SpringerPlus

CrossMark

The rapidly convergent solutions of strongly nonlinear oscillators

M. S. Alam[1*], Md. Abdur Razzak[1], Md. Alal Hosen[1] and Md. Riaz Parvez[2]

*Correspondence:
msalam1964@yahoo.com
[1] Department
of Mathematics, Rajshahi
University of Engineering
and Technology (RUET),
Kazla, Rajshahi 6204,
Bangladesh
Full list of author information
is available at the end of the
article

Abstract

Based on the harmonic balance method (HBM), an approximate solution is determined from the integral expression (i.e., first order differential equation) of some strongly nonlinear oscillators. Usually such an approximate solution is obtained from second order differential equation. The advantage of the new approach is that the solution converges significantly faster than that obtained by the usual HBM as well as other analytical methods. By choosing some well known nonlinear oscillators, it has been verified that an n-th ($n \geq 2$) approximate solution (concern of this article) is very close to $(2n - 1)$-th approximations obtained by usual HBM.

Keywords: Nonlinear oscillation, Harmonic balance method, Duffing equation

Background

The harmonic balance method (HBM) (Mickens 1996; West 1960; Mickens 1984, 1986; Lim and Wu 2003; Lim et al. 2005; Wu et al. 2006; Belendez et al. 2006; Alam et al. 2007; Hu 2006a, b; Lai et al. 2009; Hosen et al. 2012) is a widely used technique for solving strongly nonlinear oscillators

$$\ddot{x} + f(x, \dot{x}) = 0, \quad [x(0) = a, \ \dot{x}(0) = 0], \tag{1}$$

where $f(x, \dot{x})$ is a nonlinear function, satisfies a condition, $f(-x, \dot{x}) = -f(x, \dot{x})$. Multiplying both sides of Eq. (1) by $2\dot{x}$ and then integrating, Eq. (1) readily becomes

$$\dot{x}^2 + F(x) = F(a), \tag{2}$$

where $dF/dx = f(x)$. In general $f(x)$ is an odd polynomial function. Therefore Eq. (2) can be written as

$$G(\dot{x}^2, x^2 - a^2, x^4 - a^4, \ldots) = 0. \tag{3}$$

When $f(x)$ is not a simple polynomial function (e.g., pendulum equation, $l \ddot{x} + g \sin x = 0$), Eq. (3) is valid for amplitude of oscillation, $a < 1$. Sometimes the nonlinear function, f depends on both x and $\dot{x}$ (e.g., $\ddot{x} + (1 + \dot{x}^2)x = 0$). In this case, the integral expression of such equations has been taken in the form of Eq. (3).

The modified Lindsted–Poincare method (Cheung et al. 1991; He 2002a, b; Ozis and Yildirim 2007), He's homotopy perturbation method (Belendez et al. 2007a, b; Belendez 2007), iterative method (Mickens 1987a, b, 2005, 2010; Lim and Wu 2002; Lim et al.

2006; Hu 2006a, b; Guo et al. 2011; Haque et al. 2013), He's energy balance method (He 2002a, b) etc. are also used to investigate nonlinear oscillators. Though, all these analytical methods (Mickens 1984, 1986, 1987a, b, 1996, 2005, 2010; West 1960; Lim and Wu 2002, 2003; Lim et al. 2005, 2006; Wu et al. 2006; Belendez et al. 2006, 2007a, b; Alam et al. 2007; Hu 2006a, b; Lai et al. 2009; Cheung et al. 1991; He 2002a, b; Ozis and Yildirim 2007; Belendez 2007; Guo et al. 2011; Haque et al. 2013) have been developed for handling nonlinear oscillator Eq. (1), they provide almost similar results for a particular approximation. Recently, HBM has been modified by truncating some higher order terms of the algebraic equations of related variables to the solution [see Hosen et al. (2012) for details] and it measures more correct result than the usual HBM solutions (derived in Wu et al. 2006; Belendez et al. 2006; Alam et al. 2007; Hu 2006a, b; Lai et al. 2009) as well as other solutions derived by several analytical methods (Belendez 2007; Belendez et al. 2007b; Mickens 1987a, b, 2005, 2010; Lim and Wu 2002; Lim et al. 2006; Hu 2006a, b; Guo et al. 2011; Haque et al. 2013; He 2002a, b). However for any approximation, the result (even the solution obtained in Hosen et al. (2012)) is not better than the next higher approximation. Moreover, the modification on HBM used in Hosen et al. (2012) is valid for some nonlinear oscillators especially when $f(x)$ contains a term, x^3. In this article, a new approach (based on the HBM) has been introduced in which the solution rapidly converges toward its exact solution. The trial solution is similar to that of Hosen et al. (2012) and the determination of the related unknowns is also similar. Yet the solution converges faster than the usual solution. Actually an nth ($n \geq 2$) approximate solution of Eq. (2) is almost similar to the $(2n - 1)$-th approximation obtained from Eq. (1). To verify this statement, the second and third approximations have been obtained from the integral expressions of some important nonlinear oscillators. The new solutions are respectively close to the third and fifth approximations determined by usual HBM which are agree with the statement.

Methods

Let us consider a periodic solution in the form (Hosen et al. 2012)

$$x(t) = a \left((1 - u_1(a) - u_2(a) - \cdots) \cos \varphi(a, t) + u_1(a) \cos 3\varphi(a, t) \right.$$
$$\left. + u_2(a) \cos 5\varphi(a, t) + \cdots \right), \tag{4}$$

where a (amplitude) and $\dot{\varphi}$ (frequency) are constants and initial phase, $\varphi_0(a) = 0$. This trial solution was early used in Hosen et al. (2012) to solve Eq. (1). In this article, Eq. (4) is used for solving Eq. (2).

Differentiating x, squaring and simplifying, we obtain

$$\dot{x}^2 = a_0^2 \dot{\varphi}^2 \sin^2 \varphi \left(1 + 4u_1 + 8u_2 + 22u_1^2 + 76u_1u_2 + 116u_2^2 + (12u_1 + 20u_2 + 24u_1^2 \right.$$
$$\left. + 148u_1u_2 + 180u_2^2) \cos 2\varphi + (20u_2 + 18u_1^2 + 100u_1u_2 + 130u_2^2) \cos 4\varphi + \cdots \right). \tag{5}$$

Then we have determined an expression for $x^2 - a^2$, as

$$x^2 - a^2 = -a^2 \sin^2 \varphi \left(1 + 4u_1 + 8u_2 - 2u_1^2 - 4u_1u_2 - 4u_2^2 + (4u_1 + 12u_2 - 4u_1u_2 \right.$$
$$\left. - 4u_2^2) \cos 2\varphi + (4u_2 + 2u_1^2 + 4u_1u_2 + 2u_2^2) \cos 4\varphi + \cdots \right). \tag{6}$$

All other expressions $x^4 - a^4, x^6 - a^6, \ldots$ of Eq. (3) have a factor $x^2 - a^2$; so that a common factor $a^2 \sin^2 \varphi$ must be cancelled when all these values are substituted in Eq. (3). It is noted that the canceling of the common (i.e., $a^2 \sin^2 \varphi$) factor makes the solution better than usual solution. Otherwise the solution does not converge fast. It also makes the solution different from that obtained by the energy balance method.

Examples

Quintic Duffing oscillator

Let us consider quintic Duffing oscillator, i.e.,

$$\ddot{x} + x + x^5 = 0. \tag{7}$$

By utilization of initial conditions, $[x(0) = a, \ \dot{x}(0) = 0]$, Eq. (7) readily takes the form

$$\dot{x}^2 + (x^2 - a^2) + (x^6 - a^6)/3 = 0. \tag{8}$$

It has already been mentioned that an analytical solution can be obtained either from Eq. (7) or from Eq. (8). The aim of this article is to find approximate solution from Eq. (8) rather than Eq. (7). A third approximate solution (in which u_1 and u_2 are non-zero) has been mainly considered. Sometimes a second approximate solution has been considered to compare it with existing solution obtained by several authors.

Substituting solution Eq. (4) (together with $u_j = 0, j > 2$) in Eq. (8), dividing by $a^2 \sin^2 \varphi$ and equating the coefficient of Constant, $\cos 2\varphi$ and $\cos 4\varphi$, the following nonlinear algebraic equations are obtained

$$
\begin{aligned}
&\dot{\varphi}^2(1 + 4u_1 + 8u_2 + 22u_1^2 + 76u_1 u_2 + 116u_2^2) - (1 + 4u_1 + 8u_2 - 2u_1^2 - 4u_1 u_2 - 4u_2^2) - 5a^4(1 + 4u_1 \\
&\quad + 10u_2 - 5u_1^2 - 18u_1 u_2 - 19u_2^2 + 8u_1^3 - 10u_1^4 + 32u_1^2 u_2 + 56u_1 u_2^2 + 36u_2^3 - 40u_1^3 u_2 + \cdots)/8 = 0, \\
&\dot{\varphi}^2(12u_1 + 20u_2 + 24u_1^2 + 148u_1 u_2 + 180u_2^2) - (4u_1 + 12u_2 - 4u_1 u_2 - 4u_2^2) \\
&\quad - a^4(4 + 45u_1 + 123u_2 - 30u_1^2 - 150u_1 u_2 - 180u_2^2 + 20u_1^3 + \cdots)/12 = 0, \\
&\dot{\varphi}^2(20u_2 + 18u_1^2 + 100u_1 u_2 + 130u_2^2) - (4u_2 + 2u_1^2 + 4u_1 u_2 + 2u_2^2) \\
&\quad - a^4(1 + 36u_1 + 132u_2 + 60u_1^2 + 120u_1 u_2 - 160u_1^3 + 225u_1^4 + \cdots)/24 = 0.
\end{aligned}
\tag{9}
$$

By elimination of $\dot{\varphi}$ from three equations of Eq. (9), we obtain two equations of u_1 and u_2 as

$$
\begin{aligned}
&24u_1 + 24u_2 + 456u_1 u_2 + 72u_1^2 - 864u_1^3 - 4320u_1^2 u_2 + 552u_2^2 - 9120u_1 u_2^2 - 7200u_2^3 \\
&\quad + a^4(-4 + 45u_1 + 27u_2 + 1440u_1 u_2 + 210u_1^2 - 2450u_1^3 - 13050u_1^2 u_2 + 1830u_2^2 \\
&\quad - 27060u_1 u_2^2 - 19950u_2^3)/4 = 0, \ 384u_1^2 + 384u_2 + 2304u_1 u_2 + 3072u_2^2 - 11520u_1^2 u_2 \\
&\quad - 38400u_1 u_2^2 - 57600u_2^3 + a^4(-1 - 36u_1 + 168u_2 + 210u_1^2 + 1380u_1 u_2 - 160u_1^3 \\
&\quad + 2505u_2^2 - 7020u_1^2 u_2 - 26880u_1 u_2^2 - 41060u_2^3) = 0.
\end{aligned}
\tag{10}
$$

In general, u_1 and u_2 are small. So, it is possible to divide the first and second of Eq. (10) respectively by $1 + 3u_1 + 19u_2 - 36u_1^2 - 180u_1 u_2 - 380u_2^2$ and $1 + 6u_1 + 8u_2 - 30u_1^2 - 100u_1 u_2 - 150u_2^2$, and then they become

$$
\begin{aligned}
&-24u_1 - 24u_2 + 72u_1 u_2 - 1080u_1^2 u_2 - 96u_2^2 - 5400u_1 u_2^2 - 96u_2^3 \\
&\quad + a^4\left(4 - 57u_1 - 103u_2 + 672u_1 u_2 + 105u_1^2 + 83u_1^3 - 4929u_1^2 u_2/2 + 1647u_2^2\right)/4 = 0, \\
&-384u_1^2 - 384u_2 + 2304u_2^3 + 3072u_1^2 u_2 + a^4(1 + 30u_1 - 176u_2 - 360u_1^2 - 464u_1 u_2 - 947u_2^2) = 0
\end{aligned}
\tag{11}
$$

Now, the above equations can written as

$$u_1 = \lambda \left(1 - \frac{103u_2}{4} + 168u_1u_2 + \frac{105u_1^2}{4} + \frac{83u_1^3}{4} - \frac{4929u_1^2u_2}{4} + \frac{1647u_2^2}{4}\right)$$

$$+ \left(\frac{1}{24} - \frac{19\lambda}{32}\right) \times \left(-24u_2 + 72u_1u_2 - 1080u_1^2u_2 - 96u_2^2 - 5400u_1u_2^2 - 96u_2^3\right),$$

$$u_2 = \left(\frac{\lambda}{16} + \frac{13\lambda^2}{64} + \frac{169\lambda^2}{256}\right)\left(1 + 30u_1 - 360u_1^2 - 464u_1u_2 - 947u_2^2\right)$$

$$+ \left(\frac{1}{384} - \frac{11\lambda}{384} + \frac{143\lambda^2}{1536}\right) \times \left(-384u_1^2 + 2304u_2^3 + 3072u_1^2u_2\right), \tag{12}$$

where $\lambda = 4a^4/(96 + 57a^4)$. It is clear that λ is much smaller than 1 for every values of a. As $a \to \infty$, λ becomes 4/57 (which is the largest). Therefore, u_1 and u_2 can be obtained in powers of λ of the forms $u_1 = l_1\lambda + l_2\lambda^2 + \cdots$ and $u_2 = m_1\lambda + m_2\lambda^2 + \cdots$ [see Hosen et al. (2012) for details]. Substituting the series of u_1 and u_2 in Eq. (12) and equating the equal powers of λ, a set of linear algebraic equations of $l_1, l_2, ..., m_1, m_2, ...,$ are obtained. Solving these algebraic equations, the unknown constants, $l_1, l_2, ..., m_1, m_2, ...$ are determined. Thus u_1 and u_2 become

$$u_1 = \frac{15\lambda}{16} - \frac{105\lambda^2}{64} + \frac{38865\lambda^3}{2048} - \cdots,$$

$$u_2 = \frac{\lambda}{16} + \frac{277\lambda^2}{256} - \frac{1153\lambda^3}{4096} - \cdots. \tag{13}$$

Now substituting these values of u_1 and u_2 in the first equation of Eq. (9) and then simplifying, the frequency (i.e., $\dot{\varphi}$) is obtained. It is noted that the series of u_1 and u_2 are valid for all values of a. For some particular values of a, the approximate frequency has been calculated and presented in Table 1. When $a < 1$, this approximate solution can be compare with some results obtained by usual HBM. In this case, λ can be expanded in powers of a and the series of $\dot{\varphi}^2$ becomes

$$\dot{\varphi}_{3(q)}^2 = 1 + \frac{5a^4}{8} - \frac{65a^8}{1536} + \frac{1055a^{12}}{36864} - \frac{129906\frac{7}{48}a^{16}}{6291456} + \cdots. \tag{14}$$

The exact value of $\dot{\varphi}^2$ is

$$\dot{\varphi}_{Ex(q)}^2 = 1 + \frac{5a^4}{8} - \frac{65a^8}{1536} + \frac{1055a^{12}}{36864} - \frac{129545a^{16}}{6291456} + \cdots, \tag{15}$$

where $\dot{\varphi}_{3(q)}^2$ and $\dot{\varphi}_{Ex(q)}^2$ denote respectively, the third approximate frequency by the present method and exact frequency of the Eq. (7).

Comparing Eqs. (14) and (15), it is clear that the first four terms of $\dot{\varphi}_{3(q)}^2$ obtained in Eq. (14) are identical to those of its exact result, $\dot{\varphi}_{Ex(q)}^2$. But the result of $\dot{\varphi}_{3(q)}^2$ is different from that of $\dot{\varphi}_{3(q,Usual)}^2$ obtained by the usual HBM [see Eq. (39) of Appendix 1: though the solution is obtained from Eq. (7) containing two higher harmonic terms u_1 and u_2]. We see that first three terms of $\dot{\varphi}_{3(q,Usual)}^2$ are identical to its exact result. It is noted that the first four terms of $\dot{\varphi}_{3(q,Usual)}^2$ would be same those of $\dot{\varphi}_{Ex(q)}^2$, when the solution is derived from Eq. (7) containing four higher harmonic terms u_1, u_2, u_3 and u_4. Certainly, it is a laborious task to determine five unknown u_1, u_2, u_3, u_4 and $\dot{\varphi}^2$ for any analytical method.

Table 1 Comparison the approximate frequencies of Eq. (7) between the present method and the usual HBM method with the exact frequency $\dot{\varphi}_{Ex}$, obtained by direct numerical integration

a	$\dot{\varphi}_{Ex}$	$\dot{\varphi}_{3(q,Usual)}$ Er (%)	$\dot{\varphi}_{3(q,Present)}$ Er (%)
0.5	1.0192663	1.0192661	1.0192663
		0.00002	0.00000
0.7	1.0714295	1.0714202	1.0714295
		0.00087	0.00000
1	1.26471	1.26446	1.26469
		0.020	0.002
2	3.16666	3.16223	3.16639
		0.140	0.008
3	6.80379	6.79391	6.80382
		0.145	0.000
4	11.9959	11.9785	11.9963
		0.145	0.003
5	18.7007	18.6736	18.7014
		0.145	0.003
10	74.6909	74.5829	74.6941
		0.145	0.004
50	1867.09	1864.39	1867.17
		0.145	0.004
100	7468.34	7457.55	7468.66
		0.145	0.004

Er (%) denotes absolute percentage error

Cubic Duffing oscillator

Let us consider cubic Duffing oscillator, i.e.,

$$\ddot{x} + x + x^3 = 0. \tag{16}$$

By utilization of initial conditions, $[x(0) = a, \; \dot{x}(0) = 0]$, Eq. (16) readily takes the form

$$\dot{x}^2 + (x^2 - a^2) + (x^4 - a^4)/2 = 0. \tag{17}$$

First of all we consider a third approximate solution in which u_1 and u_2 are non-zero. Substituting solution Eq. (4) in Eq. (17), dividing by $a^2\sin^2\varphi$ and equating the coefficient of Constant, $\cos 2\varphi$ and $\cos 4\varphi$, we obtain

$$\begin{aligned}
&\dot{\varphi}^2(1 + 4u_1 + 8u_2 + 22u_1^2 + 76u_1u_2 + 116u_2^2) - (1 + 4u_1 + 8u_2 - 2u_1^2 - 4u_1u_2 - 4u_2^2) - 3a^2(1 \\
&\quad + 4u_1 + 28u_2/3 - 12u_1u_2 - 4u_1^2 - 12u_2^2 + 4u_1^3 - 2u_1^4 + 12u_1^2u_2 - 16u_1^3u_2/3 + \cdots)/4 = 0, \\
&\dot{\varphi}^2(12u_1 + 20u_2 + 24u_1^2 + 148u_1u_2 + 180u_2^2) - 4(u_1 + 3u_2 - u_1u_2 - u_2^2) - a^2(1 + 16u_1 \\
&\quad - 6u_1^2 + 2u_1^4 + 44u_2 - 36u_1u_2 + 24u_1^2u_2 - 8u_1^3u_2 - 42u_2^2)/4 = 0, \\
&\dot{\varphi}^2(20u_2 + 18u_1^2 + 100u_1u_2 + 130u_2^2) - 2(2u_2 + u_1^2 + 2u_1u_2 + u_2^2) \\
&\quad - a^2(u_1 + 3u_1^2 - 4u_1^3 + 2u_1^4 + 5u_2 + 6u_1u_2 - 12u_1^2u_2 + 7u_1^3u_2 + 3u_2^2) = 0.
\end{aligned} \tag{18}$$

By elimination of $\dot{\varphi}$ from three equations of Eq. (18) and then simplifying (discussed in "Quintic Duffing oscillator" section), the following relations of u_1 and u_2 are obtained as

$$
\begin{aligned}
u_1 &= \mu(1 - 35u_2 + 27u_1^2 + 194u_1u_2 + 3u_2^2 - 45u_1^3 - 1971u_1^2u_2 + 245u_1^4 + 15232u_1^3u_2 \\
&\quad - 5451u_1^5) + (1 - 23\mu)(-u_2 + 3u_1u_2 - 4u_2^2 - 45u_1^2u_2 + 315u_1^3u_2), \\
u_2 &= (2\mu + 6\mu^2 + 18\mu^3 + 54\mu^4 + 162\mu^5 + 486\mu^6 + 1458\mu^7)(u_1 - 33u_1^2/2 - 17u_1u_2 \\
&\quad - 69u_2^2/2 + 125u_1^3 + 294u_1^2u_2 - 772u_1^4 - 2877u_1^3u_2 + 3588u_1^5) + (1 - 20\mu - 60\mu^2 \\
&\quad - 180\mu^3 - 540\mu^4 - 1620\mu^5 - 4860\mu^6)(-u_1^2 + 6u_1^3 + 8u_1^2u_2 - 39u_1^4 - 96u_1^3u_2 + 186u_1^5),
\end{aligned}
\tag{19}
$$

where $\mu = a^2/(32 + 23a^2)$. It is clear that μ is much smaller than 1 for every values of a. As $a \to \infty$, μ becomes $1/23$ (which is the largest). For every values of a, u_1 and u_2 can be express in terms of μ (as discuss in "Quintic Duffing oscillator" section) and that are calculated as

$$
\begin{aligned}
u_1 &= \mu - \mu^2 + 19\mu^3 - 62\mu^4 + 670\mu^5 + 1288\mu^6 + 18981\mu^7 + 384658\,\mu^8 + \cdots, \\
u_2 &= \mu^2 - \mu^3 + 45\mu^4 - 215\mu^5 + 1004\mu^6 - 13589\,\mu^7 + 7668\mu^8 + \cdots.
\end{aligned}
\tag{20}
$$

Substituting the values of u_1 and u_2 into the first equation Eq. (18), and then simplifying, it becomes

$$
\dot{\varphi}_{3(c)}^2 = 1 + \frac{3a^2}{4} - \frac{3a^4}{128} + \frac{9a^6}{512} - \frac{1779a^8}{131072} + \frac{5643a^{10}}{524288} - \frac{146542\frac{1}{8}a^{12}}{16777216} + \cdots.
\tag{21}
$$

The exact value of $\dot{\varphi}^2$ is

$$
\dot{\varphi}_{Ex(c)}^2 = 1 + \frac{3a^2}{4} - \frac{3a^4}{128} + \frac{9a^6}{512} - \frac{1779a^8}{131072} + \frac{5643a^{10}}{524288} - \frac{146661a^{12}}{16777216} + \cdots,
\tag{22}
$$

where $\dot{\varphi}_{3(c)}^2$ and $\dot{\varphi}_{Ex(c)}^2$ denote respectively, the third approximate frequency by the present method and exact frequency of the Eq. (16).

We see that first six terms of Eq. (21) are identical to the exact result in Eq. (22), and error occurs slightly in 7th term. It is noted that only the four terms of Eq. (45) [see "Appendix 2" and also (7–10)] are identical to the exact frequency when a third approximate solution is obtained from original equation Eq. (16). On the contrary, six terms of the fifth approximate solution (obtained by usual HBM) would be identical to its exact result $\dot{\varphi}_{Ex(c)}^2$. It has already been mentioned that the derivation of a fifth approximate solution is very laborious.

A strongly nonlinear oscillator containing $\dot{x}^2$

Now we consider the nonlinear oscillator

$$
\ddot{x} + (1 + \dot{x}^2)x = 0.
\tag{23}
$$

By introducing a scaling variable ε, as $x(t) = \sqrt{\varepsilon}\,y(t)$, $0 < \varepsilon < 1$, Eq. (23) can be easily transformed to a weak nonlinear equation, $\ddot{y} + y + \varepsilon\,y\dot{y}^2 = 0$ and it has a perturbation solution [see Belendez et al. (2007c) for details]. The aim of this article is to obtain another approximate solution. An integral expression of this equation is

$$
\ln(1 + \dot{x}^2) + x^2 = a^2
$$

or,

$$\dot{x}^2 = \exp(a^2 - x^2) - 1 \tag{24}$$

When $a \leq 1$, $\exp(a^2 - x^2)$ can be expanded in the Maclaurin series and Eq. (24) becomes

$$\dot{x}^2 + (x^2 - a^2) - \frac{(x^2 - a^2)^2}{2} + \frac{(x^2 - a^2)^3}{6} - \frac{(x^2 - a^2)^4}{24} + \frac{(x^2 - a^2)^5}{120} - \cdots = 0. \tag{25}$$

Substituting solution Eq. (4) in Eq. (25), dividing by $a^2 \sin^2 \varphi$ and equating the coefficient of Constant, $\cos 2\varphi$, we obtain

$$
\begin{aligned}
&\dot{\varphi}^2(1 + 4u_1 + 22u_1^2) - (1 + 4u_1 - 2u_1^2) - a^2(1 + 4u_1 + 4u_1^2 + \cdots)/4 \\
&\quad - a^4(1 + 4u_1 + 7u_1^2 + \cdots)/16 - a^6(5 + 20u_1 + 44u_1^2 + \cdots)/384 + \cdots = 0, \\
&\dot{\varphi}^2(12u_1 + 24u_1^2) - 4u_1 + a^2(1 - 6u_1^2)/4 + a^4(2 + 3u_1 - 6u_1^2 + \cdots)/24 \\
&\quad + a^6(147 + 1568u_1 - 392u_1^2 + \cdots)/37632 + \cdots = 0.
\end{aligned} \tag{26}
$$

By elimination of $\dot{\varphi}$ from two equations of Eq. (26), the equation of u_1 is obtained as

$$a^2 + a^4/3 + 5a^6/64 + 32u_1 + 16a^2u_1 + 29a^4u_1/6 + 224u_1^2 + 88a^2u_1^2 - 64u_1^3 = 0. \tag{27}$$

The coefficient of u_1 is much greater than the coefficients of a^2, a^4, a^6, $\cdots$, so Eq. (27) can be solved by choosing $u_1 = l_1 a^2 + l_2 a^4 + l_3 a^6 + \cdots$, where the unknown coefficients, $l_1, l_2, l_3, \ldots$, to be determined. Doing all these, the solution becomes

$$u_1 = -\frac{a^2}{32} - \frac{5a^4}{3072} - \frac{3a^6}{8192} - \cdots. \tag{28}$$

Now substituting the value of u_1 in the first equation Eq. (26), the approximate frequency (i.e., $\dot{\varphi}$) for small oscillation is obtained as

$$\dot{\varphi}_2^2 = 1 + \frac{a^2}{4} + \frac{5a^4}{128} + \frac{5a^6}{1536} - \cdots, \tag{29}$$

The exact value of $\dot{\varphi}^2$ is

$$\dot{\varphi}_{Ex}^2 = 1 + \frac{a^2}{4} + \frac{5a^4}{128} + \frac{5a^6}{1536} - \frac{3a^8}{131072} - \frac{91a^{10}}{2621440} - \frac{293a^{12}}{150994944} \cdots. \tag{30}$$

In a similar way, the third approximate solution of Eq. (23) can be obtained as

$$
\begin{aligned}
u_1 &= -\frac{a^2}{32} - \frac{a^4}{256} + \frac{a^6}{16384} + \cdots, \\
u_2 &= \frac{7a^4}{3072} + \frac{7a^6}{32768} + \cdots,
\end{aligned} \tag{31}
$$

and

$$\dot{\varphi}_3^2 = 1 + \frac{a^2}{4} + \frac{5a^4}{128} + \frac{5a^6}{1536} - \frac{3a^8}{131072} - \frac{91a^{10}}{2621440} - \cdots. \tag{32}$$

Comparing Eqs. (29) and (32) to Eq. (30), it is clear that second and third approximations respectively measure four and six terms in correct figures. On the contrary, the usual HBM is able to respectively measure three and four terms in correct figures (see "Appendix 3"). Thus the statement is true for nonlinear oscillator, Eq. (23) [or, Eq. (24)].

It is noted that the series given in Eq. (31) is converge only for the small amplitudes in the region $a \leq 1$.

Results and discussions

A new analytical approach based on the HBM has been presented to obtain approximate solutions of some well known nonlinear oscillators. Usually, a harmonic balance solution is obtained from the second order equations. Earlier, He (2002a, b) obtained some approximate solutions (mainly first approximation) for various nonlinear oscillators from corresponding first order differential equations (i.e., energy balance equations). But the new approach (concern of this article) is entirely different from He (2002a, b) technique. In this article, the first order equation is rewritten in such a way that every term is completely divisible by $a^2 \sin^2 \varphi$ for the proposed solution Eq. (4) (see "Methods" section). For three well known nonlinear problems, it has been verified that the solutions are better than corresponding solutions obtained by usual HBM. Recently, Hosen et al. (2012) have developed a technique based on the same method (i.e., HBM), but their solutions are significantly improved for the quadratic and cubic Duffing oscillators (see Hosen et al. (2012) details). On the contrary, the solution obtained by the new approach is better than usual harmonic solution even for the quintic Duffing oscillator.

To check the results, we have calculated the approximate frequency of Eq. (7) for some particular values of a (both small and large) by using Eq. (13) into the first Eq. (9) and compared with numerical solution together with other existing solutions (those solutions obtained by Wu et al. 2006; Belendez et al. 2006; Alam et al. 2007; Hu 2006a, b; Lai et al. 2009; Hosen et al. 2012) (see also "Appendix 1") and which is presented in Table 1. The Table 1 indicates that the approximate frequencies obtained by new approach are better than those obtained by usual HBM. Next, for some particular values of a (both small and large), we have calculated the approximate frequency of Eq. (16) by using the Eq. (20) into the first Eq. (18) and compared with numerical solution together with other existing solutions and which is presented in Table 2. The Table 2 indicates that the approximate frequencies give good agreement with the corresponding numerical result and also give better result than those obtained by the other usual HBM. Finally, for some particular values of a, we have also calculated the approximate frequency of Eq. (23) by using the Eq. (31) into the first Eq. (26) and compared with numerical solution together with other existing solutions obtained by usual HBM and which is presented in Table 3. The Table 3 indicates that the approximate frequencies give better result than those obtained by the other usual HBM. Moreover, we have determined the approximate periodic solution of Eqs. (7), (16), and (23) for different values of A and those solutions have been presented in Figs. 1a, b, 2a, b, 3a, b. All figures have been included the corresponding numerical solutions obtained by fourth-order Runge–Kutta method.

From these six figures, we see that the present method provides good agreement with the corresponding numerical solution.

Table 2 Comparison the approximate frequencies of Eq. (16) between the present method and truncation HBM Hosen et al. (2012), the usual HBM method with the exact frequency $\dot{\varphi}_{Ex}$, obtained by direct numerical integration

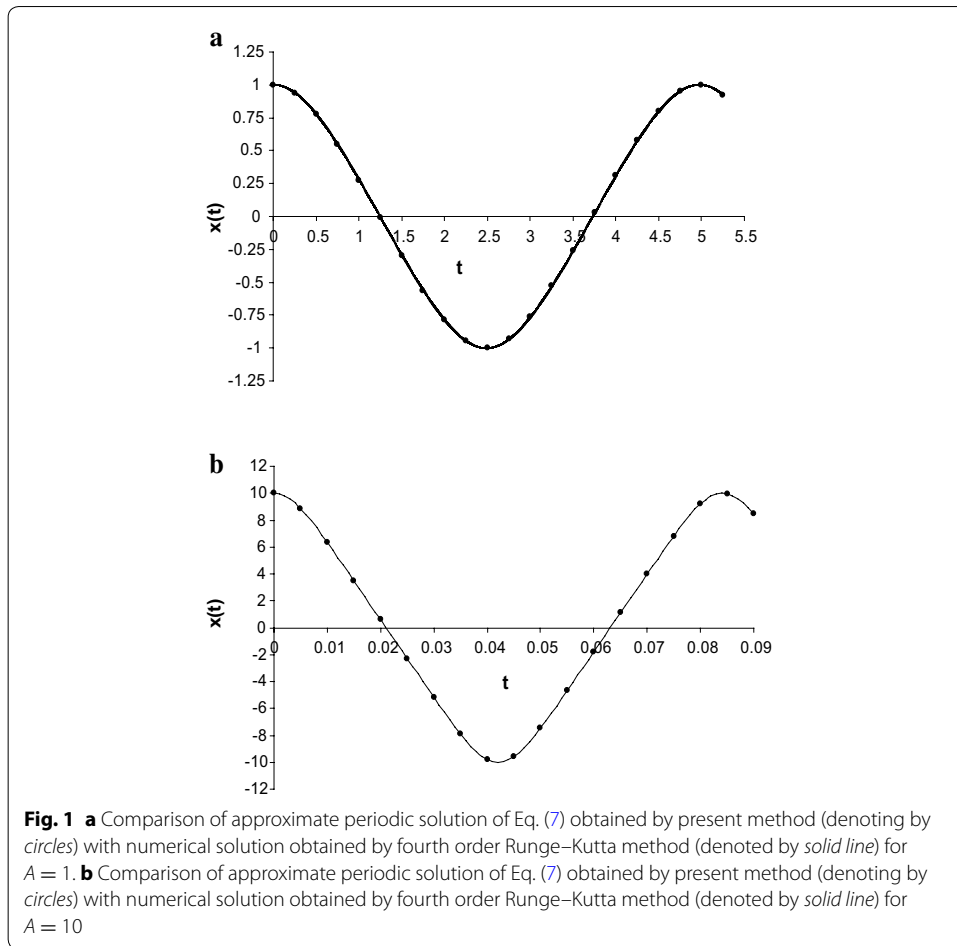| a	$\dot{\varphi}_{Ex}$	$\dot{\varphi}_{3(c,Usual(trunc))}$ (Hosen et al. 2012) Er (%)	$\dot{\varphi}_{3(c,Usual)}$ Er (%)	$\dot{\varphi}_{3(c,Present)}$ Er (%)
0.5	1.0891582	1.0891582	1.0891582	1.0891582
		0.00000	0.00000	0.00000
0.7	1.1676370	1.1676370	1.1676374	1.1676370
		0.00000	0.00004	0.00000
1	1.31778	1.31778	1.31778	1.31778
		0.000	0.000	0.000
2	1.97602	1.97601	1.97607	1.97602
		0.000	0.003	0.000
3	2.73849	2.73847	2.73862	2.73849
		0.000	0.005	0.000
4	3.53924	3.53921	3.53946	3.53926
		0.000	0.006	0.000
5	4.35746	4.35741	4.35777	4.35748
		0.001	0.007	0.000
10	8.53359	8.53347	8.5343	8.53363
		0.002	0.008	0.000
50	42.3730	42.3724	42.3767	42.3732
		0.002	0.009	0.000
100	84.7275	84.7262	84.7349	84.7279
		0.002	0.009	0.000

Er (%) denotes absolute percentage error

Table 3 Comparison the approximate frequencies of Eq. (23) between the present method and the usual HBM method with the exact frequency $\dot{\varphi}_{Ex}$, obtained by direct numerical integration

a	$\dot{\varphi}_{Ex}$	$\dot{\varphi}_{3(q,Usual)}$ Er (%)	$\dot{\varphi}_{3(q,Present)}$ Er (%)
1.8	1.52154	1.52669	1.52180
		0.339	0.017
2.0	1.67047	1.68325	1.67091
		0.765	0.026
2.2	1.84092	1.86926	1.84103
		1.540	0.006
2.4	2.03064	2.0876	2.028
		2.805	0.130
2.6	2.236	2.34108	2.22324
		4.700	0.571
2.8	2.45251	2.63242	2.41143
		7.336	1.675

Furthermore, the first-order approximate frequency obtained by usual harmonic balance method (HBM) is
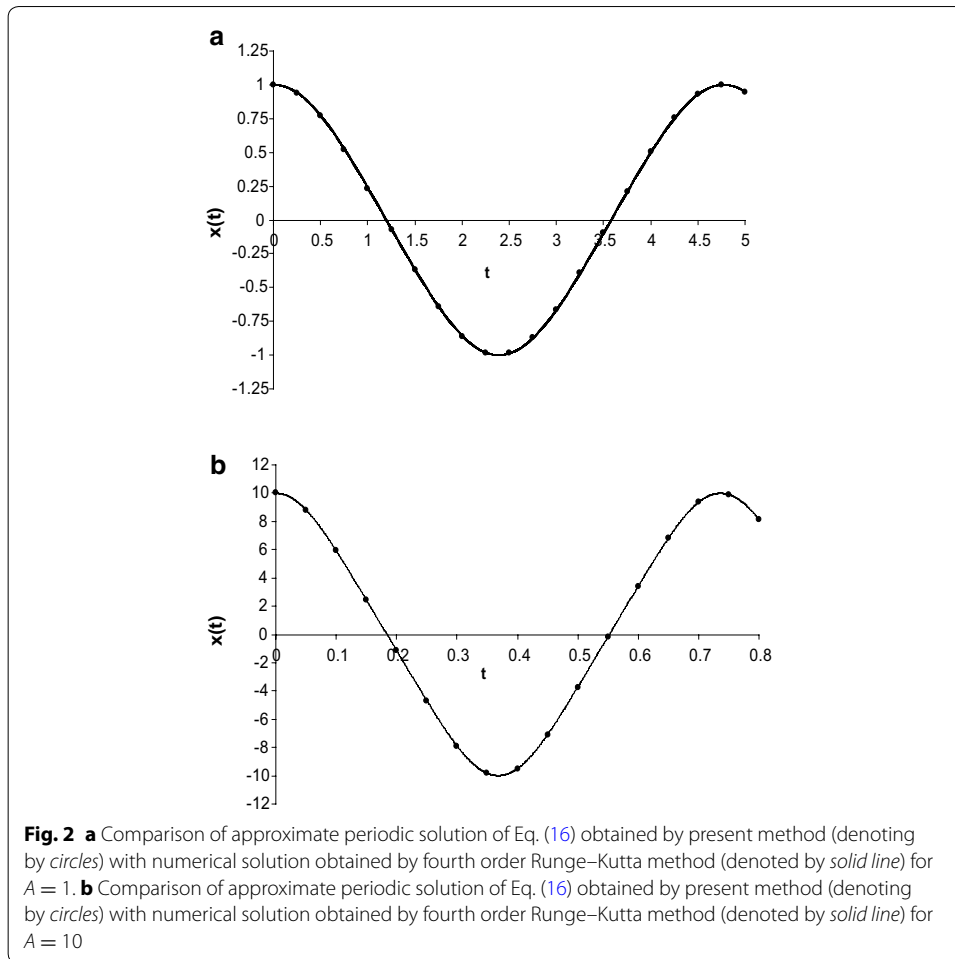
$$\dot{\varphi} = \frac{2}{\sqrt{4 - a^2}} \tag{33}$$

Fig. 1 **a** Comparison of approximate periodic solution of Eq. (7) obtained by present method (denoting by *circles*) with numerical solution obtained by fourth order Runge–Kutta method (denoted by *solid line*) for $A = 1$. **b** Comparison of approximate periodic solution of Eq. (7) obtained by present method (denoting by *circles*) with numerical solution obtained by fourth order Runge–Kutta method (denoted by *solid line*) for $A = 10$

From Eq. (33), it is observed that the approximate frequency, $\dot{\varphi}$ is undefined at $a = 2$. It is a big shortcoming of usual HBM.

On the other hand, the first-order approximate frequency becomes

$$\dot{\varphi} = \sqrt{e^{a^2/2}\left(J_0(a^2/2) - J_1(a^2/2)\right)}, \tag{34}$$

according to the present method. Here J_0 and J_1 are Bessell's functions. From Eq. (34), it is clear that the first approximate frequency is finite for all values of a. However, the relative error gradually increases as the amplitude increases.
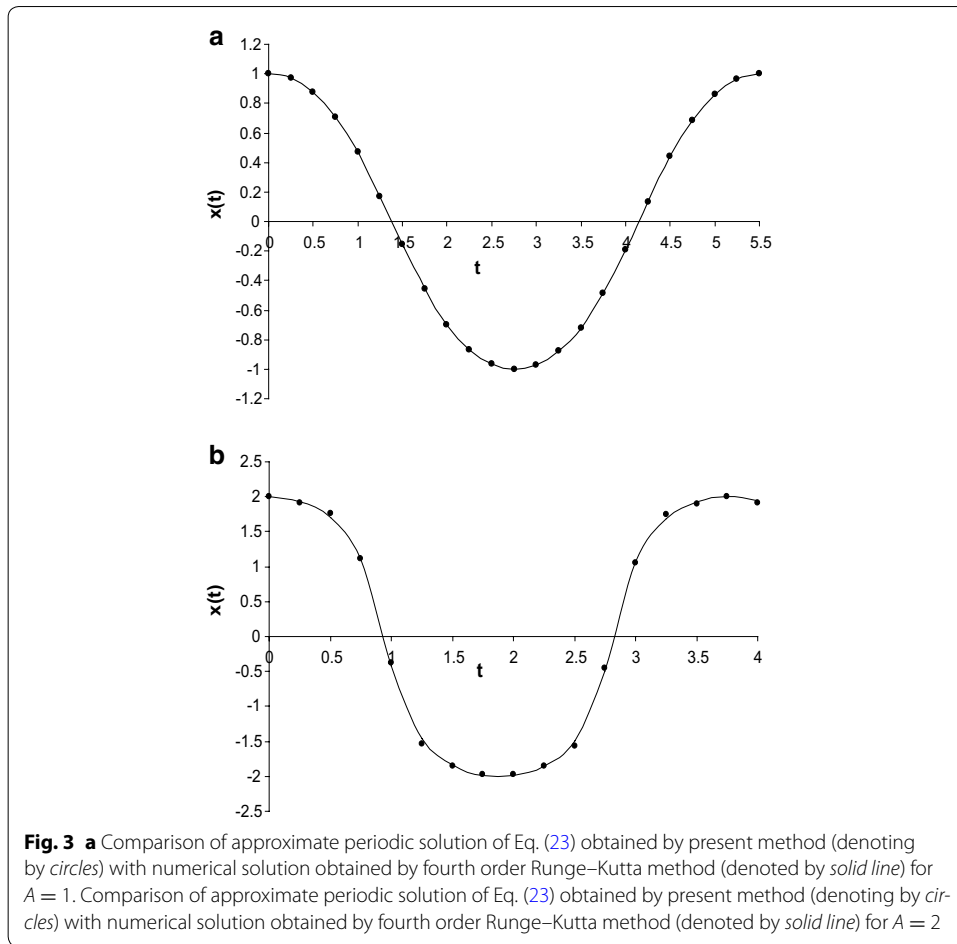
It has already been mentioned that the series Eq. (31) is mainly converged in the region $a \leq 1$, but the series also gives significant better result for obtaining approximate frequency even the amplitude increases up to $a = 2.8$ (see Table 3). On the contrary, the solution is only valid for in the region $a \leq 1$ while the amplitude increases, the solutions are deviated from the numerical solution (see Fig. 3b). Comparing the approximate frequency obtained by usual HBM with the exact approximate frequency determined numerically, it is shown from Table 3 that the relative error of the approximate value is

Fig. 2 a Comparison of approximate periodic solution of Eq. (16) obtained by present method (denoting by *circles*) with numerical solution obtained by fourth order Runge–Kutta method (denoted by *solid line*) for $A = 1$. **b** Comparison of approximate periodic solution of Eq. (16) obtained by present method (denoting by *circles*) with numerical solution obtained by fourth order Runge–Kutta method (denoted by *solid line*) for $A = 10$

less than 5, 8 % for $a < 2.6$ and $a < 2.8$, respectively while the relative error of the approximate frequency obtained in present method is less than 0.6, 2 % for $a < 2.6$ and $a < 2.8$, respectively. Therefore, the present method is faster than usual HBM.

Conclusion

Based on HBM, a new technique has been presented for solving a class of nonlinear oscillators. In the case of small values of amplitude, it has been verified that the fourth-order approximate frequency obtained by usual HBM is almost same as the third-order approximate frequency obtained by new method. For the case of large values of amplitude, the approximate frequencies obtained by new method not only gives better results than usual HBM but also gives nicely close to their exact results. Therefore, the results obtained in this paper are much better than those obtained by the usual HBM. The method also proved that it is a powerful mathematical tool for solving nonlinear oscillators.

Fig. 3 a Comparison of approximate periodic solution of Eq. (23) obtained by present method (denoting by *circles*) with numerical solution obtained by fourth order Runge–Kutta method (denoted by *solid line*) for $A = 1$. Comparison of approximate periodic solution of Eq. (23) obtained by present method (denoting by *circles*) with numerical solution obtained by fourth order Runge–Kutta method (denoted by *solid line*) for $A = 2$

Author details
[1] Department of Mathematics, Rajshahi University of Engineering and Technology (RUET), Kazla, Rajshahi 6204, Bangladesh. [2] Department of Mechanical Engineering, Rajshahi University of Engineering and Technology (RUET), Kazla, Rajshahi 6204, Bangladesh.

Competing interests
The author declares that they have no competing interests.

Appendix 1

Substituting solution Eq. (4) (together with $u_j = 0$, $j > 2$) in Eq. (7) and equating the coefficient of $\cos \varphi$, $\cos 3\varphi$ and $\cos 5\varphi$, the following nonlinear algebraic equations are obtained

$$\dot{\varphi}^2(-1 + u_1 + u_2) + (1 - u_1 - u_2) + 5a^4(2 - 5u_1 - 9u_2 + 12u_1^2 + 32u_1u_2 + 28u_2^2$$
$$- 20u_1^3 + 20u_1^4 - 60u_1^2u_2 + 60u_1^3u_2 - 146u_1u_2^2 - 50u_2^3 + \cdots)/16 = 0,$$
$$- 9u_1\dot{\varphi}^2 + u_1 + 5a^4(1 + u_1 - u_2 - 8u_1^2 - 8u_1u_2 - 2u_2^2 + 20u_1^3 - 25u_1^4 + 30u_1^2u_2$$
$$+ 30u_1u_2^2 - 40u_1^3u_2 + 8u_2^3 + \cdots)/16 = 0,$$
$$- 25u_2\dot{\varphi}^2 + u_2 + 5a^4(5 + 3u_1 + 5u_2 - 8u_1^2 - 28u_1u_2 - 22u_2^2 + 10u_1^3 - 5u_1^4 + 60u_1^2u_2$$
$$+ 84u_1u_2^2 - 64u_1^3u_2 + 46u_2^3 + \cdots)/16 = 0. \tag{35}$$

By elimination of $\dot{\varphi}$ from three equations of Eq. (35), we obtain two equations of u_1 and u_2 as

$$
\begin{aligned}
&- 8u_1 + 8u_1 u_2 + 8u_1^2 + 5a^4(1 - 18u_1 - 2u_2 + 73u_1 u_2 + 36u_1^2 - 80u_1^3 - 242u_1^2 u_2 \\
&\quad - u_2^2 - 212u_1 u_2^2 + 10u_2^3 + 135u_1^4 + 450u_1^3 u_2)/16 = 0, \\
&- 24u_2 + 24u_1 u_2 + a^4(1 + 14u_1 - 226u_2 - 55u_1^2 + 445u_1 u_2 + 90u_1^3 + 990u_2^2 \\
&\quad - 1020u_1^2 u_2 - 3330u_1 u_2^2 - 3160u_2^3 - 75u_1^4 + 1830u_1^3 u_2)/16 = 0.
\end{aligned}
\tag{36}
$$

In general, u_1 and u_2 are small. So, it is possible to divide the first and second of Eq. (36) by $1 - u_1 - u_2$ and then they become

$$
\begin{aligned}
384u_1 &= a^4(15 - 225u_1 - 15u_2 + 825u_1 u_2 + 285u_1^2 - 30u_2^2 - 915u_1^3 \\
&\quad - 2520u_1^2 u_2 - 2385u_1 u_2^2 + 120u_2^3), \\
384u_2 &= a^4(1 + 15u_1 - 225u_2 + 235u_1 u_2 - 40u_1^2 + 50u_1^3 - 825u_1^2 u_2 \\
&\quad + 765u_2^2 - 2330u_1 u_2^2 - 2395u_2^3) = 0.
\end{aligned}
\tag{37}
$$

Therefore, by choosing $\lambda_0 = 16a^4/(384 + 225a^4)$, the power series solution of these equations are obtained as

$$
\begin{aligned}
u_1 &= \frac{15\lambda_0}{16} - \frac{15\lambda_0^2}{256} + \frac{73095\lambda_0^3}{4096} - \cdots, \\
u_2 &= \frac{\lambda_0}{16} + \frac{225\lambda_0^2}{256} - \frac{4935\lambda_0^3}{4096} - \cdots.
\end{aligned}
\tag{38}
$$

Substituting the values of u_1 and u_2 into first equation of Eq. (35), and then simplifying, we obtain

$$
\dot{\varphi}_{3(q,Usual)}^2 = 1 + \frac{5a^4}{8} - \frac{65a^8}{1536} + \frac{3485a^{12}}{131072} - \frac{3953755a^{16}}{226492416} + \cdots,
\tag{39}
$$

where, $\dot{\varphi}_{3(q,Usual)}^2$ denotes the third approximate frequency of the Eq. (7) by the usual HBM.

Appendix 2

Substituting solution Eq. (4) (together with $u_j = 0$, $j > 2$) in Eq. (16) and equating the coefficient of $\cos\varphi$, $\cos 3\varphi$ and $\cos 5\varphi$, the following nonlinear algebraic equations are obtained

$$
\begin{aligned}
&\dot{\varphi}^2(-1 + u_1 + u_2) + 1 - u_1 - u_2 + 3a^2(1 - 2u_1 - 3u_2 + 3u_1^2 + 6u_1 u_2 + 5u_2^2 - 2u_1^3 \\
&\quad - 4u_1^2 u_2 - 6u_1 u_2^2 - 3u_2^3 + \cdots)/4 = 0, \\
&- 9u_1\dot{\varphi}^2 + u_1 + a^2(1 + 3u_1 - 9u_1^2 - 6u_1 u_2 - 3u_2^2 + 8u_1^3 - 25u_1^4 + 6u_1^2 u_2 \\
&\quad + 9u_1 u_2^2 + 2u_2^3 + \cdots)/4 = 0, \\
&- 25u_2\dot{\varphi}^2 + u_2 + 3a^2(u_1 + 2u_2 - u_1^2 - 6u_1 u_2 - 4u_2^2 + 5u_1^2 u_2 + 5u_1 u_2^2 + 3u_2^3 + \cdots)/4 = 0.
\end{aligned}
\tag{40}
$$

By elimination of $\dot{\varphi}$ from three equations of Eq. (40), we obtain two equations of u_1 and u_2 as $8(-u_1 + u_1 u_2 + u_1^2) + a^2(1 - 25u_1 - u_2 + 72u_1 u_2 + 42u_1^2 - 3u_2^2 - 64u_1^3$

$$
\begin{aligned}
&\qquad - 141u_1^2 u_2 - 117u_1 u_2^2 + 5u_2^3 + 46u_1^4 + 94u_1^3 u_2 + \cdots)/4 = 0, \\
&24(-u_2 + u_1 u_2 + u_2^2) + 3a^2(u_1 - 23u_2 - 2u_1^2 + 41u_1 u_2 + u_1^3 \\
&\qquad + 69u_2^2 - 63u_1^2 u_2 - 3330u_1 u_2^2 - 118u_2^3 - 135u_1 u_2^2 + \cdots)/4 = 0.
\end{aligned}
\tag{41}
$$

Dividing the first and second of Eq. (41) by $1 - u_1 - u_2$ and they are become

$$
\begin{aligned}
u_1 &= \mu_1(1 + 18u_1^2 - 46u_1^3 + 48u_1u_2 - 3u_2^2 - 75u_1^2u_2 - 27u_1^3u_2 - \cdots), \\
u_2 &= \mu(u - u^2 + 19uv - 45u^2v + 46v^2 - 70uv^2 - \cdots),
\end{aligned} \tag{42}
$$

where $\mu_1 = a^2/(32 + 24a^2)$ and $\mu = a^2/(32 + 23a^2)$.

The algebraic relation between μ_1 and μ is

$$
\mu_1 = \mu/(1 - \mu). \tag{43}
$$

Expanding the right hand side of Eq. (43) in powers of μ and substituting this value of μ_1 into the second equation of Eq. (42), it can be solved Eq. (42) in powers of μ_1 as

$$
\begin{aligned}
u_1 &= \mu_1 + 18\mu_1^3 + 2\mu_1^4 + 570\mu_1^5 + 129\mu_1^6 + 20642\mu_1^7 + 6296\mu_1^8 + \cdots, \\
u_2 &= \mu_1^2 + 37\mu_1^4 + 4\mu_1^5 + 1545\mu_1^6 + 346\mu_1^7 + 67039\mu_1^8 + \cdots.
\end{aligned} \tag{44}
$$

Substituting the values of u_1 and u_2 into first equation of Eq. (40), and then simplifying, we obtain

$$
\dot{\varphi}^2_{3(c,Usual)} = 1 + \frac{3a^2}{4} - \frac{3a^4}{128} + \frac{9a^6}{512} - \frac{1773a^8}{131072} + \frac{5589a^{10}}{524288} - \frac{144008\frac{1}{2}a^{12}}{16777216} + \cdots. \tag{45}
$$

where, $\dot{\varphi}^2_{3(c,Usual)}$ denotes the third approximate frequency of the Eq. (16) by the usual HBM.

Dividing the first equation of Eq. (40) by $1 - u_1 - u_2$, we obtain

$$
\dot{\varphi}^2 = 1 + 3a^2(1 - u + 2u^2 - 2v + 3uv + u^2v + 3v^2 + u^2v^2 + u^2v^3 + 3u^3v^3 \cdots)/4 \tag{46}
$$

The third approximate solution measures better result when Eqs. (46) and (44) are truncated as

$$
\dot{\varphi}^2 = 1 + 3a^2(1 - u_1 + 2u_1^2 + u_1^3 - 2u_2 + 2u_1^2u_2 + 3u_1^3u_2 - 2u_2^2 - 2u_1u_2^2 - 2u_2^3 - 4u_2u_2^3 + \cdots)/4. \tag{47}
$$

and

$$
\begin{aligned}
u_1 &= \lambda_1(1 + 18u_1^2 - 50u_1^3 + 51\,u_1u_2 - 58u_1^3u_2 + 15u_1^4u_2 + 54u_1u_2^2 + 48u_1^2u_2^2 + 50u_1u_2^3 + \cdots), \\
v_1 &= \mu_1(u_1 - u_1^2 + 22u_1u_2 - 50u_1^2u_2 + 50u_2^2 - 50u_1^2u_2^2 + 50u_2^3 + 50u_1u_2^3 + 50u_2^4 + 100u_1u_2^4 + \cdots),
\end{aligned} \tag{48}
$$

where λ_1 and μ_1 are given in Eqs. (42), (43).

The power series of Eq. (48) is obtained as

$$
\begin{aligned}
u_1 &= \mu_1 + 18\mu_1^3 + \mu_1^4 + 620\mu_1^5 + 7183\mu_1^6/25 + 657782\mu_1^7/25 + 819506\mu_1^8/25 + \cdots, \\
u_2 &= \mu_1^2 + 40\mu_1^4 + 5\mu_1^5 + 1874\mu_1^6 + 22583\mu_1^7/25 + 2377007\mu_1^8/25 + \cdots.
\end{aligned} \tag{49}
$$

Appendix 3

Substituting the second approximate solution Eq. (4) in Eq. (23) and equating the coefficient of $\cos\varphi$, $\cos 3\varphi$, the following nonlinear algebraic equations are obtained

$$\dot{\varphi}^2 \left(-4 + 4u_1 + a^2(1 + 2u_1 + 11u_1^2 - 14u_1^3)\right)/4 - 1 - u_1 = 0,$$
$$\dot{\varphi}^2 \left(-36u_1 - a^2(1 + 5u_1 - 7u_1^2 + 12u_1^3)\right)/4 + u_1 = 0. \tag{50}$$

By elimination of $\dot{\varphi}$ from three equations of Eq. (50), we obtain the equation of u_1 as

$$a^2 + 32u_1 - 5a^2u_1 - 32u_1^2 + 14a^2u_1^2 - 8a^2u_1^3 + \cdots = 0. \tag{51}$$

Now, the above equation can written as

$$u_1 = -(a^2 - 5a^2u_1 - 32u_1^2 + 14a^2u_1^2 - 8a^2u_1^3 + \cdots)/32. \tag{52}$$

It is clear that u_1 is much smaller than 1 when $a = \mathrm{O}(1)$. In this case Eq. (52) can be solved in powers of a^2 as $u_1 = l_1a^2 + l_2a^4 + l_3a^6 + \cdots$, where the unknown coefficients, $l_1, l_2, l_3, \ldots$, to be determined.

Thus we have the solution of Eq. (52), as

$$u_1 = -\frac{a^2}{32} - \frac{a^4}{256} - \frac{13a^6}{163884} - \cdots. \tag{53}$$

Now substituting the value of u_1 in the first equation Eq. (50), the approximate frequency (i.e., $\dot{\varphi}$) for small oscillation is obtained as

$$\dot{\varphi}^2_{2(Existing)} = 1 + \frac{a^2}{4} + \frac{5a^4}{128} + \frac{9a^6}{2048} + \frac{37a^8}{65536} + \cdots. \tag{54}$$

where $\dot{\varphi}^2_{2(Existing)}$ denotes the second approximate solution of Eq. (23) by the usual HBM.

In a similar way, the third approximate solution of Eq. (23) can be obtained as

$$u_1 = -\frac{a^2}{32} - \frac{a^4}{256} + \frac{a^6}{16384} + \cdots,$$
$$u_2 = \frac{7a^4}{3072} + \frac{31a^6}{73728} + \cdots, \tag{55}$$

and

$$\dot{\varphi}^2_{3(Existing)} = 1 + \frac{a^2}{4} + \frac{5a^4}{128} + \frac{5a^6}{1536} - \frac{16\frac{5}{9}a^6}{131072} - \cdots. \tag{56}$$

where $\dot{\varphi}^2_{3(Existing)}$ denotes the third approximate solution of Eq. (23) by the usual HBM.

References

Alam MS, Haque ME, Hossian MB (2007) A new analytical technique to find periodic solutions of nonlinear systems. Int J Non Linear Mech 42:1035–1045

Belendez A (2007) Application of He's homotopy perturbation method to the Duffing-harmonic oscillator. Int J Non Linear Sci Numer Simul 8:79–88

Belendez A, Hernandz A, Marquez A, Belendez T, Neipp C (2006) Analytical approximations for the period of a nonlinear pendulum. Eur J Phys 27:539–551

Belendez A, Belendez T, Hernandez A, Gallego S, Ortuno M, Neipp C (2007a) Comments on investigation of the properties of the period for the nonlinear oscillator $\ddot{x} + (1 + \dot{x}^2)x = 0$. J Sound Vib 303:925–930

Belendez A, Pascual C, Gallego S, Ortuno M, Neipp C (2007b) Application of a modified He's homotopy perturbation method to obtain higher-order approximation of an $x^{1/3}$ force nonlinear oscillator. Phys Lett A 371:421–426

Belendez A, Hernandz A, Marquez A, Belendez T, Neipp C (2007c) Application of He's homotopy perturbation method to nonlinear pendulum. Eur J Phys 28:93–104

Cheung YK, Mook SH, Lau SL (1991) A modified Lindstedt–Poincare method for certain strongly nonlinear oscillators. Int J Non Linear Mech 26:367–378

Guo Z, Leung AYT, Yang HX (2011) Iterative Homotopy harmonic balancing approach for conservative oscillator with strong odd nonlinearity. Appl Math Model 35(4):1717–1728

Haque BMI, Alam MS, Rahmam MM (2013) Modified solutions of some oscillators by iteration procedure. J Egyptian Math Soc 21:68–73

He JH (2002a) Modified Lindstedt–Poincare methods for some strongly nonlinear oscillations-I: expansion of a constant. Int J Non Linear Mech 37:309–314

He JH (2002b) Preliminary report on the energy balance for nonlinear oscillations. Mech Res Commun 29:107–118

Hosen MA, Rahman MS, Alam MS, Amin MR (2012) An analytical technique for solving a class of strongly nonlinear conservative systems. Appl Math Comput 218:5474–5486

Hu H (2006a) Solution of a quadratic nonlinear oscillator by the method of harmonic balance. J Sound Vib 293:462–468

Hu H (2006b) Solutions of Duffing-harmonic oscillator by an iteration procedure. J Sound Vib 298:446–452

Lai SK, Lim CW, Wu BS, Wang C, Zeng QC, He XF (2009) Newton-harmonic balancing approach for accurate solutions to nonlinear cubic-quintic Duffing oscillators. Appl Math Model 33:852–866

Lim CW, Wu BS (2002) A modified procedure for certain non-linear oscillators. J Sound Vib 257:202–206

Lim CW, Wu BS (2003) A new analytical approach to the Duffing-harmonic oscillator. Phys Lett A 311:365–373

Lim CW, Lai SK, Wu BS (2005) Accurate higher-order analytical approximate solutions to large-amplitude oscillating systems with general non-rational restoring force. J Nonlinear Dyn 42:267–281

Lim CW, Wu BS, Sun WP (2006) Higher accuracy analytical approximations to the Duffing-harmonic oscillator. J Sound Vib 296:1039–1045

Mickens RE (1984) Comments on the method of harmonic balance. J Sound Vib 94:456–460

Mickens RE (1986) A generalization of the method of harmonic balance. J Sound Vib 111:515–518

Mickens RE (1987a) Nonlinear oscillations. Cambridge University Press, New York

Mickens RE (1987b) Iteration procedure for determining approximate solutions to nonlinear oscillator equation. J Sound Vib 116:185–188

Mickens RE (1996) Oscillation in planar dynamic systems. World Scientific, Singapore

Mickens RE (2005) A general procedure for calculating approximation to periodic solutions of truly nonlinear oscillators. J Sound Vib 287:1045–1051

Mickens RE (2010) Truly nonlinear oscillations. World Scientific, Singapore

Ozis T, Yildirim A (2007) Determination of periodic solution for a $u^{1/3}$ force by He's modified Lindstedt–Poincare method. J Sound Vib 301:415–419

West JC (1960) Analytical techniques for nonlinear control systems. English University Press, London

Wu BS, Sun WP, Lim CW (2006) An analytical approximate technique for a class of strongly nonlinear oscillators. Int J Non Linear Mech 41:766–774