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Large gap asymptotics for the generating function of the sine point process

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Abstract

We consider the generating function of the sine point process on m consecutive intervals. It can be written as a Fredholm determinant with discontinuities, or equivalently as the convergent series

$$\sum_{k_1,\ldots,k_m\geqslant 0} \mathbb{P}\left(\bigcap_{j=1}^{m} \#\{\text{points in the } j\text{th interval}\} = k_j\right)\prod_{j=1}^{m} s_j^{k_j},$$

where $s_1,\ldots,s_m \in [0,+\infty)$. In particular, we can deduce from it joint probabilities of the counting function of the process. In this work, we obtain large gap asymptotics for the generating function, which are asymptotics as the size of the intervals grows. Our results are valid for an arbitrary integer m, in the cases where all the parameters $s_1,\ldots,s_m$, except possibly one, are positive. This generalizes two known results: (1) a result of Basor and Widom, which corresponds to $m = 1$ and $s_1 > 0$, and (2) the case $m = 1$ and $s_1 = 0$ for which many authors have contributed. We also present some applications in the context of thinning and conditioning of the sine process.

1. *Introduction*

Let

$$m \in \mathbb{N}\setminus\{0\}, \quad \vec{s} = (s_1,\ldots,s_m) \in [0,+\infty)^m \quad \text{and} \quad \vec{x} = (x_0,x_1,\ldots,x_m) \in \mathbb{R}^{m+1}$$

with $\vec{x}$ such that $-\infty < x_0 < x_1 < \ldots < x_m < +\infty$, and consider the Fredholm determinant

$$F(\vec{x},\vec{s}) = \det\left(1 - \sum_{k=1}^{m}(1-s_k)\mathcal{K}|_{(x_{k-1},x_k)}\right) \tag{1.1}$$

where, for a given bounded Borel set $A \subset \mathbb{R}$, $\mathcal{K}|_A$ is the (trace class) integral operator acting on $L^2(A)$ whose kernel is given by

$$K(x,y) = \frac{\sin(x-y)}{\pi(x-y)}. \tag{1.2}$$

In this paper, we obtain asymptotics for $F(r\vec{x},\vec{s})$ as $r \to +\infty$, up to and including the term of order 1, in the cases where all the parameters $s_1,\ldots,s_m$, except possibly one, are positive. $F(\vec{x},\vec{s})$ is the generating function of the well-known sine point process of random matrices and has attracted considerable attention over the years. We discuss some background on the sine process and give more motivation for the study of F in Subsection 1.2. In particular, we show

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that if $s_1, \dots, s_m \in [0,1]$, then the asymptotics of $F(r\vec{x}, \vec{s})$ as $r \to +\infty$ can be interpreted as large gap asymptotics.

Before stating our main theorems, we first briefly review the known asymptotic results available in the literature.

Large r asymptotics for $F(r\vec{x}, \vec{s})$ when $m = 1$ are already completely understood. We need to distinguish two regimes: (1) the case $s_1 = 0$ and (2) the case $s_1 \in (0, +\infty)$. For the first case $s_1 = 0$, the asymptotics are given by

$$F((rx_0, rx_1), 0) = \exp\left(-\frac{r^2(x_1 - x_0)^2}{8} - \frac{1}{4}\log(r(x_1 - x_0)) + \frac{1}{3}\log 2 + 3\zeta'(-1) + \mathcal{O}(r^{-1})\right) \tag{1.3}$$

as $r \to +\infty$, where ζ is Riemann's zeta-function. This result was first conjectured by Dyson in [**28**], then proved simultaneously and independently by Ehrhardt and Krasovsky in [**29, 42**], and then by Deift *et al.* in [**22**]. On the other hand, for the second case where $s_1 = e^{u_1} \in (0, +\infty)$, we have

$$\begin{aligned} F((rx_0, rx_1), e^{u_1}) = \exp\bigg(&\frac{ru_1(x_1 - x_0)}{\pi} + \frac{u_1^2}{2\pi^2}\log(2r(x_1 - x_0)) \\ &+2\log G\Big(1 + \frac{u_1}{2\pi i}\Big)G\Big(1 - \frac{u_1}{2\pi i}\Big) + \mathcal{O}(r^{-1})\bigg), \end{aligned} \tag{1.4}$$

as $r \to +\infty$, where G is Barnes' G-function (see, for example, [**47**, equation 5.17.2] for a definition). This result was first proved by Basor and Widom in [**2**], and then independently by Budylin and Buslaev in [**12**]. Note that the leading term for $\log F$ is of order r^2 in (1.3), while it is of order r in (1.4), and that if we naively take $u_1 \to -\infty$ (or equivalently $s_1 \to 0$) in (1.4), we do not recover (1.3). This explains heuristically why these two cases cannot be treated both at once. In fact, a critical transition takes place as $r \to +\infty$ and simultaneously $s_1 \to 0$. This transition is quite technical and is described in terms of elliptic θ-function in a series of papers by Bothner, Deift, Its and Krasovsky [**8–10**].

Less is known for $m \geqslant 2$. In [**54**], Widom has tackled the problem of finding large r asymptotics for $F(r\vec{x}, \vec{s})$ in the case where m is odd and $\vec{s} = (0, 1, 0, 1, \dots, 0, 1, 0)$. He obtained

$$\partial_r \log F(r\vec{x}, (0, 1, \dots, 1, 0)) = c_1 r + c_2(r) + o(1), \quad \text{as } r \to +\infty, \tag{1.5}$$

where c_1 is independent of r and is explicitly computable, and the function $c_2(r)$ is a bounded oscillatory function of r that requires the solution of a Jacobi inversion problem. These asymptotics were subsequently refined in [**23**], where the oscillations are described in terms of elliptic θ-function. Note that (1.5) is an asymptotic formula for the log derivative of F, which leads after integration to an asymptotic formula for $\log F(r\vec{x}, (0, 1, \dots, 1, 0))$. However, with this method, the constant of integration (the term of order 1 in the large r asymptotics) remains unknown. Using a different method, Fahs and Krasovsky in [**32, 33**] have recently obtained this constant for the case $m = 3$ and $\vec{s} = (0, 1, 0)$.

Until now, no results were available in the literature on large r asymptotics of $F(r\vec{x}, \vec{s})$ when $m \geqslant 2$ and several functions s_j are in the open intervals $(0, 1) \cup (1, +\infty)$.

The aim of this paper is to contribute to these developments on large r asymptotics of $F(r\vec{x}, \vec{s})$. We obtain our results for an arbitrary integer m, in the cases where all the parameters $s_1, \dots, s_m$, except possibly one, are positive. We distinguish two cases: in Theorem 1.1, we obtain large r asymptotics for $F(r\vec{x}, \vec{s})$ with $s_1, \dots, s_m \in (0, +\infty)$, and in Theorem 1.2, we obtain asymptotics for $F(r\vec{x}, \vec{s})$ with $s_p = 0$ and $s_1, \dots, s_{p-1}, s_{p+1}, \dots, s_m \in (0, +\infty)$ (for an arbitrary $p \in \{1, \dots, m\}$). Theorem 1.1 generalizes the result (1.4), while Theorem 1.2 generalizes (1.3). We describe several applications of our results in Subsection 1.2.

1.1. *Main results*

THEOREM 1.1. *Let*

$$m \in \mathbb{N}_{>0}, \quad \vec{s} = (s_1, \dots, s_m) \in (0, +\infty)^m, \quad \vec{x} = (x_0, \dots, x_m) \in \mathbb{R}^{m+1}$$

be such that $x_0 < x_1 < x_2 < \dots < x_m$. *As* $r \to +\infty$, *we have*

$$F(r\vec{x}, \vec{s}) = \exp\left\{ \sum_{j=1}^{m} \frac{u_j}{\pi}(x_j - x_0)r + \sum_{j=1}^{m} \frac{u_j^2}{2\pi^2} \log\left(2r(x_j - x_0)\right) \right.$$
$$+ \sum_{1 \leqslant j < k \leqslant m} \frac{u_j u_k}{2\pi^2} \log\left(\frac{2r(x_j - x_0)(x_k - x_0)}{x_k - x_j}\right) + \sum_{j=1}^{m} \log\left(G\Big(1 + \frac{u_j}{2\pi i}\Big)G\Big(1 - \frac{u_j}{2\pi i}\Big)\right)$$
$$\left. + \log\left(G\left(1 + \sum_{j=1}^{m} \frac{u_j}{2\pi i}\right) G\left(1 - \sum_{j=1}^{m} \frac{u_j}{2\pi i}\right)\right) + \mathcal{O}\Big(\frac{\log r}{r}\Big) \right\} \tag{1.6}$$

where G *is Barnes' G-function, and*

$$u_j = \log \frac{s_j}{s_{j+1}} \quad \textit{for} \quad j = 1, \dots, m, \tag{1.7}$$

with $s_{m+1} := 1$. *Furthermore, the error term in* (1.6) *is uniform in* $s_1, \dots, s_m$ *in compact subsets of* $(0, +\infty)$ *(or equivalently uniform in* $u_1, \dots, u_m$ *in compact subsets of* $\mathbb{R}$*) and uniform in* $x_0, \dots, x_m$ *in compact subsets of* $\mathbb{R}$, *as long as there exists* $\delta > 0$ *independent of* r *such that*

$$\min_{0 \leqslant j < k \leqslant m} x_k - x_j \geqslant \delta. \tag{1.8}$$

Alternatively, one can rewrite (1.6) *as follows:*

$$F(r\vec{x}, \vec{s}) = \exp\left\{ \sum_{j=1}^{m} u_j \mu_j(r) + \sum_{j=1}^{m} \frac{u_j^2}{2} \sigma_j^2(r) + \sum_{1 \leqslant j < k \leqslant m} u_j u_k \Sigma_{j,k}(r) \right.$$
$$+ \log\left(G\left(1 + \sum_{j=1}^{m} \frac{u_j}{2\pi i}\right) G\left(1 - \sum_{j=1}^{m} \frac{u_j}{2\pi i}\right)\right)$$
$$\left. + \sum_{j=1}^{m} \log\left(G\Big(1 + \frac{u_j}{2\pi i}\Big)G\Big(1 - \frac{u_j}{2\pi i}\Big)\right) + \mathcal{O}\Big(\frac{\log r}{r}\Big) \right\},$$

where μ_j, σ_j^2 *and* $\Sigma_{j,k}$ *are given by*

$$\mu_j(r) = \frac{r(x_j - x_0)}{\pi}, \tag{1.9}$$

$$\sigma_j^2(r) = \frac{\log(2r(x_j - x_0))}{\pi^2}, \tag{1.10}$$

$$\Sigma_{j,k}(r) = \frac{1}{2\pi^2} \log\left(\frac{2r(x_j - x_0)(x_k - x_0)}{|x_k - x_j|}\right). \tag{1.11}$$

THEOREM 1.2. *Let* $m \in \mathbb{N}_{>0}$, $p \in \{1, \dots, m\}$ *and*

$$s_p = 0, \quad (s_1, \dots, s_{p-1}, s_{p+1}, \dots, s_m) \in (0, +\infty)^{m-1}, \quad \vec{x} = (x_0, \dots, x_m) \in \mathbb{R}^{m+1}$$

be such that $x_0 < x_1 < x_2 < \ldots < x_m$, *and define* $\vec{s} = (s_1, \ldots, s_m)$. *As* $r \to +\infty$, *we have*

$$\begin{aligned} F(r\vec{x}, \vec{s}) = \exp\Bigg\{ & -\frac{r^2(x_p - x_{p-1})^2}{8} \\ & - \left(\sum_{j=0}^{p-2} \frac{u_j}{\pi} \sqrt{x_p - x_j}\sqrt{x_{p-1} - x_j} - \sum_{j=p+1}^{m} \frac{u_j}{\pi} \sqrt{x_j - x_p}\sqrt{x_j - x_{p-1}} \right) r \\ & + \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \frac{u_j^2}{4\pi^2} \log\left(\frac{4\sqrt{|x_j - x_p|\,|x_j - x_{p-1}|}|2x_j - x_p - x_{p-1}|r}{x_p - x_{p-1}} \right) \\ & - \frac{1}{4}\log\left(r(x_p - x_{p-1})\right) \\ & + \sum_{\substack{0 \leqslant j < k \leqslant m \\ j,k \neq p-1,p}} \frac{u_j u_k}{2\pi^2} \log\left(\frac{\sqrt{|x_k - x_p|}\sqrt{|x_j - x_{p-1}|} + \sqrt{|x_k - x_{p-1}|}\sqrt{|x_j - x_p|}}{\left|\sqrt{|x_k - x_p|}\sqrt{|x_j - x_{p-1}|} - \sqrt{|x_k - x_{p-1}|}\sqrt{|x_j - x_p|}\right|} \right) \\ & + \frac{1}{3}\log 2 + 3\zeta'(-1) + \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \log\left(G\Big(1 + \frac{u_j}{2\pi i}\Big) G\Big(1 - \frac{u_j}{2\pi i}\Big) \right) + \mathcal{O}\Big(\frac{\log r}{r}\Big) \Bigg\}, \end{aligned} \tag{1.12}$$

where G *is Barnes' G-function,* ζ *is Riemann's zeta-function, and* $u_0, \ldots, u_{p-2}, u_{p+1}, \ldots, u_m$ *are given by*

$$u_j = \log \frac{s_j}{s_{j+1}}, \quad j \in \{0, \ldots, m\} \setminus \{p-1, p\}, \tag{1.13}$$

where $s_0 := 1$, $s_{m+1} := 1$. *Furthermore, the error term in* (1.12) *is uniform in* $s_1, \ldots, s_{p-1}, s_{p+1}, \ldots, s_m$ *in compact subsets of* $(0, +\infty)$ *(or equivalently uniform in* $u_0, \ldots, u_{p-2}, u_{p+1}, \ldots, u_m$ *in compact subsets of* $\mathbb{R}$*) and uniform in* $x_0, \ldots, x_m$ *in compact subsets of* $\mathbb{R}$, *as long as there exists* $\delta > 0$ *independent of* r *such that* (1.8) *holds.*

Alternatively, one can rewrite (1.12) *as follows:*

$$\begin{aligned} F(r\vec{x}, \vec{s}) = F((rx_{p-1}, rx_p), 0) \exp\Bigg\{ & -\left(\sum_{j=0}^{p-2} u_j \hat{\mu}_j(r) - \sum_{j=p+1}^{m} u_j \hat{\mu}_j(r) \right) + \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \frac{u_j^2}{2} \hat{\sigma}_j^2(r) \\ & + \sum_{\substack{0 \leqslant j < k \leqslant m \\ j,k \neq p-1,p}} u_j u_k \hat{\Sigma}_{j,k} + \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \log\left(G\Big(1 + \frac{u_j}{2\pi i}\Big) G\Big(1 - \frac{u_j}{2\pi i}\Big) \right) + \mathcal{O}\Big(\frac{\log r}{r}\Big) \Bigg\} \end{aligned}$$

where the large r *asymptotics of* $F((rx_{p-1}, rx_p), 0)$ *are given by* (1.3), *and* $\hat{\mu}_j$, $\hat{\sigma}_j^2$ *and* $\hat{\Sigma}_{j,k}$ *are given by*

$$\hat{\mu}_j(r) = \frac{r}{\pi} \sqrt{|x_p - x_j|}\sqrt{|x_{p-1} - x_j|}, \tag{1.14}$$

$$\hat{\sigma}_j^2(r) = \frac{1}{2\pi^2} \log\left(\frac{4\sqrt{|x_j - x_p|\,|x_j - x_{p-1}|}|2x_j - x_p - x_{p-1}|r}{x_p - x_{p-1}} \right), \tag{1.15}$$

$$\hat{\Sigma}_{j,k} = \frac{1}{2\pi^2} \log\left(\frac{\sqrt{|x_k - x_p|}\sqrt{|x_j - x_{p-1}|} + \sqrt{|x_k - x_{p-1}|}\sqrt{|x_j - x_p|}}{\left|\sqrt{|x_k - x_p|}\sqrt{|x_j - x_{p-1}|} - \sqrt{|x_k - x_{p-1}|}\sqrt{|x_j - x_p|}\right|} \right). \tag{1.16}$$

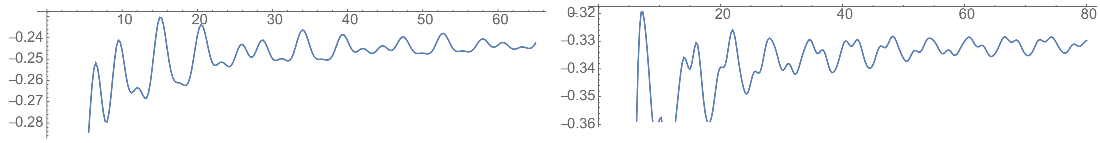

FIGURE 1 (colour online). *Numerical confirmations of Theorem* 1.1.

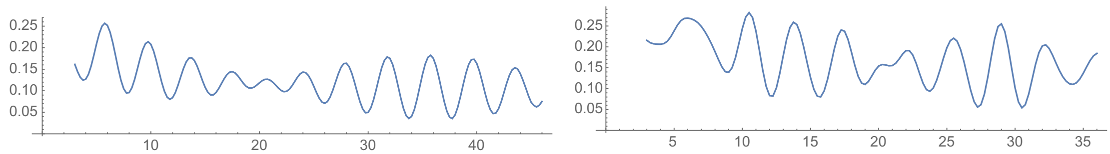

FIGURE 2 (colour online). *Numerical confirmations of Theorem* 1.2.

Numerical confirmations of Theorems 1.1 *and* 1.2. Recent progress of Bornemann [**6**] on the numerical evaluation of Fredholm determinants have allowed us to verify Theorems 1.1 and 1.2 for several choices of the parameters. Let $\mathcal{F}_1(r\vec{x},\vec{s})$ denote the right-hand side of (1.6) without the error term. Figure 1 represents the graph of the function

$$r \mapsto r(\log F(r\vec{x},\vec{s}) - \log \mathcal{F}_1(r\vec{x},\vec{s})) \tag{1.17}$$

for the following two choices of the parameters:

Left: $m=2,\ \ x_0=0,\ x_1=0.7,\ x_2=1.2,\qquad u_1=-1.1,\ u_2=-2.4,$

Right: $m=3,\ \ x_0=0,\ x_1=0.5,\ x_2=1.1,\ x_3=1.7,\ \ u_1=-0.8,\ u_2=-1.8,\ u_3=-1.32.$

Similarly, let $\mathcal{F}_2(r\vec{x},\vec{s})$ denote the right-hand side of (1.12) without the error term. Figure 2 represents the graph of the function

$$r \mapsto r(\log F(r\vec{x},\vec{s}) - \log \mathcal{F}_2(r\vec{x},\vec{s})) \tag{1.18}$$

for the following two cases:

Left: $m=3,\ p=2,\qquad x_0=0,\ x_1=0.5,\ x_2=1.1,\ x_3=1.7,$
$\qquad u_0=0.8,\ u_3=-1.32,$

Right: $m=4,\ p=3,\qquad x_0=0,\ x_1=0.5,\ x_2=1.1,\ x_3=1.7,\ x_4=2.5,$
$\qquad u_0=0.8,\ u_1=1.8,\ u_4=-1.87.$

We see in Figures 1 and 2 that the functions (1.17) and (1.18) seem to remain bounded as $r\to+\infty$. These observations are consistent with Theorems 1.1 and 1.2. In fact, Figures 1 and 2 also suggest that the error terms in Theorems 1.1 and 1.2 could be reduced from $\mathcal{O}(\frac{\log r}{r})$ to $\mathcal{O}(\frac{1}{r})$.

1.2. *Background and applications of Theorems* 1.1 *and* 1.2

The sine point process lies at the heart of random matrix theory. It has attracted a lot of attention since the seminal work of Dyson [**27**], who first proved that this process describes the local eigenvalue statistics in the bulk of the spectrum of large random Hermitian matrices taken from the Gaussian Unitary Ensemble. Dyson also conjectured that this same process also describes the bulk local eigenvalue statistics for a wide class of large random matrices. There has been much progress on this conjecture, which has now been rigorously proved for many random matrix models, see, for example, [**3, 24, 31, 40, 49, 50, 52**]. We refer to [**30, 43, 48, 53**] for recent surveys of known appearances of the sine process in random matrix theory.

The Fredholm determinant $F(\vec{x}, \vec{s})$ is a central object in the study of the sine process. For the convenience of the reader, we first briefly recall the definition of a point process, following the classical references [**7, 41, 51**].

A point process on $\mathbb{R}$ is a probability measure over the space $\{X\}$ of all locally finite point configurations on $\mathbb{R}$. In general, the process can be well understood via the study of its k-point correlation functions $\{\rho_k : \mathbb{R}^k \to [0, +\infty)\}_{k\geqslant 1}$ which are defined such that

$$\mathbb{E}\left[\sum_{\substack{\xi_1,\ldots,\xi_k\in X\\ \xi_i\neq\xi_j \text{ if } i\neq j}} f(\xi_1,\ldots,\xi_k)\right] = \int_{\mathbb{R}^k} f(u_1,\ldots,u_k)\rho_k(u_1,\ldots,u_k)du_1\ldots du_k \tag{1.19}$$

holds for any measurable symmetric function $f : \mathbb{R}^k \to \mathbb{R}$ with compact support. The sum at the left-hand side of (1.19) is taken over all (ordered) k-tuples of distinct points of the random point configuration X.

A point process on $\mathbb{R}$ is *determinantal* if all its correlation functions $\{\rho_k\}_{k\geqslant 1}$ exist and can be expressed as determinants involving a kernel $\mathbb{K} : \mathbb{R}^2 \to \mathbb{R}$ as follows

$$\rho_k(u_1,\ldots,u_k) = \det(\mathbb{K}(u_i,u_j))_{i,j=1}^k, \quad \text{for all } k\geqslant 1 \text{ and for all } u_1,\ldots,u_k\in\mathbb{R}. \tag{1.20}$$

The sine process is determinantal and corresponds to the case $\mathbb{K} = K$, where the sine kernel $K : \mathbb{R}^2 \to \mathbb{R}$ is defined in (1.2). In a determinantal point process, all quantities of interest can be expressed in terms of the kernel. For example, it is directly seen from (1.19) and (1.20) that the probability that a random point configuration X (distributed according to the sine point process) contains no points on a given bounded Borel set $A \subset \mathbb{R}$ is equal to

$$\mathbb{P}[X\cap A=\emptyset] = \mathbb{E}\left[\prod_{\xi\in X}(1-\chi_A(\xi))\right] = 1 + \sum_{k=1}^{+\infty}\frac{(-1)^k}{k!}\int_{A^k}\det(K(u_i,u_j))_{i,j=1}^k du_1\ldots du_k, \tag{1.21}$$

where $\chi_A(\xi) = 1$ if $\xi\in A$ and $\chi_A(\xi) = 0$ otherwise. Note that the right-hand side of (1.21) is, by definition, equal to the Fredholm determinant $\det(1-\mathcal{K}|_A)$. In the sine process, the expected number of points that fall in A can be computed explicitly using (1.2):

$$\mathbb{E}[\#(X\cap A)] = \mathbb{E}\left[\sum_{\xi\in X}\chi_A(\xi)\right] = \int_A K(x,x)dx = \frac{|A|}{\pi}, \tag{1.22}$$

where $|A|$ is the Lebesgue measure of A. We refer the reader to [**7, 41, 51**] for more discussions on the algebraic and probabilistic properties of determinantal point processes.

Taking $A = (x_0, x_1)$ in (1.21), we see that the probability to observe a gap in the sine process on (x_0, x_1) can be expressed in terms of F (defined in (1.1)) by

$$F((x_0,x_1),0) = \det(1-\mathcal{K}|_{(x_0,x_1)}) = \mathbb{P}[X\cap(x_0,x_1)=\emptyset]. \tag{1.23}$$

The large gap asymptotics on a single interval in the sine process are given by (1.3).

More generally, taking m odd and $A = (x_0,x_1)\cup(x_2,x_3)\cup\ldots\cup(x_{m-1},x_m)$, we infer from (1.21) and (1.1) that the probability to find no points on $\frac{m+1}{2}$ disjoint intervals is given by

$$F(\vec{x},(0,1,0,1,0,\ldots,1,0)) = \mathbb{P}[X\cap((x_0,x_1)\cup(x_2,x_3)\cup\ldots\cup(x_{m-1},x_m))=\emptyset], \tag{1.24}$$

where $s_j = 0$ if j is odd and $s_j = 1$ otherwise. The known results on the asymptotics of (1.24) as the size of the intervals gets large have been discussed below (1.5).

We now discuss the meaning of $F(\vec{x},\vec{s})$ in terms of probabilities for general values of $s_1,\ldots,s_m$. As before, let X be a random point configuration distributed according to the sine process. Given a Borel set A, we define $N_A = \#(X\cap A)$. In other words, N_A is the random variable that counts the number of points in X that falls in A; N_A is also called the counting function on A. It is known [**51**, Theorem 2] that $F(\vec{x},\vec{s})$ is an entire function of $s_1,\ldots,s_m$ which can be rewritten as follows

$$F(\vec{x},\vec{s}) = \mathbb{E}\left[\prod_{j=1}^{m} s_j^{N_{(x_{j-1},x_j)}}\right] = \sum_{k_1,\ldots,k_m\geqslant 0} \mathbb{P}\left(\bigcap_{j=1}^{m} N_{(x_{j-1},x_j)} = k_j\right)\prod_{j=1}^{m} s_j^{k_j}. \tag{1.25}$$

The above expression motivates why F is called the generating function of the sine point process; it shows in particular that we can deduce a lot of information from F. Any quantity of the form 0^0 in (1.25) should be interpreted as being equal to 1. More precisely,

$$s_j^{N_{(x_{j-1},x_j)}} = 1 \text{ if } s_j = 0 \text{ and } N_{(x_{j-1},x_j)} = 0, \quad \text{ and } \quad s_j^{k_j} = 1 \text{ if } s_j = 0 \text{ and } k_j = 0.$$

For example, for an odd integer m and for $\vec{s}$ such that $s_j = 0$ if j is odd and $s_j = 1$ if j is even, we get from (1.25) that

$$F(\vec{x},(0,1,0,\ldots,1,0)) = \mathbb{P}\big(N_{(x_0,x_1)} = 0 \cap N_{(x_2,x_3)} = 0 \cap \ldots \cap N_{(x_{m-1},x_m)} = 0\big),$$

which is equivalent to (1.24), as it must.

We mention that there is a well-known connection between F and the theory of Painlevé equations. Using monodromy preserving deformations, Jimbo, Miwa, Môri and Sato in [**39**, equation (2.27)] have established the following remarkable identity

$$F((x_0,x_1),s_1) = \exp\left(\int_0^{x_1-x_0}\frac{\sigma(x)}{x}dx\right)$$

where $s_1\in[0,+\infty)$ and σ is the solution to the Painlevé V equation

$$(x\sigma'')^2 + 4(x\sigma'-\sigma)(x\sigma'-\sigma+(\sigma')^2) = 0$$

$$\text{which satisfies} \qquad \sigma(x) = -\frac{s_1}{\pi}x - \frac{s_1^2}{\pi^2}x^2 - \frac{s_1^3}{\pi^3}x^3 + \mathcal{O}(x^4), \quad \text{as } x\to 0.$$

For general values of the parameters $m\geqslant 1$, $\vec{s}$ and $\vec{x}$, the determinant $F(\vec{x},\vec{s})$ is related to a more involved system of partial differential equations which generalizes the Painlevé V equation [**39**] (see also [**1**, Theorem 3.6.1 and Subsection 3.6.3]). The solution of this system of equations involves transcendental functions.

Thinning. The operation of thinning consists of randomly removing a fraction of points and was introduced in random matrix theory by Bohigas and Pato in [**4, 5**]. Given a function

$$s:\mathbb{R}\to[0,1], \quad x\mapsto s(x),$$

we say that a point configuration $\widetilde{X}$ is distributed according to *the thinned sine point process* if

$$\widetilde{X} = \{x\in X : \mathcal{B}(x) = 1\}, \tag{1.26}$$

where X is distributed according to the sine process, $\mathcal{B} = \{\mathcal{B}(x) : x\in\mathbb{R}\}$ is a random field of independent Bernoulli variables with $\mathbb{P}[\mathcal{B}(x) = 1] = 1 - s(x)$, and furthermore $\mathcal{B}$ is independent of X. It is well known [**46**, Proposition A.2], and also easy to see from (1.19), that the thinned sine process is also determinantal and that its kernel is given by

$$\widetilde{K}(x,y) = \sqrt{1-s(x)}K(x,y)\sqrt{1-s(y)} = \sqrt{1-s(x)}\frac{\sin(x-y)}{\pi(x-y)}\sqrt{1-s(y)}. \tag{1.27}$$

If the function $x \mapsto s(x) = s_1$ is constant, then each point is removed with the same probability $s_1 \in [0,1]$.[†] The thinned sine point process already presents interesting features in this case (see, for example, [**8, 11**]), as it describes a crossover between the original process (when $s_1 = 0$), and an uncorrelated Poisson process (when $s_1 \to 1$ at a certain speed). It follows directly from (1.21) (with X and K replaced by $\widetilde{X}$ and $\widetilde{K}$) and the definition (1.1) of F that

$$F((x_0,x_1),s_1) = \det(1-(1-s_1)\mathcal{K}|_{(x_0,x_1)}) = \mathbb{P}[\widetilde{X} \cap (x_0,x_1) = \emptyset].$$

In particular, (1.4) can be interpreted as large gap asymptotics in the (constant) thinned sine process. By comparing the leading terms of (1.4) and (1.3), we see that it is significantly more likely to observe a large gap in the (constant) thinned sine point process than in the usual sine point process.

More generally, let s be a piecewise constant function, say

$$s(x) = \begin{cases} s_j, & \text{if } x \in (x_{j-1},x_j),\ j=1,\dots,m, \\ 0, & \text{otherwise,} \end{cases} \tag{1.28}$$

where $s_1,\dots,s_m \in [0,1]$. It follows directly from (1.1), (1.21) and (1.27) that the probability of observing no points on (x_0,x_m) in the (piecewise constant) thinned point process is simply given by

$$F(\vec{x},\vec{s}) = \det\left(1 - \sum_{k=1}^{m}(1-s_k)\mathcal{K}|_{(x_{k-1},x_k)}\right) = \mathbb{P}[\widetilde{X} \cap (x_0,x_m) = \emptyset]. \tag{1.29}$$

Therefore, Theorems 1.1 and 1.2 give large gap asymptotics in any piecewise thinned sine process, as long as at most one of the parameters $s_1,\dots,s_m$ is 0.

Conditioning. Now, following [**16**], we consider a situation where we have information about the thinned process, and we try to deduce from it some information about the initial process. More precisely, let X be a random point configuration distributed according to the sine point process, let $\widetilde{X}$ be as in (1.26), and assume that $\#(\widetilde{X} \cap B) = 0$, where $B = (x_0,x_m)$. We are interested in the conditional random variable

$$\widehat{\mathcal{N}}_B := \#(X \cap B)|\Big(\#(\widetilde{X} \cap B) = 0\Big). \tag{1.30}$$

If s is piecewise constant and given by (1.28), then using first Bayes' formula and then (1.23) and (1.29), we obtain

$$\mathbb{P}\Big(\widehat{\mathcal{N}}_B = 0\Big) = \mathbb{P}\Big(\#(X \cap B) = 0\Big|\#(\widetilde{X} \cap B) = 0\Big) = \frac{\mathbb{P}(\#(X \cap B) = 0)}{\mathbb{P}(\#(\widetilde{X} \cap B) = 0)} = \frac{F((x_0,x_m),0)}{F(\vec{x},\vec{s})}.$$

Therefore, if at most one of the parameters $s_1,\dots,s_m$ is 0, we can obtain large r asymptotics for $\mathbb{P}(\widehat{\mathcal{N}}_{(rx_0,rx_m)} = 0)$ by combining (1.3) with either Theorem 1.1 or Theorem 1.2. The conditional random variable (1.30) is relevant in, for example, nuclear physics [**4, 5**]. Indeed, it is now well known (from the work of Dyson) that the energy levels of heavy atoms feature a similar repulsive structure as the points of the sine point process. However, high-quality data are often not available, and in practice one usually observes only a fraction of the energy levels. It is then natural to wonder if one can retrieve some missing energy levels given the available information.

The random variable (1.30) is conditioned on $\#(\widetilde{X} \cap B) = 0$. We mention that different types of conditioning of the sine process have been studied in great depth in the literature. In particular, it is known [**35**, Theorem 4.2] that for almost all point configurations X,[‡] if A is

---

[†] If $s_1 = 0$, no particle are removed and the thinned process coincides with the initial point process.

[‡] Here, 'almost all X' means 'almost all X with respect to the sine process'.

a compact interval, then $X \setminus A$ determines almost surely $\#(X \cap A)$. The conditional measure of the sine process on $\{X|X \setminus A\}$ admits an explicit density, see [**13**, Theorems 1.1 and 1.4]. Furthermore, the correlation kernel of this conditional sine process converges to the usual sine kernel as the size of the interval A gets large, see [**45**, Theorems 1.3 and 1.4].

Asymptotics for the variance and covariance of the sine counting function. We first briefly review some known results on the counting function of the sine process.

The formula (1.22) implies, in particular, that $\mathbb{E}[N_{(rx_0,rx_1)}] = \mu_1(r)$, where μ_1 is given by (1.9). There is no such explicit expression for $\mathrm{Var}[N_{(rx_0,rx_1)}]$, but we can compute its large r asymptotics as follows. We know from (1.25) with $m = 1$ that

$$\begin{aligned} F((rx_0, rx_1), e^u) &= \mathbb{E}\big[e^{uN_{(rx_0,rx_1)}}\big] \\ &= 1 + u\,\mathbb{E}[N_{(rx_0,rx_1)}] + \frac{u^2}{2}\mathbb{E}[N_{(rx_0,rx_1)}^2] + \mathcal{O}(u^3) \quad \text{as } u \to 0. \end{aligned} \tag{1.31}$$

Recall that the asymptotics of $F((rx_0, rx_1), e^u)$ as $r \to +\infty$ are given by (1.4) (and were obtained in [**2**]) and are uniform for u in compact subsets of $\mathbb{R}$. In particular, these asymptotics can be expanded as $u \to 0$. A comparison of this expansion with (1.31) yields

$$\mathrm{Var}[N_{(rx_0,rx_1)}] = \sigma_1^2(r) + \frac{1+\gamma_{\mathrm{E}}}{\pi^2} + \mathcal{O}(r^{-1}), \tag{1.32}$$

as $r \to +\infty$, where σ_1^2 is given by (1.10). Here $\gamma_{\mathrm{E}} \approx 0.5772$ is Euler's gamma constant and is part of the definition of the Barnes' G function, see [**47**, formula 5.17.3]. The leading term of (1.32) was also obtained in [**20**] without relying on [**2**]. In a slightly different direction, Holcomb and Paquette in [**36**] have studied the maximum deviation of the sine-β process. For $\beta = 2$, this process coincides with the determinantal sine point process, and their result states that for any $\epsilon > 0$, we have

$$\lim_{r \to +\infty} \mathbb{P}\left(\frac{\sqrt{2}}{\pi} - \epsilon \leqslant \frac{\max_{0 \leqslant r' \leqslant r}(N_{(r'x_0, r'x_1)} - \mu_1(r'))}{\log r} \leqslant \frac{\sqrt{2}}{\pi} + \epsilon\right) = 1.$$

Theorem 1.1 allows to obtain precise large r asymptotics for the covariance between $N_{(rx_0,rx_1)}$ and $N_{(rx_0,rx_2)}$. To see this, we first rewrite the expression (1.25) for F as follows

$$F(r\vec{x}, \vec{s}) = \mathbb{E}\left[\prod_{j=1}^{m} s_j^{N_{(rx_{j-1},rx_j)}}\right] = \mathbb{E}\left[\prod_{j=1}^{m} e^{u_j N_{(rx_0,rx_j)}}\right], \tag{1.33}$$

where $u_1, \ldots, u_m$ are given by (1.7). In particular, using (1.33) with $m = 1$ and $m = 2$, we obtain

$$\begin{aligned} \frac{F\big(r(x_0,x_1,x_2),(e^{2u},e^u)\big)}{F(r(x_0,x_1),e^u)F(r(x_0,x_2),e^u)} &= \frac{\mathbb{E}\big[e^{uN_{(rx_0,rx_1)}}e^{uN_{(rx_0,rx_2)}}\big]}{\mathbb{E}\big[e^{uN_{(rx_0,rx_1)}}\big]\mathbb{E}\big[e^{uN_{(rx_0,rx_2)}}\big]} \\ &= 1 + \mathrm{Cov}(N_{(rx_0,rx_1)}, N_{(rx_0,rx_2)})u^2 + \mathcal{O}(u^3), \quad \text{as } u \to 0. \end{aligned} \tag{1.34}$$

The large r asymptotics for the left-hand side of (1.34) can be deduced from Theorem 1.1 and are uniform for u in compact subsets of $\mathbb{R}$. By expanding these asymptotics as $u \to 0$, and then comparing with the right-hand side of (1.34), we obtain

$$\mathrm{Cov}[N_{(rx_0,rx_1)}, N_{(rx_0,rx_2)}] = \Sigma_{1,2}(r) + \frac{1+\gamma_{\mathrm{E}}}{2\pi^2} + \mathcal{O}\left(\frac{\log r}{r}\right), \quad \text{as } r \to +\infty, \tag{1.35}$$

where $\Sigma_{1,2}$ is given by (1.11). Note that the leading term in (1.35) is proportional to $\log r$. Interestingly, this contrasts with the asymptotics of the covariances of the Airy and Bessel counting functions which remain bounded, see [**17**, below Remark 1] and [**15**, equation (1.17)].

Asymptotics for the mean, variance and covariance of a conditional counting function. If $s_p = 0$ for a certain $p \in \{1,\dots,m\}$, then we can rewrite (1.25) as follows:

$$F(r\vec{x},\vec{s}) = \mathbb{P}\big(N_{(rx_{p-1},rx_p)} = 0\big)\mathbb{E}\left[\prod_{j\neq p} s_j^{N_{(rx_{j-1},rx_j)}} \middle| N_{(rx_{p-1},rx_p)} = 0\right]$$
$$= F((rx_{p-1},rx_p),0)\,\mathbb{E}\left[\prod_{j=0}^{p-2} e^{-u_j \widehat{N}_{r\mathcal{I}_j}} \prod_{j=p+1}^{m} e^{u_j \widehat{N}_{r\mathcal{I}_j}}\right],$$

where $\widehat{N}_{r\mathcal{I}_j}$ is the conditional random variable defined by

$$\widehat{N}_{r\mathcal{I}_j} := N_{r\mathcal{I}_j}|\big(N_{(rx_{p-1},rx_p)} = 0\big), \quad \mathcal{I}_j = \begin{cases} (x_p,x_j), & \text{if } j \in \{p+1,\dots,m\}, \\ (x_j,x_{p-1}), & \text{if } j \in \{0,\dots,p-2\}. \end{cases}$$

Then, proceeding as in the derivations of (1.32) and (1.35), we obtain the following new asymptotic formulas

$$\mathbb{E}\Big[\widehat{N}_{r\mathcal{I}_j}\Big] = \hat{\mu}_j(r) + \mathcal{O}\Big(\frac{\log r}{r}\Big),$$
$$\mathrm{Var}\Big[\widehat{N}_{r\mathcal{I}_j}\Big] = \hat{\sigma}_j^2(r) + \frac{1+\gamma_{\mathrm{E}}}{\pi^2} + \mathcal{O}\Big(\frac{\log r}{r}\Big),$$
$$\mathrm{Cov}[\widehat{N}_{r\mathcal{I}_j},\widehat{N}_{r\mathcal{I}_k}] = \hat{\Sigma}_{j,k} + \mathcal{O}\Big(\frac{\log r}{r}\Big),$$

as $r \to +\infty$, for any $j,k \in \{0,\dots,m\} \setminus \{p-1,p\}$, and where $\hat{\mu}_j$, $\hat{\sigma}_j^2$ and $\hat{\Sigma}_{j,k}$ are given by (1.14), (1.15) and (1.16), respectively. Note that, in contrast to (1.35), $\mathrm{Cov}[\widehat{N}_{r\mathcal{I}_j},\widehat{N}_{r\mathcal{I}_k}]$ remains bounded as $r \to +\infty$.

Outline. The rest of the paper is organized as follows. In Section 2, using the fact the sine kernel is *integrable* in the sense of Its, Izergin, Korepin and Slavnov (IIKS) [**37**], we express the kernel $K_{\vec{x},\vec{s}}$ of the resolvent operator associated to F in terms of an RH problem whose solution is denoted Y (following [**8–10, 23**]). Next, we transform the RH problem for Y into a new RH problem with constant jumps whose solution is denoted Φ. In Section 3, we obtain a differential identity which expresses $\partial_{s_k} \log F(r\vec{x},\vec{s})$ (for an arbitrary $k \in \{1,\dots,m\}$) in terms of Φ. We obtain large r asymptotics for Φ with $s_1,\dots,s_m \in (0,+\infty)$ in Section 4 via the Deift/Zhou steepest descent method. In Section 5, we substitute the asymptotics of Φ in the differential identity to obtain large r asymptotics for $\partial_{s_k} \log F(r\vec{x},\vec{s})$. Then, we proceed with the successive integrations of these asymptotics in $s_1,\dots,s_m$, which finishes the proof of Theorem 1.1. Sections 6 and 7 are devoted to the proof of Theorem 1.2 (with $s_p = 0$), and are organized in the same way as Sections 4 and 5.

2. *Model RH problem*

Let us denote $\mathcal{K}_{\vec{x},\vec{s}}$ for the integral operator that appears in the definition (1.1) of $F(\vec{x},\vec{s})$, that is,

$$\mathcal{K}_{\vec{x},\vec{s}} = \sum_{j=1}^{m}(1-s_j)\mathcal{K}|_{(x_{j-1},x_j)}. \tag{2.1}$$

In Section 3, we will express $\partial_{s_k} \log F(\vec{x},\vec{s})$, $k=1,\dots,m$, in terms of the resolvent operator

$$\mathcal{R}_{\vec{x},\vec{s}} = (1-\mathcal{K}_{\vec{x},\vec{s}})^{-1} - 1 = (1-\mathcal{K}_{\vec{x},\vec{s}})^{-1}\mathcal{K}_{\vec{x},\vec{s}}. \tag{2.2}$$

The goal of this section is to relate $\mathcal{R}_{\vec{x},\vec{s}}$ to a convenient model RH problem.

We will proceed in three steps.

(1) The kernel $K_{\vec{x},\vec{s}}$ of the operator $\mathcal{K}_{\vec{x},\vec{s}}$ is *integrable* in the sense of IIKS [37], which means that it can be written in the form

$$K_{\vec{x},\vec{s}}(u,v) = \frac{f^T(u)g(v)}{u-v}, \tag{2.3}$$

for suitable vector valued functions f and g which are written down in (2.4). This fact will allow us to use a result of Deift, Its and Zhou [23] to express the resolvent operator in terms of an RH problem whose solution is denoted Y.

(2) As a preparation for the third step, we will consider another RH problem, whose solution $\Phi_{\sin}$ can be explicitly written in terms of elementary functions.

(3) Finally, using the properties of $\Phi_{\sin}$, we will transform the RH problem for Y into a new RH problem with constant jumps. The solution to this RH problem is denoted Φ and will play a central role in the next sections.

REMARK 1. The above steps 2 and 3 will allow us to work with Φ instead of Y. In Section 3, we will take advantage of the fact that Φ has constant jumps to simplify the differential identity using a Lax pair. In the same spirit, other RH problems with constant jumps related to the Airy and Bessel processes have also been used in [18, 19] to simplify the analysis. However, we mention that if $m = 1$ our RH problem for Y reduces to the RH problem considered by Bothner *et al.* in [8, 9], and that their approach is different from ours and does not rely on the steps 2 and 3 above; instead they have successfully performed a Deift–Zhou steepest descent analysis directly on Y (though in a different regime of the parameters than in this paper).

It is directly seen from (2.1) and (1.2) that $K_{\vec{x},\vec{s}}$ can be written in the form (2.3) with

$$f(u) = \begin{pmatrix} \sin(u) \\ -\cos(u) \end{pmatrix}, \quad g(v) = \frac{1}{\pi}\begin{pmatrix} \sum_{j=1}^{m} \chi_{(x_{j-1},x_j)}(v)(1-s_j)\cos v \\ \sum_{j=1}^{m} \chi_{(x_{j-1},x_j)}(v)(1-s_j)\sin v \end{pmatrix}, \tag{2.4}$$

where we recall that for any Borel set $A \subset \mathbb{R}$, $\chi_A(u) = 1$ if $u \in A$ and $\chi_A(u) = 0$ otherwise.

In the sine point process, for all bounded Borel set B with non-zero Lebesgue measure, we have $\mathbb{P}(N_B = 0) > 0$. Therefore, from (1.1) and (1.25), we have

$$F(\vec{x},\vec{s}) = \det(1-\mathcal{K}_{\vec{x},\vec{s}}) \geqslant \mathbb{P}(N_{(x_0,x_1)} = 0) > 0, \tag{2.5}$$

which implies in particular that $1-\mathcal{K}_{\vec{x},\vec{s}}$ is invertible and that $\mathcal{R}_{\vec{x},\vec{s}}$ exists. Let us now define the matrix Y by

$$Y(z) = I - \int_{x_0}^{x_m} \frac{\widetilde{f}(u)g^T(u)}{u-z}du, \quad \widetilde{f}(u) = \left((1-\mathcal{K}_{\vec{x},\vec{s}})^{-1}f\right)(u). \tag{2.6}$$

The function Y satisfies the following RH problem [23, Lemma 2.12].

RH problem for Y.

(a) $Y : \mathbb{C}\setminus[x_0,x_m] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) For $u \in (x_0,x_m)\setminus\{x_1,\dots,x_{m-1}\}$, the limits $\lim_{\epsilon\to 0_+} Y(u\pm i\epsilon)$ exist, are denoted $Y_+(u)$ and $Y_-(u)$, respectively, are continuous as functions of u, and satisfy furthermore the jump relation

$$Y_+(u) = Y_-(u)J_Y(u), \quad J_Y(u) = I - 2\pi i f(u)g^T(u).$$

(c) $Y(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

(d) $Y(z) = \mathcal{O}(\log(z-x_j))$ as $z \to x_j$, for each $j = 0,\dots,m$.

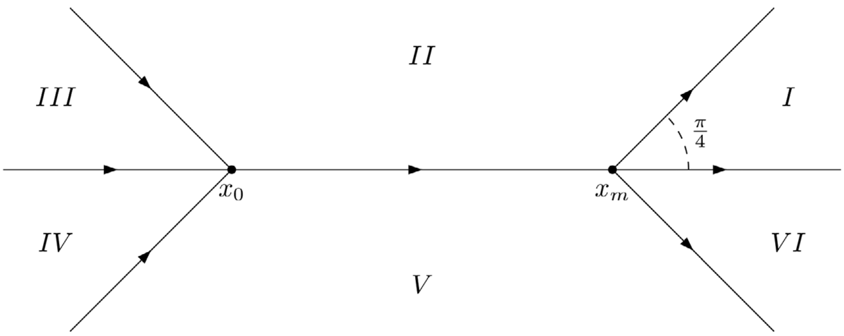

FIGURE 3. *Jump contours $\Sigma_{\sin}$ for the RH problem for $\Phi_{\sin}$.*

From [23], the kernel $R_{\vec{x},\vec{s}}$ of the resolvent operator $\mathcal{R}_{\vec{x},\vec{s}}$ can be written as

$$R_{\vec{x},\vec{s}}(u,v) = \frac{\widetilde{f}^T(u)\widetilde{g}(v)}{u-v}, \quad u,v \in (x_0, x_m), \tag{2.7}$$

with $\widetilde{f}$ and $\widetilde{g}$ expressed in terms of Y as follows:

$$\widetilde{f}(u) = Y_+(u)f(u) \quad \text{and} \quad \widetilde{g}(v) = (Y_+^{-1}(v))^T g(v).$$

Let $I, \dots, VI$ be the six regions shown in Figure 3. We consider the following RH problem, whose solution is denoted $\Phi_{\sin}$.

RH problem for $\Phi_{\sin}$

(a) $\Phi_{\sin} : \mathbb{C} \setminus \Sigma_{\sin} \to \mathbb{C}^{2\times 2}$ is analytic, where

$$\Sigma_{\sin} = \mathbb{R} \cup (x_m + e^{\pm\frac{\pi i}{4}}\mathbb{R}^+) \cup (x_0 + e^{\pm\frac{3\pi i}{4}}\mathbb{R}^+)$$

is oriented as shown in Figure 3 and $\mathbb{R}^+ := (0, +\infty)$.

(b) The jumps are given by

$$\Phi_{\sin,+}(z) = \Phi_{\sin,-}(z)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad z \in (x_0, x_m),$$

$$\Phi_{\sin,+}(z) = \Phi_{\sin,-}(z)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad z \in (x_m + e^{\pm\frac{\pi i}{4}}\mathbb{R}^+) \cup (x_0 + e^{\pm\frac{3\pi i}{4}}\mathbb{R}^+),$$

$$\Phi_{\sin,+}(z) = \Phi_{\sin,-}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in (-\infty, x_0) \cup (x_m, +\infty).$$

(c) As $z \to \infty$, we have

$$\Phi_{\sin}(z) = Ne^{-\frac{\pi i}{4}\sigma_3}\Big(I + \mathcal{O}\Big(e^{-2|\Im z|}\chi_{II\cup V}(z)\Big)\Big)e^{-iz\sigma_3} \times \begin{cases} I, & \text{if } \Im z > 0, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{if } \Im z < 0, \end{cases}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \chi_{II\cup V}(z) = \begin{cases} 1, & \text{if } z \in II \cup V, \\ 0, & \text{otherwise.} \end{cases}$$

As $z \to x_0$ and as $z \to x_m$, we have $\Phi_{\sin}(z) = \mathcal{O}(1)$.

The unique solution to the above RH problem is explicitly given by

$$\Phi_{\sin}(z) = Ne^{-\frac{\pi i}{4}\sigma_3} \times \begin{cases} \begin{pmatrix} e^{-iz} & 0 \\ 0 & e^{iz} \end{pmatrix}, & z \in I \cup III, \\ \begin{pmatrix} e^{-iz} & 0 \\ e^{iz} & e^{iz} \end{pmatrix}, & z \in II, \\ \begin{pmatrix} 0 & -e^{-iz} \\ e^{iz} & 0 \end{pmatrix}, & z \in IV \cup VI, \\ \begin{pmatrix} e^{-iz} & -e^{-iz} \\ e^{iz} & 0 \end{pmatrix}, & z \in V. \end{cases} \tag{2.8}$$

Now, we use $\Phi_{\sin}$ to transform the RH problem for Y. Let us consider

$$\Phi(z) = Y(z)\Phi_{\sin}(z). \tag{2.9}$$

Since Y is analytic on $\mathbb{C} \setminus [x_0, x_m]$, the jumps $J_\Phi := \Phi_-^{-1}\Phi_+$ coincide with $\Phi_{\sin,-}^{-1}\Phi_{\sin,+}$ on $\Sigma_{\sin} \setminus [x_0, x_m]$. On (x_0, x_m), we have

$$J_\Phi(u) = \Phi_{\sin,-}^{-1}(u)J_Y(u)\Phi_{\sin,+}(u), \quad u \in (x_0, x_m).$$

which, using the explicit expression for $\Phi_{\sin}$ given by (2.8), simplifies to

$$J_\Phi(u) = \begin{pmatrix} 1 & \sum_{j=1}^m s_j\chi_{(x_{j-1},x_j)}(u) \\ 0 & 1 \end{pmatrix}, \quad u \in (x_0, x_m).$$

Now, it directly follows from the properties of Y and $\Phi_{\sin}$ that Φ satisfies the following RH problem. For convenience, we define $s_0 := 1$, $s_{m+1} := 1$.

RH problem for Φ

(a) $\Phi : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ is analytic, where $\Sigma = \Sigma_{\sin}$ is shown in Figure 3.
(b) The jumps are given by

$$\Phi_+(z) = \Phi_-(z)\begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix}, \quad z \in (x_{j-1}, x_j),\ j = 1, \dots, m, \tag{2.10}$$

$$\Phi_+(z) = \Phi_-(z)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z \in (x_m + e^{\pm\frac{\pi i}{4}}\mathbb{R}^+) \cup (x_0 + e^{\pm\frac{3\pi i}{4}}\mathbb{R}^+),$$

$$\Phi_+(z) = \Phi_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, x_0) \cup (x_m, +\infty).$$

(c) As $z \to \infty$, we have

$$\Phi(z) = \left(I + \frac{\Phi_1}{z} + \mathcal{O}(z^{-2})\right)Ne^{-\frac{\pi i}{4}\sigma_3}e^{-iz\sigma_3} \times \begin{cases} I, & \text{if } \Im z > 0, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{if } \Im z < 0, \end{cases} \tag{2.11}$$

for a certain traceless matrix $\Phi_1 = \Phi_1(\vec{x}, \vec{s})$ independent of z.
As $z \to x_j$, $j \in \{0, 1, \dots, m\}$, we have

$$\Phi(z) = G_j(z; \vec{x}, \vec{s})\begin{pmatrix} 1 & -\frac{s_{j+1}-s_j}{2\pi i}\log(z - x_j) \\ 0 & 1 \end{pmatrix} V_j(z)H(z), \tag{2.12}$$

where G_j is analytic in a neighborhood of x_j, satisfies $\det G_j \equiv 1$ and

$$V_j(z) = \begin{cases} I, & \Im z > 0, \\ \begin{pmatrix} 1 & -s_{j+1} \\ 0 & 1 \end{pmatrix}, & \Im z < 0, \end{cases} \qquad H(z) = \begin{cases} I, & z \in II \cup V, \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & z \in I \cup III, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in IV \cup VI. \end{cases}$$

Remark 2. The solution to the RH problem for Φ is unique; this follows from standard arguments and the fact that the jumps for Φ have determinant 1 (see, for example, [**21**, Theorem 7.18]). Proving existence of a given RH problem is in general a more difficult task than proving uniqueness, but in our case this directly follows from the fact that $(1-\mathcal{K}_{\vec{x},\vec{s}})^{-1}$ exists (see (2.5)), and from the explicit formulas (2.9) and (2.6).

After substituting (2.4), (2.8) and (2.9) in (2.7), we obtain

$$R_{\vec{x},\vec{s}}(u,u) = \frac{1-s_k}{2\pi i}[\Phi^{-1}(u;\vec{x},\vec{s})\partial_u(\Phi(u;\vec{x},\vec{s}))]_{21}, \quad u \in (x_{k-1},x_k), \quad k=1,\dots,m. \tag{2.13}$$

Remark 3. In (2.13), we do not indicate whether we use the $+$ or $-$ boundary values of Φ, but this is without ambiguity. Indeed, from the jumps for Φ on (x_0,x_m) given by (2.10), we easily verify that

$$[\Phi_+^{-1}(u;\vec{x},\vec{s})\partial_u\Phi_+(u;\vec{x},\vec{s})]_{21} = [\Phi_-^{-1}(u;\vec{x},\vec{s})\partial_u\Phi_-(u;\vec{x},\vec{s})]_{21}.$$

3. *Differential identity*

By standard properties of trace class operators, for any $k \in \{1,\dots,m\}$, we have

$$\begin{aligned} \partial_{s_k} \log F(\vec{x},\vec{s}) &= \partial_{s_k} \log\det(1-\mathcal{K}_{\vec{x},\vec{s}}) = -\mathrm{Tr}\big((1-\mathcal{K}_{\vec{x},\vec{s}})^{-1}\partial_{s_k}\mathcal{K}_{\vec{x},\vec{s}}\big) \\ &= \frac{1}{1-s_k}\mathrm{Tr}\big((1-\mathcal{K}_{\vec{x},\vec{s}})^{-1}\mathcal{K}_{\vec{x},\vec{s}}|_{(x_{k-1},x_k)}\big) \\ &= \frac{1}{1-s_k}\mathrm{Tr}\big(\mathcal{R}_{\vec{x},\vec{s}}|_{(x_{k-1},x_k)}\big) = \frac{1}{1-s_k}\int_{x_{k-1}}^{x_k} R_{\vec{x},\vec{s}}(u,u)du, \end{aligned} \tag{3.1}$$

where we recall that $\mathcal{R}_{\vec{x},\vec{s}}$ is defined by (2.2). Substituting the expression (2.13) for $R_{\vec{x},\vec{s}}$ in (3.1), we obtain the following differential identity

$$\partial_{s_k} \log F(\vec{x},\vec{s}) = \frac{1}{2\pi i}\int_{x_{k-1}}^{x_k}[\Phi^{-1}(u;\vec{x},\vec{s})\partial_u\Phi(u;\vec{x},\vec{s})]_{21}du. \tag{3.2}$$

We implicitly assumed $s_k \neq 1$ in (3.1). However, recall that $F(\vec{x},\vec{s})$ is an entire function of s_k (see [**51**, Theorem 2]) and that $\det(1-\mathcal{K}_{\vec{x},\vec{s}})|_{s_k=1} > 0$ (see (2.5)). Therefore, the left-hand side of (3.2) is well defined at $s_k = 1$, and (3.2) also holds for $s_k = 1$ by continuity.

After replacing $\vec{x}$ by $r\vec{x}$ in (3.2), we obtain

$$\partial_{s_k} \log F(r\vec{x},\vec{s}) = \frac{1}{2\pi i}\int_{x_{k-1}}^{x_k}[\Phi^{-1}(ru;r\vec{x},\vec{s})\partial_u\Phi(ru;r\vec{x},\vec{s})]_{21}du. \tag{3.3}$$

Our goal for the rest of this section is to simplify the integral on the right-hand side of (3.3). For this, we will study a Lax pair associated to Φ. We first focus on some properties of $\partial_z\Phi(rz;r\vec{x},\vec{s})$. Since the jumps for Φ are independent of z, we have

$$\partial_z\Phi(rz;r\vec{x},\vec{s}) = A(z)\Phi(rz;r\vec{x},\vec{s}), \tag{3.4}$$

where $z \mapsto A(z)$ is analytic for $z \in \mathbb{C} \setminus \{x_0, \dots, x_m\}$. A also depends on r, $\vec{x}$ and $\vec{s}$, even though this is not indicated in the notation. Furthermore, since $\det \Phi \equiv 1$, A is traceless. From the asymptotics for Φ at $x_0, \dots, x_m$ and at ∞, we conclude that A takes the form

$$A(z) = \begin{pmatrix} 0 & -r \\ r & 0 \end{pmatrix} + \sum_{j=0}^{m} \frac{A_j}{z - x_j}, \tag{3.5}$$

where the matrices $A_j = A_j(r, \vec{x}, \vec{s})$ are traceless and given by

$$\begin{aligned} A_j &= -\frac{s_{j+1} - s_j}{2\pi i} (G_j \sigma_+ G_j^{-1})(r x_j; r\vec{x}, \vec{s}) \\ &= -\frac{s_{j+1} - s_j}{2\pi i} \begin{pmatrix} -G_{j,11} G_{j,21} & G_{j,11}^2 \\ -G_{j,21}^2 & G_{j,11} G_{j,21} \end{pmatrix}, \quad \text{where } \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.6}$$

The integrand on the right-hand side of (3.3) can now be rewritten using (3.4). Since A is traceless and $\det \Phi \equiv 1$, we obtain

$$\begin{aligned} [\Phi^{-1}(rz; r\vec{x}, \vec{s}) \partial_z (\Phi(rz; r\vec{x}, \vec{s}))]_{21} &= [\Phi^{-1}(rz; r\vec{x}, \vec{s}) A(z) \Phi(rz; r\vec{x}, \vec{s})]_{21} \\ &= \Phi_{11}^2 A_{21} - \Phi_{21}^2 A_{12} - 2\Phi_{11}\Phi_{21} A_{11}. \end{aligned}$$

By substituting (3.5) in the above equation, we infer that

$$\begin{aligned} [\Phi^{-1}(rz; r\vec{x}, \vec{s}) \partial_z (\Phi(rz; r\vec{x}, \vec{s}))]_{21} &= (\Phi \sigma_+ \Phi^{-1})_{12}(rz; r\vec{x}, \vec{s}) \left[r + \sum_{j=0}^{m} \frac{A_{j,21}}{z - x_j} \right] \\ &+ (\Phi \sigma_+ \Phi^{-1})_{21}(rz; r\vec{x}, \vec{s}) \left[-r + \sum_{j=0}^{m} \frac{A_{j,12}}{z - x_j} \right] + 2(\Phi \sigma_+ \Phi^{-1})_{11}(rz; r\vec{x}, \vec{s}) \sum_{j=0}^{m} \frac{A_{j,11}}{z - x_j}. \end{aligned} \tag{3.7}$$

Let us define

$$B(z) = \partial_{s_k} \Phi(rz; r\vec{x}, \vec{s}) \Phi(rz; r\vec{x}, \vec{s})^{-1}.$$

From the RH problem for Φ, we deduce that B satisfies the following RH problem.

RH problem for B.

(a) $B : \mathbb{C} \setminus [x_{k-1}, x_k] \to \mathbb{C}^{2\times 2}$ is analytic.
(b) B satisfies the jumps

$$B_+(z) = B_-(z) + (\Phi_- \sigma_+ \Phi_-^{-1})(rz; r\vec{x}, \vec{s}), \quad z \in (x_{k-1}, x_k). \tag{3.8}$$

(c) B satisfies the following asymptotic behaviors

$$B(z) = \frac{\partial_{s_k} \Phi_1(r\vec{x}, \vec{s})}{rz} + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty, \tag{3.9}$$

$$B(z) = \frac{\partial_{s_k}(s_{j+1} - s_j)}{s_{j+1} - s_j} A_j \log(r(z - x_j)) + B_j + o(1), \qquad \text{as } z \to x_j,\ j = 0, \dots, m, \tag{3.10}$$

where $B_j = (\partial_{s_k} G_j G_j^{-1})(r x_j; r\vec{x}, \vec{s})$.

Using the jumps (2.10) for Φ on (x_{k-1}, x_k), we note that

$$(\Phi_- \sigma_+ \Phi_-^{-1})(rz; r\vec{x}, \vec{s}) = (\Phi_+ \sigma_+ \Phi_+^{-1})(rz; r\vec{x}, \vec{s}), \quad z \in (x_{k-1}, x_k),$$

which implies that $z \mapsto (\Phi\sigma_+\Phi^{-1})(rz;r\vec{x},\vec{s})$ is analytic for $z \in (x_{k-1},x_k)$. In particular, we can replace $\Phi_-\sigma_+\Phi_-^{-1}$ in (3.8) by $\Phi\sigma_+\Phi^{-1}$ without ambiguity. By (3.8) and Cauchy's formula, we have

$$B(z) = \frac{1}{2\pi i}\int_{x_{k-1}}^{x_k} \frac{(\Phi\sigma_+\Phi^{-1})(ru;r\vec{x},\vec{s})}{u-z}du. \tag{3.11}$$

Expanding (3.11) as $z \to \infty$ and then comparing with (3.9), we obtain

$$-\frac{1}{2\pi i}\int_{x_{k-1}}^{x_k}(\Phi\sigma_+\Phi^{-1})(ru;r\vec{x},\vec{s})du = \frac{\partial_{s_k}\Phi_1(r\vec{x},\vec{s})}{r}.$$

Now, we substitute (3.7) in (3.3), and then use the expansions of B at ∞ and at x_j, $j = 0,1,\ldots,m$ (given by (3.9)) to simplify the integral. Since $\det A_j \equiv 0$ for $j = 0,\ldots,m$, we infer that the logarithmic terms in the expansions (3.10) of $B(z)$ as $z \to x_j$ for $j = 0,\ldots,m$ do not contribute in (3.3), and after a computation we obtain

$$\partial_{s_k}\log F(r\vec{x},\vec{s}) = \partial_{s_k}\Phi_{1,21}(r\vec{x},\vec{s}) - \partial_{s_k}\Phi_{1,12}(r\vec{x},\vec{s}) + \sum_{j=0}^{m}(A_{j,21}B_{j,12} + A_{j,12}B_{j,21} + 2A_{j,11}B_{j,11}).$$

The above formula can be further simplified by using the explicit expressions for the functions A_j and B_j given by (3.6) and below (3.10). After some simplifications, which use $\det G_j \equiv 1$, we get

$$\partial_{s_k}\log F(r\vec{x},\vec{s}) = K_\infty + \sum_{j=0}^{m} K_{x_j}, \tag{3.12}$$

where

$$K_\infty = \partial_{s_k}\Phi_{1,21}(r\vec{x},\vec{s}) - \partial_{s_k}\Phi_{1,12}(r\vec{x},\vec{s}), \tag{3.13}$$

$$K_{x_j} = -\frac{s_{j+1}-s_j}{2\pi i}(G_{j,11}\partial_{s_k}G_{j,21} - G_{j,21}\partial_{s_k}G_{j,11})(rx_j;r\vec{x},\vec{s}). \tag{3.14}$$

4. *Riemann–Hilbert analysis for* $s_1,\ldots,s_m \in (0,+\infty)$

In this section, we employ the Deift/Zhou steepest descent method to obtain large r asymptotics for $\Phi(rz;r\vec{x},\vec{s})$ uniformly for z in different regions of the complex z-plane.

At the level of the parameters, we assume that:

- $s_1,\ldots,s_m$ are in a compact subset of $(0,+\infty)$;
- $x_0,\ldots,x_m$ are in a compact subset of $\mathbb{R}$;
- there exists $\delta > 0$ independent of r such that

$$\min_{0\leqslant j<k\leqslant m} x_k - x_j \geqslant \delta. \tag{4.1}$$

4.1. *Normalization of the RH problem with g-function*

The main purpose of the first transformation is to obtain a new RH problem whose solution T remains bounded at ∞. Let us define

$$T(z) = \Phi(rz;r\vec{x},\vec{s})e^{-rg(z)\sigma_3}, \tag{4.2}$$

where the g-function is given by

$$g(z) = \begin{cases} -iz, & \text{if } \Im z > 0, \\ iz, & \text{if } \Im z < 0. \end{cases} \tag{4.3}$$

One easily verifies from (2.11), (4.2), and (4.3) that T remains bounded at ∞, as desired. More precisely, we have

$$T(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2})\right) N e^{-\frac{\pi i}{4}\sigma_3} \begin{cases} I, & \Im z > 0, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \Im z < 0, \end{cases} \qquad \text{as } z \to \infty,$$

where

$$T_1 = \frac{\Phi_1(r\vec{x}, \vec{s})}{r}. \tag{4.4}$$

We can obtain the jumps for T straightforwardly from those of Φ and the relation $g_+(z) + g_-(z) = 0$ for $z \in \mathbb{R}$. For $z \in (x_{j-1}, x_j)$, $j = 1, \ldots, m$, we have

$$T_-(z)^{-1} T_+(z) = \begin{pmatrix} e^{-2rg_+(z)} & s_j \\ 0 & e^{-2rg_-(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{-2rg_-(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{-2rg_+(z)} & 1 \end{pmatrix}. \tag{4.5}$$

where we have used $s_j \neq 0$ to factorize the jump matrix.

4.2. *Opening of the lenses*

For $j = 1, \ldots, m$, we let the lenses $\gamma_{j,+}$ and $\gamma_{j,-}$ be open curves starting at x_{j-1}, ending at x_j and lying in the upper and lower half plane, respectively. The region inside $\gamma_{j,+} \cup (x_{j-1}, x_j)$ is denoted $\Omega_{j,+}$, and the region inside $\gamma_{j,-} \cup (x_{j-1}, x_j)$ is denoted $\Omega_{j,-}$. In particular, $\Omega_{j,+}$ and $\Omega_{j,-}$ are subsets of the upper and lower half plane, respectively. The next transformation is defined by

$$S(z) = T(z) \prod_{j=1}^{m} \begin{cases} \begin{pmatrix} 1 & 0 \\ -s_j^{-1} e^{-2rg(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,+}, \\ \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{-2rg(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,-}, \\ I, & \text{if } z \in \mathbb{C} \setminus (\Omega_{j,+} \cup \Omega_{j,-}). \end{cases} \tag{4.6}$$

Using the factorization (4.5) and the properties of the RH problem for Φ, we easily verify that S satisfies the following RH problem.

RH problem for S.

(a) $S : \mathbb{C} \backslash \Sigma_S \to \mathbb{C}^{2\times 2}$ is analytic, with

$$\Sigma_S = \mathbb{R} \cup \gamma_+ \cup \gamma_-, \qquad \gamma_\pm = \bigcup_{j=0}^{m+1} \gamma_{j,\pm},$$

where Σ_S is oriented as shown in Figure 4 and

$$\gamma_{0,\pm} := x_0 + e^{\pm\frac{3\pi i}{4}}(0, +\infty), \qquad \gamma_{m+1,\pm} := x_m + e^{\pm\frac{\pi i}{4}}(0, +\infty).$$

(b) The jumps for S are given by

$$S_+(z) = S_-(z) \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix}, \qquad z \in (x_{j-1}, x_j),\ j = 0, \ldots, m+1,$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{-2rg(z)} & 1 \end{pmatrix}, \qquad z \in \gamma_{j,\pm},\ j = 0, \ldots, m+1, \tag{4.7}$$

where $x_{-1} := -\infty$ and $x_{m+1} := +\infty$ (recall that $s_0 = s_{m+1} = 1$).

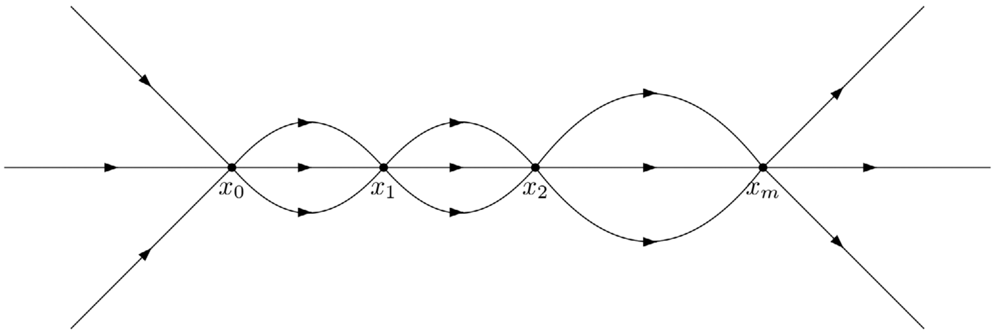

FIGURE 4. *Jump contours Σ_S for the RH problem for S with $m = 3$.*

(c) As $z \to \infty$, we have

$$S(z) = \left(I + \frac{T_1}{z} + \mathcal{O}\left(z^{-2}\right)\right) N e^{-\frac{\pi i}{4}\sigma_3} \begin{cases} I, & \Im z > 0, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \Im z < 0. \end{cases} \tag{4.8}$$

As $z \to x_j$ from outside the lenses, $j = 0, \ldots, m$, we have

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z - x_j)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z - x_j)) \end{pmatrix}.$$

Since $\Re g(z) > 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, it follows from (4.7) that for any $\epsilon > 0$, there exists $c > 0$ such that

$$S_-(z)^{-1} S_+(z) - I = \mathcal{O}(e^{-c|z|r}), \quad \text{as } r \to +\infty, \tag{4.9}$$

uniformly for $z \in \gamma_+ \cup \gamma_-$ such that $\min_{j \in \{0,\ldots,m\}} |z - x_j| \geqslant \epsilon$. The estimate (4.9) does not hold for z close to x_j because $\Re g_\pm(z) = 0$ for $z \in \mathbb{R}$. More precisely, for $z \in \gamma_+ \cup \gamma_-$ such that $\min_{j \in \{0,\ldots,m\}} |z - x_j| \leqslant \epsilon$, we only have

$$S_-(z)^{-1} S_+(z) - I = o(1), \quad \text{as } r \to +\infty.$$

4.3. *Global parametrix*

Ignoring the jumps for S on the lenses $\gamma_+ \cup \gamma_-$, we are led to the following RH problem, whose solution is denoted $P^{(\infty)}$. We will show in Subsection 4.5 that $P^{(\infty)}$ is a good approximation for S away from neighborhoods of x_j, $j = 0, 1, \ldots, m$.

RH problem for $P^{(\infty)}$.

(a) $P^{(\infty)} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.
(b) The jumps for $P^{(\infty)}$ are given by

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix}, \qquad z \in (x_{j-1}, x_j),\ j = 0, \ldots, m+1.$$

(c) As $z \to \infty$, we have

$$P^{(\infty)}(z) = \left(I + \frac{P_1^{(\infty)}}{z} + \mathcal{O}\left(z^{-2}\right)\right) N e^{-\frac{\pi i}{4}\sigma_3} \begin{cases} I, & \Im z > 0, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \Im z < 0. \end{cases} \tag{4.10}$$

for a certain matrix $P_1^{(\infty)}$ independent of z.
As $z \to x_j$, $j \in \{0, \dots, m\}$, we have $P^{(\infty)}(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}$.

The behavior of $P^{(\infty)}$ near x_j, $j \in \{0, \dots, m\}$ is added to ensure uniqueness of the solution to the RH problem for $P^{(\infty)}$. The construction of $P^{(\infty)}$ relies on the following function:

$$D(z) = \exp\left(\frac{\theta(z)}{2\pi i}\sum_{j=1}^{m}\log s_j \int_{x_{j-1}}^{x_j}\frac{du}{u-z}\right), \quad \text{where} \quad \theta(z) = \begin{cases} +1, & \text{if } \Im z > 0, \\ -1, & \text{if } \Im z < 0. \end{cases} \tag{4.11}$$

D satisfies

$$D_+(z)D_-(z) = s_j, \qquad \text{for } z \in (x_{j-1}, x_j),\ j = 0, \dots, m+1,$$

and has the following behavior at ∞:

$$D(z) = \exp\left(\theta(z)\sum_{\ell=1}^{k}\frac{d_\ell}{z^\ell} + \mathcal{O}(z^{-k-1})\right), \quad \text{as } z \to \infty,$$

where $k \in \mathbb{N}_{>0}$ is arbitrary and

$$d_\ell = -\frac{1}{2\pi i}\sum_{j=1}^{m}\log s_j \int_{x_{j-1}}^{x_j} u^{\ell-1}du = -\frac{1}{2\pi i\ell}\sum_{j=1}^{m}\log s_j \left(x_j^\ell - x_{j-1}^\ell\right). \tag{4.12}$$

Using the above properties of D, we verify that

$$P^{(\infty)}(z) = Ne^{-\frac{\pi i}{4}\sigma_3}\left\{\begin{matrix} I, & \Im z > 0 \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \Im z < 0 \end{matrix}\right\} D(z)^{-\sigma_3} \tag{4.13}$$

satisfies the RH problem for $P^{(\infty)}$ with

$$P_1^{(\infty)} = \begin{pmatrix} 0 & id_1 \\ -id_1 & 0 \end{pmatrix}. \tag{4.14}$$

In preparation for the analysis of Section 5, we now compute the first terms in the asymptotics of $D(z)$ as $z \to x_j$, $j = 0, 1, \dots, m$. It is straightforward to see from (4.11) that D can be rewritten as

$$D(z) = \prod_{j=0}^{m}(z - x_j)^{\theta(z)\beta_j}, \tag{4.15}$$

where $\beta_0, \dots, \beta_m$ are defined by

$$\beta_j = \frac{1}{2\pi i}\log\frac{s_j}{s_{j+1}} \quad \text{or equivalently} \quad e^{-2i\pi\beta_j} = \frac{s_{j+1}}{s_j}, \quad j = 0, \dots, m. \tag{4.16}$$

Note that, since $s_0 = s_{m+1} = 1$, we have

$$\beta_0 + \dots + \beta_m = 0.$$

As $z \to x_j$, $j \in \{0, 1, \dots, m\}$, $\Im z > 0$, we infer from (4.15) that

$$D(z) = \sqrt{s_{j+1}}\,(z - x_j)^{\beta_j}\prod_{\substack{k=0 \\ k \neq j}}^{m}|x_j - x_k|^{\beta_k}(1 + \mathcal{O}(z - x_j)). \tag{4.17}$$

Finally, we note that the constants d_ℓ defined in (4.12) can be rewritten in terms of the functions β_j as follows

$$d_\ell = -\frac{1}{\ell}\sum_{j=0}^{m}\beta_j x_j^\ell. \tag{4.18}$$

4.4. *Local parametrices*

Let $\mathcal{D}_{x_j}$ be small open disks centered at x_j, $j=0,1,\ldots,m$ whose radii are equal to $\frac{\delta}{3}$, where δ is defined in (4.1). The definition of the radii ensures that the disks do not intersect each other.

The local parametrix $P^{(x_j)}$ is defined in $\mathcal{D}_{x_j}$ and satisfies an RH problem with the same jumps as S inside $\mathcal{D}_{x_j}$. On the boundary of the disk, $P^{(x_j)}$ 'matches' with $P^{(\infty)}$ in the sense that

$$P^{(x_j)}(z) = (I + o(1))P^{(\infty)}(z), \qquad \text{as } r \to +\infty, \tag{4.19}$$

uniformly for $z \in \partial\mathcal{D}_{x_j}$. Furthermore, we require that

$$S(z)P^{(x_j)}(z)^{-1} = \mathcal{O}(1), \qquad \text{as } z \to x_j. \tag{4.20}$$

The construction of $P^{(x_j)}$ is similar for all $j \in \{0,1,\ldots,m\}$ and relies on the confluent hypergeometric functions. This type of local parametrix is well understood [**14, 34, 38**] and involves the solution Φ_{HG} to a model RH problem, which we recall in Subsection A.2. The function

$$f_{x_j}(z) = -2\begin{cases} g(z) - g_+(x_j), & \text{if } \Im z > 0 \\ -(g(z) - g_-(x_j)), & \text{if } \Im z < 0 \end{cases} = 2i(z - x_j) \tag{4.21}$$

is a conformal map from $\mathcal{D}_{x_j}$ to a neighborhood of 0 which maps the real line on the imaginary axis, that is, $f_{x_j}(\mathbb{R}\cap\mathcal{D}_{x_j}) \subset i\mathbb{R}$. Let us deform the lenses in a small neighborhood of x_j such that

$$f_{x_j}((\gamma_{j,+}\cup\gamma_{j+1,+})\cap\mathcal{D}_{x_j}) \subset \Gamma_3\cup\Gamma_2, \quad f_{x_j}((\gamma_{j,-}\cup\gamma_{j+1,-})\cap\mathcal{D}_{x_j}) \subset \Gamma_5\cup\Gamma_6,$$

where Γ_3, Γ_2, Γ_5 and Γ_6 are as shown in Figure A.2. In this way, f_{x_j} maps the jump contour for $P^{(x_j)}$ in a subset of the jump contour for Φ_{HG}. We seek for $P^{(x_j)}$ in the form

$$P^{(x_j)}(z) = E_{x_j}(z)\Phi_{\mathrm{HG}}(rf_{x_j}(z);\beta_j)(s_js_{j+1})^{-\frac{\sigma_3}{4}}e^{-rg(z)\sigma_3}, \tag{4.22}$$

where we recall that β_j is given by (4.16). If E_{x_j} is analytic, then it is straightforward to verify from the jumps for Φ_{HG} (given by (A.4)) that $P^{(x_j)}$ satisfies the same jumps as S inside $\mathcal{D}_{x_j}$. From the asymptotics (A.5) of Φ_{HG}, we see that in order to satisfy (4.19), we are forced to define E_{x_j} by

$$E_{x_j}(z) = P^{(\infty)}(z)(s_js_{j+1})^{\frac{\sigma_3}{4}}\begin{Bmatrix} \sqrt{\dfrac{s_{j+1}}{s_j}}^{\sigma_3}, & \Im z > 0 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \Im z < 0 \end{Bmatrix} e^{rg_+(x_j)\sigma_3}(rf_{x_j}(z))^{\beta_j\sigma_3}. \tag{4.23}$$

Using the jumps for $P^{(\infty)}$, we verify that E_{x_j} has no jump inside $\mathcal{D}_{x_j}$. Also, since $P^{(\infty)}(z)$ remains bounded as $z \to x_j$, and since $\beta_j \in i\mathbb{R}$, it is directly seen from (4.23) that $E_{x_j}(z)$ remains bounded as $z \to x_j$. We conclude that E_{x_j} is analytic in the whole disk $\mathcal{D}_{x_j}$, as desired. Also, since the jumps of $P^{(x_j)}$ coincide with those of S on $(\mathbb{R}\cup\gamma_+\cup\gamma_-)\cap\mathcal{D}_{x_j}$, $S(z)P^{(x_j)}(z)^{-1}$ is analytic in $\mathcal{D}_{x_j}\setminus\{x_j\}$. Using (A.7) and condition (d) of the RH problem for S, we obtain that $S(z)P^{(x_j)}(z)^{-1} = \mathcal{O}(\log(z - x_j))$, which means that x_j is a removable singularity and in

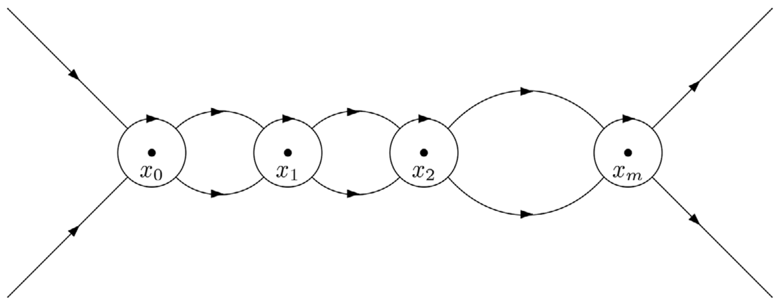

FIGURE 5. *Jump contours Σ_R for the RH problem for R with $m = 3$.*

particular (4.20) holds. In Section 5, we will need more precise information about the matching condition (4.19). Using (A.5), we obtain

$$P^{(x_j)}(z)P^{(\infty)}(z)^{-1} = I + \frac{1}{rf_{x_j}(z)}E_{x_j}(z)\Phi_{\mathrm{HG},1}(\beta_j)E_{x_j}(z)^{-1} + \mathcal{O}(r^{-2}), \tag{4.24}$$

as $r \to +\infty$, uniformly for $z \in \partial\mathcal{D}_{x_j}$, where $\Phi_{\mathrm{HG},1}(\beta_j)$ is given by (A.6). Also, using (4.13), (4.17), and (4.21), we obtain

$$E_{x_j}(x_j) = N\Lambda_j^{\sigma_3}, \qquad \text{where} \qquad \Lambda_j = e^{-\frac{\pi i}{4}}e^{rg_+(x_j)}(2r)^{\beta_j}\prod_{\substack{k=0\\k\neq j}}^{m}|x_j - x_k|^{-\beta_k}. \tag{4.25}$$

4.5. *Small norm problem*

In this subsection, we show that $P^{(\infty)}$ and $P^{(x_j)}$ approximate S as $r \to +\infty$. For this, we define

$$R(z) = \begin{cases} S(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C}\setminus\bigcup_{j=0}^m \mathcal{D}_{x_j}, \\ S(z)P^{(x_j)}(z)^{-1}, & \text{for } z \in \mathcal{D}_{x_j},\ j \in \{0,1,\ldots,m\}. \end{cases} \tag{4.26}$$

It follows from the analysis of Subsection 4.4 that R is analytic inside the $m+1$ disks. Since the jumps of $P^{(\infty)}$ and of S are the same on (x_{j-1}, x_j), $j = 1,\ldots,m$, we conclude that R is analytic on $\mathbb{C}\setminus\Sigma_R$, where

$$\Sigma_R = \bigcup_{j=0}^m \partial\mathcal{D}_{x_j} \cup \left((\gamma_+ \cup \gamma_-)\setminus\bigcup_{j=0}^m \mathcal{D}_{x_j}\right),$$

see Figure 5. Also, from (4.9) and (4.13), we infer that the jumps $J_R := R_-^{-1}R_+$ satisfy

$$J_R(z) = P^{(\infty)}(z)S_-(z)^{-1}S_+(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}(e^{-c|z|r}), \quad \text{as } r \to +\infty, \tag{4.27}$$

uniformly for $z \in \Sigma_R \cap (\gamma_+ \cup \gamma_-)$, for a certain $c > 0$ independent of z and r. As shown in Figure 5, we orient the boundaries of the disks in the clockwise direction. It follows from (4.24) that

$$J_R(z) = P^{(x_j)}(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}\left(\frac{1}{r}\right), \qquad \text{as } r \to +\infty \tag{4.28}$$

uniformly $z \in \bigcup_{j=0}^m \partial\mathcal{D}_{x_j}$. Furthermore, from the behavior of $S(z)$ and $P^{(\infty)}(z)$ as $z \to \infty$ given by (4.8) and (4.10), we have

$$R(z) = I + \mathcal{O}(z^{-1}), \quad \text{as } z \to \infty. \tag{4.29}$$

By standard theory for RH problems [24–26], it follows that R exists for sufficiently large r and satisfies

$$R(z) = I + \frac{R^{(1)}(z)}{r} + \mathcal{O}(r^{-2}), \quad R^{(1)}(z) = \mathcal{O}(1), \qquad \text{as } r \to +\infty \tag{4.30}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$. Note that the presence of $r^{\pm\beta_j}$ in the entries of E_{x_j} (see (4.23)) implies, by (4.24), that the entries of J_R contain factors of the form $r^{\pm 2\beta_j}$. Thus, a standard analysis of the Cauchy operator associated to R (see, for example, [17, 38] for similar situations) shows that

$$\partial_{\beta_j} R(z) = \frac{\partial_{\beta_j} R^{(1)}(z)}{r} + \mathcal{O}\left(\frac{\log r}{r^2}\right), \quad \partial_{\beta_j} R^{(1)}(z) = \mathcal{O}(\log r), \quad \text{as } r \to +\infty. \tag{4.31}$$

Moreover, since the asymptotics (4.27) and (4.28) are uniform in $x_0, \dots, x_m$ in compact subsets of $\mathbb{R}$ such that (4.1) holds, and uniform for $\beta_1, \dots, \beta_m$ in compact subsets of $i\mathbb{R}$, the asymptotics (4.30) and (4.31) also hold uniformly in $x_0, \dots, x_m, \beta_1, \dots, \beta_m$ in the same way.

Now, we compute explicitly $R^{(1)}(z)$ for $z \in \mathbb{C} \setminus \bigcup_{j=0}^m \mathcal{D}_{x_j}$. For this, we first note that

$$R(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{R_-(s)(J_R(s) - I)}{s - z} ds. \tag{4.32}$$

The above formula directly follows from $R_+ = R_- J_R$ and the asymptotics (4.29). Also, we know from (4.24) that for

$$J_R(z) = I + \frac{J_R^{(1)}(z)}{r} + \mathcal{O}(r^{-2}), \quad J_R^{(1)}(z) = \frac{1}{f_{x_j}(z)} E_{x_j}(z) \Phi_{\mathrm{HG},1}(\beta_j) E_{x_j}(z)^{-1}, \tag{4.33}$$

as $r \to \infty$ uniformly for $z \in \mathcal{D}_{x_j}$, $j = 0, \dots, m$. By substituting (4.33) in (4.32), we obtain

$$R^{(1)}(z) = \frac{1}{2\pi i} \int_{\bigcup_{j=0}^m \partial \mathcal{D}_{x_j}} \frac{J_R^{(1)}(s)}{s - z} ds. \tag{4.34}$$

Note that the expression (4.33) for $J_R^{(1)}$ can be analytically continued from $\partial \mathcal{D}_{x_j}$ to $\overline{\mathcal{D}_{x_j}} \setminus \{x_j\}$, and that $J_R^{(1)}$ has a simple pole at x_j. Recalling that the disks are oriented in the clockwise direction, and using (4.21), (4.24)–(4.25) and (A.6), we obtain

$$R^{(1)}(z) = \sum_{j=0}^m \frac{1}{z - x_j} \mathrm{Res}(J_R^{(1)}(s), s = x_j), \qquad \text{for } z \in \mathbb{C} \setminus \bigcup_{j=0}^m \mathcal{D}_{x_j}, \tag{4.35}$$

where for $j \in \{0, \dots, m\}$ we have

$$\begin{aligned} \mathrm{Res}\Big(J_R^{(1)}(s), s = x_j\Big) &= \frac{\beta_j^2}{2i} N \begin{pmatrix} -1 & \widetilde{\Lambda}_{j,1} \\ -\widetilde{\Lambda}_{j,2} & 1 \end{pmatrix} N^{-1} \\ &= \frac{\beta_j^2}{4} \begin{pmatrix} -\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} & -i(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} + 2i) \\ -i(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} - 2i) & \widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2} \end{pmatrix}, \end{aligned}$$

with

$$\widetilde{\Lambda}_{j,1} = \tau(\beta_j) \Lambda_j^2 \quad \text{and} \quad \widetilde{\Lambda}_{j,2} = \tau(-\beta_j) \Lambda_j^{-2}.$$

5. *Proof of Theorem* 1.1

We prove Theorem 1.1 in two steps. First, we use the RH analysis of Section 4 to obtain large r asymptotics for the quantities K_∞ and K_{x_j} defined in (3.13)–(3.14). By substituting these asymptotics in the differential identity

$$\partial_{s_k} \log F(r\vec{x}, \vec{s}) = K_\infty + \sum_{j=0}^{m} K_{x_j}, \quad k \in \{1, \dots, m\},$$

we obtain large r asymptotics for $\partial_{s_k} \log F(r\vec{x}, \vec{s})$. Second, we integrate these asymptotics over the parameters $s_1, \dots, s_m$ to obtain large r asymptotics for $\log F(r\vec{x}, \vec{s})$.

5.1. *Large r asymptotics for $\partial_{s_k} \log F(r\vec{x}, \vec{s})$*

Asymptotics for K_∞. By (4.26), (4.8), and (4.10), we have

$$R(z) = S(z)P^{(\infty)}(z)^{-1} = I + \frac{R_1}{z} + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty,$$

for a certain matrix R_1 satisfying $T_1 = R_1 + P_1^{(\infty)}$. Hence, by (4.30),

$$T_1 = P_1^{(\infty)} + \frac{R_1^{(1)}}{r} + \mathcal{O}(r^{-2}), \qquad \text{as } r \to +\infty,$$

where $R_1^{(1)}$ is the z^{-1} coefficient in the large z expansion of $R^{(1)}(z)$. Using (4.14) and (4.35), we find the following large r asymptotics for T_1:

$$T_1 = \begin{pmatrix} 0 & id_1 \\ -id_1 & 0 \end{pmatrix} + \sum_{j=0}^{m} \frac{\beta_j^2}{4r} \begin{pmatrix} -\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} & -i(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} + 2i) \\ -i(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} - 2i) & \widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2} \end{pmatrix} + \mathcal{O}(r^{-2}). \quad (5.1)$$

By (3.13), (4.4), (4.31) and (5.1), we obtain

$$K_\infty = r(\partial_{s_k} T_{1,21} - \partial_{s_k} T_{1,12}) = -2i\partial_{s_k} d_1 r - \sum_{j=0}^{m} \partial_{s_k}(\beta_j^2) + \mathcal{O}\left(\frac{\log r}{r}\right), \qquad \text{as } r \to +\infty. \quad (5.2)$$

Asymptotics for K_{x_j} with $j \in \{0, \dots, m\}$. For z outside the lenses and inside $\mathcal{D}_{x_j}$, by (4.6), (4.26) and (4.22), we have

$$T(z) = R(z)E_{x_j}(z)\Phi_{\mathrm{HG}}(rf_{x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} e^{-rg(z)\sigma_3}, \quad (5.3)$$

and by (4.21) and (A.8), we also have

$$\Phi_{\mathrm{HG}}(rf_{x_j}(z); \beta_j) = \widehat{\Phi}_{\mathrm{HG}}(rf_{x_j}(z); \beta_j), \quad \text{for } \Im z > 0.$$

Using (4.16) and Euler's reflection formula for the Γ-function (see e.g. [**47**, equation 5.5.3]), we verify that

$$\frac{\sin(\pi\beta_j)}{\pi} = \frac{1}{\Gamma(\beta_j)\Gamma(1-\beta_j)} = -\frac{s_{j+1} - s_j}{2\pi i\sqrt{s_j s_{j+1}}}. \quad (5.4)$$

This relation, combined with (4.21) and (A.9), allows to verify that

$$\Phi_{\mathrm{HG}}(rf_{x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} = \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix} (I + \mathcal{O}(z - x_j)) \begin{pmatrix} 1 & -\frac{s_{j+1}-s_j}{2\pi i} \log(r(z - x_j)) \\ 0 & 1 \end{pmatrix}, \quad (5.5)$$

as $z \to x_j$ from $\Im z > 0$ and z outside the lenses, where the principal branch is taken for the log and

$$\Psi_{j,11} = \frac{\Gamma(1-\beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, \quad \Psi_{j,12} = \frac{(s_j s_{j+1})^{\frac{1}{4}}}{\Gamma(\beta_j)}\left(\log 2 - \frac{i\pi}{2} + \frac{\Gamma'(1-\beta_j)}{\Gamma(1-\beta_j)} + 2\gamma_{\mathrm{E}}\right),$$

$$\Psi_{j,21} = \frac{\Gamma(1+\beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, \quad \Psi_{j,22} = \frac{-(s_j s_{j+1})^{\frac{1}{4}}}{\Gamma(-\beta_j)}\left(\log 2 - \frac{i\pi}{2} + \frac{\Gamma'(-\beta_j)}{\Gamma(-\beta_j)} + 2\gamma_{\mathrm{E}}\right). \tag{5.6}$$

In particular, we note that

$$\Psi_{j,11}\Psi_{j,21} = -\beta_j \frac{2\pi i}{s_{j+1}-s_j}, \quad j = 0,\dots,m, \tag{5.7}$$

where we have used the well-known formula $\Gamma(1+z) = z\Gamma(z)$ and (5.4). We deduce from (2.12), (4.2), (5.3) and (5.5) that

$$G_j(rx_j; r\vec{x}, \vec{s}) = R(x_j)E_{x_j}(x_j)\begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix}. \tag{5.8}$$

Also, from (4.25), we have

$$\partial_{s_k} E_{x_j,11}(x_j) = E_{x_j,11}(x_j)\partial_{s_k}\log\Lambda_j, \quad \partial_{s_k} E_{x_j,12}(x_j) = -E_{x_j,12}(x_j)\partial_{s_k}\log\Lambda_j,$$

$$\partial_{s_k} E_{x_j,21}(x_j) = E_{x_j,21}(x_j)\partial_{s_k}\log\Lambda_j, \quad \partial_{s_k} E_{x_j,22}(x_j) = -E_{x_j,22}(x_j)\partial_{s_k}\log\Lambda_j.$$

Substituting (5.8) in the formula for K_{x_j} given by (3.14), and using (4.30), (4.31), the above relations and the fact that $\det E_{x_j}(x_j) = 1$, we obtain the following large r asymptotics

$$\sum_{j=0}^m K_{x_j} = \sum_{j=0}^m -\frac{s_{j+1}-s_j}{2\pi i}(\Psi_{j,11}\partial_{s_k}\Psi_{j,21} - \Psi_{j,21}\partial_{s_k}\Psi_{j,11}) - \sum_{j=0}^m 2\beta_j\partial_{s_k}\log\Lambda_j + \mathcal{O}\left(\frac{\log r}{r}\right). \tag{5.9}$$

Asymptotics for $\partial_{s_k}\log F(r\vec{x},\vec{s})$. By summing the large r asymptotics of K_{x_j}, $j = 0,\dots,m$ and K_∞ using (5.2) and (5.9), we obtain

$$\partial_{s_k}\log F(r\vec{x},\vec{s}) = -2i\partial_{s_k}d_1\, r - \sum_{j=0}^m \left(2\beta_j\partial_{s_k}\log\Lambda_j + \partial_{s_k}(\beta_j^2)\right)$$
$$+ \sum_{j=0}^m -\frac{s_{j+1}-s_j}{2\pi i}(\Psi_{j,11}\partial_{s_k}\Psi_{j,21} - \Psi_{j,21}\partial_{s_k}\Psi_{j,11}) + \mathcal{O}\left(\frac{\log r}{r}\right), \quad \text{as } r \to +\infty. \tag{5.10}$$

This last sum can be simplified using the expressions (5.6) for $\Psi_{j,11}$ and $\Psi_{j,21}$ together with (5.7):

$$\sum_{j=0}^m -\frac{s_{j+1}-s_j}{2\pi i}(\Psi_{j,11}\partial_{s_k}\Psi_{j,21} - \Psi_{j,21}\partial_{s_k}\Psi_{j,11}) = \sum_{j=0}^m \beta_j\partial_{s_k}\log\frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j)}. \tag{5.11}$$

Also, using (4.25), we have

$$\sum_{j=0}^m -2\beta_j\partial_{s_k}\log\Lambda_j = -2\sum_{j=0}^m \beta_j\partial_{s_k}(\beta_j)\log(2r) - 2\sum_{j=0}^m \beta_j \sum_{\substack{\ell=0\\ \ell\neq j}}^m \partial_{s_k}(\beta_\ell)\log|x_j - x_\ell|^{-1}. \tag{5.12}$$

For convenience, we will integrate with respect to $\beta_1,\dots,\beta_m$ rather than in the variables $s_1,\dots,s_m$ (recall that $\beta_0=-\beta_1-\ldots-\beta_m$). Let us define

$$\widetilde{F}(r\vec{x},\vec{\beta})=F(r\vec{x},\vec{s}), \tag{5.13}$$

where $\vec{\beta}=(\beta_1,\dots,\beta_m)$ and $\vec{s}=(s_1,\dots,s_m)$ are related via (4.16). Substituting (5.11) and (5.12) into (5.10), and taking the derivative with respect to β_k instead of s_k, $k\in\{1,\dots,m\}$, we obtain

$$\partial_{\beta_k}\log\widetilde{F}(r\vec{x},\vec{\beta})=-2i\partial_{\beta_k}d_1\,r-2\sum_{j=0}^{m}\beta_j\partial_{\beta_k}(\beta_j)\log(2r)-2\sum_{j=0}^{m}\beta_j\sum_{\substack{\ell=0\\ \ell\neq j}}^{m}\partial_{\beta_k}(\beta_\ell)\log|x_j-x_\ell|^{-1}$$
$$-\sum_{j=0}^{m}\partial_{\beta_k}(\beta_j^2)+\sum_{j=0}^{m}\beta_j\partial_{\beta_k}\log\frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j)}+\mathcal{O}\Big(\frac{\log r}{r}\Big),\quad \text{as } r\to+\infty. \tag{5.14}$$

Using $\beta_0=-\beta_1-\ldots-\beta_m$ and the explicit expression (4.18) of d_1, we can simplify the different terms that appear on the right-hand side of (5.14):

$$-2i\partial_{\beta_k}d_1r=2i(x_k-x_0)r,$$

$$-2\sum_{j=0}^{m}\beta_j\partial_{\beta_k}(\beta_j)\log(2r)=-2\beta_k\log(2r)-2(\beta_1+\cdots+\beta_m)\log(2r),$$

$$-2\sum_{j=0}^{m}\beta_j\sum_{\substack{\ell=0\\ \ell\neq j}}^{m}\partial_{\beta_k}(\beta_\ell)\log|x_j-x_\ell|^{-1}=-2\sum_{\substack{j=1\\ j\neq k}}^{m}\beta_j\log\left|\frac{(x_j-x_0)(x_k-x_0)}{x_j-x_k}\right|-4\beta_k\log|x_k-x_0|,$$

$$-\sum_{j=0}^{m}\partial_{\beta_k}(\beta_j^2)=-2\beta_k-2\sum_{j=1}^{m}\beta_j,$$

$$\sum_{j=0}^{m}\beta_j\partial_{\beta_k}\log\frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j)}=\beta_k\partial_{\beta_k}\log\frac{\Gamma(1+\beta_k)}{\Gamma(1-\beta_k)}+(\beta_1+\cdots+\beta_m)\partial_{\beta_k}\log\frac{\Gamma(1+\beta_1+\cdots+\beta_m)}{\Gamma(1-\beta_1-\cdots-\beta_m)}.$$

These identities allow to rewrite (5.14) as follows

$$\begin{aligned}\partial_{\beta_k}\log\widetilde{F}(r\vec{x},\vec{\beta})&\\ &=2i(x_k-x_0)r-4\beta_k\log\left(2r(x_k-x_0)\right)-2\sum_{\substack{j=1\\ j\neq k}}^{m}\beta_j\log\left(\frac{2r(x_j-x_0)(x_k-x_0)}{|x_j-x_k|}\right)\\ &\quad-2\beta_k-2\sum_{j=1}^{m}\beta_j+\beta_k\partial_{\beta_k}\log\frac{\Gamma(1+\beta_k)}{\Gamma(1-\beta_k)}\\ &\quad+(\beta_1+\cdots+\beta_m)\partial_{\beta_k}\log\frac{\Gamma(1+\beta_1+\cdots+\beta_m)}{\Gamma(1-\beta_1-\cdots-\beta_m)}+\mathcal{O}\Big(\frac{\log r}{r}\Big),\\ &\qquad\text{as } r\to+\infty.\end{aligned} \tag{5.15}$$

The analysis of Subsection 4.5 implies that the error term in (5.15) is uniform in $x_0,\dots,x_m$ in compact subsets of $\mathbb{R}$ such that (4.1) holds, and uniform in $\beta_1,\dots,\beta_m$ in compact subsets of $i\mathbb{R}$.

5.2. *Integration of the differential identity*

We first find an explicit formula for

$$I_\ell(\beta_\ell;\beta_1,\ldots,\beta_{\ell-1}) = \int_0^{\beta_\ell} (\beta_1+\cdots+\beta_{\ell-1}+x)\partial_x \log\frac{\Gamma(1+\beta_1+\cdots+\beta_{\ell-1}+x)}{\Gamma(1-\beta_1-\cdots-\beta_{\ell-1}-x)}dx, \quad (5.16)$$

with $\ell \in \{1,\ldots,m\}$. Integrating (5.16) by parts, we obtain

$$\begin{aligned} I_\ell(\beta_\ell;\beta_1,\ldots,\beta_{\ell-1}) &= (\beta_1+\cdots+\beta_\ell)\log\frac{\Gamma(1+\beta_1+\cdots+\beta_\ell)}{\Gamma(1-\beta_1-\cdots-\beta_\ell)} - (\beta_1+\cdots+\beta_{\ell-1})\log\frac{\Gamma(1+\beta_1+\cdots+\beta_{\ell-1})}{\Gamma(1-\beta_1-\cdots-\beta_{\ell-1})} \\ &- \int_0^{\beta_\ell}\log\Gamma(1+\beta_1+\cdots+\beta_{\ell-1}+x)dx + \int_0^{\beta_\ell}\log\Gamma(1-\beta_1-\cdots-\beta_{\ell-1}-x)dx. \quad (5.17)\end{aligned}$$

We recall the following integral relation for the Γ function (see e.g. [**47**, formula 5.17.4]):

$$\int_0^z \log\Gamma(1+x)dx = \frac{z}{2}\log 2\pi - \frac{z(z+1)}{2} + z\log\Gamma(z+1) - \log G(z+1), \quad (5.18)$$

where G is Barnes' G-function. Using twice (5.18) with suitable changes of variables, we obtain

$$\begin{aligned} \int_0^{\beta_\ell} \log\frac{\Gamma(1-\beta_1-\cdots-\beta_{\ell-1}-x)}{\Gamma(1+\beta_1+\cdots+\beta_{\ell-1}+x)}dx \\ = \beta_\ell^2 + 2\beta_\ell(\beta_1+\cdots+\beta_{\ell-1}) + (\beta_1+\cdots+\beta_\ell)\log\frac{\Gamma(1-\beta_1-\cdots-\beta_\ell)}{\Gamma(1+\beta_1+\cdots+\beta_\ell)} \\ -(\beta_1+\cdots+\beta_{\ell-1})\log\frac{\Gamma(1-\beta_1-\cdots-\beta_{\ell-1})}{\Gamma(1+\beta_1+\cdots+\beta_{\ell-1})} \\ +\log\frac{G(1+\beta_1+\cdots+\beta_\ell)G(1-\beta_1-\cdots-\beta_\ell)}{G(1+\beta_1+\cdots+\beta_{\ell-1})G(1-\beta_1-\cdots-\beta_{\ell-1})}. \end{aligned}$$

Substituting this identity in (5.17) and simplifying, we arrive at

$$\begin{aligned} I_\ell(\beta_\ell;\beta_1,\ldots,\beta_{\ell-1}) = \beta_\ell^2 + 2\beta_\ell(\beta_1+\cdots+\beta_{\ell-1}) \\ +\log\frac{G(1+\beta_1+\cdots+\beta_\ell)G(1-\beta_1-\cdots-\beta_\ell)}{G(1+\beta_1+\cdots+\beta_{\ell-1})G(1-\beta_1-\cdots-\beta_{\ell-1})}. \quad (5.19)\end{aligned}$$

Now, we will use the identity (5.15) for $k = 1,\ldots,m$. We start with $k=1$ and $\beta_2 = 0 = \beta_3 = \cdots = \beta_m$. With the notation $\vec{\beta}_1 = (\beta_1, 0, \ldots, 0)$, (5.15) becomes

$$\begin{aligned} \partial_{\beta_1}\log\widetilde{F}(r\vec{x},\vec{\beta}_1) = 2i(x_1-x_0)r - 4\beta_1\log(2r|x_1-x_0|) \\ -4\beta_1 + 2\beta_1\partial_{\beta_1}\log\frac{\Gamma(1+\beta_1)}{\Gamma(1-\beta_1)} + \mathcal{O}\left(\frac{\log r}{r}\right), \end{aligned}$$

as $r \to +\infty$. Since the above asymptotics are uniform for β_1 in compact subsets of $i\mathbb{R}$, we can integrate over β_1 from $\beta_1 = 0$ to an arbitrary $\beta_1 \in i\mathbb{R}$ without changing the order of the error term. Using formula (5.19) with $\ell = 1$, we obtain

$$\begin{aligned} \log\frac{\widetilde{F}(r\vec{x},\vec{\beta}_1)}{\widetilde{F}(r\vec{x},\vec{0})} = 2i\beta_1(x_1-x_0)r - 2\beta_1^2\log(2r(x_1-x_0)) \\ +2\log(G(1+\beta_1)G(1-\beta_1)) + \mathcal{O}\left(\frac{\log r}{r}\right) \end{aligned}$$

as $r \to +\infty$, where $\vec{0} = (0, \ldots, 0)$. This result matches with the known asymptotics (1.4), with a slightly worse error term. Now, we use (5.15) with $k = 2$, $\beta_3 = \cdots = \beta_m = 0$, and with $\beta_1 \in i\mathbb{R}$ fixed. With the notation $\vec{\beta}_2 = (\beta_1, \beta_2, 0, \ldots, 0)$, we first rewrite (5.15) as follows

$$\partial_{\beta_2} \log \widetilde{F}(r\vec{x}, \vec{\beta}_2) = 2i(x_2 - x_0)r - 4\beta_2 \log (2r(x_2 - x_0)) - 2\beta_1 \log \left(\frac{2r(x_1 - x_0)(x_2 - x_0)}{x_2 - x_1} \right)$$
$$- 2\beta_2 - 2(\beta_1 + \beta_2) + \beta_2 \partial_{\beta_2} \log \frac{\Gamma(1 + \beta_2)}{\Gamma(1 - \beta_2)} + (\beta_1 + \beta_2)\partial_{\beta_2} \log \frac{\Gamma(1 + \beta_1 + \beta_2)}{\Gamma(1 - \beta_1 - \beta_2)} + \mathcal{O}\left(\frac{\log r}{r} \right).$$

Again, since the above asymptotics are uniform in β_2 in compact subsets of $i\mathbb{R}$, they can be integrated over β_2 from $\beta_2 = 0$ to an arbitrary $\beta_2 \in i\mathbb{R}$ without changing the order of the error term. Using twice formula (5.19) with $\ell = 2$ (once for $I_2(\beta_2; 0)$ and once for $I_2(\beta_2; \beta_1)$), we obtain

$$\log \frac{\widetilde{F}(r\vec{x}, \vec{\beta}_2)}{\widetilde{F}(r\vec{x}, \vec{\beta}_1)} = 2i\beta_2(x_2 - x_0)r - 2\beta_2^2 \log (2r(x_2 - x_0)) - 2\beta_1\beta_2 \log \left(\frac{2r(x_1 - x_0)(x_2 - x_0)}{x_2 - x_1} \right)$$
$$+ \log (G(1 + \beta_2)G(1 - \beta_2)) + \log \frac{G(1 + \beta_1 + \beta_2)G(1 - \beta_1 - \beta_2)}{G(1 + \beta_1)G(1 - \beta_1)} + \mathcal{O}\left(\frac{\log r}{r} \right).$$

We proceed similarly to integrate over the variables $\beta_3, \ldots, \beta_m$. At the last step, we use (5.15) with $k = m$ and $\beta_1, \ldots, \beta_{m-1}$ arbitrary. The integration of (5.15) over β_m gives

$$\log \frac{\widetilde{F}(r\vec{x}, \vec{\beta})}{\widetilde{F}(r\vec{x}, \vec{\beta}_{m-1})} = 2i\beta_m(x_m - x_0)r - 2\beta_m^2 \log (2r(x_m - x_0))$$
$$-2 \sum_{j=1}^{m-1} \beta_j\beta_m \log \left(\frac{2r(x_j - x_0)(x_m - x_0)}{x_m - x_j} \right) + \log (G(1 + \beta_m)G(1 - \beta_m))$$
$$+ \log \frac{G(1 + \beta_1 + \cdots + \beta_m)G(1 - \beta_1 - \cdots - \beta_m)}{G(1 + \beta_1 + \cdots + \beta_{m-1})G(1 - \beta_1 - \cdots - \beta_{m-1})} + \mathcal{O}\left(\frac{\log r}{r} \right),$$

as $r \to +\infty$, where we have used the notation $\vec{\beta}_{m-1} = (\beta_1, \ldots, \beta_{m-1}, 0)$. By summing the asymptotics obtained after each integration, we obtain

$$\log \frac{\widetilde{F}(r\vec{x}, \vec{\beta})}{\widetilde{F}(r\vec{x}, \vec{0})} = 2i \sum_{j=1}^{m} \beta_j(x_j - x_0)r - \sum_{j=1}^{m} 2\beta_j^2 \log (2r(x_j - x_0))$$
$$-2 \sum_{1 \leqslant j < k \leqslant m} \beta_j\beta_k \log \left(\frac{2r(x_j - x_0)(x_k - x_0)}{x_k - x_j} \right) + \sum_{j=1}^{m} \log (G(1 + \beta_j)G(1 - \beta_j))$$
$$+ \log (G(1 + \beta_1 + \cdots + \beta_m)G(1 - \beta_1 - \cdots - \beta_m)) + \mathcal{O}\left(\frac{\log r}{r} \right),$$

as $r \to +\infty$. Note from (1.7) and (4.16) that $u_j = 2\pi i\beta_j$. Since $\widetilde{F}(r\vec{x}, \vec{0}) = F(r\vec{x}, \vec{1}) = 1$ (by (5.13) and (1.1)), this finishes the proof of Theorem 1.1.

6. *Riemann–Hilbert analysis for* $s_p = 0$

In this section, $p \in \{1, \ldots, m\}$ is fixed and we obtain large r asymptotics for $\Phi(rz; r\vec{x}, \vec{s})$, in the case where the parameters are such that:

- $s_p = 0$ and $s_1, \dots, s_{p-1}, s_{p+1}, \dots, s_m$ are in a compact subset of $(0, +\infty)$;
- $x_0, \dots, x_m$ are in a compact subset of $\mathbb{R}$;
- there exists $\delta > 0$ independent of r such that

$$\min_{0 \leqslant j < k \leqslant m} x_k - x_j \geqslant \delta. \tag{6.1}$$

6.1. *Normalization of the RH problem with g-function*

Let us define

$$g(z) = -i\theta(z)\sqrt{z - x_{p-1}}\sqrt{z - x_p}, \tag{6.2}$$

where the principal branches are taken for the square roots, and θ is defined as in (4.11). The g-function satisfies the jumps

$$g_+(z) + g_-(z) = 0 \quad \text{for} \quad z \in (-\infty, x_{p-1}) \cup (x_p, +\infty), \tag{6.3}$$

with an asymptotic behavior at ∞ given by

$$g(z) \sim \begin{cases} -iz, & \Im z > 0, \\ iz, & \Im z < 0, \end{cases} \quad \text{as } z \to \infty. \tag{6.4}$$

We define the first transformation by

$$T(z) = \begin{pmatrix} \cos\left(\frac{r}{2}(x_{p-1} + x_p)\right) & \sin\left(\frac{r}{2}(x_{p-1} + x_p)\right) \\ -\sin\left(\frac{r}{2}(x_{p-1} + x_p)\right) & \cos\left(\frac{r}{2}(x_{p-1} + x_p)\right) \end{pmatrix} \Phi(rz; r\vec{x}, \vec{s}) e^{-rg(z)\sigma_3}. \tag{6.5}$$

The purpose of the constant prefactor matrix in (6.5) is to simplify the behavior of T at ∞. After a computation using the behavior of $\Phi(rz; r\vec{x}, \vec{s})$ as $z \to \infty$ given by (2.11), we obtain

$$T(z) = \left(I + \frac{T_1}{z} + \mathcal{O}\left(z^{-2}\right)\right) N e^{-\frac{\pi i}{4}\sigma_3} \begin{cases} I, & \Im z > 0, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \Im z < 0, \end{cases} \quad \text{as } z \to \infty,$$

where

$$\Phi_{1,21}(r\vec{x}, \vec{s}) - \Phi_{1,12}(r\vec{x}, \vec{s}) = -\frac{r^2}{4}(x_p - x_{p-1})^2 + r(T_{1,21} - T_{1,12}). \tag{6.6}$$

The jumps for T are obtained straightforwardly from those of Φ and from (6.3). Since $s_j \neq 0$ for $j \neq p$, we verify that the jump matrix $T_-^{-1}T_+$ on (x_{j-1}, x_j), $j \neq p$, can be factorized as in (4.5).

6.2. *Opening of the lenses*

Around each interval (x_{j-1}, x_j), $j = 1, \dots, m$, $j \neq p$, we let the lenses $\gamma_{j,+}$ and $\gamma_{j,-}$ denote open curves in the upper and lower half plane, respectively, as shown in Figure 6. We also let $\Omega_{j,+}$ denote the region inside $\gamma_{j,+} \cup (x_{j-1}, x_j)$, and we let $\Omega_{j,-}$ denote the region inside $\gamma_{j,-} \cup (x_{j-1}, x_j)$. We define the $T \mapsto S$ transformation by

$$S(z) = T(z) \prod_{\substack{j=1 \\ j \neq p}}^{m} \begin{cases} \begin{pmatrix} 1 & 0 \\ -s_j^{-1} e^{-2rg(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,+}, \\ \begin{pmatrix} 1 & 0 \\ s_j^{-1} e^{-2rg(z)} & 1 \end{pmatrix}, & \text{if } z \in \Omega_{j,-}, \\ I, & \text{if } z \in \mathbb{C} \setminus (\Omega_{j,+} \cup \Omega_{j,-}). \end{cases} \tag{6.7}$$

In a similar way in Subsection 4.2, we verify that S satisfies the following RH problem.

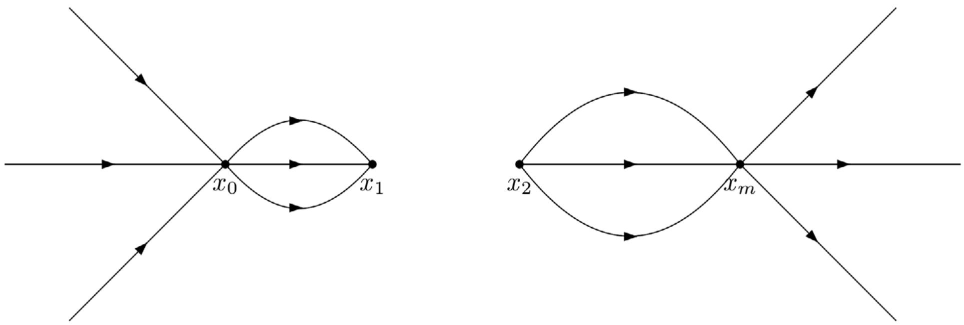

FIGURE 6. *Jump contours Σ_S for the RH problem for S with $m = 3$ and $p = 2$.*

RH problem for S.

(a) $S : \mathbb{C}\backslash\Sigma_S \to \mathbb{C}^{2\times 2}$ is analytic, with

$$\Sigma_S = (-\infty, x_{p-1}] \cup [x_p, +\infty) \cup \gamma_+ \cup \gamma_-, \quad \gamma_\pm = \bigcup_{\substack{j=0\\ j\neq p}}^{m+1} \gamma_{j,\pm},$$

where Σ_S is oriented as shown in Figure 6 and

$$\gamma_{0,\pm} := x_0 + e^{\pm\frac{3\pi i}{4}}(0, +\infty), \quad \gamma_{m+1,\pm} := x_m + e^{\pm\frac{\pi i}{4}}(0, +\infty).$$

(b) The jumps for S are given by

$$S_+(z) = S_-(z)\begin{pmatrix} 0 & s_j \\ -s_j^{-1} & 0 \end{pmatrix}, \qquad z \in (x_{j-1}, x_j),\ j = 0, \ldots, p-1, p+1, \ldots, m+1,$$

$$S_+(z) = S_-(z)\begin{pmatrix} 1 & 0 \\ s_j^{-1}e^{-2rg(z)} & 1 \end{pmatrix}, \qquad z \in \gamma_{j,\pm},\ j = 0, \ldots, p-1, p+1, \ldots, m+1,$$

where $x_{-1} := -\infty$, $x_{m+1} := +\infty$, and we recall that $s_0 = s_{m+1} = 1$.

(c) As $z \to \infty$, we have

$$S(z) = \left(I + \frac{T_1}{z} + \mathcal{O}(z^{-2})\right) N e^{-\frac{\pi i}{4}\sigma_3} \begin{cases} I, & \Im z > 0, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \Im z < 0. \end{cases} \tag{6.8}$$

As $z \to x_j$ from outside the lenses, $j = 0, \ldots, m$, we have

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log(z - x_j)) \\ \mathcal{O}(1) & \mathcal{O}(\log(z - x_j)) \end{pmatrix}.$$

LEMMA 6.1. *The g-function defined in (6.2) satisfies*

$$\{z : \Re(g(z)) > 0\} = \mathbb{C} \setminus ((-\infty, x_{p-1}] \cup [x_p, +\infty)).$$

Proof. Clearly, $\Re(g(z)) = 0$ if and only if $g(z)^2 = -(z - x_{p-1})(z - x_p) \leqslant 0$. Since $g(z)^2 \leqslant 0$ for $z \in (-\infty, x_{p-1}] \cup [x_p, +\infty)$, this proves $(-\infty, x_{p-1}] \cup [x_p, +\infty) \subseteq \{z : \Re(g(z)) = 0\}$. On the other hand, for each $c \in \mathbb{R}^-$, the equation $-(z - x_{p-1})(z - x_p) = c$ admits exactly two solutions (counting multiplicities), and from the graph of $g(z)^2$ for $z \in \mathbb{R}$, it is immediate to verify that these two solutions lie on $(-\infty,x_{p-1}]\cup[x_p,+\infty)$, which proves $(-\infty,x_{p-1}]\cup[x_p,+\infty)\supseteq\{z:\Re(g(z))=0\}$. Since $\Re(g(z))$ is continuous, all what remains is to determine the sign of $\Re(g(z))$ on $\mathbb{C}\setminus((-\infty,x_{p-1}]\cup[x_p,+\infty))$. From the behavior of $g(z)$ as $z\to i\infty$, see (6.4), we conclude that this sign is positive. ☐

We deduce from Lemma 6.1 that the jump matrices for S tend to the identity matrix exponentially fast as $r\to+\infty$ on the lenses. This convergence is uniform for z outside of fixed neighborhoods of x_j, $j\in\{0,1,\dots,m\}$, but is not uniform as $r\to+\infty$ and simultaneously $z\to x_j$, $j\in\{0,1,\dots,m\}$.

6.3. *Global parametrix*

By ignoring the jumps for S on the lenses, we are led to consider the following RH problem, whose solution is denoted $P^{(\infty)}$. We will show in Subsection 6.5 that $P^{(\infty)}$ is a good approximation for S outside neighborhoods of x_j, $j=0,1,\dots,m$.

RH problem for $P^{(\infty)}$.

(a) $P^{(\infty)}:\mathbb{C}\setminus((-\infty,x_{p-1}]\cup[x_p,+\infty))\to\mathbb{C}^{2\times2}$ is analytic.
(b) The jumps for $P^{(\infty)}$ are given by

$$P_+^{(\infty)}(z)=P_-^{(\infty)}(z)\begin{pmatrix}0 & s_j\\ -s_j^{-1} & 0\end{pmatrix},\qquad z\in(x_{j-1},x_j),\ j=0,\dots,p-1,p+1,\dots,m+1.$$

(c) As $z\to\infty$, we have

$$P^{(\infty)}(z)=\left(I+\frac{P_1^{(\infty)}}{z}+\mathcal{O}(z^{-2})\right)Ne^{-\frac{\pi i}{4}\sigma_3}\begin{cases}I, & \Im z>0,\\ \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, & \Im z<0.\end{cases}\tag{6.9}$$

for a certain matrix $P_1^{(\infty)}$ independent of z.

(d) As $z\to x_j$, $j\in\{0,\dots,m\}\setminus\{p-1,p\}$, we have $P^{(\infty)}(z)=\begin{pmatrix}\mathcal{O}(1) & \mathcal{O}(1)\\ \mathcal{O}(1) & \mathcal{O}(1)\end{pmatrix}$.

As $z\to x_j$, $j\in\{p-1,p\}$, we have $P^{(\infty)}(z)=\begin{pmatrix}\mathcal{O}((z-x_j)^{-1/4}) & \mathcal{O}((z-x_j)^{-1/4})\\ \mathcal{O}((z-x_j)^{-1/4}) & \mathcal{O}((z-x_j)^{-1/4})\end{pmatrix}$.

The construction of $P^{(\infty)}$ relies on the following function D:

$$D(z)=\exp\left(\theta(z)\sqrt{z-x_{p-1}}\sqrt{z-x_p}\left[-\sum_{j=1}^{p-1}\frac{\log s_j}{2\pi i}\int_{x_{j-1}}^{x_j}\frac{1}{\sqrt{x_{p-1}-u}\sqrt{x_p-u}}\frac{du}{u-z}\right.\right.$$
$$\left.\left.+\sum_{j=p+1}^{m}\frac{\log s_j}{2\pi i}\int_{x_{j-1}}^{x_j}\frac{1}{\sqrt{u-x_{p-1}}\sqrt{u-x_p}}\frac{du}{u-z}\right]\right),$$

where the principal branches are taken for $\sqrt{z-x_{p-1}}$ and $\sqrt{z-x_p}$. D satisfies the following jumps

$$D_+(z)D_-(z)=s_j,\qquad \text{for } z\in(x_{j-1},x_j),\ j\in\{0,\dots,m+1\}\setminus\{p\}.$$

Using primitives, one can rewrite D as follows

$$D(z) = \prod_{j=0}^{p-2} \left(\frac{\sqrt{z-x_{p-1}}\sqrt{x_p-x_j} - \sqrt{z-x_p}\sqrt{x_{p-1}-x_j}}{\sqrt{z-x_{p-1}}\sqrt{x_p-x_j} + \sqrt{z-x_p}\sqrt{x_{p-1}-x_j}} \right)^{\beta_j \theta(z)}$$

$$\times \prod_{j=p+1}^{m} \left(\frac{\sqrt{z-x_p}\sqrt{x_j-x_{p-1}} - \sqrt{z-x_{p-1}}\sqrt{x_j-x_p}}{\sqrt{z-x_p}\sqrt{x_j-x_{p-1}} + \sqrt{z-x_{p-1}}\sqrt{x_j-x_p}} \right)^{\beta_j \theta(z)},$$

where again the principal branches are taken for $\sqrt{z-x_p}$ and $\sqrt{z-x_{p-1}}$, and

$$\beta_j = \frac{1}{2\pi i} \log \frac{s_j}{s_{j+1}}, \quad j \in \{0, \ldots, m\} \setminus \{p-1, p\} \tag{6.10}$$

$$s_0 = s_{m+1} = 1.$$

As $z \to \infty$, $\Im z > 0$, $D(z) = D_\infty(1 + d_1 z^{-1} + \mathcal{O}(z^{-2}))$, where

$$D_\infty = \prod_{j=0}^{p-2} \left(\frac{\sqrt{x_p-x_j} - \sqrt{x_{p-1}-x_j}}{\sqrt{x_p-x_j} + \sqrt{x_{p-1}-x_j}} \right)^{\beta_j} \times \prod_{j=p+1}^{m} \left(\frac{\sqrt{x_j-x_{p-1}} - \sqrt{x_j-x_p}}{\sqrt{x_j-x_{p-1}} + \sqrt{x_j-x_p}} \right)^{\beta_j},$$

and

$$d_1 = \sum_{j=0}^{p-2} \beta_j \sqrt{x_p-x_j}\sqrt{x_{p-1}-x_j} - \sum_{j=p+1}^{m} \beta_j \sqrt{x_j-x_p}\sqrt{x_j-x_{p-1}}. \tag{6.11}$$

Let us define

$$P^{(\infty)}(z) = \widehat{D} \begin{pmatrix} \frac{\beta(z)}{\sqrt{2}} & -\frac{\beta(z)^{-1}}{\sqrt{2}} \\ \frac{\beta(z)}{\sqrt{2}} & \frac{\beta(z)^{-1}}{\sqrt{2}} \end{pmatrix} N D(z)^{-\sigma_3}, \quad \widehat{D} = \begin{pmatrix} \frac{D_\infty + D_\infty^{-1}}{2} & -\frac{i(D_\infty - D_\infty^{-1})}{2} \\ \frac{i(D_\infty - D_\infty^{-1})}{2} & \frac{D_\infty + D_\infty^{-1}}{2} \end{pmatrix}, \tag{6.12}$$

where $\beta(z) = \sqrt[4]{\frac{z-x_p}{z-x_{p-1}}}$ has branch cuts on $(-\infty, x_{p-1}) \cup (x_p, +\infty)$ and satisfies

$$\beta(z) \sim 1 \quad \text{as } z \to \infty, \Im z > 0 \qquad \text{and} \qquad \beta(z) \sim i \quad \text{as } z \to \infty, \Im z < 0.$$

We verify that $P^{(\infty)}$ satisfies the properties (a), (b) and (c) of the RH problem for $P^{(\infty)}$. Furthermore, after a computation we obtain an explicit expression for $P_1^{(\infty)}$:

$$P_1^{(\infty)} = \begin{pmatrix} \frac{i}{8}(x_p - x_{p-1})(D_\infty^2 - D_\infty^{-2}) & id_1 - \frac{1}{8}(x_p - x_{p-1})(D_\infty^2 + D_\infty^{-2}) \\ -id_1 - \frac{1}{8}(x_p - x_{p-1})(D_\infty^2 + D_\infty^{-2}) & -\frac{i}{8}(x_p - x_{p-1})(D_\infty^2 - D_\infty^{-2}) \end{pmatrix}. \tag{6.13}$$

In the rest of this subsection, we compute the leading terms in the asymptotics of $D(z)$ as $z \to x_j$, $j = 0, \ldots, m$. As $z \to x_j$, $j \neq p, p-1$, $\Im z > 0$, we have

$$D(z) = \sqrt{s_{j+1}}\,(z - x_j)^{\beta_j} \prod_{\substack{k=0 \\ k \neq p-1, p}}^{m} T_{k,j}^{-\beta_k}(1 + \mathcal{O}(z - x_j)), \tag{6.14}$$

where

$$T_{k,j} = \frac{\sqrt{|x_k-x_p|}\sqrt{|x_j-x_{p-1}|} + \sqrt{|x_k-x_{p-1}|}\sqrt{|x_j-x_p|}}{\left|\sqrt{|x_k-x_p|}\sqrt{|x_j-x_{p-1}|} - \sqrt{|x_k-x_{p-1}|}\sqrt{|x_j-x_p|}\right|}, \quad k \neq j,$$

$$T_{j,j} = \frac{4|x_j - x_{p-1}|\,|x_j - x_p|}{x_p - x_{p-1}}. \tag{6.15}$$

As $z \to x_p$, $\Im z > 0$, we have

$$D(z) = \sqrt{s_{p+1}}\bigg(1 - \frac{2d_{x_p}}{\sqrt{x_p - x_{p-1}}}\sqrt{z - x_p} + \mathcal{O}(z - x_p)\bigg), \tag{6.16}$$

with

$$d_{x_p} = \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \beta_j \frac{\sqrt{|x_j - x_{p-1}|}}{\sqrt{|x_j - x_p|}}, \tag{6.17}$$

and as $z \to x_{p-1}$, $\Im z > 0$, we have

$$D(z) = \sqrt{s_{p-1}}\bigg(1 + \frac{2id_{x_{p-1}}}{\sqrt{x_p - x_{p-1}}}\sqrt{z - x_{p-1}} + \mathcal{O}(z - x_{p-1})\bigg), \tag{6.18}$$

with

$$d_{x_{p-1}} = \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \beta_j \frac{\sqrt{|x_j - x_p|}}{\sqrt{|x_j - x_{p-1}|}}. \tag{6.19}$$

6.4. *Local parametrices*

In this subsection, we construct local parametrices $P^{(x_j)}$ around x_j, $j \in \{0, \dots, m\}$. To be more precise, let $\mathcal{D}_{x_j}$ be small open disks centered at x_j, $j = 0, 1, \dots, m$ whose radii are equal to $\frac{\delta}{3}$, where δ is defined in (6.1). The definition of the radii ensures that the disks do not intersect each other. We require $P^{(x_j)}$ to satisfy the same jumps as S in $\mathcal{D}_{x_j}$, and to match with $P^{(\infty)}$ on $\partial\mathcal{D}_{x_j}$ in the sense that

$$P^{(x_j)}(z) = (I + o(1))P^{(\infty)}(z), \qquad \text{as } r \to +\infty, \tag{6.20}$$

is required to hold uniformly for $z \in \partial\mathcal{D}_{x_j}$. Finally, we also require

$$S(z)P^{(x_j)}(z)^{-1} = \mathcal{O}(1), \qquad \text{as } z \to x_j. \tag{6.21}$$

6.4.1. *Local parametrices around x_j, $j \in \{0, \dots, m\} \setminus \{p-1, p\}$.* The construction of $P^{(x_j)}$ for $j \in \{0, 1, \dots, m\} \setminus \{p-1, p\}$ is similar to the one done in Subsection 4.4, and involves confluent hypergeometric functions. The function

$$f_{x_j}(z) = -2\begin{cases} g(z) - g_+(x_j), & \text{if } \Im z > 0, \\ -(g(z) - g_-(x_j)), & \text{if } \Im z < 0, \end{cases}$$

is a conformal map from $\mathcal{D}_{x_j}$ to a neighborhood of 0 whose expansion as $z \to x_j$ is given by

$$f_{x_j}(z) = ic_{x_j}(z - x_j)(1 + \mathcal{O}(z - x_j)),$$
$$c_{x_j} = \begin{cases} \dfrac{x_{p-1} + x_p - 2x_j}{\sqrt{x_{p-1} - x_j}\sqrt{x_p - x_j}}, & \text{if } j = 0, \dots, p-2, \\ \dfrac{2x_j - x_{p-1} - x_p}{\sqrt{x_j - x_{p-1}}\sqrt{x_j - x_p}}, & \text{if } j = p+1, \dots, m. \end{cases} \tag{6.22}$$

Note that $f_{x_j}(\mathbb{R} \cap \mathcal{D}_{x_j}) \subset i\mathbb{R}$. In order to use the model RH problem for Φ_{HG}, we deform the lenses is a small neighborhood of x_j such that the jump contour for $P^{(x_j)}$ is mapped by f_{x_j} to a subset of Σ_{HG} (see Figure A.2), that is,

$$f_{x_j}((\gamma_{j,+} \cup \gamma_{j+1,+}) \cap \mathcal{D}_{x_j}) \subset \Gamma_3 \cup \Gamma_2, \quad f_{x_j}((\gamma_{j,-} \cup \gamma_{j+1,-}) \cap \mathcal{D}_{x_j}) \subset \Gamma_5 \cup \Gamma_6,$$

where Γ_3, Γ_2, Γ_5 and Γ_6 are as shown in Figure A.2. We seek for $P^{(x_j)}$ in the form

$$P^{(x_j)}(z) = E_{x_j}(z)\Phi_{\rm HG}(rf_{x_j}(z);\beta_j)(s_js_{j+1})^{-\frac{\sigma_3}{4}}e^{-rg(z)\sigma_3}, \tag{6.23}$$

for a suitable analytic matrix valued function E_{x_j}. We recall that the parameter β_j in (6.23) is given by (6.10). Since E_{x_j} is analytic, it is straightforward to see from the jumps of $\Phi_{\rm HG}$ (given by (A.4)) that $P^{(x_j)}$ given by (6.23) satisfies the same jumps as S inside $\mathcal{D}_{x_j}$. In view of (A.5), we see that to satisfy the matching condition (6.20), we are forced to define E_{x_j} by

$$E_{x_j}(z) = P^{(\infty)}(z)(s_js_{j+1})^{\frac{\sigma_3}{4}}\begin{Bmatrix} \sqrt{\dfrac{s_{j+1}}{s_j}}^{\sigma_3}, & \Im z>0 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, & \Im z<0 \end{Bmatrix} e^{rg_+(x_j)\sigma_3}(rf_{x_j}(z))^{\beta_j\sigma_3}. \tag{6.24}$$

It can be verified from the jumps for $P^{(\infty)}$ that E_{x_j} defined by (6.24) has no jump at all inside $\mathcal{D}_{x_j}$. Furthermore, using (6.14), we verify that $E_{x_j}(z)$ is bounded as $z\to x_j$ and E_{x_j} is then analytic in the whole disk $\mathcal{D}_{x_j}$, as desired. Since $P^{(x_j)}$ and S have exactly the same jumps on $(\mathbb{R}\cup\gamma_+\cup\gamma_-)\cap\mathcal{D}_{x_j}$, $S(z)P^{(x_j)}(z)^{-1}$ is analytic in $\mathcal{D}_{x_j}\setminus\{x_j\}$. As $z\to x_j$ from outside the lenses, by condition (d) in the RH problem for S and by (A.7), $S(z)P^{(x_j)}(z)^{-1}$ behaves as $\mathcal{O}(\log(z-x_j))$. This means that x_j is a removable singularity of $S(z)P^{(x_j)}(z)^{-1}$ and therefore (6.21) holds. We will need later a more detailed matching condition than (6.20), which can be obtained from (A.5):

$$P^{(x_j)}(z)P^{(\infty)}(z)^{-1} = I + \frac{1}{rf_{x_j}(z)}E_{x_j}(z)\Phi_{{\rm HG},1}(\beta_j)E_{x_j}(z)^{-1} + \mathcal{O}(r^{-2}), \tag{6.25}$$

as $r\to+\infty$, uniformly for $z\in\partial\mathcal{D}_{x_j}$, where $\Phi_{{\rm HG},1}(\beta_j)$ is given by (A.6). Also, using (6.12), (6.14)–(6.15) and (6.22), we note for later use that

$$E_{x_j}(x_j) = \frac{1}{\sqrt{2}}\widehat{D}\begin{pmatrix} 1 & -1 \\ 1 & 1\end{pmatrix}\left(\frac{|x_j-x_p|}{|x_j-x_{p-1}|}\right)^{\frac{\sigma_3}{4}}N\Lambda_j^{\sigma_3}, \tag{6.26}$$

where

$$\Lambda_j = e^{rg_+(x_j)}(T_{j,j}c_{x_j}r)^{\beta_j}\prod_{\substack{k=0\\ k\neq j,p-1,p}}^{m}T_{k,j}^{\beta_k}. \tag{6.27}$$

6.4.2. *Local parametrix around x_p.* For the local parametrix $P^{(x_p)}$, we need to use another model RH problem whose solution $\Phi_{\rm Be}$ is expressed in terms of Bessel functions. This model RH problem is well known, see, for example, [**44**], and is recalled in Subsection A.1 for the convenience of the reader. Consider the function

$$f_{x_p}(z) = -\frac{g(z)^2}{4} = \frac{(z-x_{p-1})(z-x_p)}{4}.$$

This is a conformal map from $\mathcal{D}_{x_p}$ to a neighborhood of 0 whose expansion as $z\to x_p$ is given by

$$f_{x_p}(z) = c_{x_p}^2(z-x_p)\left(1+\frac{z-x_p}{x_p-x_{p-1}}+\mathcal{O}\big((z-x_p)^2\big)\right), \quad c_{x_p} = \frac{\sqrt{x_p-x_{p-1}}}{2}>0. \tag{6.28}$$

We choose the lenses such that they are mapped by $-f_{x_p}$ onto a subset of $\Sigma_{\rm Be}$ ($\Sigma_{\rm Be}$ is the jump contour for $\Phi_{\rm Be}$, see Figure A.1):

$$-f_{x_p}(\gamma_{p+1,+})\subset e^{-\frac{2\pi i}{3}}\mathbb{R}^+, \quad -f_{x_p}(\gamma_{p+1,-})\subset e^{\frac{2\pi i}{3}}\mathbb{R}^+.$$

If we take $P^{(x_p)}$ of the form

$$P^{(x_p)}(z) = E_{x_p}(z)\sigma_3\Phi_{\mathrm{Be}}(-r^2 f_{x_p}(z))\sigma_3 s_{p+1}^{-\frac{\sigma_3}{2}} e^{-rg(z)\sigma_3}, \tag{6.29}$$

with E_{x_p} analytic in $\mathcal{D}_{x_p}$, then it is straightforward to verify from (A.1) that $P^{(x_p)}$ has the same jumps as S in $\mathcal{D}_{x_p}$. To satisfy the matching condition, by (A.2), we need to define E_{x_p} by

$$E_{x_p}(z) = P^{(\infty)}(z)s_{p+1}^{\frac{\sigma_3}{2}} N\Big(2\pi r(-f_{x_p}(z))^{1/2}\Big)^{\frac{\sigma_3}{2}},$$

where we take the principal branches for the square roots. We verify from the jumps for $P^{(\infty)}$ that E_{x_p} has no jumps in $\mathcal{D}_{x_p}$, and has a removable singularity at x_p; therefore E_{x_p} is analytic in $\mathcal{D}_{x_p}$, as required. We will need later a more detailed matching condition than (6.20), which can be obtained using (A.2):

$$P^{(x_p)}(z)P^{(\infty)}(z)^{-1} = I + \frac{1}{r(-f_{x_p}(z))^{1/2}} P^{(\infty)}(z)s_{p+1}^{\frac{\sigma_3}{2}}\sigma_3\Phi_{\mathrm{Be},1}\sigma_3 s_{p+1}^{-\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1} + \mathcal{O}(r^{-2}), \tag{6.30}$$

as $r \to +\infty$ uniformly for $z \in \partial\mathcal{D}_{x_p}$, where $\Phi_{\mathrm{Be},1}$ is given below (A.2). Using (6.12), (6.16), (6.28) and the expansion

$$(-f_{x_p}(z))^{1/2} = -i\, c_{x_p}\sqrt{z - x_p}(1 + \mathcal{O}(z - x_p)), \quad \text{as } z \to x_p,\ \Im z > 0, \tag{6.31}$$

we obtain $E_{x_p}(x_p)$ by taking the limit of $E_{x_p}(z)$ as $z \to x_p$ from the upper half plane:

$$E_{x_p}(x_p) = \frac{1}{\sqrt{2}}\widehat{D}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & i \\ i & -2d_{x_p} \end{pmatrix} e^{-\frac{\pi i}{4}\sigma_3}(\pi(x_p - x_{p-1})r)^{\frac{\sigma_3}{2}}. \tag{6.32}$$

6.4.3. *Local parametrix around* x_{p-1}. The local parametrix $P^{(x_{p-1})}$ is also constructed in terms of Bessel functions, and relies on the model RH problem Φ_{Be}. The function

$$f_{x_{p-1}}(z) = \frac{g(z)^2}{4} = -\frac{(z - x_{p-1})(z - x_p)}{4} \tag{6.33}$$

is a conformal map from $\mathcal{D}_{x_{p-1}}$ to a neighborhood of 0 whose expansion as $z \to x_{p-1}$ is given by

$$f_{x_{p-1}}(z) = c_{x_{p-1}}^2(z - x_{p-1})\Big(1 - \frac{z - x_{p-1}}{x_p - x_{p-1}} + \mathcal{O}\big((z - x_{p-1})^2\big)\Big),$$
$$c_{x_{p-1}} = \frac{\sqrt{x_p - x_{p-1}}}{2} > 0. \tag{6.34}$$

In a neighborhood of x_{p-1}, we deform the lenses such that

$$f_{x_{p-1}}(\gamma_{p-1,+}) \subset e^{\frac{2\pi i}{3}}\mathbb{R}^+, \quad f_{x_{p-1}}(\gamma_{p-1,-}) \subset e^{-\frac{2\pi i}{3}}\mathbb{R}^+.$$

In this way, the jump contour for $P^{(x_{p-1})}$ is mapped by $f_{x_{p-1}}$ onto a subset of Σ_{Be}. We take $P^{(x_{p-1})}$ of the form

$$P^{(x_{p-1})}(z) = E_{x_{p-1}}(z)\Phi_{\mathrm{Be}}(r^2 f_{x_{p-1}}(z))s_{p-1}^{-\frac{\sigma_3}{2}} e^{-rg(z)\sigma_3}, \tag{6.35}$$

where $E_{x_{p-1}}$ is analytic in $\mathcal{D}_{x_{p-1}}$. Using (A.1), it is straightforward to see that $P^{(x_{p-1})}$ has the same jumps as S in $\mathcal{D}_{x_{p-1}}$. To satisfy the matching condition (6.20), using (A.2) we conclude that $E_{x_{p-1}}$ needs to be defined as follows:

$$E_{x_{p-1}}(z) = P^{(\infty)}(z)s_{p-1}^{\frac{\sigma_3}{2}} N^{-1}\Big(2\pi r(f_{x_{p-1}}(z))^{1/2}\Big)^{\frac{\sigma_3}{2}}.$$

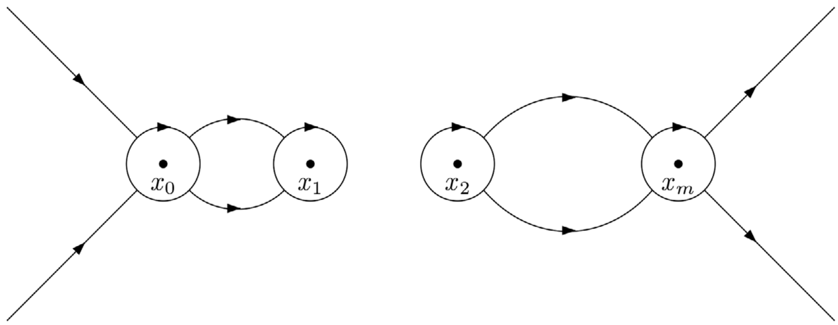

FIGURE 7. *Jump contours Σ_R for the RH problem for R with $m = 3$ and $p = 2$.*

It can be verified from the jumps for $P^{(\infty)}$ that $E_{x_{p-1}}$ has no jumps in $\mathcal{D}_{x_{p-1}}$ and has a removable singularity at x_{p-1}. We conclude that $E_{x_{p-1}}$ is analytic in $\mathcal{D}_{x_{p-1}}$, as required. We will need later a more detailed matching condition than (6.20), which can be obtained using (A.2):

$$P^{(x_{p-1})}(z)P^{(\infty)}(z)^{-1} = I + \frac{1}{r(f_{x_{p-1}}(z))^{1/2}} P^{(\infty)}(z) s_{p-1}^{\frac{\sigma_3}{2}} \Phi_{\mathrm{Be},1} s_{p-1}^{-\frac{\sigma_3}{2}} P^{(\infty)}(z)^{-1} + \mathcal{O}(r^{-2}), \tag{6.36}$$

as $r \to +\infty$ uniformly for $z \in \partial\mathcal{D}_{x_{p-1}}$, where $\Phi_{\mathrm{Be},1}$ is given below (A.2). Furthermore, using (6.12), (6.18) and (6.33), one shows that

$$E_{x_{p-1}}(x_{p-1}) = \frac{1}{\sqrt{2}} \widehat{D} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2id_{x_{p-1}} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i}{4}\sigma_3} (\pi(x_p - x_{p-1})r)^{\frac{\sigma_3}{2}}. \tag{6.37}$$

6.5. *Small norm problem*

The last transformation of the steepest descent is defined by

$$R(z) = \begin{cases} S(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus \bigcup_{j=0}^{m} \mathcal{D}_{x_j}, \\ S(z)P^{(x_j)}(z)^{-1}, & \text{for } z \in \mathcal{D}_{x_j},\ j \in \{0, 1, \ldots, m\}. \end{cases} \tag{6.38}$$

It follows from the analysis of Subsection 6.4 that R is analytic inside the $m+1$ disks. Since the jumps of $P^{(\infty)}$ and of S are the same on (x_{j-1}, x_j), $j = 1, \ldots, m$, we conclude that R is analytic on $\mathbb{C} \setminus \Sigma_R$, where

$$\Sigma_R = \bigcup_{j=0}^{m} \partial\mathcal{D}_{x_j} \cup \left((\gamma_+ \cup \gamma_-) \setminus \bigcup_{j=0}^{m} \mathcal{D}_{x_j} \right),$$

see Figure 7. From Lemma 6.1 and the fact that $P^{(\infty)}$ is independent of r (see (6.12)), we infer that the jumps $J_R := R_-^{-1} R_+$ satisfy

$$J_R(z) = P^{(\infty)}(z) S_-(z)^{-1} S_+(z) P^{(\infty)}(z)^{-1} = I + \mathcal{O}(e^{-c|z|r}), \quad \text{as } r \to +\infty, \tag{6.39}$$

uniformly for $z \in \Sigma_R \cap (\gamma_+ \cup \gamma_-)$, for a certain $c > 0$ independent of z and r. Let us orient the boundaries of the disks in the clockwise direction as shown in Figure 7. For $z \in \bigcup_{j=0}^{m} \partial\mathcal{D}_{x_j}$, from (6.25), (6.30) and (6.36), we have

$$J_R(z) = P^{(\infty)}(z) P^{(x_j)}(z)^{-1} = I + \mathcal{O}\left(\frac{1}{r}\right), \quad \text{as } r \to +\infty. \tag{6.40}$$

Therefore, R satisfies a small norm RH problem. By standard theory for small norm RH problems [**24, 25**], R exists for sufficiently large r and satisfies

$$R(z) = I + \frac{R^{(1)}(z)}{r} + \mathcal{O}(r^{-2}), \quad R^{(1)}(z) = \mathcal{O}(1), \qquad \text{as } r \to +\infty \tag{6.41}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$. For any $j \in \{0, \ldots, m\} \setminus \{p-1, p\}$, we see from (6.24) that some factors $r^{\pm\beta_j}$ are present in the entries of E_{x_j}. Hence, by (6.25), some factors of the form $r^{\pm 2\beta_j}$ also appear in the entries of J_R, and therefore

$$\partial_{\beta_j} R(z) = \frac{\partial_{\beta_j} R^{(1)}(z)}{r} + \mathcal{O}\left(\frac{\log r}{r^2}\right), \quad \partial_{\beta_j} R^{(1)}(z) = \mathcal{O}(\log r), \quad \text{as } r \to +\infty. \tag{6.42}$$

Furthermore, since the asymptotics (6.39) and (6.40) hold uniformly for $\beta_0, \ldots, \beta_{p-2}, \beta_{p+1}, \ldots, \beta_m$ in compact subsets of $i\mathbb{R}$, and uniformly in $x_0, x_1, \ldots, x_m$ in compact subsets of $\mathbb{R}$ such that (6.1) holds, the asymptotics (6.41) and (6.42) also hold uniformly in $\beta_0, \ldots, \beta_{p-2}, \beta_{p+1}, \ldots, \beta_m, x_0, \ldots, x_m$ in the same way.

Now, we compute explicitly $R^{(1)}(x_p)$, $R^{(1)}(x_{p-1})$ and $R^{(1)}(z)$ for $z \in \mathbb{C} \setminus \bigcup_{j=0}^m \mathcal{D}_{x_j}$. As in (4.34), $R^{(1)}$ admits the following integral representation

$$R^{(1)}(z) = \frac{1}{2\pi i} \int_{\bigcup_{j=0}^m \partial \mathcal{D}_{x_j}} \frac{J_R^{(1)}(s)}{s - z} ds,$$

where $J_R^{(1)}$ is defined via the expansion

$$J_R(z) = I + \frac{J_R^{(1)}(z)}{r} + \mathcal{O}(r^{-2}), \quad J_R^{(1)}(z) = \mathcal{O}(1), \quad \text{as } r \to +\infty, \quad z \in \bigcup_{j=0}^m \partial \mathcal{D}_{x_j}.$$

Recall that $J_R^{(1)}(z)$ is defined only for z on the boundaries of the disks. However, from the explicit expressions for $J_R^{(1)}$ given by (6.25), (6.30), and (6.36), we see that $J_R^{(1)}$ can be analytically continued on $\bigcup_{j=0}^m \overline{\mathcal{D}_{x_j}} \setminus \{x_j\}$, and that $J_R^{(1)}$ has a pole of order 1 at each of the functions x_j. Since the disks are oriented in the clockwise direction, a direct residue calculation shows that

$$R^{(1)}(z) = \sum_{j=0}^m \frac{1}{z - x_j} \mathrm{Res}(J_R^{(1)}(s), s = x_j), \qquad \text{for } z \in \mathbb{C} \setminus \bigcup_{j=0}^m \mathcal{D}_{x_j}, \tag{6.43}$$

$$R^{(1)}(x_p) = \sum_{\substack{j=0 \\ j \neq p}}^m \frac{1}{x_p - x_j} \mathrm{Res}(J_R^{(1)}(s), s = x_j) - \mathrm{Res}\left(\frac{J_R^{(1)}(s)}{s - x_p}, s = x_p\right), \tag{6.44}$$

$$R^{(1)}(x_{p-1}) = \sum_{\substack{j=0 \\ j \neq p-1}}^m \frac{1}{x_{p-1} - x_j} \mathrm{Res}(J_R^{(1)}(s), s = x_j) - \mathrm{Res}\left(\frac{J_R^{(1)}(s)}{s - x_{p-1}}, s = x_{p-1}\right). \tag{6.45}$$

Using (6.22) and (6.25)–(6.26), for $j \in \{0, \ldots, m\} \setminus \{p-1, p\}$, we obtain

$$\begin{aligned} &\mathrm{Res}\left(J_R^{(1)}(s), s = x_j\right) \\ &= \frac{\beta_j^2}{2ic_{x_j}} \widehat{D} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \left(\frac{|x_j - x_p|}{|x_j - x_{p-1}|}\right)^{\frac{\sigma_3}{4}} N \begin{pmatrix} -1 & \widetilde{\Lambda}_{j,1} \\ -\widetilde{\Lambda}_{j,2} & 1 \end{pmatrix} N^{-1} \\ &\quad \times \left(\frac{|x_j - x_p|}{|x_j - x_{p-1}|}\right)^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \widehat{D}^{-1} \end{aligned}$$

$$= \frac{\beta_j^2}{4c_{x_j}} \widehat{D} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \left(\frac{|x_j - x_p|}{|x_j - x_{p-1}|} \right)^{\frac{\sigma_3}{4}} \begin{pmatrix} -\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} & -i(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} + 2i) \\ -i(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} - 2i) & \widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2} \end{pmatrix}$$

$$\times \left(\frac{|x_j - x_p|}{|x_j - x_{p-1}|} \right)^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \widehat{D}^{-1},$$

where

$$\widetilde{\Lambda}_{j,1} = \tau(\beta_j)\Lambda_j^2 \quad \text{and} \quad \widetilde{\Lambda}_{j,2} = \tau(-\beta_j)\Lambda_j^{-2}. \tag{6.46}$$

Using (6.12), (6.16), (6.28), (6.30), and (6.31), we obtain

$$\text{Res}\Big(J_R^{(1)}(s), s = x_p\Big) = \frac{1}{16} \widehat{D} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \widehat{D}^{-1}, \tag{6.47}$$

and by (6.12), (6.18), (6.34), and (6.36), we have

$$\text{Res}\Big(J_R^{(1)}(s), s = x_{p-1}\Big) = \frac{1}{16} \widehat{D} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \widehat{D}^{-1}. \tag{6.48}$$

In the same way as we derived the residues (6.47) and (6.48), but with more efforts, we also obtain

$$\text{Res}\left(\frac{J_R^{(1)}(s)}{s - x_p}, s = x_p \right) = \frac{\widehat{D}}{16(x_p - x_{p-1})} \begin{pmatrix} 3 + 16d_{x_p}^2 & -3 + 16d_{x_p}^2 + 16id_{x_p} \\ 3 - 16d_{x_p}^2 + 16id_{x_p} & -3 - 16d_{x_p}^2 \end{pmatrix} \widehat{D}^{-1}$$

and

$$\text{Res}\left(\frac{J_R^{(1)}(s)}{s - x_{p-1}}, s = x_{p-1} \right)$$

$$= \frac{\widehat{D}}{16(x_p - x_{p-1})} \begin{pmatrix} 3 + 16d_{x_{p-1}}^2 & 3 - 16d_{x_{p-1}}^2 + 16id_{x_{p-1}} \\ -3 + 16d_{x_{p-1}}^2 + 16id_{x_{p-1}} & -3 - 16d_{x_{p-1}}^2 \end{pmatrix} \widehat{D}^{-1}.$$

7. *Proof of Theorem* 1.2

We prove Theorem 1.2 using the same strategy as in Section 5. First, we use the RH analysis done in Section 6 to find large r asymptotics for

$$\partial_{s_k} \log F(r\vec{x}, \vec{s}) = K_\infty + \sum_{j=0}^{m} K_{x_j}, \quad k = 1, \ldots, p-1, p+1, \ldots, m.$$

The above identity was obtained in (3.12) and the quantities K_∞ and K_{x_j} are defined in (3.13)–(3.14). Then, we integrate these asymptotics over the parameters $s_1, \ldots, s_{p-1}, s_{p+1}, \ldots, s_m$.

7.1. *Large r asymptotics for $\partial_{s_k} \log F(r\vec{x}, \vec{s})$*

Asymptotics for K_∞. Using (6.8), (6.38), and (6.9), we obtain

$$T_1 = R_1 + P_1^{(\infty)},$$

where R_1 is the z^{-1} coefficient in the large z expansion of $R(z)$. Hence, by (6.41), we have

$$T_1 = P_1^{(\infty)} + \frac{R_1^{(1)}}{r} + \mathcal{O}(r^{-2}), \quad \text{as } r \to +\infty,$$

where $R_1^{(1)}$ is defined through the expansion

$$R^{(1)}(z) = \frac{R_1^{(1)}}{z} + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty.$$

Hence, using (6.13) and (6.43), we get

$$\begin{aligned} T_1 = & \begin{pmatrix} \frac{i}{8}(x_p - x_{p-1})(D_\infty^2 - D_\infty^{-2}) & id_1 - \frac{1}{8}(x_p - x_{p-1})(D_\infty^2 + D_\infty^{-2}) \\ -id_1 - \frac{1}{8}(x_p - x_{p-1})(D_\infty^2 + D_\infty^{-2}) & -\frac{i}{8}(x_p - x_{p-1})(D_\infty^2 - D_\infty^{-2}) \end{pmatrix} \\ & + \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \frac{\beta_j^2}{4c_{x_j} r} \widehat{D} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \left(\frac{|x_j - x_p|}{|x_j - x_{p-1}|} \right)^{\frac{\sigma_3}{4}} \\ & \times \begin{pmatrix} -\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} & -i(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} + 2i) \\ -i(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} - 2i) & \widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2} \end{pmatrix} \left(\frac{|x_j - x_p|}{|x_j - x_{p-1}|} \right)^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \widehat{D}^{-1} \\ & + \frac{1}{16r} \widehat{D} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \widehat{D}^{-1} + \frac{1}{16r} \widehat{D} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \widehat{D}^{-1} + \mathcal{O}(r^{-2}), \qquad \text{as } r \to +\infty, \end{aligned}$$

which implies, by (3.13), (6.6) and (6.42), that

$$\begin{aligned} K_\infty = & \, r(\partial_{s_k} T_{1,21} - \partial_{s_k} T_{1,12}) = -2i \partial_{s_k} d_1 r \\ & + \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \frac{|x_j - x_{p-1}| \partial_{s_k} \left(\beta_j^2 (\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} - 2i) \right) - |x_j - x_p| \partial_{s_k} \left(\beta_j^2 (\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} + 2i) \right)}{2ic_{x_j} \sqrt{|x_j - x_{p-1}| \, |x_j - x_p|}} \\ & + \mathcal{O}\left(\frac{\log r}{r} \right) \end{aligned} \tag{7.1}$$

as $r \to +\infty$.

Asymptotics for K_{x_j} with $j \in \{0, \ldots, p-2, p+1, \ldots, m\}$. For z outside the lenses and inside $\mathcal{D}_{x_j}$, by (6.7), (6.38), and (6.23), we have

$$T(z) = R(z) E_{x_j}(z) \Phi_{\mathrm{HG}}(r f_{x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} e^{-rg(z)\sigma_3}, \tag{7.2}$$

and by (6.22) and (A.8), we also have

$$\Phi_{\mathrm{HG}}(r f_{x_j}(z); \beta_j) = \widehat{\Phi}_{\mathrm{HG}}(r f_{x_j}(z); \beta_j), \qquad \text{for } \Im z > 0.$$

Using (6.10) and Euler's reflection formula (see, for example, [**47**, equation 5.5.3]), we note that

$$\frac{\sin(\pi \beta_j)}{\pi} = \frac{1}{\Gamma(\beta_j)\Gamma(1-\beta_j)} = -\frac{s_{j+1} - s_j}{2\pi i \sqrt{s_j s_{j+1}}}, \qquad j = 0, \ldots, p-2, p+1, \ldots, m. \tag{7.3}$$

This identity, combined with (6.22) and (A.9), implies

$$\begin{aligned} & \Phi_{\mathrm{HG}}(r f_{x_j}(z); \beta_j)(s_j s_{j+1})^{-\frac{\sigma_3}{4}} \\ & = \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix} (I + \mathcal{O}(z - x_j)) \begin{pmatrix} 1 & -\frac{s_{j+1} - s_j}{2\pi i} \log(r(z - x_j)) \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{7.4}$$

as $z \to x_j$ from $\Im z > 0$ and outside the lenses, where the principal branch is taken for the log and

$$\Psi_{j,11} = \frac{\Gamma(1-\beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, \quad \Psi_{j,12} = \frac{(s_j s_{j+1})^{\frac{1}{4}}}{\Gamma(\beta_j)} \left(\log c_{x_j} - \frac{i\pi}{2} + \frac{\Gamma'(1-\beta_j)}{\Gamma(1-\beta_j)} + 2\gamma_{\mathrm{E}} \right),$$

$$\Psi_{j,21} = \frac{\Gamma(1+\beta_j)}{(s_j s_{j+1})^{\frac{1}{4}}}, \quad \Psi_{j,22} = \frac{-(s_j s_{j+1})^{\frac{1}{4}}}{\Gamma(-\beta_j)} \left(\log c_{x_j} - \frac{i\pi}{2} + \frac{\Gamma'(-\beta_j)}{\Gamma(-\beta_j)} + 2\gamma_{\mathrm{E}} \right), \tag{7.5}$$

and using $\Gamma(1+z) = z\Gamma(z)$ and (7.3), we verify that

$$\Psi_{j,11}\Psi_{j,21} = -\beta_j \frac{2\pi i}{s_{j+1} - s_j}, \quad j = 0, \ldots, p-2, p+1, \ldots, m. \tag{7.6}$$

From (2.12), (6.5), (7.2) and (7.4), we get

$$G_j(rx_j; r\vec{x}, \vec{s}) = \begin{pmatrix} \cos\left(\frac{r}{2}(x_{p-1}+x_p)\right) & -\sin\left(\frac{r}{2}(x_{p-1}+x_p)\right) \\ \sin\left(\frac{r}{2}(x_{p-1}+x_p)\right) & \cos\left(\frac{r}{2}(x_{p-1}+x_p)\right) \end{pmatrix} R(x_j) E_{x_j}(x_j) \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix}.$$

Also, from (6.12), we see that

$$\widehat{D}_{11}\partial_{s_k}\widehat{D}_{21} - \widehat{D}_{21}\partial_{s_k}\widehat{D}_{11} = i\partial_{s_k}\log D_\infty.$$

Therefore, using (6.41), (6.42), and $\det E_{x_j}(x_j) = 1$ in the definition of K_{x_j} given by (3.14), we obtain after a long calculation that

$$\begin{aligned} \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} K_{x_j} = & \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} -\frac{s_{j+1}-s_j}{2\pi i}\left(\Psi_{j,11}\partial_{s_k}\Psi_{j,21} - \Psi_{j,21}\partial_{s_k}\Psi_{j,11}\right) \\ & - \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} 2\beta_j \partial_{s_k}\log\Lambda_j + \frac{i}{2}\partial_{s_k}\log D_\infty \\ & \times \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} -\frac{s_{j+1}-s_j}{2\pi i}\left(\frac{\sqrt{|x_j-x_p|}}{\sqrt{|x_j-x_{p-1}|}}\left(\Lambda_j\Psi_{j,11} + i\Lambda_j^{-1}\Psi_{j,21}\right)^2 \right. \\ & \left. - \frac{\sqrt{|x_j-x_{p-1}|}}{\sqrt{|x_j-x_p|}}\left(\Lambda_j\Psi_{j,11} - i\Lambda_j^{-1}\Psi_{j,21}\right)^2 \right) \\ & + \mathcal{O}\left(\frac{\log r}{r}\right), \quad \text{as } r \to +\infty. \end{aligned} \tag{7.7}$$

Using (7.3), (7.5), (7.6), and the definitions (6.46) of $\widetilde{\Lambda}_{j,1}$ and $\widetilde{\Lambda}_{j,2}$, for $j = 1, \ldots, p-2, p+1, \ldots, m$, we get

$$-\frac{s_{j+1}-s_j}{2\pi i}\left(\Lambda_j\Psi_{j,11} + i\Lambda_j^{-1}\Psi_{j,21}\right)^2 = \beta_j^2(\widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2}) + 2i\beta_j,$$

$$-\frac{s_{j+1}-s_j}{2\pi i}\left(\Lambda_j\Psi_{j,11} - i\Lambda_j^{-1}\Psi_{j,21}\right)^2 = \beta_j^2(\widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2}) - 2i\beta_j,$$

$$-\frac{s_{j+1}-s_j}{2\pi i}\left(\Psi_{j,11}\partial_{s_k}\Psi_{j,21} - \Psi_{j,21}\partial_{s_k}\Psi_{j,11}\right) = \beta_j\partial_{s_k}\log\frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j)}.$$

Substituting the above identities in (7.7), we finally arrive at

$$\sum_{\substack{j=0\\ j\neq p-1,p}}^{m} K_{x_j} = \sum_{\substack{j=0\\ j\neq p-1,p}}^{m} \beta_j \partial_{s_k} \log \frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j)} - \sum_{\substack{j=0\\ j\neq p-1,p}}^{m} 2\beta_j \partial_{s_k} \log \Lambda_j$$
$$+\frac{i}{2}\partial_{s_k} \log D_\infty \sum_{\substack{j=0\\ j\neq p-1,p}}^{m} \beta_j^2(\widetilde{\Lambda}_{j,1}+\widetilde{\Lambda}_{j,2})\left(\frac{\sqrt{|x_j-x_p|}}{\sqrt{|x_j-x_{p-1}|}} - \frac{\sqrt{|x_j-x_{p-1}|}}{\sqrt{|x_j-x_p|}}\right)$$
$$+\frac{i}{2}\partial_{s_k} \log D_\infty \sum_{\substack{j=0\\ j\neq p-1,p}}^{m} 2i\beta_j\left(\frac{\sqrt{|x_j-x_p|}}{\sqrt{|x_j-x_{p-1}|}} + \frac{\sqrt{|x_j-x_{p-1}|}}{\sqrt{|x_j-x_p|}}\right) + \mathcal{O}\left(\frac{\log r}{r}\right),$$

as $r \to +\infty$.

Asymptotics for K_{x_p}. It follows from (6.7), (6.38), and (6.29) that for $z \in \mathcal{D}_{x_p}$, z outside the lenses, we have

$$T(z) = R(z)E_{x_p}(z)\sigma_3\Phi_{\mathrm{Be}}(-r^2 f_{x_p}(z))\sigma_3\sqrt{s_{p+1}}^{-\sigma_3}e^{-rg(z)\sigma_3}. \tag{7.8}$$

Using (A.3) and (6.28), we obtain

$$\sigma_3\Phi_{\mathrm{Be}}(-r^2 f_{x_p}(z))\sqrt{s_{p+1}}^{-\sigma_3}\sigma_3 = \begin{pmatrix} \Psi_{p,11} & \Psi_{p,12} \\ \Psi_{p,21} & \Psi_{p,22} \end{pmatrix}(I+\mathcal{O}(z-x_p))\begin{pmatrix} 1 & -\frac{s_{p+1}}{2\pi i}\log(r(z-x_p)) \\ 0 & 1 \end{pmatrix}$$

as $z \to x_p$ from $\Im z > 0$ and outside the lenses, where

$$\Psi_{p,11} = s_{p+1}^{-1/2}, \qquad \Psi_{p,12} = -s_{p+1}^{1/2}\left(\frac{\gamma_{\mathrm{E}}}{\pi i} + \frac{\log(c_{x_p}^2 r) - \pi i}{2\pi i}\right),$$
$$\Psi_{p,21} = 0, \qquad \Psi_{p,22} = s_{p+1}^{1/2}.$$

On the other hand, using (2.12) and (6.5), as $z \to x_p$, $\Im z > 0$, we also have

$$T(z) = \begin{pmatrix} \cos\left(\frac{r}{2}(x_{p-1}+x_p)\right) & \sin\left(\frac{r}{2}(x_{p-1}+x_p)\right) \\ -\sin\left(\frac{r}{2}(x_{p-1}+x_p)\right) & \cos\left(\frac{r}{2}(x_{p-1}+x_p)\right) \end{pmatrix} G_p(rz; r\vec{x}, \vec{s})$$
$$\times \begin{pmatrix} 1 & -\frac{s_{p+1}}{2\pi i}\log(r(z-x_p)) \\ 0 & 1 \end{pmatrix} e^{-rg(z)\sigma_3}. \tag{7.9}$$

By combining (7.8) with (7.9), we arrive at

$$G_p(rx_p; r\vec{x}, \vec{s}) = \begin{pmatrix} \cos\left(\frac{r}{2}(x_{p-1}+x_p)\right) & -\sin\left(\frac{r}{2}(x_{p-1}+x_p)\right) \\ \sin\left(\frac{r}{2}(x_{p-1}+x_p)\right) & \cos\left(\frac{r}{2}(x_{p-1}+x_p)\right) \end{pmatrix} R(x_p)E_{x_p}(x_p)$$
$$\times \begin{pmatrix} \Psi_{p,11} & \Psi_{p,12} \\ \Psi_{p,21} & \Psi_{p,22} \end{pmatrix}.$$

From the definition (3.14) of K_{x_p} and the explicit expressions for $E_{x_p}(x_p)$ and $R^{(1)}(x_p)$ given by (6.32) and (6.44), we find after a computation that

$$\begin{aligned} K_{x_p} &= -\frac{ir}{2}(x_p - x_{p-1})\partial_{s_k} \log D_\infty + \partial_{s_k} \log D_\infty \\ &\quad \times \left(d_{x_p} + \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \frac{x_p - x_{p-1}}{2ic_{x_j}(x_p - x_j)} \beta_j^2 (\widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2}) \right) \\ &\quad + \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \frac{\sqrt{|x_j - x_p|}(x_p - x_{p-1})}{4i\sqrt{|x_j - x_{p-1}|}(x_p - x_j)c_{x_j}} \partial_{s_k}\left(\beta_j^2(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} + 2i)\right) + \mathcal{O}\left(\frac{\log r}{r}\right) \end{aligned} \tag{7.10}$$

as $r \to +\infty$.

Asymptotics for $K_{x_{p-1}}$. For z outside the lenses and inside $\mathcal{D}_{x_{p-1}}$, we deduce from (6.7), (6.35) and (6.38) that

$$T(z) = R(z) E_{x_{p-1}}(z) \Phi_{\mathrm{Be}}(r^2 f_{x_{p-1}}(z)) \sqrt{s_{p-1}}^{-\sigma_3} e^{-rg(z)\sigma_3}. \tag{7.11}$$

Also, using (A.3) and (6.34), we get

$$\begin{aligned} \Phi_{\mathrm{Be}}(r^2 f_{x_{p-1}}(z)) \sqrt{s_{p-1}}^{-\sigma_3} &= \begin{pmatrix} \Psi_{p-1,11} & \Psi_{p-1,12} \\ \Psi_{p-1,21} & \Psi_{p-1,22} \end{pmatrix} (I + \mathcal{O}(z - x_{p-1})) \\ &\quad \times \begin{pmatrix} 1 & \frac{s_{p-1}}{2\pi i} \log(r(z - x_{p-1})) \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

as $z \to x_{p-1}$ from $\Im z > 0$ and outside the lenses, where

$$\Psi_{p-1,11} = s_{p-1}^{-1/2}, \qquad \Psi_{p-1,12} = s_{p-1}^{1/2}\left(\frac{\gamma_{\mathrm{E}}}{\pi i} + \frac{\log(c_{x_{p-1}}^2 r)}{2\pi i}\right),$$

$$\Psi_{p-1,21} = 0, \qquad \Psi_{p-1,22} = s_{p-1}^{1/2}.$$

On the other hand, using (2.12) and (6.5), as $z \to x_{p-1}$ from $\Im z > 0$ we also have that

$$\begin{aligned} T(z) &= \begin{pmatrix} \cos\left(\frac{r}{2}(x_{p-1} + x_p)\right) & \sin\left(\frac{r}{2}(x_{p-1} + x_p)\right) \\ -\sin\left(\frac{r}{2}(x_{p-1} + x_p)\right) & \cos\left(\frac{r}{2}(x_{p-1} + x_p)\right) \end{pmatrix} \\ &\quad \times G_{p-1}(rz; r\vec{x}, \vec{s}) \begin{pmatrix} 1 & \frac{s_{p-1}}{2\pi i} \log(r(z - x_{p-1})) \\ 0 & 1 \end{pmatrix} e^{-rg(z)\sigma_3}. \end{aligned}$$

Combining (7.11) with (7.12), we arrive at

$$\begin{aligned} G_{p-1}(rx_{p-1}; r\vec{x}, \vec{s}) &= \begin{pmatrix} \cos\left(\frac{r}{2}(x_{p-1} + x_p)\right) & -\sin\left(\frac{r}{2}(x_{p-1} + x_p)\right) \\ \sin\left(\frac{r}{2}(x_{p-1} + x_p)\right) & \cos\left(\frac{r}{2}(x_{p-1} + x_p)\right) \end{pmatrix} \\ &\quad \times R(x_{p-1}) E_{x_{p-1}}(x_{p-1}) \begin{pmatrix} \Psi_{p-1,11} & \Psi_{p-1,12} \\ \Psi_{p-1,21} & \Psi_{p-1,22} \end{pmatrix}. \end{aligned} \tag{7.12}$$

Using (3.14) and the explicit expressions for $E_{x_{p-1}}(x_{p-1})$ and $R^{(1)}(x_{p-1})$ given by (6.37) and (6.45), we find after a computation that

$$\begin{aligned} K_{x_{p-1}} &= \frac{ir}{2}(x_p - x_{p-1})\partial_{s_k} \log D_\infty + \partial_{s_k} \log D_\infty \\ &\times \left(d_{x_{p-1}} + \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \frac{x_p - x_{p-1}}{2ic_{x_j}(x_{p-1}-x_j)} \beta_j^2 (\widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2}) \right) \\ &+ \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \frac{\sqrt{|x_j - x_{p-1}|}(x_p - x_{p-1})}{4i\sqrt{|x_j - x_p|}(x_{p-1}-x_j)c_{x_j}} \partial_{s_k}\Big(\beta_j^2(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2} - 2i)\Big) + \mathcal{O}\left(\frac{\log r}{r}\right) \end{aligned} \tag{7.13}$$

as $r \to +\infty$.

Asymptotics for $\partial_{s_k} \log F(r\vec{x}, \vec{s})$. After substituting the explicit expression (6.22) for c_{x_j}, $j = 0, \ldots, p-2, p+1, \ldots, m$ into (7.1), (7.7), (7.10), and (7.13) and simplifying, we obtain the following asymptotics as $r \to +\infty$:

$$\begin{aligned} K_\infty &= -2i\partial_{s_k} d_1\, r + \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \\ &\times \left(\frac{1}{2i} \frac{|x_j - x_{p-1}| - |x_j - x_p|}{|x_{p-1} + x_p - 2x_j|} \partial_{s_k}\Big(\beta_j^2(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2})\Big) - \partial_{s_k}(\beta_j^2) \right) + \mathcal{O}\left(\frac{\log r}{r}\right), \end{aligned}$$

$$\begin{aligned} \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} K_{x_j} &= -\partial_{s_k} \log D_\infty \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \left(\frac{|x_j - x_p| - |x_j - x_{p-1}|}{\sqrt{|x_j - x_p|\,|x_j - x_{p-1}|}} \frac{\beta_j^2}{2i}(\widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2}) \right. \\ &\left. + \beta_j \frac{|x_j - x_p| + |x_j - x_{p-1}|}{\sqrt{|x_j - x_p|\,|x_j - x_{p-1}|}} \right) \\ &+ \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \left(\beta_j \partial_{s_k} \log \frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j} - 2\beta_j \partial_{s_k} \log \Lambda_j \right) + \mathcal{O}\left(\frac{\log r}{r}\right), \end{aligned}$$

$$\begin{aligned} K_{x_p} + K_{x_{p-1}} &= \partial_{s_k} \log D_\infty \left(d_{x_{p-1}} + d_{x_p} + \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \frac{|x_j - x_p| - |x_j - x_{p-1}|}{\sqrt{|x_j - x_p|\,|x_j - x_{p-1}|}} \frac{\beta_j^2}{2i}(\widetilde{\Lambda}_{j,1} + \widetilde{\Lambda}_{j,2}) \right) \\ &+ \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \frac{x_p - x_{p-1}}{2i(x_p + x_{p-1} - 2x_j)} \partial_{s_k}\Big(\beta_j^2(\widetilde{\Lambda}_{j,1} - \widetilde{\Lambda}_{j,2})\Big) + \mathcal{O}\left(\frac{\log r}{r}\right). \end{aligned}$$

Also, from the definitions of d_{x_p} and $d_{x_{p-1}}$ given by (6.17) and (6.19), we have

$$d_{x_{p-1}} + d_{x_p} = \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \beta_j \frac{|x_j - x_p| + |x_j - x_{p-1}|}{\sqrt{|x_j - x_p|\,|x_j - x_{p-1}|}}.$$

Therefore, summing the above asymptotics and simplifying, we obtain

$$\partial_{s_k} \log F(r\vec{x},\vec{s}) = -2i\partial_{s_k} d_1\, r + \sum_{\substack{j=0 \\ j\neq p-1,p}}^{m} \left(\beta_j \partial_{s_k} \log \frac{\Gamma(1+\beta_j)}{\Gamma(1-\beta_j} - 2\beta_j \partial_{s_k} \log \Lambda_j - \partial_{s_k}(\beta_j^2)\right) + \mathcal{O}\left(\frac{\log r}{r}\right), \tag{7.14}$$

as $r \to +\infty$. We also note from (6.27) that

$$\partial_{s_k} \log \Lambda_j = \partial_{s_k}(\beta_j) \log \left(\frac{4\sqrt{|x_j - x_p|\,|x_j - x_{p-1}|}|2x_j - x_p - x_{p-1}|r}{x_p - x_{p-1}} \right) + \sum_{\substack{\ell=0 \\ \ell\neq j,p-1,p}}^{m} \partial_{s_k}(\beta_\ell) \log T_{\ell,j}. \tag{7.15}$$

It is clear from (6.10) that there is a one-to-one correspondence between

$$\vec{s} = (s_1,\ldots,s_{p-1},0,s_{p+1},\ldots,s_m) \in (\mathbb{R}^+)^{p-1} \times \{0\} \times (\mathbb{R}^+)^{m-p}$$

and $\vec{\beta} := (\beta_0,\ldots,\beta_{p-2},\beta_{p+1},\ldots,\beta_m) \in (i\mathbb{R})^{m-1}$. Let us define $\widetilde{F}(r\vec{x},\vec{\beta}) := F(r\vec{x},\vec{s})$. By substituting (7.15) in (7.14) and then writing the derivatives with respect to β_k instead of s_k, we obtain

$$\begin{aligned}
&\partial_{\beta_k} \log \widetilde{F}(r\vec{x},\vec{\beta}) \\
&= -2i\partial_{\beta_k} d_1\, r + \beta_k \partial_{\beta_k} \log \frac{\Gamma(1+\beta_k)}{\Gamma(1-\beta_k} - 2\beta_k - \sum_{\substack{j=0 \\ j\neq k,p-1,p}}^{m} 2\beta_j \log T_{k,j} \\
&\quad -2\beta_k \log \left(\frac{4\sqrt{|x_k - x_p|\,|x_k - x_{p-1}|}|2x_k - x_p - x_{p-1}|r}{x_p - x_{p-1}} \right) + \mathcal{O}\left(\frac{\log r}{r}\right), \quad \text{as } r \to +\infty.
\end{aligned} \tag{7.16}$$

It follows from the analysis of Subsection 6.5 that the asymptotics (7.16) are valid uniformly for $\beta_0,\ldots,\beta_{p-2},\beta_{p+1},\ldots,\beta_m$ in compact subsets of $i\mathbb{R}$, and uniformly in $x_0,\ldots,x_m$ in compact subsets of $\mathbb{R}$ such that (6.1) holds.

7.2. *Integration of the differential identity*

For convenience, we define $\vec{\beta}_j \in (i\mathbb{R})^{m-1}$ by

$$\vec{\beta}_j = \begin{cases} (\beta_0,\ldots,\beta_j,0,\ldots,0), & \text{if } j \in \{0,\ldots,p-2\}, \\ (\beta_0,\ldots,\beta_{p-2},\beta_{p+1},\ldots,\beta_j,0,\ldots,0), & \text{if } j \in \{p+1,\ldots,m\}. \end{cases}$$

For $k = 0$ and $\beta_1 = \cdots = \beta_{p-2} = \beta_{p+1} = \cdots = \beta_m = 0$, the asymptotics (7.16) are as follows:

$$\begin{aligned}
\partial_{\beta_0} \log \widetilde{F}(r\vec{x},\vec{\beta}_0) = {} & -2i\sqrt{x_p - x_0}\sqrt{x_{p-1} - x_0}\, r + \beta_0 \partial_{\beta_0} \log \frac{\Gamma(1+\beta_0)}{\Gamma(1-\beta_0} - 2\beta_0 \\
& -2\beta_0 \log \left(\frac{4\sqrt{|x_p - x_0|\,|x_{p-1} - x_0|}(x_p + x_{p-1} - 2x_0)r}{x_p - x_{p-1}} \right) \\
& + \mathcal{O}\left(\frac{\log r}{r}\right), \quad \text{as } r \to +\infty,
\end{aligned} \tag{7.17}$$

where we have used the definition (6.11) of d_1. Since these asymptotics are uniform for β_0 in compact subsets of $i\mathbb{R}$, we can integrate (7.17) from $\beta_0 = 0$ to an arbitrary $\beta_0 \in i\mathbb{R}$ without worsening the order of the error term. Recalling from (5.19) that

$$\int_0^{\beta_0} x\partial_x \log \frac{\Gamma(1+x)}{\Gamma(1-x)} dx = \beta_0^2 + \log\left(G(1+\beta_0)G(1-\beta_0)\right), \tag{7.18}$$

an integration of (7.17) yields

$$\begin{aligned}\log \frac{\widetilde{F}(r\vec{x},\vec{\beta}_0)}{\widetilde{F}(r\vec{x},\vec{0})} &= -2i\beta_0\sqrt{x_p - x_0}\sqrt{x_{p-1}-x_0}\, r + \log\left(G(1+\beta_0)G(1-\beta_0)\right) \\ &- \beta_0^2 \log\left(\frac{4\sqrt{|x_p-x_0|\,|x_{p-1}-x_0|}(x_p+x_{p-1}-2x_0)r}{x_p-x_{p-1}}\right) + \mathcal{O}\left(\frac{\log r}{r}\right), \quad \text{as } r \to +\infty,\end{aligned}$$

where $\vec{0} = (0,\dots,0)$. In a similar way, we integrate successively in the variables $\beta_1,\dots,\beta_{p-2}$. At the last step, we use (7.16) with $k = p-2$, and with $\beta_0,\dots,\beta_{p-3}$ fixed but arbitrary:

$$\begin{aligned}&\partial_{\beta_{p-2}} \log \widetilde{F}(r\vec{x},\vec{\beta}_{p-2}) \\ &= -2i\sqrt{x_p-x_{p-2}}\sqrt{x_{p-1}-x_{p-2}}\, r + \beta_{p-2}\partial_{\beta_{p-2}} \log\frac{\Gamma(1+\beta_{p-2})}{\Gamma(1-\beta_{p-2})} - 2\beta_{p-2} \\ &\quad -\sum_{j=0}^{p-3} 2\beta_j \log T_{p-2,j} - 2\beta_{p-2}\log\left(\frac{4\sqrt{|x_p-x_{p-2}|\,|x_{p-1}-x_{p-2}|}(x_p+x_{p-1}-2x_{p-2})r}{x_p-x_{p-1}}\right) \\ &\quad +\mathcal{O}\left(\frac{\log r}{r}\right),\end{aligned}$$

as $r \to +\infty$. Since the above asymptotics are uniform for β_{p-2} in compact subsets of $i\mathbb{R}$, an integration over β_{p-2} from $\beta_{p-2} = 0$ to an arbitrary $\beta_{p-2} \in i\mathbb{R}$ let the order of the error term unchanged, and using again the formula (7.18) (with β_0 now replaced by β_{p-2}), we obtain

$$\begin{aligned}&\log \frac{\widetilde{F}(r\vec{x},\vec{\beta}_{p-2})}{\widetilde{F}(r\vec{x},\vec{\beta}_{p-3})} \\ &= -2i\beta_{p-2}\sqrt{x_p-x_{p-2}}\sqrt{x_{p-1}-x_{p-2}}\, r + \log\left(G(1+\beta_{p-2})G(1-\beta_{p-2})\right) \\ &\quad -\sum_{j=0}^{p-3} 2\beta_j\beta_{p-2}\log T_{p-2,j} - \beta_{p-2}^2\log\left(\frac{4\sqrt{|x_p-x_{p-2}|\,|x_{p-1}-x_{p-2}|}(x_p+x_{p-1}-2x_{p-2})r}{x_p-x_{p-1}}\right) \\ &\quad +\mathcal{O}\left(\frac{\log r}{r}\right),\end{aligned} \tag{7.19}$$

as $r \to +\infty$. The successive integrations in $\beta_{p+1},\dots,\beta_m$ can be done similarly. At the last step, we use (7.16) with $k = m$ and $\vec{\beta}_{m-1}$ arbitrary but fixed:

$$\begin{aligned}&\partial_{\beta_m} \log \widetilde{F}(r\vec{x},\vec{\beta}_m) \\ &= 2i\sqrt{x_m-x_p}\sqrt{x_m-x_{p-1}}\, r + \beta_m\partial_{\beta_m}\log\frac{\Gamma(1+\beta_m)}{\Gamma(1-\beta_m} - 2\beta_m\end{aligned}$$

$$- \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m-1} 2\beta_j \log T_{m,j} - 2\beta_m \log \left(\frac{4\sqrt{|x_p - x_m|\,|x_{p-1} - x_m|}|x_p + x_{p-1} - 2x_m|r}{x_p - x_{p-1}} \right)$$
$$+ \mathcal{O}\left(\frac{\log r}{r} \right),$$

as $r \to +\infty$. After an integration of the above asymptotics from $\beta_m = 0$ to an arbitrary $\beta_m \in i\mathbb{R}$, we obtain an asymptotic formula similar to (7.19). Finally, summing all the successive asymptotic formulas for the ratios

$$\log \frac{\widetilde{F}(r\vec{x}, \vec{\beta}_0)}{\widetilde{F}(r\vec{x}, \vec{0})}, \log \frac{\widetilde{F}(r\vec{x}, \vec{\beta}_1)}{\widetilde{F}(r\vec{x}, \vec{\beta}_0)}, \ldots, \log \frac{\widetilde{F}(r\vec{x}, \vec{\beta}_{p-2})}{\widetilde{F}(r\vec{x}, \vec{\beta}_{p-3})}, \log \frac{\widetilde{F}(r\vec{x}, \vec{\beta}_{p+1})}{\widetilde{F}(r\vec{x}, \vec{\beta}_{p-2})}, \ldots, \log \frac{\widetilde{F}(r\vec{x}, \vec{\beta})}{\widetilde{F}(r\vec{x}, \vec{\beta}_{m-1})},$$

we obtain

$$\log \frac{\widetilde{F}(r\vec{x}, \vec{\beta})}{\widetilde{F}(r\vec{x}, \vec{0})} = -2id_1 r + \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \log \left(G(1+\beta_j)G(1-\beta_j) \right) - 2 \sum_{\substack{0 \leqslant j < k \leqslant m \\ j,k \neq p-1,p}} \beta_j \beta_k \log T_{k,j}$$
$$- \sum_{\substack{j=0 \\ j \neq p-1,p}}^{m} \beta_j^2 \log \left(\frac{4\sqrt{|x_j - x_p|\,|x_j - x_{p-1}|}|2x_j - x_p - x_{p-1}|r}{x_p - x_{p-1}} \right) + \mathcal{O}\left(\frac{\log r}{r} \right) \tag{7.20}$$

as $r \to +\infty$. Note from (1.13) and (6.10) that $u_j = 2\pi i \beta_j$. We obtain (1.12) after substituting in (7.20) the known large r asymptotics of $\widetilde{F}(r\vec{x}, \vec{0}) = F((rx_{p-1}, rx_p), 0)$ given by (1.3). This finishes the proof of Theorem 1.2.

Appendix. *Model RH problems*

In this section, we recall two well-known RH problems.

A.1. *Bessel model RH problem.*

(a) $\Phi_{\mathrm{Be}} : \mathbb{C} \setminus \Sigma_{\mathrm{Be}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{Be} is shown in Figure A.1.

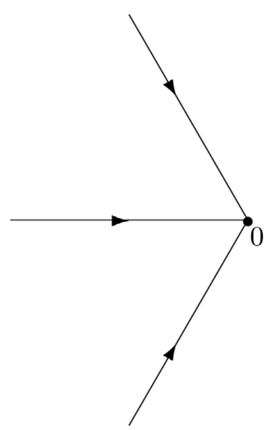

FIGURE A.1. *The jump contour* Σ_{Be} *for* Φ_{Be}.

(b) Φ_{Be} satisfies the jump conditions

$$\begin{aligned} \Phi_{\mathrm{Be},+}(z) &= \Phi_{\mathrm{Be},-}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in \mathbb{R}^-, \\ \Phi_{\mathrm{Be},+}(z) &= \Phi_{\mathrm{Be},-}(z)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z \in e^{\frac{2\pi i}{3}}\mathbb{R}^+, \\ \Phi_{\mathrm{Be},+}(z) &= \Phi_{\mathrm{Be},-}(z)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z \in e^{-\frac{2\pi i}{3}}\mathbb{R}^+. \end{aligned} \tag{A.1}$$

(c) As $z \to \infty$, $z \notin \Sigma_{\mathrm{Be}}$, we have

$$\Phi_{\mathrm{Be}}(z) = (2\pi z^{\frac{1}{2}})^{-\frac{\sigma_3}{2}} N\left(I + \frac{\Phi_{\mathrm{Be},1}}{z^{\frac{1}{2}}} + \mathcal{O}(z^{-1})\right) e^{2z^{\frac{1}{2}}\sigma_3}, \tag{A.2}$$

where $\Phi_{\mathrm{Be},1} = \frac{1}{16}\begin{pmatrix} -1 & -2i \\ -2i & 1 \end{pmatrix}$.

(d) As z tends to 0, the behavior of $\Phi_{\mathrm{Be}}(z)$ is

$$\Phi_{\mathrm{Be}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi. \end{cases}$$

The unique solution to the above RH problem was obtained in [44] and is given by

$$\Phi_{\mathrm{Be}}(z) = \begin{cases} \begin{pmatrix} I_0(2z^{\frac{1}{2}}) & \frac{i}{\pi}K_0(2z^{\frac{1}{2}}) \\ 2\pi i z^{\frac{1}{2}} I_0'(2z^{\frac{1}{2}}) & -2z^{\frac{1}{2}}K_0'(2z^{\frac{1}{2}}) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\ \begin{pmatrix} \frac{1}{2}H_0^{(1)}(2(-z)^{\frac{1}{2}}) & \frac{1}{2}H_0^{(2)}(2(-z)^{\frac{1}{2}}) \\ \pi z^{\frac{1}{2}}\left(H_0^{(1)}\right)'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}}\left(H_0^{(2)}\right)'(2(-z)^{\frac{1}{2}}) \end{pmatrix}, & \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \frac{1}{2}H_0^{(2)}(2(-z)^{\frac{1}{2}}) & -\frac{1}{2}H_0^{(1)}(2(-z)^{\frac{1}{2}}) \\ -\pi z^{\frac{1}{2}}\left(H_0^{(2)}\right)'(2(-z)^{\frac{1}{2}}) & \pi z^{\frac{1}{2}}\left(H_0^{(1)}\right)'(2(-z)^{\frac{1}{2}}) \end{pmatrix}, & -\pi < \arg z < -\frac{2\pi}{3}, \end{cases}$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of the first and second kind (of order 0), and I_0 and K_0 are the modified Bessel functions of the first and second kind.

It is easy to see from the properties (b) and (d) of the RH problem for Φ_{Be} that in a neighborhood of 0, we have

$$\Phi_{\mathrm{Be}}(z) = \Phi_{\mathrm{Be},0}(z)\begin{pmatrix} 1 & \frac{1}{2\pi i}\log z \\ 0 & 1 \end{pmatrix}\widetilde{H}_0(z), \tag{A.3}$$

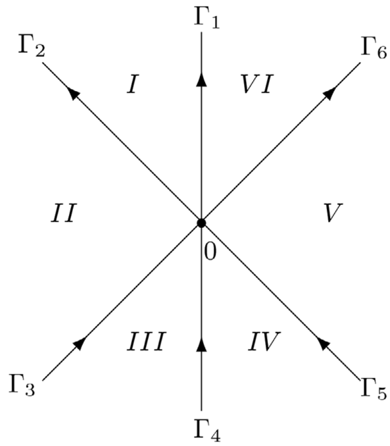

FIGURE A.2. *The jump contour* Σ_{HG} *for* Φ_{HG}*. The ray* Γ_k *is oriented from* 0 *to* ∞*, and forms an angle with* $\mathbb{R}^+$ *which is a multiple of* $\frac{\pi}{4}$*.*

where $\Phi_{\mathrm{Be},0}$ is analytic in a neighborhood of 0 and $\widetilde{H}_0$ is given by

$$\widetilde{H}_0(z) = \begin{cases} I, & \text{for } -\frac{2\pi}{3} < \arg(z) < \frac{2\pi}{3}, \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg(z) < \pi, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } -\pi < \arg(z) < -\frac{2\pi}{3}. \end{cases}$$

Using the asymptotics of the Bessel functions near the origin (see, for example, [**47**, Chapter 10.30(i)]), we obtain after a computation that

$$\Phi_{\mathrm{Be},0}(0) = \begin{pmatrix} 1 & \frac{\gamma_{\mathrm{E}}}{\pi i} \\ 0 & 1 \end{pmatrix},$$

where γ_{E} is Euler's gamma constant.

A.2. *Confluent hypergeometric model RH problem.*

(a) $\Phi_{\mathrm{HG}} : \mathbb{C} \setminus \Sigma_{\mathrm{HG}} \to \mathbb{C}^{2\times 2}$ is analytic, where Σ_{HG} is shown in Figure A.2.
(b) For $z \in \Gamma_k$ (see Figure A.2), $k = 1, \ldots, 6$, Φ_{HG} satisfies the jump relations

$$\Phi_{\mathrm{HG},+}(z) = \Phi_{\mathrm{HG},-}(z) J_k, \tag{A.4}$$

where

$$J_1 = \begin{pmatrix} 0 & e^{-i\pi\beta} \\ -e^{i\pi\beta} & 0 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 0 & e^{i\pi\beta} \\ -e^{-i\pi\beta} & 0 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\beta} & 1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\beta} & 1 \end{pmatrix}, \quad J_5 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi\beta} & 1 \end{pmatrix}, \quad J_6 = \begin{pmatrix} 1 & 0 \\ e^{i\pi\beta} & 1 \end{pmatrix}.$$

(c) As $z \to \infty$, $z \notin \Sigma_{\mathrm{HG}}$, we have

$$\Phi_{\mathrm{HG}}(z) = \left(I + \frac{\Phi_{\mathrm{HG},1}(\beta)}{z} + \mathcal{O}(z^{-2}) \right) z^{-\beta\sigma_3} e^{-\frac{z}{2}\sigma_3} \begin{cases} e^{i\pi\beta\sigma_3}, & \frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & -\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \end{cases} \tag{A.5}$$

where $z^\beta = |z|^\beta e^{i\beta \arg z}$ with $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{3})$ and

$$\Phi_{\mathrm{HG},1}(\beta) = \beta^2 \begin{pmatrix} -1 & \tau(\beta) \\ -\tau(-\beta) & 1 \end{pmatrix}, \quad \tau(\beta) = \frac{-\Gamma(-\beta)}{\Gamma(\beta+1)}. \tag{A.6}$$

As $z \to 0$, we have

$$\Phi_{\mathrm{HG}}(z) = \begin{cases} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log z) \\ \mathcal{O}(1) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in II \cup V, \\ \begin{pmatrix} \mathcal{O}(\log z) & \mathcal{O}(\log z) \\ \mathcal{O}(\log z) & \mathcal{O}(\log z) \end{pmatrix}, & \text{if } z \in I \cup III \cup IV \cup VI. \end{cases} \tag{A.7}$$

This model RH problem was solved explicitly in [38]. Define

$$\widehat{\Phi}_{\mathrm{HG}}(z) = \begin{pmatrix} \Gamma(1-\beta) G(\beta; z) & -\frac{\Gamma(1-\beta)}{\Gamma(\beta)} H(1-\beta; z e^{-i\pi}) \\ \Gamma(1+\beta) G(1+\beta; z) & H(-\beta; z e^{-i\pi}) \end{pmatrix},$$

where G and H are related to the Whittaker functions:

$$G(a; z) = \frac{M_{\kappa,\mu}(z)}{\sqrt{z}}, \quad H(a; z) = \frac{W_{\kappa,\mu}(z)}{\sqrt{z}}, \quad \mu = 0, \quad \kappa = \frac{1}{2} - a.$$

The solution Φ_{HG} is given by

$$\Phi_{\mathrm{HG}}(z) = \begin{cases} \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1}, & \text{for } z \in I, \\ \widehat{\Phi}_{\mathrm{HG}}(z), & \text{for } z \in II, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_3^{-1}, & \text{for } z \in III, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_6^{-1} J_5, & \text{for } z \in IV, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1} J_6^{-1}, & \text{for } z \in V, \\ \widehat{\Phi}_{\mathrm{HG}}(z) J_2^{-1} J_1^{-1}, & \text{for } z \in VI. \end{cases} \tag{A.8}$$

The asymptotics of $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ as $z \to 0$ given by [47, Subsection 13.14 (iii)] allow to obtain a more precise version of (A.7). Using also $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} = -\Gamma(-z)\Gamma(1+z)$, we get

$$\Phi_{\mathrm{HG}}(z) = \widehat{\Phi}_{\mathrm{HG}}(z) = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} (I + \mathcal{O}(z)) \begin{pmatrix} 1 & \frac{\sin(\pi\beta)}{\pi} \log z \\ 0 & 1 \end{pmatrix}, \quad \text{as } z \to 0,\ z \in II, \tag{A.9}$$

where

$$\log z = \log|z| + i \arg z, \quad \arg z \in \left(-\frac{\pi}{2}, \frac{3\pi}{2} \right),$$

and

$$\Psi_{11} = \Gamma(1-\beta), \quad \Psi_{12} = \frac{1}{\Gamma(\beta)} \left(\frac{\Gamma'(1-\beta)}{\Gamma(1-\beta)} + 2\gamma_{\mathrm{E}} - i\pi \right),$$

$$\Psi_{21} = \Gamma(1+\beta), \quad \Psi_{22} = \frac{-1}{\Gamma(-\beta)} \left(\frac{\Gamma'(-\beta)}{\Gamma(-\beta)} + 2\gamma_{\mathrm{E}} - i\pi \right),$$

and where γ_{E} is Euler's gamma constant.

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