\mathcal{W} = (0.3, 0.5, 0.2)^T$ be the WV of $\Upsilon_j (j = 1, 2, 3)$. If we take $\Lambda = 2$, then

$$\left(\sum_{j=1}^{3} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda} = \left(0.3\left(-\ln\left(1-0.8\right)\right)^2 + 0.5\left(-\ln\left(1-0.5\right)\right)^2 + 0.2\left(-\ln\left(1-0.8\right)\right)^2\right)^{1/2} = 1.2391$$

$$\left(\sum_{j=1}^{3} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda} = \left(0.3\left(-\ln\left(1-0.3\right)\right)^2 + 0.5\left(-\ln\left(1-0.8\right)\right)^2 + 0.2\left(-\ln\left(1-0.7\right)\right)^2\right)^{1/2} = 1.2741$$

$$\left(\sum_{j=1}^{3} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda} = \left(0.3\left(-\ln 0.7\right)^2 + 0.5\left(-\ln 0.9\right)^2 + 0.2\left(-\ln 0.3\right)^2\right)^{1/2} = 0.5776$$

$$\left(\sum_{j=1}^{3} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda} = \left(0.3\left(-\ln 0.5\right)^2 + 0.5\left(-\ln 0.1\right)^2 + 0.2\left(-\ln 0.2\right)^2\right)^{1/2} = 1.8202$$

Therefore,

$$LDFAAWA(\Upsilon_1, \Upsilon_2, \Upsilon_3) = \bigoplus_{j=1}^{3} (\varpi_j \Upsilon_j)$$

$$= \left(\begin{array}{c} \left\langle 1 - e^{-\left(\sum_{j=1}^{3} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{3} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\sum_{j=1}^{3} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{3} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{array}\right)$$

$$= \left(\begin{array}{c} \left\langle 1 - e^{-1.2391}, e^{-0.5776} \right\rangle, \\ \left\langle 1 - e^{-1.2741}, e^{-1.8202} \right\rangle \end{array}\right) = (\langle 0.7103, 0.5612\rangle, \langle 0.7203, 0.1619\rangle).$$

From Theorem 4.2, it is witnessed that the designed LDFAAWA operator exhibit the following features:

Theorem 4.4. *(Idempotency) If all $\Upsilon_j (j = 1, 2, \cdots, k)$ are identical, i.e., $\Upsilon_j = \Upsilon$ for all j, then*

$$LDFAAWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) = \Upsilon. \tag{4.3}$$

Proof. Since $\Upsilon_j = \Upsilon$ for all j, and $\sum_{j=1}^{k} \varpi_j = 1$ so according to Theorem 4.2, we have

$$LDFAAWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) = \bigoplus_{j=1}^{k} (\varpi_j \Upsilon_j)$$

$$= \begin{pmatrix} \left\langle 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \left\langle 1 - e^{-\left(\left(-\ln\left(1-f\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\left(-\ln g\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\left(-\ln\left(1-\alpha\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\left(-\ln \beta\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{pmatrix} = \left(\left\langle f, g\right\rangle, \left\langle \alpha, \beta\right\rangle\right) = \Upsilon. \quad \square$$

Theorem 4.5. *(Monotonicity) Let* $\Upsilon_j = \left(\left\langle f_j, g_j\right\rangle, \left\langle \alpha_j, \beta_j\right\rangle\right)$ *and* $\Upsilon_j^{\star} = \left(\left\langle f_j^{\star}, g_j^{\star}\right\rangle, \left\langle \alpha_j^{\star}, \beta_j^{\star}\right\rangle\right)$ $(j = 1, 2, \cdots, k)$ *are two collections of LD-FNs such that* $f_j \geq f_j^{\star}$, $g_j \leq g_j^{\star}$, *and* $\alpha_j \geq \alpha_j^{\star}$, $\beta_j \leq \beta_j^{\star}$, *for all* j, *then*

$$LDFAAWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) \geq LDFAAWA(\Upsilon_1^{\star}, \Upsilon_2^{\star}, \cdots, \Upsilon_k^{\star}). \tag{4.4}$$

Proof. Since $f_j \geq f_j^{\star}$, $g_j \leq g_j^{\star}$, and $\alpha_j \geq \alpha_j^{\star}$, $\beta_j \leq \beta_j^{\star}$, for all j. Based upon these observation, we have the following inequalities,

$$1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} \geq 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-f_j^{\star}\right)\right)^{\Lambda}\right)^{1/\Lambda}},$$

$$e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \leq e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln g_j^{\star}\right)^{\Lambda}\right)^{1/\Lambda}},$$

$$1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} \geq 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j^{\star}\right)\right)^{\Lambda}\right)^{1/\Lambda}},$$

and

$$e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda}} \leq e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln \beta_j^{\star}\right)^{\Lambda}\right)^{1/\Lambda}}$$

which gives

$$= \begin{pmatrix} \left\langle 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-f_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln g_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln \beta_j\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{pmatrix}$$

$$\geq \begin{pmatrix} \left\langle 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-f_j^{\star}\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln g_j^{\star}\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\alpha_j^{\star}\right)\right)^{\Lambda}\right)^{1/\Lambda}}, e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln \beta_j^{\star}\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{pmatrix}.$$

Therefore, $LDFAAWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) \geq LDFAAWA(\Upsilon_1^{\star}, \Upsilon_2^{\star}, \cdots, \Upsilon_k^{\star})$. $\square$

Theorem 4.6. *(Boundedness) Let* $\Upsilon_j = \left(\left\langle f_j, g_j\right\rangle, \left\langle \alpha_j, \beta_j\right\rangle : j = 1, 2, \cdots, k\right)$ *be an assembling of LD-FNs over* $\wp$. *If*

$$\Upsilon^{-} = \bigwedge_{j=1}^{k} \Upsilon_j = \left(\left\langle \bigwedge_{j=1}^{k} f_j, \bigvee_{j=1}^{k} g_j\right\rangle, \left\langle \bigwedge_{j=1}^{k} \alpha_j, \bigvee_{j=1}^{k} \beta_j\right\rangle\right)$$

and

$$\Upsilon^{+} = \bigvee_{j=1}^{k} \Upsilon_j = \left(\left\langle \bigvee_{j=1}^{k} f_j, \bigwedge_{j=1}^{k} g_j\right\rangle, \left\langle \bigvee_{j=1}^{k} \alpha_j, \bigwedge_{j=1}^{k} \beta_j\right\rangle\right),$$

then

$$\Upsilon^- \preceq LDFHWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) \preceq \Upsilon^+. \tag{4.5}$$

Proof. The proof is straightforward. □

Theorem 4.7. *(Homogeneity) Let $\gamma > 0$. Then*

$$LDFHWA(\gamma\Upsilon_1, \gamma\Upsilon_2, \cdots, \gamma\Upsilon_k) = \gamma LDFHWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k). \tag{4.6}$$

Proof. Straightforward. □

Theorem 4.8. *(Shift Invariance) Consider any LD-FN $\mathcal{T} = \left(\left\langle f, g\right\rangle, \left\langle \vartheta, \varrho\right\rangle\right)$, then*

$$LDFHWA(\Upsilon_1 \oplus \mathcal{T}, \Upsilon_2 \oplus \mathcal{T}, \cdots, \Upsilon_k \oplus \mathcal{T}) = LDFHWA(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) \oplus \mathcal{T}. \tag{4.7}$$

Proof. Straightforward. □

4.2. Linear Diophantine fuzzy Aczel-Alsina weighted geometric (LDFAAWG) operator

In the following, we shall present the LDFAAWG operator and examines its important characteristics.

Definition 4.9. Let $\Upsilon_j = \left(\left\langle f_j, g_j\right\rangle, \left\langle \alpha_j, \beta_j\right\rangle : j = 1, 2, \cdots, k\right)$ be a family of LD-FNs over $\wp$ and $\mathcal{W} = (\varpi_1, \varpi_2, \cdots, \varpi_k)^T$ be a WV such that $\varpi_j > 0$ and $\sum_{j=1}^{k} \varpi_j = 1$. Then, the map

$$LDFAAWG : \mathfrak{LDFN}(\wp)^k \longrightarrow \mathfrak{LDFN}(\wp)$$

is term as LDFAAWG operator and is defined as:

$$LDFAAWG(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) = \bigotimes_{j=1}^{k} \left(\Upsilon_j\right)^{\varpi_j}. \tag{4.8}$$

Theorem 4.10. *Let $\Upsilon_j = \left(\left\langle f_j, g_j\right\rangle, \left\langle \alpha_j, \beta_j\right\rangle : j = 1, 2, \cdots, k\right)$ be an assembling of LD-FNs over $\wp$. Then, the aggregated value via the LDFAAWG operator is also an LD-FN which is given by*

$$LDFAAWG(\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_k) = \bigotimes_{j=1}^{k} \left(\Upsilon_j\right)^{\varpi_j}$$

$$= \begin{pmatrix} \left\langle e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln f_j\right)^{\Lambda}\right)^{1/\Lambda}}, 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-g_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle, \\ \left\langle e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln \alpha_j\right)^{\Lambda}\right)^{1/\Lambda}}, 1 - e^{-\left(\sum_{j=1}^{k} \varpi_j \left(-\ln\left(1-\beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda}} \right\rangle \end{pmatrix} \tag{4.9}$$

Proof. Same as Theorem 4.2. □

Example 4.11. Let $\Upsilon_1 = \left(\left\langle 0.8, 0.7\right\rangle, \left\langle 0.3, 0.6\right\rangle\right)$, $\Upsilon_2 = \left(\left\langle 0.6, 0.5\right\rangle, \left\langle 0.3, 0.7\right\rangle\right)$ and $\Upsilon_3 = \left(\left\langle 0.5, 0.8\right\rangle, \left\langle 0.8, 0.1\right\rangle\right)$ be three LD-FNs and $\mathcal{W} = (0.3, 0.5, 0.2)^T$ be the associated WV of $\Upsilon_j (j = 1, 2, 3)$. If we take $\Lambda = 2$, then

$$\left(\sum_{j=1}^{3} \varpi_j \left(-\ln f_j\right)^{\Lambda}\right)^{1/\Lambda} = \left(0.3\left(-\ln 0.8\right)^2 + 0.5\left(-\ln 0.6\right)^2 + 0.2\left(-\ln 0.5\right)^2\right)^{1/2} = 0.4914$$

$$\left(\sum_{j=1}^{3} \varpi_j \left(-\ln \alpha_j\right)^{\Lambda}\right)^{1/\Lambda} = \left(0.3\left(-\ln 0.3\right)^2 + 0.5\left(-\ln 0.3\right)^2 + 0.2\left(-\ln 0.8\right)^2\right)^{1/2} = 1.0815$$

$$\left(\sum_{j=1}^{3} \varpi_j \left(-\ln\left(1 - g_j\right)\right)^{\Lambda}\right)^{1/\Lambda} = \left(0.3\left(-\ln\left(1 - 0.7\right)\right)^2 + 0.5\left(-\ln\left(1 - 0.5\right)\right)^2 + 0.2\left(-\ln\left(1 - 0.8\right)\right)^2\right)^{1/2} = 1.0923$$

$$\left(\sum_{j=1}^{3} \varpi_j \left(-\ln\left(1 - \beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda} = \left(0.3\left(-\ln\left(1 - 0.6\right)\right)^2 + 0.5\left(-\ln\left(1 - 0.7\right)\right)^2 + 0.2\left(-\ln\left(1 - 0.1\right)\right)^2\right)^{1/2} = 0.9894$$

Therefore,

$$LDFAAWG(\Upsilon_1,\Upsilon_2,\Upsilon_3)=\bigotimes_{j=1}^{3}\left(\Upsilon_j\right)^{\varpi_j}$$

$$=\left(\begin{array}{c}\left\langle e^{-\left(\sum_{j=1}^{3}\varpi_j\left(-\ln f_j\right)^{\Lambda}\right)^{1/\Lambda}}, 1-e^{-\left(\sum_{j=1}^{3}\varpi_j\left(-\ln\left(1-g_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}\right\rangle,\\ \left\langle e^{-\left(\sum_{j=1}^{3}\varpi_j\left(-\ln \alpha_j\right)^{\Lambda}\right)^{1/\Lambda}}, 1-e^{-\left(\sum_{j=1}^{3}\varpi_j\left(-\ln\left(1-\beta_j\right)\right)^{\Lambda}\right)^{1/\Lambda}}\right\rangle\end{array}\right)$$

$$=\left(\begin{array}{c}\left\langle e^{-0.4914},1-e^{-1.0923}\right\rangle,\\ \left\langle e^{-1.0815},1-e^{-0.9894}\right\rangle\end{array}\right)=\left(\left\langle 0.6118,0.6646\right\rangle,\left\langle 0.3391,0.6282\right\rangle\right).$$

From Theorem 4.10, one can easily verified that the LDFHWG operator satisfies the subsequent features.

Theorem 4.12. *(Idempotency) If all $\Upsilon_j (j=1,2,\cdots,k)$ are identical, i.e., $\Upsilon_j=\Upsilon$ for all j, then*

$$LDFAAWG(\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_k)=\Upsilon. \tag{4.10}$$

Proof. Identical to the proof of Theorem 4.4. □

Theorem 4.13. *(Monotonicity) Let $\Upsilon_j=\left(\left\langle f_j,g_j\right\rangle,\left\langle \alpha_j,\beta_j\right\rangle\right)$ and $\Upsilon_j^{\star}=\left(\left\langle f_j^{\star},g_j^{\star}\right\rangle,\left\langle \alpha_j^{\star},\beta_j^{\star}\right\rangle\right)$ $(j=1,2,\cdots,k)$ are two collections of LD-FNs such that $f_j\geq f_j^{\star}$, $g_j\leq g_j^{\star}$, and $\alpha_j\geq\alpha_j^{\star}$, $\beta_j\leq\beta_j^{\star}$, for all j, then*

$$LDFAAWG(\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_k)\geq LDFAAWG(\Upsilon_1^{\star},\Upsilon_2^{\star},\cdots,\Upsilon_k^{\star}). \tag{4.11}$$

Proof. Similar to the proof of Theorem 4.5. □

Theorem 4.14. *(Boundedness) Let $\Upsilon_j=\left(\left\langle f_j,g_j\right\rangle,\left\langle \alpha_j,\beta_j\right\rangle : j=1,2,\cdots,k\right)$ be an assembling of LD-FNs over $\wp$. If*

$$\Upsilon^{-}=\bigwedge_{j=1}^{k}\Upsilon_j=\left(\left\langle\bigwedge_{j=1}^{k}f_j,\bigvee_{j=1}^{k}g_j\right\rangle,\left\langle\bigwedge_{j=1}^{k}\alpha_j,\bigvee_{j=1}^{k}\beta_j\right\rangle\right)$$

and

$$\Upsilon^{+}=\bigvee_{j=1}^{k}\Upsilon_j=\left(\left\langle\bigvee_{j=1}^{k}f_j,\bigwedge_{j=1}^{k}g_j\right\rangle,\left\langle\bigvee_{j=1}^{k}\alpha_j,\bigwedge_{j=1}^{k}\beta_j\right\rangle\right),$$

then

$$\Upsilon^{-}\preceq LDFHWG(\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_k)\preceq\Upsilon^{+}. \tag{4.12}$$

Proof. Obvious. □

Theorem 4.15. *(Homogeneity) Let $\gamma>0$. Then*

$$LDFHWG(\gamma\Upsilon_1,\gamma\Upsilon_2,\cdots,\gamma\Upsilon_k)=\gamma LDFHWG(\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_k). \tag{4.13}$$

Proof. Straightforward. □

Theorem 4.16. *(Shift Invariance) Consider any LD-FN $\mathcal{T}=\left(\left\langle f,g\right\rangle,\left\langle\vartheta,\varrho\right\rangle\right)$, then*

$$LDFHWG(\Upsilon_1\otimes\mathcal{T},\Upsilon_2\otimes\mathcal{T},\cdots,\Upsilon_k\otimes\mathcal{T})=LDFHWG(\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_k)\otimes\mathcal{T}. \tag{4.14}$$

Proof. Straightforward. □

5. MCDM approach based on proposed aggregation operators

MCDM is a process used to assess and choose among multiple alternatives, taking into account numerous criteria that are important to the DMrs. It is often applied in various domains, such as industry, economics, human sciences, engineering technologies, mission appraisal, business decision-making, etc. Within this section, we will construct a novel MCDM scheme that makes use of LDFAAWA and LDFAAWG operators to cope with situations where the data supplied by the DMrs is in the form of LD-FNs.

5.1. Problem statement

Let $\wp = \{k_1, k_2, \cdots, k_m\}$ denotes the set of m objects, and $\tilde{\mathfrak{C}} = \{\tilde{\mathscr{C}_1}, \tilde{\mathscr{C}_2}, \cdots, \tilde{\mathscr{C}_n}\}$ denotes an assembling of n criteria linked with each object. The associated WV of criteria is represented as $\mathcal{W} = (\varpi_1, \varpi_2, \cdots, \varpi_k)$ such that $0 \leq \varpi_1, \varpi_2, \cdots, \varpi_j \leq 1$ and $\sum_{j=1}^{n} \varpi_j = 1$. A DMr is recruited to assess all the alternatives. Suppose that the DMr provides his/her evaluations of the alternatives k_i $(i = 1, 2, \cdots, m)$ w.r.t. the criteria $\tilde{\mathscr{C}_j}$ $(j = 1, 2, \cdots, n)$ as LD-FNs, i.e., $\tilde{\mathscr{C}_j}(k_i) = (\langle f_{ij}, g_{ij} \rangle, \langle \alpha_{ij}, \beta_{ij} \rangle)$, such that $0 \leq \alpha_{ij} f_{ij} + \beta_{ij} g_{ij} \leq 1$ and $0 \leq \alpha_{ij} + \beta_{ij} \leq 1$. Here, f_{ij} means the degree to which k_i satisfies $\tilde{\mathscr{C}_j}$, g_{ij} means the degree to which k_i dissatisfies $\tilde{\mathscr{C}_j}$, and α_{ij} is the degree of corresponding reference parameter to f_{ij}, and β_{ij} is the degree of corresponding reference parameter to g_{ij}. Hence, the linear Diophantine fuzzy decision matrix is represented as $\mathscr{D} = \left(\tilde{\mathscr{C}_j}(k_i)\right)_{m \times n}$.

In the framework of the above strategy, we computed the procedure of ranking alternatives, whose procedure is described below.

5.2. Decision-making algorithm

The following steps outline the recommended strategy:

Step 1: Organize the data denoted by LD-FNs regarding the alternatives related to criteria in the decision matrix as follows:

$$\mathscr{D} = \left(d_{ij} = \tilde{\mathfrak{c}}_j(k_i)\right)_{m \times n} = \begin{pmatrix} (\langle f_{11}, g_{11} \rangle, \langle \alpha_{11}, \beta_{11} \rangle) & (\langle f_{12}, g_{12} \rangle, \langle \alpha_{12}, \beta_{12} \rangle) & \cdots & (\langle f_{1n}, g_{1n} \rangle, \langle \alpha_{1n}, \beta_{1n} \rangle) \\ (\langle f_{21}, g_{21} \rangle, \langle \alpha_{21}, \beta_{21} \rangle) & (\langle f_{22}, g_{22} \rangle, \langle \alpha_{22}, \beta_{22} \rangle) & \cdots & (\langle f_{2n}, g_{2n} \rangle, \langle \alpha_{2n}, \beta_{2n} \rangle) \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ (\langle f_{m1}, g_{m1} \rangle, \langle \alpha_{m1}, \beta_{m1} \rangle) & (\langle f_{m2}, g_{m2} \rangle, \langle \alpha_{m2}, \beta_{m2} \rangle) & \cdots & (\langle f_{mn}, g_{mn} \rangle, \langle \alpha_{mn}, \beta_{mn} \rangle) \end{pmatrix} \tag{5.1}$$

Step 2: If there is any cost criteria $\tilde{\mathscr{C}_j} \in \mathfrak{C}$, then construct the normalized decision matrix $\mathcal{N} = [\tilde{d_{ij}}]_{m \times n}$ as follows:

$$\mathcal{N} = [\tilde{d_{ij}}]_{m \times n} = \begin{cases} d_{ij}; \text{ if } \tilde{\mathscr{C}_j} \text{ is benefit criteria ,} \\ d_{ij}^c; \text{ if } \tilde{\mathscr{C}_j} \text{ is cost criteria ,} \end{cases} \tag{5.2}$$

where d_{ij}^c represent the complement of d_{ij}.

Step 3: In MCDM situations, allocating appropriate weights to different criteria is crucial. These weights play a pivotal role in achieving a balanced evaluation of criteria, enabling effective comparison of alternatives, and ensuring objective DM process. By quantifying the relative importance of criteria and incorporating the preferences of DMrs, the weight allocation procedure in MCDM contributes to a structured, informed, and comprehensive DM scheme that aligns with the targets and nature of the decision issues under consideration. Calculate the entropy for each criterion $\tilde{\mathscr{C}_j}$ postulated in Eq. (5.3).

$$\mathcal{E}_j(\tilde{\mathscr{C}_j}) = 1 - \left[\frac{1}{2m} \sum_{i=1}^{m} \left(\left(f_{ij} - g_{ij}\right)^2 + \left(\alpha_{ij} - \beta_{ij}\right)^2 \right)\right]^{1/2}. \tag{5.3}$$

Entropy is a degree of randomness related to a given criteria. By calculating the entropy of every criterion, we can understand the level of diversity or homogeneity within the available alternatives for that criterion. Determine the criteria weight with the formulation stated in Eq. (5.4).

$$\varpi_j = \frac{1 - \mathcal{E}_j}{n - \sum_{j=1}^{n} \mathcal{E}_j}. \tag{5.4}$$

Step 4: Based on the normalize decision matrix and entropy weights, obtain the over all aggregated value of the objects k_i $(i = 1, 2, \cdots, m)$ w.r.t. criteria $\tilde{\mathscr{C}_j}$ $(j = 1, 2, \cdots, n)$ by using LDFAAWA (LDFAAWG) operator, stated in Eqs. (4.2) and (4.9).

Step 5: By Eq. (2.7), compute the SVs of the aggregated LD-FNs acquired in the step 4.

Step 6: The ranking outcomes of objects k_i $(i = 1, 2, \cdots, m)$ may be determined based on their respective SVs. The higher the SV, the optimal the alternative is.

The Fig. 2 depicts the flowchart of invented MCDM mechanism.

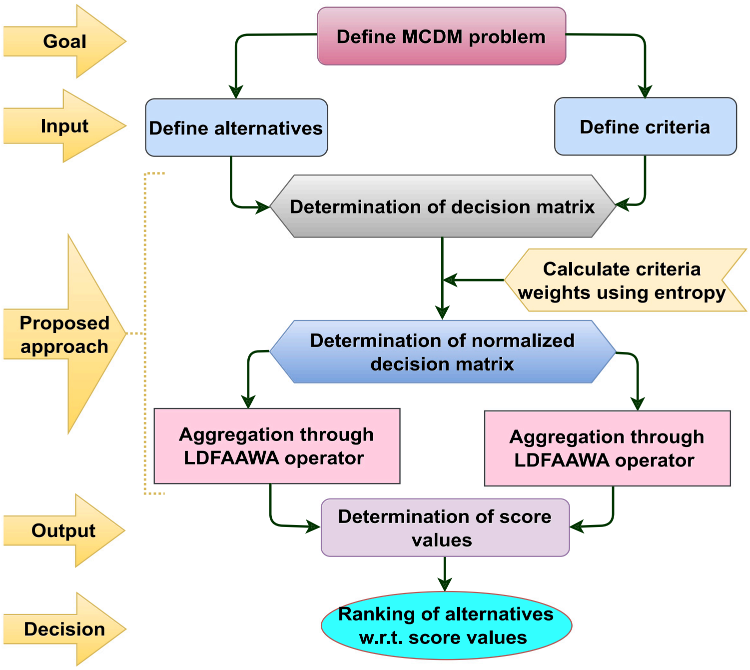


Fig. 2. Procedural flowchart of the suggested MCDM approach.

Table 2
Decision matrix $\mathscr{D} = [d_{ij}]_{5\times 6}$.

$\wp \setminus \tilde{\mathfrak{C}}$	$\tilde{\mathscr{C}_1}$	$\tilde{\mathscr{C}_2}$	$\tilde{\mathscr{C}_3}$	$\tilde{\mathscr{C}_4}$	$\tilde{\mathscr{C}_5}$	$\tilde{\mathscr{C}_6}$
k_1	$(\langle 0.81, 0.47\rangle, \langle 0.52, 0.39\rangle)$	$(\langle 0.47, 0.81\rangle, \langle 0.32, 0.51\rangle)$	$(\langle 0.56, 0.71\rangle, \langle 0.79, 0.15\rangle)$	$(\langle 0.36, 0.73\rangle, \langle 0.31, 0.63\rangle)$	$(\langle 0.79, 0.36\rangle, \langle 0.16, 0.56\rangle)$	$(\langle 0.36, 0.43\rangle, \langle 0.57, 0.31\rangle)$
k_2	$(\langle 0.56, 0.27\rangle, \langle 0.37, 0.41\rangle)$	$(\langle 0.56, 0.31\rangle, \langle 0.25, 0.61\rangle)$	$(\langle 0.16, 0.81\rangle, \langle 0.71, 0.28\rangle)$	$(\langle 0.27, 0.67\rangle, \langle 0.11, 0.32\rangle)$	$(\langle 0.37, 0.56\rangle, \langle 0.30, 0.67\rangle)$	$(\langle 0.37, 0.71\rangle, \langle 0.37, 0.21\rangle)$
k_3	$(\langle 0.32, 0.56\rangle, \langle 0.11, 0.81\rangle)$	$(\langle 0.23, 0.73\rangle, \langle 0.25, 0.75\rangle)$	$(\langle 0.79, 0.33\rangle, \langle 0.11, 0.21\rangle)$	$(\langle 0.19, 0.39\rangle, \langle 0.32, 0.56)$	$(\langle 0.71, 0.81\rangle, \langle 0.24, 0.76\rangle)$	$(\langle 0.56, 0.63\rangle, \langle 0.27, 0.32\rangle)$
k_4	$(\langle 0.16, 0.79\rangle, \langle 0.14, 0.76\rangle)$	$(\langle 0.45, 0.37\rangle, \langle 0.56, 0.37\rangle)$	$(\langle 0.52, 0.71\rangle, \langle 0.21, 0.56\rangle)$	$(\langle 0.27, 0.81\rangle, \langle 0.45, 0.35\rangle)$	$(\langle 0.56, 0.79\rangle, \langle 0.23, 0.51\rangle)$	$(\langle 0.71, 0.53\rangle, \langle 0.37, 0.21\rangle)$
k_5	$(\langle 0.81, 0.31\rangle, \langle 0.81, 0.13\rangle)$	$(\langle 0.71, 0.22\rangle, \langle 0.67, 0.29\rangle)$	$(\langle 0.32, 0.89\rangle, \langle 0.33, 0.56\rangle)$	$(\langle 0.69, 0.23\rangle, \langle 0.67, 0.13\rangle)$	$(\langle 0.93, 0.63\rangle, \langle 0.53, 0.21\rangle)$	$(\langle 0.83, 0.21\rangle, \langle 0.76, 0.13\rangle)$

5.3. Numerical example

Weather forecasts are vital to organize outdoor activities and detect potential weather predictions. Agriculture depends on accurate weather forecasting for planting, irrigation, and harvesting. Weather stations use various wind speed measuring instruments.

Meteorologists utilize a computer-generated prediction as a reference to create a weather forecast for a specific location. They integrate it with other information acquired from recent radar and satellite pictures. They also depend upon their grasp of weather processes. Accurate forecasts have the potential to save industries worldwide billions of dollars every year.

It involves reviewing data from multiple dynamic models and historical weather data to create an accurate decision. The weather is not stationary. It varies according to local conditions from day to day or hour to hour. There are four primary ingredients of weather: temperature, wind, humidity, and precipitation. When combined, these factors provide a description of the current weather.

Selecting appropriate weather forecasting options is essential to enhance the efficiency of weather forecasting. This procedure needs a comprehensive DM tool for the complexities and challenges of weather variables.

In this model, we consider five main types of weather $k_i (i = 1, 2, \cdots, 5)$, where $t_1 =$ sunny, $t_2 =$ windy, $t_3 =$ rainy, $t_4 =$ cloudy, and $t_4 =$ thunder and lighting, and six main weather conditions $\mathscr{C}_j (j = 1, 2, \cdots, 6)$ as criteria affecting the selection of most appropriate weather, where $\tilde{\mathscr{C}_1} =$ temperature, $\tilde{\mathscr{C}_2} =$ atmospheric pressure, $\tilde{\mathscr{C}_3} =$ wind, $\tilde{\mathscr{C}_4} =$ humidity, $\tilde{\mathscr{C}_5} =$ cloudiness, and $\tilde{\mathscr{C}_6} =$ precipitation.

Step 1: Consider $\wp = \{k_1, k_2, k_3, k_4, k_5\}$ and $\tilde{\mathfrak{C}} = \{\tilde{\mathscr{C}_1}, \tilde{\mathscr{C}_2}, \tilde{\mathscr{C}_3}, \tilde{\mathscr{C}_4}, \tilde{\mathscr{C}_5}, \tilde{\mathscr{C}_6}\}$. Construct the decision matrix $\mathscr{D} = [d_{ij}]_{5\times 6}$ by a DMr, which is displayed in Table 2, where the data values d_{ij} are assumed as LD-FNs $d_{ij} = \left(\left\langle f_{ij}, g_{ij}\right\rangle, \left\langle \alpha_{ij}, \beta_{ij}\right\rangle\right)$. In Table 2, f_{ij} represents “the degree of association of weather k_i w.r.t. the corresponding weather condition $\tilde{\mathscr{C}_j}$”, g_{ij} represents “the degree of non-affiliation of weather k_i w.r.t. the corresponding weather condition $\tilde{\mathscr{C}_j}$”, and α_{ij} means “highly effect on weather” and β_{ij} means “not highly effects on the weather”.

Step 2: Since there is no cost criteria, so the normalized decision matrix $\mathscr{N} = [\tilde{d}_{ij}]_{5\times 6}$ will remain same as in Table 2.

Table 3
Aggregated LD-FNs using LDFAAWA operator.

Alternatives	Aggregated LD-FNs
k_1	$(\langle 0.6955, 0.4957\rangle, \langle 0.6182, 0.2987\rangle)$
k_2	$(\langle 0.4609, 0.4009\rangle, \langle 0.5208, 0.3323\rangle)$
k_3	$(\langle 0.6416, 0.4651\rangle, \langle 0.2412, 0.3732\rangle)$
k_4	$(\langle 0.5388, 0.5554\rangle, \langle 0.4054, 0.3723\rangle)$
k_5	$(\langle 0.8206, 0.2872\rangle, \langle 0.7085, 0.1762\rangle)$

Table 4
SVs.

$\tilde{\mathfrak{S}}(k_i)$	
$\tilde{\mathfrak{S}}(k_1)$	0.2596
$\tilde{\mathfrak{S}}(k_2)$	0.1243
$\tilde{\mathfrak{S}}(k_3)$	0.0223
$\tilde{\mathfrak{S}}(k_4)$	0.0083
$\tilde{\mathfrak{S}}(k_5)$	0.5328

Table 5
Aggregated LD-FNs by LDFAAWG operator.

Alternatives	Aggregated LD-FNs
k_1	$(\langle 0.4700, 0.6833\rangle, \langle 0.3200, 0.4987\rangle)$
k_2	$(\langle 0.2923, 0.6764\rangle, \langle 0.2461, 0.5150\rangle)$
k_3	$(\langle 0.3123, 0.6651\rangle, \langle 0.1687, 0.7099\rangle)$
k_4	$(\langle 0.2958, 0.7434\rangle, \langle 0.2362, 0.6034\rangle)$
k_5	$(\langle 0.5149, 0.7256\rangle, \langle 0.5059, 0.38829\rangle)$

Step 3: According to Eq. (5.3), we can calculate the entropy values for each criteria $\tilde{\mathscr{C}}_j$ as follows:

$$\mathcal{E}_1(\tilde{\mathscr{C}}_1) = 0.525, \mathcal{E}_2(\tilde{\mathscr{C}}_2) = 0.644, \mathcal{E}_3(\tilde{\mathscr{C}}_3) = 0.576, \mathcal{E}_4(\tilde{\mathscr{C}}_4) = 0.633, \mathcal{E}_5(\tilde{\mathscr{C}}_5) = 0.665, \mathcal{E}_6(\tilde{\mathscr{C}}_6) = 0.674$$

Now, from Eq. (5.4), we can compute the weight of each criterion as follows:

$$\varpi_1 = \frac{1-0.525}{6-3.171} = 0.208,$$
$$\varpi_2 = \frac{1-0.644}{6-3.171} = 0.156,$$
$$\varpi_3 = \frac{1-0.576}{6-3.171} = 0.186,$$
$$\varpi_3 = \frac{1-0.633}{6-3.171} = 0.161,$$
$$\varpi_5 = \frac{1-0.665}{6-3.171} = 0.147,$$
$$\varpi_6 = \frac{1-0.674}{6-3.171} = 0.142.$$

Step 4: We use LDFAAWA operator to calculate the aggregated values exhibited in Table 2, with $\Lambda = 3$, and criteria weights are $\mathcal{W} = (0.208, 0.156, 0.186, 0.161, 0.147, 0.142)$. The aggregated LD-FNs for all alternatives k_i under criteria $\tilde{\mathscr{C}}_j$ are displayed in Table 3.

Step 5: Using Eq. (2.7), determine the SVs for all aggregated LD-FNs, which are listed in Table 4.

Step 6: In the view of SVs, we get the following ranking outcome of alternatives:

$$k_5 \geq k_1 \geq k_2 \geq k_3 \geq k_4.$$

Thus, k_5 is the most desirable alternative.

On the other hand, if we employ LDFAAWG operator (fixing $\Lambda = 3$) by using Eq. (4.9), the aggregated LD-FNs for all alternatives related criteria $\tilde{\mathscr{C}}_j$ are given in Table 5.

Applying Eq. (2.7), we obtain the SVs for all alternatives, which are tabulated in Table 6.

In the light of SVs, we get the following ranking result:

$$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3.$$

Table 6
SVs by LDFAAWG operator.

$\widetilde{\mathfrak{S}}(t_i)$	
$\widetilde{\mathfrak{S}}(k_1)$	−0.1960
$\widetilde{\mathfrak{S}}(k_2)$	−0.3265
$\widetilde{\mathfrak{S}}(k_3)$	−0.4470
$\widetilde{\mathfrak{S}}(k_4)$	−0.4074
$\widetilde{\mathfrak{S}}(k_5)$	−0.0439

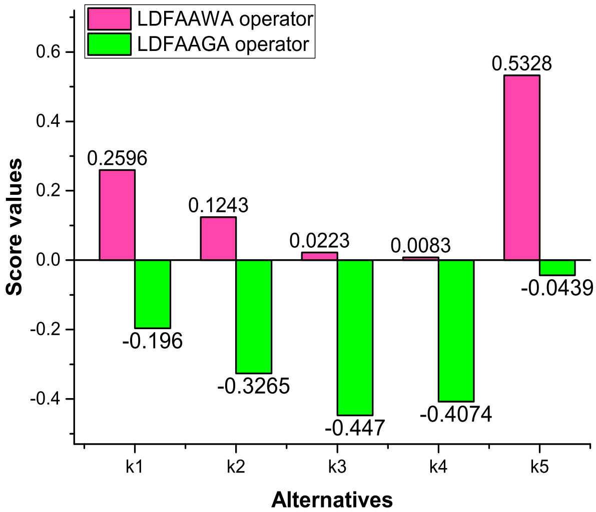


Fig. 3. Ranking order under LDFAAWA and LDFAAGA operators.

Hence, again, the candidate k_5 is the optimal choice.

We observe that the SVs and the ranking results of alternatives fluctuate when we implement two different AOs. However, the optimal choice k_5 is same for both the LDFAAWA and LDFAAWG operators. Moreover, the ranking results of each of the recommended LDFAAWA and the LDFAAWG operators are portrayed in Fig. 3.

6. Sensitivity analysis

The current section conducts a detailed sensitivity analysis of the invented AOs to confirm the validity and supremacy of our recommended approach and operators. Depending on the choice of DMr, multiple values might be allocated to the parameter Λ. To demonstrate the impact of the parameter Λ on the decision outcomes, we perform a comprehensive analysis using various inputs of Λ.

To showcase the effect of Λ on the ranking outcomes, we assign different inputs to Λ from 1 to 100 within our devised MCDM strategy. The resultant SVs and the associated ranking outcomes based on LDFHWA and LDFHWG operators related to these inputs of Λ have been tabulated in Tables 7 and 8. Additionally, the influence of the parameter Λ has been visually depicted in Fig. 4. Moreover, in light of Tables 7 and 8, it is clear that the optimal alternatives remain consistent for both LDFHWA and LDFHWG operators across different inputs of Λ, with K_5 as the optimal choice.

At the same time, from Tables 7 and 8, it has been revealed that as the values of Λ increase for the LDFHWA operator, the SVs of alternatives also increase. On the other hand, as the Λ values increase for LDFHWG operator, the SVs of alternatives decrease. However, the optimal choice remains consistent in both scenarios, revealing that the invented scheme always possessed the isotonicity property, permitting DMr to select the appropriate value according to his preferences.

From this discussion, it is concluded that a DMr can select the value of Λ depending on his behavior towards the decision procedure. For example, if the DMr has an optimistic behavior towards the decision, then he can pick the LDFHWG operator with the lower value of Λ. On the other hand, if he utilizes the LDFHWA operator to aggregate the decision process, then he can select the higher value of the parameter Λ. Similarly, if a DMr implements the LDFHWA operator to aggregation procedure to acquire the most pessimistic decision, he may assign the smaller value of Λ. This influence of Λ values on the decision outcomes makes our designed scheme more flexible. DMrs can choose the parameter Λ according to their preferences and requirements in practical situations.

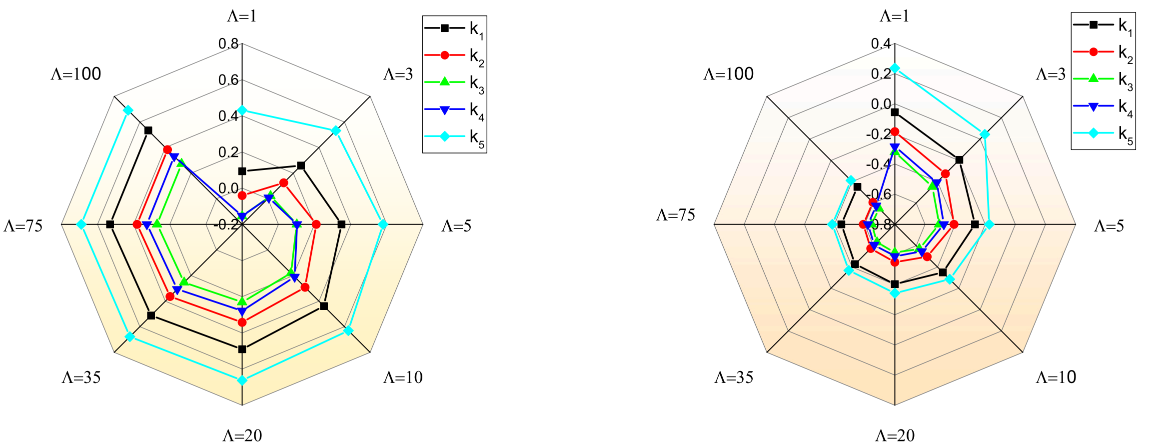
From the above discussion, we conclude that the proposed MCDM scheme under LDFAAWA and LDFAAWG operators have a high degree of stability across different values of Λ.

Table 7
Ranking results with various values of Λ under LDFAAWA operator.

Λ	Score values					Ranking result
	$\widetilde{\mathfrak{S}}(k_1)$	$\widetilde{\mathfrak{S}}(k_2)$	$\widetilde{\mathfrak{S}}(k_3)$	$\widetilde{\mathfrak{S}}(k_4)$	$\widetilde{\mathfrak{S}}(k_5)$	
$\Lambda = 1$	0.0934	−0.0404	−0.1544	−0.1537	0.4301	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 3$	0.2596	0.1243	0.0223	0.0083	0.5328	$k_5 \geq k_1 \geq k_2 \geq k_3 \geq k_4$
$\Lambda = 5$	0.3493	0.2091	0.1021	0.1031	0.5790	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 10$	0.4388	0.2922	0.1810	0.2111	0.6303	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 20$	0.4900	0.3412	0.2304	0.2773	0.6627	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 35$	0.5131	0.3640	0.2535	0.3064	0.6777	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 75$	0.5301	0.3805	0.2703	0.3270	0.6890	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 100$	0.5338	0.3842	0.2740	0.3315	0.6917	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$

Table 8
Ranking results with various values of Λ under LDFAAWG operator.

Λ	Score values					Ranking result
	$\widetilde{\mathfrak{S}}(k_1)$	$\widetilde{\mathfrak{S}}(k_2)$	$\widetilde{\mathfrak{S}}(k_3)$	$\widetilde{\mathfrak{S}}(k_4)$	$\widetilde{\mathfrak{S}}(k_5)$	
$\Lambda = 1$	−0.0564	−0.1855	−0.3203	−0.2841	0.2362	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 3$	−0.1960	−0.3265	−0.4470	−0.4074	0.0439	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 5$	−0.2686	−0.4070	−0.5065	−0.4749	−0.1737	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 10$	−0.3499	−0.4963	−0.5707	−0.5456	−0.2861	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 20$	−0.4029	−0.5497	−0.6123	−0.5876	−0.3434	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 35$	−0.4274	−0.5736	−0.6325	−0.6069	−0.3678	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 75$	−0.4449	−0.5904	−0.6472	−0.6216	−0.3850	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
$\Lambda = 100$	−0.4487	−0.5941	−0.6504	−0.6249	−0.3888	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$



(a) Impact of Λ under LDFAAWA operator (b) Impact of Λ under LDFAAWG operator

Fig. 4. Sensitivity analysis of LDFAAWA and LDFAAWG operators for various inputs of Λ.

7. Comparative analysis and discussion

In this segment, we undertake a comparative study using quantitative and qualitative aspects and specific prevailing schemes to highlight the efficacy and supremacy of the devised MCDM approach. The particular comparison process is as follows.

7.1. Quantitative comparison

To highlight the efficacy and validity of our deliberated scheme, in this subsection, we undertake an in-depth comparative study between our offered method and various prevailing methods, including Pythagorean fuzzy AA-AOs [62], picture fuzzy AA-AOs [63], q-ROF AA-AOs [65], p, q-QOFSs AA-AOs [69], LD-FSs [6], LDFWA operator [48], LDFWG operator [48], LDF-TOPSIS [77], LDFEWA operator [49] and LDFEWG operator [49]. The SVs and the ranking results of the alternatives using the established approach and the prevailing techniques are exhibited in Table 9. Fig. 5 demonstrates a graphical depiction of the SVs and the ranking of alternatives using various methods.

Table 9
Ranking results based on different methods.

Methods	Score values					Ranking order
	$\widetilde{\mathfrak{S}}(k_1)$	$\widetilde{\mathfrak{S}}(k_2)$	$\widetilde{\mathfrak{S}}(k_3)$	$\widetilde{\mathfrak{S}}(k_4)$	$\widetilde{\mathfrak{S}}(k_5)$	
Senapati et al. [62] (Pythagorean fuzzy AA-AOs)	⋯	⋯	⋯	⋯	⋯	Unable to determine
Hussain et al. [63] (picture fuzzy AA-AOs)	⋯	⋯	⋯	⋯	⋯	Unable to determine
Senapati et al. [65] (q-ROF AA-AOs)	⋯	⋯	⋯	⋯	⋯	Unable to determine
Ali and Naeem [69] (p, q-QOFSs AA-AOs)	⋯	⋯	⋯	⋯	⋯	Unable to determine
Riaz and Hashmi [6] (using QSF)	0.0273	0.0273	0.1613	0.1105	0.3144	$k_5 \geq k_3 \geq k_4 \geq k_1 \approx k_2$
Riaz and Hashmi [6] (using ESF)	0.5788	0.5788	0.6941	0.6662	0.7800	$k_5 \geq k_3 \geq k_4 \geq k_1 \approx k_2$
Riaz et al. [48] (LDFWA operator)	0.1868	−0.0808	−0.3087	−0.3072	0.8625	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
Riaz et al. [48] (LDFWG operator)	−0.1128	−0.3710	−0.6405	−0.5682	0.4724	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$
Gül and Aydougdu [77] (LDF-TOPSIS)	0.4890	0.3882	0.3423	0.2819	0.6330	$k_5 \geq k_1 \geq k_2 \geq k_3 \geq k_4$
Iampan et al. [49] (LDFEWA operator)	−0.0202	0.0775	−0.1579	0.0371	0.4355	$k_5 \geq k_2 \geq k_4 \geq k_1 \geq k_3$
Iampan et al. [49] (LDFEWG operator)	−0.2351	−0.1082	−0.3879	−0.1899	0.1950	$k_5 \geq k_2 \geq k_4 \geq k_1 \geq k_3$
LDFAAWA operator (Proposed)	0.2596	0.1243	0.0223	0.0083	0.5328	$k_5 \geq k_1 \geq k_2 \geq k_3 \geq k_4$
LDFAAWG operator (Proposed)	−0.1960	−0.3265	−0.4470	−0.4074	−0.0439	$k_5 \geq k_1 \geq k_2 \geq k_4 \geq k_3$

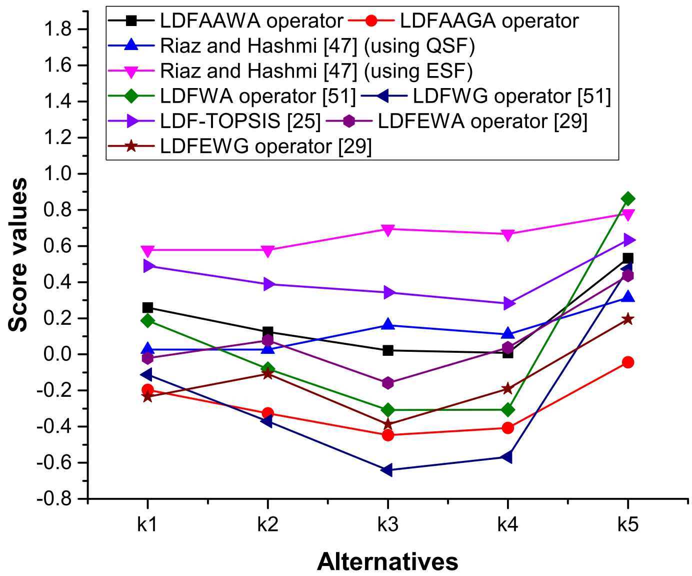


Fig. 5. Comparison with existing approaches.

In the light of the data displayed in Table 9 revealed that the ranking order of the alternatives displays some fluctuations. Nonetheless, it is notable that k_5 maintains its position as an optimal choice. Therefore, according to the above discussion, the results acquired by the designed method are highly reliable.

7.2. Qualitative comparison

This subsection is devoted to the characteristics comparison of the suggested methodology and some existing studies, and the comparison results are exhibited in Table 10. We perform a qualitative comparison from five aspects: handle MD, handle NMD, Aczel-Alsina operation' flexibility, lower computational complexity, and entropy weights characteristics to legitimate its supremacy. According to Table 10, it becomes evident that the projected method has all listed features, but the mentioned methods do not have all of them.

7.3. Advantages

The expounded technique offers several benefits, which are summarized as follows:

1. The methods presented in [69,71,72,38,39,41,59,70] are incapable of addressing linear Diophantine fuzzy information while our established AOs can logically accommodate linear Diophantine fuzzy information.
2. Including parameter Λ in the expounded AOs offers DMrs a better level of flexibility. They can fix this parameter based on the demands of real-life dilemmas.

Table 10
Characteristics comparison of different approaches.

Methods	Characteristics				
	Handle MD	Handle NMD	Aczel-Alsina operations' felxibility	Lower computational complexity	Use entropy to calculate criteria weights
Merigo and Casanovas [78]	✓	✗	✗	✓	✗
Riaz and Hashmi [6]	✓	✓	✗	✓	✗
Riaz et al. [48]	✓	✓	✗	✓	✗
Riaz et al. [52]	✓	✓	✗	✗	✗
Iampan et al. [49]	✓	✓	✗	✗	✗
Ayub et al. [24]	✓	✓	✗	✗	✗
Farid et al. [79]	✓	✓	✗	✗	✗
Zhang et al. [70]	✓	✓	✓	✗	✗
Proposed method	✓	✓	✓	✓	✓

3. The symmetry of the invented AOs regarding the parameter Λ certifies that the ranking order of alternatives remains comparatively consistent for various parameter inputs. This objective is inevitable in the DM process, as it prevents outcomes from being influenced by DMrs' pessimistic or optimistic behavior.
4. The expounded AOs obey the characteristic of isotonicity. The values of LDFAAWA and LDFAAWG operators rise (drop) moronically with an increase (decrease) of the parameter Λ input, allowing DMrs to allocate the appropriate value according to their risk preferences. If the DMr is risk preference, he might use the parameter value as small as practically possible; if the DMr is risk aversion, he may select the parameter value as large as reasonably attainable in the situation of the LDFAAWA operator and, conversely, for the LDFAAWG operator. As a result, the DMr can assign suitable parametric input according to risk tolerance and actual circumstances.
5. The existing studies of LD-FSs [49,6,48] cannot deal with MCDM scenarios with unknown criteria weight information. In contrast, the presented scheme provides smooth aggregation by using the concept of entropy measure to determine criteria weights, which reduces the loss of information during the decision-making procedure. As a result, the suggested approach is practical for handling linear Diophantine fuzzy information in MCDM issues.

7.4. Limitations

Although the developed scheme has numerous merits, it is also essential to acknowledge its shortcomings:

1. Even though the parametric structure of the devised scheme delivers flexibility, it also introduces the necessity to allocate appropriate input of Λ. The sensitivity of this parameter might be a drawback, as choosing an inappropriate value might yield less reliable or biased results.
2. The devised approach neglects the interconnections among the criteria. By considering this, the DMrs may feel more confident in their assessments.

8. Concluding remarks and future research plans

The LD-FS approach is a remarkable generalization of the IFSs, PyFSs, and q-ROFSs to circumvent ambiguity and uncertainty in complex decision-making paradigms. It expands the space of MD and NMD by adding RPs. Moreover, AA-AOs are more flexible and potent in many issues involving uncertainty, as they incorporate a controlling parameter that can play a vital role in managing extreme values. Having a generic nature, both LD-FSs and AA-AOs require further study to be conducted. This article establishes some innovative operational laws in the context of LD-FSs using Aczel-Alsina TN and TCN and examines their related features. Based on these operational rules, we developed two novel AOs, namely LDFAAWA and LDFAAWG operators. These operators offer a parametric mechanism that permits customized aggregation procedures to tackle the choices of DMrs. Moreover, this study investigates many interesting cardinal characteristics of the devised AOs. Besides, we utilized these AOs to construct a realistic MCDM technique within the LD-FS setting, which is more generic and flexible. To authenticate the practicability of the interpreted scheme, we provide a real-life case study. Further, by performing a detailed sensitivity analysis, we have examined the role of parameter Λ on the decision results. Finally, we show the supremacy and effectiveness of the developed approach through a comparison analysis between the proposed method and some prevailing schemes.

The recommended work exhibits a broad spectrum of potential applications. In the future, we plan to employ our proposed research in the following domains:

1. To enhance the application spectrum of our recommended approach, our future research plans include expanding the invented AOs to the Archimedean norm and defining other AOs, such as the Bonferroni mean operator and power Maclaurin symmetric mean operator.
2. The devised work can be extended in the framework of CLD-FS [29], IVLD-FSs [53,54], spherical LD-FSs [30], q-LD-FSs [56], and (p, q)-Rung LD-FSs [57].

3. In the near future, our work will integrate with various MCDM approaches such as TOPSIS, VIKOR, MARCOS, COPRAS, ELECTRE, CODAS, TODIM method, etc.
4. The current work can be extended to hybrid theories involving the generalizations of soft set theories.
5. We can also develop other AOs like those based on Yager's and Dombi's norms.

CRediT authorship contribution statement

Rizwan Gul: Writing – original draft, Validation, Investigation, Formal analysis, Conceptualization. **Tareq M. Al-shami:** Writing – original draft, Validation, Investigation, Formal analysis, Conceptualization. **Saba Ayub:** Validation, Investigation, Formal analysis. **Muhammad Shabir:** Validation, Investigation, Formal analysis. **M. Hosny:** Validation, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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