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Analysis of a diffusive epidemic system with spatial heterogeneity and lag effect of media impact

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Abstract

We considered an SIS functional partial differential model cooperated with spatial heterogeneity and lag effect of media impact. The wellposedness including existence and uniqueness of the solution was proved. We defined the basic reproduction number and investigated the threshold dynamics of the model, and discussed the asymptotic behavior and monotonicity of the basic reproduction number associated with the diffusion rate. The local and global Hopf bifurcation at the endemic steady state was investigated theoretically and numerically. There exists numerical cases showing that the larger the number of basic reproduction number, the smaller the final epidemic size. The meaningful conclusion generalizes the previous conclusion of ordinary differential equation.

Keywords Media impact · Functional partial differential model · Spatial heterogeneity · Hopf bifurcation

1 Introduction

Infectious diseases can have a great impact on the development of human society as they can negatively bring morbidity, mortality, unemployment and inequality. Therefore, prevention and control of infectious diseases are of significance for public health and welfare. Recent outbreaks of infectious diseases, such as Ebola, severe acute respiratory syndrome (SARS), the 2009 novel influenza A(H1N1) pandemic, Covid-19 have highlighted an important role played by global public health systems of surveillance and response which help to quickly curb an emerging disease and reduce its

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influence on socioeconomic activities (Lau et al. 2004; Tang et al. 2012; Winters et al. 2018; Lai et al. 2020). However, the impact of infectious disease, the massive news coverage and fast information flow on the emerging diseases are all subject to behavioral changes of human trying to minimize the effect of the disease onto themselves. In recent years, several emerging infectious diseases confirmed the existence of a so-called behavioural immune system (Schaller 2011). For example, during the 2003 SARS and Covid-19 outbreak, people took precautionary actions such as wearing face masks, hand-washing, avoiding public contact (Lau et al. 2004; Beutels et al. 2009; Lai et al. 2020). Morover, the 2009 A/H1N1 influenza pandemic and Covid-19 had induced a significant proportion of the population to adapt their behaviour and take preventive measures such as social distancing, home isolation or school closure (Tang et al. 2012; Lai et al. 2020).

The exact impact of media coverage can have on the infectious disease, however, is difficult to quantify and often subject to speculation in mathematical modelling. A number of mathematical models were developed to investigate the impact of media coverage on the spreading and control of infectious diseases (Verelst et al. 2016; Funk 2010; Cui et al. 2008; Xiao et al. 2013, 2015; Yan et al. 2016; Tang et al. 2010; Liu et al. 2007; Li and Cui 2009; Tchuenche et al. 2011; Sun et al. 2011; Song and Xiao 2018, 2019). In order to characterize the media impact on disease, a media function, decreasing in the number of infected individuals, was often included. For example, in Liu et al. (2007), a media function $\beta e^{-\alpha_1 E-\alpha_2 I-\alpha_3 H}$ was introduced into the transmission coefficient, where E, I, and H are the numbers of reported exposed, infectious, and hospitalized individuals, respectively. Li and Cui (2009) used the media function $\beta_1 - \beta_2 \frac{I}{m+I}$ (or $\beta_2 \frac{I}{m+I}$) to reflect the reduced amount of contact rate due to media coverage. Xiao et al. (2015) extended these media functions by assuming that the function depends on both the case number and its rate of change, and obtained that media impact switches on and off in a highly nonlinear fashion. Yan et al. (2016) further extended extended a class epidemic model of SEIR type by including extra compartment, i.e., the level of media coverage M, and characterize the media impact by including the function $e^{-\mu M}$ with $\mu > 0$ in the incidence. It was found that although the media coverage itself is not a determined fact to eradicate the infection of the disease, media coverage can greatly delay the epidemic peak and decrease severity of outbreak (Liu et al. 2007; Song and Xiao 2019).

However, these models built on ordinary differential equations ignore two important factors: the lag effect of media impact and human mobility in heterogeneous environment. The lag of media impact are induced directly by the mass media's response to the disease infection, and indirectly by the time from for individuals' response to the media coverage such as symptom onset to hospitalization. In fact, by analyzing the case data and media coverage on A/H1N1 in Shaanxi province in 2009, Yan et al. (2016) obtained the correlation between the case number and media coverage, confirmed the existence of time lags and further identified the time lags. Hence, including time delay in the incidence rate is more reasonable. Recently, we (Song and Xiao 2018, 2019) initially included time delay in the media function and explored the lag effect of media impact on infectious disaes. However, human mobility in heterogeneous environment were ignored in Song and Xiao (2018, 2019). There is increasing evidence which show that environmental heterogeneity and human mobility have significant impact on the spread of infectious diseases (Cantrell and Cosner 2003; Murray 2002; Riley 2007; y Piontti et al. 2018). In recent years, numerous reaction-diffusion models have been proposed to investigate the roles of diffusion and spatial heterogeneity on the transmission of diseases (Allen et al. 2008; Wang and Zhao 2012; Zhao 2017; Peng and Zhao 2012; Cui and Lou 2016; Cui et al. 2017; Deng and Wu 2016; Wu and Zou 2016; Li et al. 2017; Ge et al. 2015; Li et al. 2020). Among these works, Allen et al. (2008) proposed a susceptible-infected- susceptible (SIS) reaction-diffusion system cooperated with spatial heterogeneity as follows:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S - \dfrac{\beta(x)SI}{S+I} + \gamma(x)I, & x \in \Omega, t > 0, \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + \dfrac{\beta(x)SI}{S+I} - \gamma(x)I, & x \in \Omega, t > 0, \\ \dfrac{\partial S}{\partial n} = \dfrac{\partial I}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{1}$$

The main results of Allen et al. (2008) concern with the definition, monotonicity and asymptotic properties of basic production number ($\mathcal{R}_0$), threshold-type results on the global dynamics in terms of $\mathcal{R}_0$ and particularly the existence, uniqueness and asymptotic behaviors of the endemic steady state (EE) as the diffusion rate of the susceptible individuals (d_S) approaches to zero. Peng and Zhao (Peng and Zhao 2012) recently considered the same SIS reaction-diffusion model, but the rates of disease transmission and recovery are assumed to be spatially heterogeneous and temporally periodic. In Deng and Wu (2016), Wu and Zou (2016), the authors investigated an SIS model with mass action infection mechanism. In Li et al. (2017), Li et al. provided qualitative analysis on an SIS reaction diffusion system with a linear source term. Ge et al. introduced a free boundary model for characterizing the spreading front of the disease in Ge et al. (2015). The effects of diffusion and advection for SIS epidemic reaction-diffusion model in heterogeneous environments were studied in Cui and Lou (2016), Cui et al. (2017). Dynamics and asymptotic profiles of endemic steady state for two frequency-dependent SIS epidemic models with cross-diffusion was studied in Li et al. (2020). Moreover, to grasp the impact of the media coverage and heterogenous environment on preventing and controlling the transmission of infectious diseases, Ge et al. (2017) consider an SIS reaction-diffusion equation with media impact. However, these reaction diffusion models only consider human mobility in heterogeneous environment or the media impact without lag effect. The report delay and response time for individuals to the current infection were ignored in these models.

1.1 Model description

In order to investigate the lag effect of media impact and human mobility in heterogeneous environment on the transmission dynamics of infectious diseases, we devide the population into two groups: susceptible (S), infected (I), and consider the following system:

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \Delta S + \Lambda(x) - \dfrac{\beta(x) e^{-m(x)I(x,t-r)} SI}{S+I} + \gamma(x)I - \alpha(x)S, & x \in \Omega, t > 0, \\ \dfrac{\partial I}{\partial t} = d_I \Delta I + \dfrac{\beta(x) e^{-m(x)I(x,t-r)} SI}{S+I} - \gamma(x)I - \alpha(x)I, & x \in \Omega, t > 0, \\ \dfrac{\partial S}{\partial n} = \dfrac{\partial I}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{2}$$

Here, Ω is a bounded domain in $\mathbb{R}^{n_0}$ with smooth boundary $\partial\Omega$, where n_0 is a positive integer and the homogeneous Neumann boundary conditions assumed in model (2) mean that no population flux crosses the boundary $\partial\Omega$. $S(x,t)$ and $I(x,t)$ denote the density of susceptible and infected individuals at location x and time t, respectively. d_S and d_I represent the diffusion coefficients associated with susceptible and infected individuals, respectively. The positive Hölder continuous functions $\Lambda(x)$, $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ on $\overline{\Omega}$ represent the natural birth rate, the natural death rate and transmission rate and recovery rate at x, respectively.

Media coverage and fast information flow induce a profound psychological impact on the public, a reduction in the incidence rate at the position x, is represented by $e^{-m(x)I(x,t-r)}$. Here time delay r denotes the report delay and response time for individuals to the current infection, and the positive Hölder continuous functions $m(x)$ stands for the weight of media effect sensitive to number of infected population at the position x. Here we assume that people care more about the prevalence of infectious disease in the place they stay, therefore we choose the media impact function $e^{-m(x)I(x,t-r)}$ rather than $e^{-\int_\Omega m(x)I(x,t-r)dx}$. We also point out here that the disease-related death was not included in model (2), since, on one hand, we focus our model on the effect of spatial heterogeneity and media impact on prevalence, incidence or accumulated cases rather than case fatal rate of infectious disease, and the disease-induced death rate is far smaller than recovery rate. On the other hand, we compromise here to make further bifurcation analysis not too complicated.

It is easy to verify that $\frac{\beta SI}{S+I}$ is a Lipschitz continuous function of S and I, therefore we define it to be zero whenever $S = 0$ or $I = 0$. For notation convenience, we denote

$$\underline{g} = \min_{x \in \overline{\Omega}} \{g(x)\}, \quad \overline{g} = \max_{x \in \overline{\Omega}} \{g(x)\},$$

where g can be $\Lambda(x)$, $\alpha(x)$, $\beta(x)$, $\gamma(x)$ and $m(x)$. Moreover, throughout this paper, we assume

A1: $S(x,0) \geq 0$, $I(x,0) \geq 0$ for $x \in \overline{\Omega}$ and $\int_\Omega I(x,0)dx > 0$;

A2: $H^+ = \{x \in \Omega | \beta(x) > \alpha(x) + \gamma(x)\}$ and $H^- = \{x \in \Omega | \beta(x) < \alpha(x) + \gamma(x)\}$ are nonempty.

For further purposes, we also define two conditions as follows:

B1: $\int_\Omega \beta dx < \int_\Omega (\gamma(x) + \alpha(x))dx$;

B2: $\int_\Omega \beta dx > \int_\Omega (\gamma(x) + \alpha(x))dx$.

1.2 Steady state problems

For further purposes, we define non-negative steady state solutions of model (2):

$$\begin{cases} d_S \Delta \tilde{S} + \Lambda(x) - \dfrac{\beta(x) e^{-m(x)\tilde{I}} \tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} + \gamma(x)\tilde{I} - \alpha(x)\tilde{S} = 0, & x \in \Omega, \\ d_I \Delta \tilde{I} + \dfrac{\beta(x) e^{-m(x)\tilde{I}} \tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} - \gamma(x)\tilde{I} - \alpha(x)\tilde{I} = 0, & x \in \Omega, \\ \dfrac{\partial \tilde{S}}{\partial n} = \dfrac{\partial \tilde{I}}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3}$$

Here, $\tilde{S}(x)$ and $\tilde{I}(x)$ denote the density of susceptible and infected individuals, respectively, at $x \in \Omega$. A disease-free steady state (DFE) is a solution of (3) so that $\tilde{I}(x) = 0$ for every $x \in \Omega$. An endemic steady state (EE) of (3) is a solution in which $\tilde{I}(x) > 0$ for some $x \in \Omega$. By direct calculation, the disease free steady state (DFE) is

$$E_0 := (\widetilde{N}, 0) \tag{4}$$

and it is unique, where $\widetilde{N}$ is the unique solution of

$$d_S \Delta \widetilde{N} + \Lambda(x) - \alpha(x)\widetilde{N} = 0 \text{ in } \Omega; \quad \frac{\partial S}{\partial n}\bigg|_{\partial\Omega} = 0.$$

Denote the endemic steady state (EE) by $(\tilde{S}, \tilde{I})$. By the strong maximum principle, any endemic steady state, $(\tilde{S}, \tilde{I})$ are positive for any $x \in \Omega$.

The rest of this paper is organized as follows. In Sect. 2, we study the wellposedness, define the basic reproduction number and investigate the threshold dynamics of model (2). In Sect. 3, we assume that $d_S = d_I$ and explore the local Hopf bifurcation at the endemic steady state. Section 4 is devoted to global existence of periodic solutions. Numerical simulations are presented in Sect. 5 to graphically illustrate the effect of delayed media impact and human mobility in heterogeneous environment on the transmission dynamics of infectious diseases. The paper ends with a conclusion section.

2 Wellposedness, Basic reproduction number and threshold dynamics

Let $\mathcal{X} = C(\Omega, \mathbb{R}^2)$ be the Banach space of continuous functions with the supremum norm $\|\|_{\mathcal{X}}$. Set $C_r = C([-r, 0], \mathcal{X})$. For any $\phi \in C_r$, define $\|\phi\| = \max_{\theta\in[-r,0]} \|\phi(\theta)\|_{\mathcal{X}}$. Then C_r is an ordered Banach space with the cone C_r^+.

Denote $\mathcal{Y} = C(\Omega, \mathbb{R})$. Let $T_1(t)$ and $T_2(t) : \mathcal{Y} \to \mathcal{Y}, t \geq 0$, be the semigroups associated with $d_S\Delta$ and $d_I\Delta$ with the homogeneous Neumann boundary conditions, respectively, and let $A_i : D(A_i) \to Y$ be the generator of $T_i(t), i = 1, 2$. Clearly, $T(t) = (T_1(t), T_2(t)) : \mathcal{X} \to \mathcal{X}, t \geq 0$, is a semigroup generated by the operator

$A = (A_1, A_2)$ defined on $D(A) = D(A_1) \times D(A_2)$. Then for each $t > 0$, $T(t) : \mathcal{X} \to \mathcal{X}$ is compact and positive (see, e.g., (Smith 1995, Section 7.1 and Corollary 7.2.3)). Define $F = (F_1, F_2) : C_r^+ \to C_r^+$ by

$$
\begin{aligned}
F_1(\phi_1, \phi_2)(x) &= \Lambda(x) - \frac{\beta(x)e^{-m(x)\phi_2(-r,x)}\phi_1(0,x)\phi_2(0,x)}{\phi_1(0,x) + \phi_2(0,x)} \\
&\quad + \gamma(x)\phi_2(0,x) - \alpha(x)\phi_1(0,x), \\
F_2(\phi_1, \phi_2)(x) &= \frac{\beta(x)e^{-m(x)\phi_2(-r,x)}\phi_1(0,x)\phi_2(0,x)}{\phi_1(0,x) + \phi_2(0,x)} \\
&\quad - \gamma(x)\phi_2(0,x) - \alpha(x)\phi_2(0,x),
\end{aligned}
\tag{5}
$$

for all $\phi = (\phi_1, \phi_2) \in C_r^+$, $x \in \overline{\Omega}$. Given a function $u : [-r, \sigma) \to \mathcal{X}(\sigma > 0)$, define $u_t \in C_r$ by $u_t(\theta) = u(t + \theta)$ with $\theta \in [-r, 0]$. Then we can rewrite system (2) as an abstract functional differential form:

$$
\frac{du(t)}{dt} = Au(t) + F(u_t), t > 0, \tag{6}
$$

with the initial condition $u_0 \in C_r^+$.

For further purposes to obtain the wellposedness of model (2), we give the following lemma:

Lemma 1 *Du and Peng (2016) Consider the parabolic system*

$$
\begin{cases}
\dfrac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i(x, t, u), & x \in \Omega, t > 0, i = 1, \ldots, l, \\
\dfrac{\partial u_i}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\
u_i(x, 0) = u_i^0(x), & x \in \Omega,
\end{cases}
$$

where $u = (u_1, \ldots, u_l)$, $u_i^0 \in C(\overline{\Omega})$ and $d_i > 0 (i = 1, \ldots, l)$ are constants, and assume that, for each $k = 1, \ldots, l$, the functions f_k satisfy the polynomial growth condition:

$$
|f_k(x, t, u)| \le c_1 \sum_{i=1}^{l} |u_i|^q + c_2
$$

for some nonnegative constants c_1 and c_2, and positive constant q. Let p_0 be a positive constant such that $p_0 > \frac{n}{2} max\{0, (q-1)\}$ and $r(u^0)$ be the maximal existence time of the solution u corresponding to the initial data u^0. Suppose that there exists a positive constant $C_{p_0}(u^0)$ such that $\| u(\cdot, t) \|_{L^{p_0}(\Omega)} \le C_{p_0}(u^0), \forall t \in (0, r(u^0))$, then the solution u exists for all time and there is a positive constant C_∞ such that $\| u(\cdot, t) \|_{L^{p_0}(\Omega)} \le C_\infty(u^0), \forall t \in (0, \infty)$. Moreover, if there exist finite numbers ρ and K_ρ independent of initial data such that$\| u(\cdot, t) \|_{L^{p_0}(\Omega)} \le K_{p_0}(\rho), \forall t \in$

$[\rho, \infty)$, then there is a positive number $K_\infty(\rho)$ independent of initial data such that $\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq K_\infty(\rho), \forall t \in [\rho, \infty)$.

Theorem 1 *For any initial value $\phi \in C_r^+$, system (2) admits a unique nonnegative uniformly bounded solution $u(t, \phi)$ on $[0, \infty)$ with $u_0 = \varphi$, and $u_t(\phi) := (u_{1t}(\phi), u_{2t}(\phi)) \in C_r^+$ for all $t \geq 0$, and there exists a positive constant C_1 depending on initial values such that the solution $(S, I) \in \mathcal{X}^+$ of system (2) satisfies*

$$\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1, \forall t \geq 0, \tag{7}$$

and there exists a positive constant C_2 independent of initial values such that for some large time $T_0 > 0$,

$$\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2, \forall t \geq T_0. \tag{8}$$

Moreover, the solution semiflow, denoted by $\Phi(t) = u_t(\cdot) : C_r^+ \to C_r^+, t \geq 0$, has a strong global attractor.

Proof It is easy to verify that

$$\lim_{h \to 0_+} \frac{1}{h} dist(\phi(0) + hF(\phi), \mathcal{X}_+) = 0, \quad \forall \phi \in C_r^+.$$

By (Martin and Smith 1990, Proposition 3 and Remark 2.4), it then follows that for every $\phi \in C_r^+$, system (2) admits a unique noncontinuable mild solution $u(t, \phi) \in \mathcal{X}^+$ in its maximal interval of existence $[0, \sigma_\phi)$ with $u_0 = \phi$. Integrating the first and second equations of (2) and adding the resulting two identities yield

$$\frac{d}{dt} \int_\Omega (S(x, t) + I(x, t))dx \leq \int_\Omega \Lambda(x)dx - \underline{\alpha} \int_\Omega (S(x, t) + I(x, t))dx. \tag{9}$$

Then the well-known Gronwall's inequality applied to (9) asserts that there exists some constant $C_0 > 0$, such that

$$\int_\Omega (S(x, t) + I(x, t))dx \leq C_0, \quad \forall t \in (0, \sigma_\phi).$$

Set $h(x, t) = I(x, t - r), t < \sigma_\phi + r$, system (2) becomes

$$\begin{cases} \dfrac{\partial \tilde{S}}{\partial t} = d_S \Delta \tilde{S} + \Lambda(x) - \dfrac{\beta(x) e^{-m(x)h(x,t)} \tilde{S} \tilde{I}}{\tilde{S} + \tilde{I}} + \gamma(x) \tilde{I} - \alpha(x) \tilde{S}, & x \in \Omega, t > 0, \\ \dfrac{\partial \tilde{I}}{\partial t} = d_I \Delta \tilde{I} + \dfrac{\beta(x) e^{-m(x)h(x,t)} \tilde{S} \tilde{I}}{\tilde{S} + \tilde{I}} - \gamma(x) \tilde{I} - \alpha(x) \tilde{I}, & x \in \Omega, t > 0, \\ \dfrac{\partial \tilde{S}}{\partial n} = \dfrac{\partial \tilde{I}}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{10}$$

By (9) and (Du and Peng 2016, Lemma 2.1) (due to Le (1997)) with $\sigma = p_0 = 1$, along with the positiveness of $\tilde{S}$, $\tilde{I}$, we have

$$\|\tilde{S}(\cdot,t)\|_{L^\infty(\Omega)} + \|\tilde{I}(\cdot,t)\|_{L^\infty(\Omega)} \le C_1, \forall 0 \le t < \sigma_\phi + r,$$

and the maximal interval of existence extends to $[0, \sigma_\phi + r)$. Reset $h(x,t) = I(x, t - r), t < \sigma_\phi + 2r$ and repeat the above procedure again, we can obtain

$$\|\tilde{S}(\cdot,t)\|_{L^\infty(\Omega)} + \|\tilde{I}(\cdot,t)\|_{L^\infty(\Omega)} \le C_1, \forall 0 \le t < \sigma_\phi + 2r,$$

and the maximal interval of existence extends to $[0, \sigma_\phi + 2r)$. Repeating the procedures yields (7) and (8).

Therefore, the solution semiflow $\Phi(t) = u_t(\cdot) : C_r^+ \to C_r^+$ is point dissipative. By (Wu 1996, Theorem 2.2.6), $\Phi(t) = u_t(\cdot) : C_r^+ \to C_r^+$ is compact for $t > r$. Thus, it follows from (Zhao 2017, Theorem 1.1.3) (see also (Hale 1988, Theorem 3.4.8)) that $\Phi(t)$ has a strong global attractor on C_r^+. □

2.1 Definition of basic reproduction number

For infectious disease models, the basic reproduction number, defined as the expected number of secondary cases produced in a completely susceptible population by an infective individual, is one of the most significant concepts in studying the transmission of infectious disease (Diekmann and Heesterbeek 2000; Anderson and May 1991). More importantly, it often determines the threshold behavior for many epidemic models. It is often the case that a disease dies out if the basic reproduction number is less than unity and the disease is established in the population if it is greater than unity. We refer to Diekmann et al. (1990) for the approach of next genenumbern operators for the basic reproduction number and to Zhao (2017), Wang and Zhao (2012), Thieme (2009), Liang et al. (2019) for related works.

We now make use of the theory developed in Liang et al. (2019) to derive the basic reproduction number of system (2).

Lemma 2 *Let μ_0 denote the unique positive eigenvalue with a positive eigenfunction corresponding to the following problem:*

$$d_I \Delta\phi + \mu\beta(x)\phi - (\alpha(x) + \gamma(x))\phi = 0 \ \ in \ \ \Omega; \ \ \frac{\partial\phi}{\partial n}\Big|_{\partial\Omega} = 0, \tag{11}$$

then the basic reproduction number of system (2) satifies

$$\mathcal{R}_0 = \frac{1}{\mu_0} = \sup_{\substack{\varphi\in H^1(\Omega)\\ \varphi\neq 0}} \frac{\int_\Omega \beta\varphi^2 dx}{\int_\Omega (d_I|\nabla\varphi|^2 + (\gamma+\alpha)\varphi^2)dx}. \tag{12}$$

For further purposes to study the local stability of disease-free steady state, we consider the following eigenvalue problem:

$$-d_I \Delta\psi + (\alpha(x) + \gamma(x) - \beta(x))\psi = \lambda\phi \text{ in } \Omega; \quad \frac{\partial\phi}{\partial n}\Big|_{\partial\Omega} = 0, \tag{13}$$

Let λ_1 be the principal eigenvalue of (13) with the positive eigenfunction ϕ_I. Then, we have the following properties of $\mathcal{R}_0$, the proof of which resembles that of (Allen et al. 2008, Lemma 2.3) and hence is omitted.

Lemma 3 *Suppose that* **A1** − **A2** *hold.*

(i) $\mathcal{R}_0$ *is a monotone decreasing function of* d_I *with* $\mathcal{R}_0 \to \max\left\{\frac{\beta(x)}{\gamma(x)+\alpha(x)}, x \in \Omega\right\}$ *as* $d_I \to 0$ *and* $\mathcal{R}_0 \to \frac{\int_\Omega \beta(x)dx}{\int_\Omega(\gamma(x)+\alpha(x))dx}$ *as* $d_I \to \infty$.

(ii) *If* $\int_\Omega \beta dx < \int_\Omega(\gamma(x) + \alpha(x))dx$, *there exists a threshold value* $d_I^* \in (0, \infty)$ *such that* $\mathcal{R}_0 > 1$ *for* $d_I < d_I^*$ *and* $\mathcal{R}_0 < 1$ *for* $d_I > d_I^*$, *where*

$$d_I^* = \sup\left\{\frac{\int_\Omega(\beta - \gamma - \alpha)\varphi^2 dx}{\int_\Omega |\nabla\varphi|^2 dx}\Big|\varphi \in W^{1,2}(\Omega) \text{ and } \int_\Omega(\beta - \gamma - \alpha)\varphi^2 dx > 0\right\};$$

(iii) *If* $\int_\Omega \beta(x)dx > \int_\Omega(\gamma(x) + \alpha(x))dx$, *we have* $\mathcal{R}_0 > 1$ *for all* $d_I > 0$;

(iv) $\text{sign}(1 - \mathcal{R}_0) = \text{sign}(\lambda_1)$.

2.2 Threshold dynamics

The following lemma concerned with the local stability of DFE is a direct result of Lemma 3(iv) and we omit the proof.

Lemma 4 *The disease-free steady state* E_0 *in system (2) is locally asymptotically stable if* $\mathcal{R}_0 < 1$, *unstable if* $\mathcal{R}_0 > 1$.

Theorem 2 (i) *If the basic reproduction number* $\mathcal{R}_0 < 1$, *then DFE is globally asymptotically stable.*

(ii) *If* $\mathcal{R}_0 > 1$, *there exists a small positive constant* ϵ_0 *such that any positive solution of system (2) satisfies*

$$\liminf_{t\to\infty} \|(S(\cdot, t), I(\cdot, t)) - (\widetilde{N}, 0)\| > \epsilon_0.$$

Besides, system (2) admits at least one endemic steady state.

Proof See Appendix. □

2.3 Dynamics at the endemic steady state without delay

In this part, we assume $r = 0$ and $d_S = d_I = d$, and system (2) becomes the following reaction-diffusion system:

$$\begin{cases} \frac{\partial S}{\partial t} = d\Delta S + \Lambda(x) - \frac{\beta(x)e^{-m(x)I(x}SI}{S+I} + \gamma(x)I - \alpha(x)S, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} = d\Delta I + \frac{\beta(x)e^{-m(x)I(x)}SI}{S+I} - \gamma(x)I - \alpha(x)I, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{14}$$

Denote $N = S + I$, then system (14) is equivalent to the following system:

$$\begin{cases} \frac{\partial N}{\partial t} = d\Delta N + \Lambda(x) - \alpha(x)N, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} = d\Delta I + \frac{\beta(x)e^{-m(x)I(x)}(N-I)I}{N} - \gamma(x)I - \alpha(x)I, & x \in \Omega, t > 0, \\ \frac{\partial N}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{15}$$

Theorem 3 *Let $r = 0$ and suppose that $\mathcal{R}_0 > 1$ and $d_S = d_I$. Then system (2) admits a unique endemic steady state, denoted by $E_1 = (\widetilde{N} - I^*, I^*)$, which is globally asymptotically stable.*

Proof By (15), we can regard $N(t, x)$ as a fixed function on $\mathcal{R}^+ \times \overline{\Omega}$ and $\lim_{t\to\infty} N(t, \cdot) = \widetilde{N}$. Then system is asymptotic to the following system:

$$\begin{cases} \frac{\partial I}{\partial t} = d\Delta I + \frac{\beta(x)e^{-m(x)I(x)}(\widetilde{N}-I)I}{\widetilde{N}} - \gamma(x)I - \alpha(x)I, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{16}$$

If $\mathcal{R}_0 > 1$, by (Zhao 2017, Theorem 3.1.5) (See also (Freedman and Zhao 1997, Colorrary 2.2)), system (16) admits a unique positive steady state, denoted by I^*, which is globally asymptotically stable. Therefore, system (15) admits a unique endemic steady state, denoted by $\hat{E}_1 = (\widetilde{N}, I^*)$, and by (Zhao 2017, Theorem 1.2.1 with Remark 1.3.2) (see also (Thieme 1992, Theorem 4.1)), $\hat{E}_1$ is globally asymptotically stable. Thus Theorem 4 follows. □

3 Local hopf bifurcation at the endemic steady state

Throughout this section, we assume that $\mathcal{R}_0 > 1$, $d_S = d_I = d$ and **A1**, **A2**, **B1** hold, consider the limit system of (2)

$$\begin{cases} \dfrac{\partial I}{\partial t} = d\Delta I + \beta(x)e^{-m(x)I(x,t-r)}\left(1 - \dfrac{I}{\widetilde{N}}\right)I - (\gamma(x) + \alpha(x))I, & x \in \Omega, t > 0, \\ \dfrac{\partial I}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{17}$$

Letting $\tilde{I} = I, \tilde{t} = td$, dropping the tilde since no confusion occurs, and denoting $\lambda = \frac{1}{d}$, $\tau = dr$, system (17) can be rewritten as

$$\begin{cases} \dfrac{\partial I}{\partial t} = \Delta I + \lambda\beta(x)e^{-m(x)I(x,t-\tau)}\left(1 - \dfrac{I}{\widetilde{N}}\right)I - \lambda(\gamma(x) + \alpha(x))I, & x \in \Omega, t > 0, \\ \dfrac{\partial I}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{18}$$

The wellposedness (existence, uniqueness, and positivity) of solutions to systems (17) and (18) can be obtained by similar arguments as Theorem 1, so we omit the proof here. We refer to Su et al. (2012); Chen and Shi (2012); Chen et al. (2018); Faria et al. (2002) for some related works on Hopf bifurcation of functional partial differential equations.

Denote

$$X = \left\{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0, x \in \partial\Omega\right\}, Y = L^2(\Omega), \mathcal{C} = C([-\tau, 0], Y).$$

Moreover, we denote the the complexification of a linear real-valued vector space Z to be $Z_{\mathbb{C}} = Z \oplus iZ$, and the positive cone of Z if it exists by Z^+, the domain of a linear operator L by $\mathcal{D}(L)$, the kernel of L by $\mathcal{N}(L)$ and the range of L by $\mathcal{R}(L)$. For Hilbert space $Y_{\mathbb{C}}$, we use the standard inner product $< u, v >= \int_\Omega \overline{u}(x)v(x)dx$. For a nonlinear mapping F, we denote by $D_u F$ the Fréchet derivative with respect to variable u.

For further purposes, set

$$L := \Delta + \lambda_*(\beta - \gamma - \alpha), \tag{19}$$

where λ_* is the unique positive principal eigenvalue of the following problem with positive eigenfunction ϕ under conditions **A2**, **B1** (Cantrell and Cosner 2003, Theorem 2.4):

$$-\Delta\phi = \lambda_*(\beta - \gamma - \alpha)\phi \text{ in } \Omega; \quad \left.\frac{\partial\phi}{\partial n}\right|_{\partial\Omega} = 0. \tag{20}$$

Then the endemic steady state, denoted by I_λ, of (18) can be written as

$$\begin{aligned} &LI_\lambda + (\lambda - \lambda_*)(\beta - \alpha - \gamma)I_\lambda + \lambda\beta I_\lambda \left(e^{-mI_\lambda}\left(1 - \frac{I_\lambda}{\widetilde{N}}\right) - 1\right) \\ &= 0 \ \text{ in } \ \Omega; \quad \left.\frac{\partial I_\lambda}{\partial n}\right|_{\partial\Omega} = 0. \end{aligned} \tag{21}$$

Note that $X = \mathcal{N}(L) \oplus X_1$, $Y = \mathcal{N}(L) \oplus Y_1$, where $N(L) = \text{span}\{\phi\}$,

$$\begin{aligned} X_1 &= X \cap \mathcal{R}(L) = \left\{\varphi \in X : \int_\Omega \varphi\phi dx = 0\right\}, Y_1 \\ &= \mathcal{R}(L) = \left\{\varphi \in Y : \int_\Omega \varphi\phi dx = 0\right\}. \end{aligned}$$

By using the implicit function theorem, we can calculate the unique positive steady state I_λ near λ_*, which will be used later.

Lemma 5 *Assumed that* **A2**, **B1** *hold. There exist $\lambda^* > \lambda_*$ and a continuously differential mapping $\lambda \longmapsto (\xi_\lambda, A_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_1 \times \mathbb{R}^+$ such that, for $\lambda \in [\lambda_*, \lambda^*]$, the unique positive steady state of (21) has the following form:*

$$I_\lambda = A_\lambda(\lambda - \lambda_*)(\phi + (\lambda - \lambda_*)\xi_\lambda). \tag{22}$$

Moreover, for $\lambda = \lambda_*$,

$$A_{\lambda_*} = \frac{\int_\Omega (\beta - \alpha - \gamma)\phi^2 dx}{\int_\Omega \lambda_*\beta(m + 1/\widetilde{N})\phi^3 dx} \tag{23}$$

and $\xi_{\lambda_*} \in X_1$ is the unique solution of the following equation:

$$L\xi_{\lambda_*} + (\beta - \alpha - \gamma)\phi - \lambda_* A_{\lambda_*}\beta(m + 1/\widetilde{N})\phi^2 = 0, \tag{24}$$

where L is defined in (19).

Proof To start with, we show that A_{λ_*} and ξ_{λ_*} are well defined. Note that

$$\lambda_* \int_\Omega (\beta - \alpha - \gamma)\phi^2 dx = \int_\Omega |\nabla\phi|^2 dx,$$

then A_{λ_*} is well defined and positive. Note that L is bijective from X_1 to $\mathcal{R}(L)$ and

$$(\beta - \alpha - \gamma)\phi - \lambda_* A_{\lambda_*}\beta(m + 1/\widetilde{N})\phi^2 \in \mathcal{R}(L),$$

hence ξ_{λ_*} is well defined.

Substituting $I_\lambda = A_\lambda(\lambda-\lambda_*)(\phi+(\lambda-\lambda_*)\xi_\lambda)$ into (21), we see that (A_λ, ξ_λ) satisfies

$$F(\xi_\lambda, A_\lambda, \lambda) := L\xi_\lambda + (\beta-\alpha-\gamma)(\phi+(\lambda-\lambda_*)\xi_\lambda) + \lambda\beta(\phi+(\lambda-\lambda_*)\xi_\lambda)M_\lambda = 0,$$

where

$$M_\lambda = \frac{e^{-mI_\lambda}(1-I_\lambda/\widetilde{N})-1}{\lambda-\lambda_*}.$$

It is easy to verify by standard Soblev embedding theory that $F(\xi_\lambda, A_\lambda, \lambda)$ is a function from $X_1 \times \mathbb{R}^2$ to Y. Note from (23) and (24) that $F(\xi_{\lambda_*}, A_{\lambda_*}, \lambda_*) = 0$ and the Fréchet derivative of F with respect to (ξ, A) yields $(\lambda_*, \xi_{\lambda_*}, A_{\lambda_*})$

$$D_{(\xi,A)}F(\xi_{\lambda_*}, A_{\lambda_*}, \lambda_*)(\psi, \epsilon) = L\psi - \epsilon\lambda_*\beta(m+1/\widetilde{N})\phi^2.$$

Note that $D_{(\xi,A)}F(\xi_{\lambda_*}, A_{\lambda_*}, \lambda_*)$ is a bijection from $X_1 \times R^2$ to Y, which together with the implicit function theorem imply that there exist $\lambda^* > \lambda_*$ and a continuously differential mapping $\lambda \longmapsto (A_\lambda, \xi_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X_1 \times \mathcal{R}^+$ such that

$$F(\xi_\lambda, A_\lambda, \lambda) = 0, \lambda \in [\lambda_*, \lambda^*].$$

Therefore, $A_\lambda(\lambda-\lambda_*)(\phi+(\lambda-\lambda_*)\xi_\lambda)$ is a positive solution of (21). □

3.1 Eigenvalue problems

In this part, we assume $\lambda \in [\lambda_*, \lambda^*]$, derive the eigenvalue problem relative to the positive steady state I_λ in system (18) and investigate the existence of purely imaginary roots. Linearizing system (18) at I_λ yields

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \lambda K_\lambda u - \lambda N_\lambda u(t-\tau), & x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, t > 0, \end{cases} \tag{25}$$

where

$$K_\lambda = \beta e^{-mI_\lambda}(1-2I_\lambda/\widetilde{N}) - (\alpha+\gamma), N_\lambda = \beta m e^{-mI_\lambda} I_\lambda(1-I_\lambda/\widetilde{N}).$$

It follows from (Wu 1996, Theorem 3.1.5) that the solution semigroup of (25) has the infinitesimal generator $\mathcal{A}_\tau(\lambda)$ given by

$$\mathcal{A}_\tau(\lambda)\Psi = \dot{\Psi},$$

where

$$\mathcal{D}(\mathcal{A}_\tau(\lambda)) = \{\Psi \in \mathcal{C}_\mathbb{C} \cap \mathcal{C}^1_\mathbb{C} : \dot{\Psi}(0) = \Delta\Psi(0) + \lambda K_\lambda \Psi(0) - \lambda N_\lambda \Psi(-\tau)\}$$

and $\mathcal{C}^1_{\mathbb{C}} = C^1([-\tau, 0], Y_{\mathbb{C}})$. Note that $\mu \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}_\tau$, if and only if there exists $\psi \in X_{\mathbb{C}} \backslash \{0\}$ such that $\Delta(\lambda, \mu, \tau)\psi = 0$, where

$$\Delta(\lambda, \mu, \tau)\psi := \Delta\psi + \lambda K_\lambda \psi - \lambda N_\lambda e^{-\mu\tau}\psi - \mu\psi = 0. \tag{26}$$

Moreover, the eigenvalues of $\mathcal{A}_\tau$ depend continuously on τ ((Chow and Hale 1982, Chapter 14)). It can be seen from (26) that $\mathcal{A}_\tau(\lambda)$ has a pair of purely imaginary eigenvalue $\mu = \pm iw$ for some $w > 0$, if and only if

$$\Delta\psi + \lambda K_\lambda \psi - \lambda N_\lambda e^{-i\theta}\psi - iw\psi = 0 \tag{27}$$

is solvable for some value of $w > 0, \theta \in [0, 2\pi)$, and $\psi \in X_{\mathbb{C}} \backslash \{0\}$.

Solving (27) for any $\lambda > \lambda_*$ is still a challenging problem. In what follows, we will solve (27) for $\lambda \in [\lambda_*, \lambda^*]$ by using the implicit function theorem. It follows from $X = N(L) + X_1$ that if (w, θ, ψ) solves (27), then ignoring a scalar factor, $\psi \in X_{\mathbb{C}} \backslash \{0\}$ can be represented as

$$\psi = \kappa\phi + (\lambda - \lambda_*)z, \parallel \psi \parallel^2_{Y_{\mathbb{C}}} = \parallel \phi \parallel^2_{Y_{\mathbb{C}}}, \tag{28}$$

where $z \in (X_1)_{\mathbb{C}}$ and $\kappa \geq 0$. Now for $\lambda \in [\lambda_*, \lambda^*]$, substituting (22), (28) and $w = (\lambda - \lambda_*)h$ into (27), we obtain that (w, θ, ψ) with $w > 0, \theta \in [0, 2\pi)$, $\psi \in X_{\mathbb{C}} \backslash \{0\}$ and $\parallel \psi \parallel^2_{Y_{\mathbb{C}}} = \parallel \phi \parallel^2_{Y_{\mathbb{C}}}$ solves (27), if and only if the following system:

$$\begin{cases} g_1 = Lz + \dfrac{\lambda K_\lambda - \lambda_*(\beta - \gamma - \alpha)}{\lambda - \lambda_*}(\kappa\phi + (\lambda - \lambda_*)z) \\ -\dfrac{\lambda N_\lambda}{\lambda - \lambda_*}(\kappa\phi + (\lambda - \lambda_*)z)e^{-i\theta} - ih(\kappa\phi + (\lambda - \lambda_*)z) = 0, \\ g_2 = (\kappa^2 - 1) \parallel \phi \parallel^2_{Y_C} + (\lambda - \lambda_*)^2 \parallel z_\lambda \parallel^2_{Y_C} = 0, \end{cases} \tag{29}$$

is solvable for some value $z \in (X_1)_{\mathbb{C}}, h > 0, \kappa \geq 0, \theta \in [0, 2\pi)$. Define $G : (X_1)_{\mathbb{C}} \times \mathbb{R}^4 \to Y_{\mathbb{C}} \times \mathbb{R}$ by $G = (g_1, g_2)$. Then we have the following the following results where the proof is in Appendix:

Theorem 4 *Assumed that* **A2**, **B1** *hold. There exist* $\tilde{\lambda}^* > \lambda_*$ *and a continuously differentiable mapping* $\lambda \to (z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda)$ *from* $[\lambda_*, \tilde{\lambda}^*]$ *to* $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$ *such that* $G(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$. *Moreover, for* $\lambda \in [\lambda_*, \tilde{\lambda}^*]$,

$$\begin{cases} G(z, \kappa, h, \theta, \lambda) = 0, \\ z \in (X_1)_{\mathbb{C}}, h \geq 0, \kappa \geq 0, \theta \in [0, 2\pi) \end{cases} \tag{30}$$

has a unique solution $(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda)$.

If Theorem 4 holds true, then the following results can be directly derived.

Corollary 1 *Assumed that* **A2**, **B1** *hold. For each* $\lambda \in [\lambda_*, \tilde{\lambda}^*]$, *the following equation*

$$\Delta(\lambda, iw, \tau)\psi = 0, w > 0, \tau \geq 0, \psi \in X_{\mathbb{C}} \backslash \{0\}$$

has a nontrivial solution (w, τ, ψ)*, if and only if*

$$w = w_\lambda = (\lambda - \lambda_*)h_\lambda, \psi = \psi_\lambda = c(\kappa_\lambda \phi + (\lambda - \lambda_*)z_\lambda), \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{w_\lambda}, \tag{31}$$

where $n = 0, 1, 2, \cdots$*,* c *is a nonzero constant and* $(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda)$ *is defined in Theorem 4.*

3.2 Hopf bifurcation

In what follows, we will always assume $\lambda \in [\lambda_*, \tilde{\lambda}^*]$ for simplicity and take the delay τ as a bifurcation parameter to investigate the stability of the positive steady state I_λ in system (18). Particularly, we will explore the existence of local Hopf bifurcation at I_λ for system (18). By Theorem 3, we have

Lemma 6 *Assume that* $\lambda \in [\lambda_*, \tilde{\lambda}^*]$*. If* $\tau = 0$*, then all eigenvalues of* $\mathcal{A}_\tau(\lambda)$ *have negative real parts; if* $\tau > 0$*, then* 0 *is not an eigenvalue of* $\lambda \in [\lambda_*, \tilde{\lambda}^*]$*.*

We show iw_λ is a simple eigenvalue of $\mathcal{A}_{\tau_n}(\lambda)$ for $n = 0, 1, 2\cdots$ in the subsequent lemma, where τ_n is defined in (31). Thus by the implicit function theorem, there exists a neighborhood $O_n \times P_n \times Q_n \subset \mathbb{R} \times \mathbb{C} \times X_{\mathbb{C}}$ of $(\tau_n, iw_\lambda, \psi_\lambda)$ and a continuously differential function $(\mu(\tau), \psi(\tau)) : O_n \to P_n \times Q_n$ such that for any $\tau \in O_n$, the only eigenvalue of $\mathcal{A}_\tau(\lambda)$ in P_n is $\mu(\tau)$, i.e.,

$$\Delta(\lambda, \mu(\tau), \tau)\psi(\tau) = 0, \tau \in O_n \tag{32}$$

with $\mu(\tau_n) = iw_\lambda, \psi(\tau_n) = \psi_\lambda$.

Lemma 7 *Assume that* $\lambda \in [\lambda_*, \tilde{\lambda}^*]$*. Then* $\mu = iw_\lambda$ *is a simple eigenvalue of* $\mathcal{A}_{\tau_n}(\lambda)$ *for* $n = 0, 1, 2, \cdots$ *, where* iw_λ *and* τ_n *are defined as in Corollary 1.*

Proof See Appendix. □

Moreover, we have the following transversality condition:

Lemma 8 *Assume that* $\lambda \in [\lambda_*, \tilde{\lambda}^*]$*. Then*

$$\mathrm{Re}\left(\frac{d\mu(\tau)}{d\tau}\Big|_{\tau=\tau_n}\right) > 0, n = 0, 1, 2, \cdots.$$

Proof See Appendix. □

Remark 1 Here $0 < \tilde{\lambda}^* - \lambda_* \ll 1$ and the value of $\tilde{\lambda}^*$ may be chosen smaller than the one in Theorem 4, since perturbation arguments are used in the proof of Lemma 8.

From Corollary 1, Lemmas 7 and 8, we have the results on the distribution of eigenvalues of $\mathcal{A}_\tau(\lambda)$ for $\lambda \in [\lambda_*, \tilde{\lambda}^*]$.

Theorem 5 *For $\lambda \in [\lambda_*, \tilde{\lambda}^*]$, the infinitesimal generator $\mathcal{A}_\tau(\lambda)$ has exactly $2(n+1)$ eigenvalues with positive real parts when $\tau \in (\tau_n, \tau_{n+1}]$, $n = 0, 1, 2, \cdots$.*

Then by the local Hopf bifurcation theorem for partial functional differential equations ((Wu 1996, Theorem 2.1 in Chapter 6)), Lemmas 6, 7 and 8, we obtain the stability and associated local Hopf bifurcations of the positive steady state solution I_λ in system (18).

Theorem 6 *Assumed that $\mathcal{R}_0 > 1$, $d_S = d_I = d$, and* **A1**, **A2**, **B1** *hold, and $\lambda \in [\lambda_*, \tilde{\lambda}^*]$. Then the positive steady state I_λ of system (18) is locally asymptotically stable when $\tau \in [0, \tau_0)$, and unstable when $\tau \in (\tau_0, \infty)$. Moreover, when $\tau = \tau_n$, system (18) occurs Hopf bifurcation at the positive steady state I_λ.*

If **B2**($\int_\Omega \beta dx > \int_\Omega (\gamma(x) + \alpha(x))dx$) rather than **B1**($\int_\Omega \beta dx < \int_\Omega (\gamma(x) + \alpha(x))dx$) holds, we can similarly obtain stability and local Hopf bifurcation at the positive steady state I_λ and the proof for **B2** is slightly different from that for **B1**. For **B2**, $\lambda_* = 0$ and ϕ become constant, and L, X_1, Y_1 should be made some adjustment, and other calculations are similar, so we omit the proof here.

Theorem 7 *Assumed that $\mathcal{R}_0 > 1$, $d_S = d_I = d$, and* **A1**, **A2**, **B2** *hold, and $\lambda \in [0, \tilde{\lambda}^*]$. Then the positive steady state I_λ of system (18) is locally asymptotically stable when $\tau \in [0, \tau_0)$, and unstable when $\tau \in (\tau_0, \infty)$. Moreover, when $\tau = \tau_n$, system (18) occurs Hopf bifurcation at the positive steady state I_λ.*

4 Global existence of periodic solutions

Throughout this section, we assume that $\mathcal{R}_0 > 1$, $d_S = d_I = d$, $\lambda \in [\lambda_*, \tilde{\lambda}^*]$ and study the global continuation of periodic solutions bifurcating from the point (I_λ, τ_n), $n = 1, 2, \cdots$ for system (18) by using global Hopf bifurcation theorem developed in (Wu 1996, Section 6.5). For convenience, we use the notations in (Wu 1996, Section 6.5). Let $\tilde{T}(t)$ be the semigroup on Y associated with Δ under Neumann boundary condition and set $A_T : \mathcal{D}(A_T) \to Y$ to be the generator of $\tilde{T}(t)$. Denoting $u(\cdot, t) = I(\cdot, \tau t)$, we can rewrite system (18) as the following semilinear functional differential equation:

$$\dot{u} = \tau A_T u + \tau f(\tau, u_t), \tag{33}$$

where $u_t(\theta) = u(\cdot, t + \theta), \theta \in [-\tau, 0]$ and

$$f(\tau, u_t)(x) = \lambda\beta(x)e^{-m(x)u(x,t-1)}\left(1 - \frac{u(x,t)}{\tilde{N}(x)}\right)u(x,t) - \lambda(\gamma(x) + \alpha(x))u(x,t).$$

It follows from (Wu 1996, Theorem 3.1.5) that the solution semigroup of (33) has the infinitesimal generator $\tilde{\mathcal{A}}_\tau(\lambda)$ given by

$$\tilde{\mathcal{A}}_\tau(\lambda)\Psi = \dot{\Psi},$$

with

$$\mathcal{D}(\tilde{\mathcal{A}}_\tau(\lambda)) = \{\Psi \in \mathcal{C}_\mathbb{C} \cap \mathcal{C}^1_\mathbb{C} : \dot{\Psi}(0) = \tau\Delta\Psi(0) + \tau\lambda K_\lambda\Psi(0) - \tau\lambda N_\lambda\Psi(-1)\}$$

and $\mathcal{C}^1_\mathbb{C} = C^1([-1, 0], Y_\mathbb{C})$. To state the global Hopf bifurcation theorem, similar to (Wu 1996, Section 6.5), we define
(i) $E = C(S_1; X)$ is a real isometric Banach representation of the group $G = S_1 := \{z \in C : |z| = 1\}$;
(ii) Let $E^G := \{x \in E : gx = x \ \text{ for all } \ g \in G\}$. Then $E^G = X$, and E has an isotypical direct sum decomposition $E = E^G \bigoplus_{k=1}^{\infty} E_k$ where $E_k = \{e^{ikt}x : x \in X\}$ for $k \geq 1$
Then from (Wu 1996, Section 6.5), system (33) can be casted into an integral equation which is continuously differentiable, completely continuous, and G-invariant. Now we verify the three conditions **H1-3** in (Wu 1996, Section 6.5).
H1: Note that $I_\lambda \in \mathcal{D}(A_T)$, $\tau_n \in \mathbb{R}$ satisfies $A_T I_\lambda + f(\tau_n, I_\lambda) = 0$. From Lemma 6, for any $\tau \geq 0$, 0 is not an eigenvalue of $\tilde{\mathcal{A}}_\tau(\lambda)$, hence the assumption (H1) in (Wu 1996, Section 6.5) is satisfied.
H2: When $\tau = \tau_n$, $\tilde{\mathcal{A}}_\tau(\lambda)$ has a unique pair of purely imaginary eigenvalues $iw_\lambda\tau_n$, hence the assumption (H2) in (Wu 1996, Section 6.5) holds.
H3: We choose sufficiently small ϵ_0, $\eta_0 > 0$, and define the local steady state manifold

$$M_\lambda = \{(I_\lambda, \tau, \zeta) : |\tau - \tau_n| < \epsilon_0, |\zeta - w_\lambda\tau_n| < \eta_0\} \subset E^G \times \mathbb{R} \times \mathbb{R}_+.$$

Then for $(\tau, \zeta) \in [\tau_n - \epsilon_0, \tau_n + \epsilon_0] \times [w_\lambda\tau_n - \eta_0, w_\lambda\tau_n + \eta_0]$, $i\zeta$ is an eigenvalue of $\tilde{\mathcal{A}}_\tau(\lambda)$ if and only if $\tau = \tau_n$ and $\zeta = w_\lambda\tau_n$ from Lemma 7. This verifies the assumption (H3) in (Wu 1996, Section 6.5). Thus by (Wu 1996, Lemma 6.5.3), $(I_\lambda, \tau_n, w_\lambda\tau_n)$ is an isolated singular point in M_λ.

Let $\mu_k(I_\lambda, \tau_n, w_\lambda\tau_n)(k = 1, 2, \cdots)$ be the generalized crossing number defined in (Wu 1996, Section 6.5). Then from Lemma 8, if $\zeta(\tau) = \alpha(\tau) \pm i\beta(\tau)$ are the eigenvalues of $\tilde{\mathcal{A}}_\tau(\lambda)$ satisfying $\zeta(\tau_n) = iw_\lambda\tau_n$, then $\alpha'(\tau_n) > 0$. This implies that $\mu_1(I_\lambda, \tau_n, w_\lambda\tau_n) = 1$.

Then by (Wu 1996, Theorem 6.5.4), we obtains the local topological Hopf bifurcation for system (18) at $\tau = \tau_n$.

Next we consider the global continuation of the local Hopf bifurcation. Let

$$\begin{aligned} S :=&\text{cl}\{(z, \tau, \zeta) \in E \times \mathbb{R} \times \mathbb{R}_+ : u(\cdot, t) = z(\cdot, \omega t) \\ &\text{is a nontrivial } \ 2\pi/\zeta \ \text{ periodic solution of system (1.4)}\}^{\cdot} \end{aligned}$$

We also define the complete steady state manifold:

$$M^*_\lambda = \{(I_\lambda, \tau)\} \subset E^G \times \mathbb{R}.$$

Let $\mathbb{C}_n = \mathbb{C}(I_\lambda, \tau_n, w_\lambda\tau_n)$ denote the connected component of S with respect to the local bifurcation $(I_\lambda, \tau_n, w_\lambda\tau_n)$. Then it follows from the global Hopf bifurcation theorem ((Wu 1996, Theorem 6.5.5)) that

Theorem 8 *For each* $n = 1, 2, \cdots$ *,,* $\mathbb{C}_n$ *is unbounded, i.e.,*

$$\sup\{\max_{t\in R} |z(t)| + |\tau| + \zeta + \zeta^{-1} : (z, \tau, \zeta) \in \mathbb{C}_n\} = \infty.$$

Proof It follows from the global Hopf bifurcation theorem ((Wu 1996, Theorem 6.5.5)), one of the following assertions holds:
(i) $\mathbb{C}_n$ is unbounded; or
(ii) $\mathbb{C}_n \cap (M^*_\lambda \times \mathcal{R}^+)$ is finite and for all $k \geq 1$,

$$\sum_{(z,\tau,\zeta)\in\mathbb{C}_n\cap(M^*_\lambda\times\mathcal{R}^+)} \mu_k(z, \tau, \zeta) = 0,$$

where μ_k is the k-th generalized crossing number. By Lemma 8, if $\zeta(\tau) = \alpha(\tau)\pm i\beta(\tau)$ are the eigenvalues of $\tilde{\mathcal{A}}_\tau(\lambda)$ satisfying $\zeta(\tau_n) = iw_\lambda\tau_n$, then $\alpha'(\tau_n) > 0$, which implies $\mu_1(I_\lambda, \tau_n, w_\lambda\tau_n) = 1, n = 1, 2, \cdots$. Thus for $k = 1$, we have

$$\sum_{(z,\tau,\zeta)\in\mathbb{C}_n\cap(M^*_\lambda\times\mathcal{R}^+)} \mu_1(z, \tau, \zeta) > 0.$$

Hence, (ii) fails and (i) holds. □

Now we have the connected component $\mathbb{C}_n$ is unbounded. Therefore, if we verify the following three assertions:
Claim one: The projection of $\mathbb{C}(I_\lambda, \tau_n, w_\lambda\tau_n)$ onto T-space is bounded,
Claim two: The projection of $\mathbb{C}(I_\lambda, \tau_n, w_\lambda\tau_n)$ onto the τ-space does not intersect with $\tau = 0$,
Claim three: The projection of $\mathbb{C}(I_\lambda, \tau_n, w_\lambda\tau_n)$ onto z-space is bounded,
then we can conclude that the projection of $\mathbb{C}(I_\lambda, \tau_n, w_\lambda\tau_n)$ onto the τ-space can be extended to ∞. Therefore, we reap the final results of global Hopf bifurcation branches.

To start with, we prove **Claim one**. (31) yields that

$$\frac{1}{n+1} < \frac{2\pi}{\tau_n w_\lambda} < 1. \tag{34}$$

If we exclude the existence of periodic solutions of period 1, then system (33) has no periodic solutions of period $\frac{1}{n}$ for any positive integer n. Then, we can obtain the projection of $\mathbb{C}(I_\lambda, \tau_n, w_\lambda\tau_n)$ onto T-space is bounded.

Lemma 9 *If* $\mathcal{R}_0 > 1, \lambda \in [\lambda_*, \tilde{\lambda}^*]$*, then the system (33) has no periodic solutions of period* 1.

Proof Assume by contradiction that system (33) has periodic solutions of period 1. Then the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \tau \Delta u + \lambda\tau \dfrac{\beta(x)e^{-m(x)u}(\widetilde{N} - u)u}{\widetilde{N}} - \lambda\tau\gamma(x)u - \lambda\tau\alpha(x)u, & x \in \Omega, t > 0, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{35}$$

admits some periodic solution of period 1, which contradicts Theorem 3. This contradiction completes the proof. □

From (Friesecke 1993, Theorem 2) (see also (Wu 1996, Section 10.2)), for small enough $\tau > 0$, the unique positive steady state solution I_λ is globally asymptotically stable for all positive initial values for system (33). Thus **Claim two** holds.

Lemma 10 *If $\mathcal{R}_0 > 1, \lambda \in [\lambda_*, \tilde{\lambda}^*]$, then system has no positive nontrivial periodic orbit for small $\tau > 0$.*

Now we proceed to prove **Claim three**. Theorem 1 yields the following lemma, implying that the projection of $\mathbb{C}(I_\lambda, \tau_n, w_\lambda\tau_n)$ onto z-space has upper bound.

Lemma 11 *For any initial value $\phi > 0$, the solution of system (33) are uniformly bounded.*

Lemma 12 *For any $(z, \tau, \zeta) \in \mathbb{C}_n$, let $u(x, t)$ be the ω-periodic solution of system (33) with delay τ and u is a representation of z. Then we have $u(x, t) > 0$ for $t \in \mathbb{R}$ and $x \in \Omega$.*

Proof If $(z, \tau, \zeta) \in \mathbb{C}_n$, then we obtain from Lemma 10 that $\tau > 0$, and from (31) that $\zeta > 0$. Note that $(I_\lambda, \tau_n, w_\lambda\tau_n) \in \mathbb{C}_n$ then any $(z, \tau, \zeta) \in \mathbb{C}_n$ near $(I_\lambda, \tau_n, w_\lambda\tau_n)$ satisfies $u(x, t) > 0$ for $t \in \mathbb{R}$ and $x \in \Omega$, where $u(x, t)$ is an ζ-periodic solution of (33) with delay τ and u is a representation of z. Suppose by contradiction that Lemma 12 does not hold for all $(z, \tau, \zeta) \in \mathbb{C}_n$. Then there exists a $(z^*, \tau^*, \zeta^*) \in \mathbb{C}_n$ such that if $u^*(x, t)$ is an ζ^*-periodic solution of (1.4) with delay τ^* and u^* is a representation of z^*, and $u^*(x^*, t^*) = 0$ for some $x^* \in \Omega$ and $t^* \in \mathbb{R}$, which by the strong maximum principle of parabolic equations (Protter and Weinberger 1984, Chapter 3, Theorem 5, P.173) implying that $u^*(x, t) \equiv 0$. Thus system (33) occurs Hopf bifurcation at $u = 0$. This contradiction completes the proof. □

Finally we verify the conditions **H1-3** in the global Hopf bifurcation theorem ((Wu 1996, Section 6.5)) and finish the proof of **Claims one, two, three**, thus we obtain

Theorem 9 *Assume that $\mathcal{R}_0 > 1, \lambda \in [\lambda_*, \tilde{\lambda}^*]$, then for any $\tau > \tau_1$ system (33) has at least one nontrivial periodic solution.*

5 Numerical simulations and discussions

In this section, we present some numerical simulations to demonstrate the analytic results in previous sections and investigate the effect of delayed media impact and human motility in heterogeneous environment on the transmission dynamics of infectious diseases. Particularly, we will explore the effect of delayed media impact and human mobility on persistence and final epidemic size of infectious disease.

In epidemiology, disease persistence and epidemic size are mostly cared about. Disease persistence is directly related to the basic reproduction number $\mathcal{R}_0$ (Theorem 2). It is often the case that a disease dies out if the basic reproduction number is less than unity and the disease is established in the population if it is greater than unity. Here we mention that if $\mathcal{R}_0 > 1$ (endemic is established), we characterize the epidemic size by

$$H_I := \int_\Omega \widetilde{I} dx;$$

if $R_0 < 1$ (disease dies out), and the epidemic size is characterized by

$$H_I := \int_0^\infty \int_\Omega \frac{\beta(x)e^{-m(x)I}SI}{S+I} dxdt. \tag{36}$$

Note that if $R_0 < 1$, then $I(x,t)$ is exponentially decay to zero. Thus

$$H_I = \int_0^\infty \int_\Omega \frac{\beta(x)e^{-m(x)I}SI}{S+I} dxdt < \infty$$

and well defined. Here we mention that for $\mathcal{R}_0 > 1$, we can not calculate the total infection anymore since it will tend to infinity, and we use $H_I := \int_\Omega \widetilde{I} dx$ to represent epidemic size. The definition is motivated by population size in ecology theory (Lou 2006).

Throughout this section, we fixed $\Omega = (0, 1)$ and the total population $\widetilde{N}$ as 1, then $I(x,t)$ represents the fraction of infected individuals at the position x. Here we mention that some parameters fixed in this section is purposely for demonstrating our theoretical results, such as the occurrence of local Hopf bifurcation.

5.1 The effect of human mobility in heterogeneous environment

By Lemma 3, we obtain that $\mathcal{R}_0$ is a monotone decreasing function of d_I with $\mathcal{R}_0 \to \frac{\int_\Omega \beta dx}{\int_\Omega (\gamma(x)+\alpha(x))dx}$ as $d_I \to \infty$. On one hand, the basic reproduction number in spatial heterogeneous model is bigger than spatial homogeneous model, which implies spatial heterogeneity enhance the disease persistence. On the other hand, it seems nonintuitive that the bigger the diffusion rate, the smaller the basic reproduction number. An explanation is that the diffusion pattern is medical resources oriented, since diffusion has no effect on transmission rate ($\beta(x)$).

For spatial homogeneous ordinary differential models, it is often the case that the bigger the basic reproduction number $\mathcal{R}_0$, the larger the epidemic size. Thus $\mathcal{R}_0$ is also a commonly used measure of the effort needed to control an infectious disease. However, for model with human mobility in heterogeneous environment, the situation

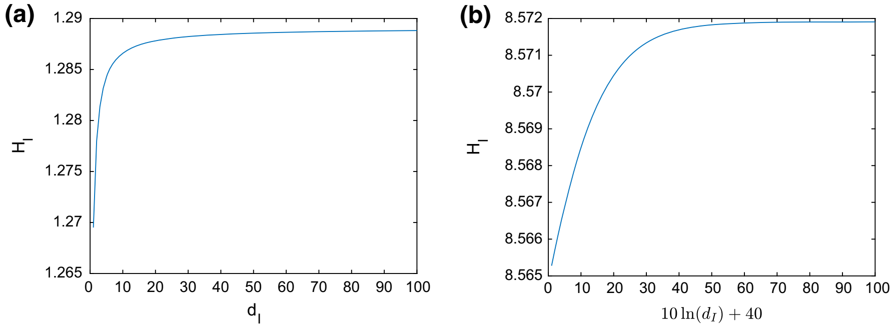

Fig. 1 **(a)** The epidemic size H_I of system (2) with respect to d_I under $\mathcal{R}_0 < 1$. Parameters are fixed as (37). **(b)** The epidemic size H_I with respect to d_I under $\mathcal{R}_0 > 1$ Parameters are fixed as (38)

may change. In what follows, we will show that the case that the larger the basic reproduction number $\mathcal{R}_0$, the larger the epidemic size, may be not true.

The following two special case shows that the smaller the basic reproduction number, the larger the epidemic size under the condition $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$, respectively. Fix other parameters and let d change

$$
\begin{aligned}
& r = 0, \Omega = (0,1), d_S = 0.2, \lambda = 2 + 0.01\sin(2\pi x), \alpha = 0.2 + 0.01\sin(2\pi x),\\
& \beta(x) = 2(3+\sin(2\pi x)), \gamma = 2(4+\sin(\pi x))(\text{day}^{-1}),\\
& \alpha = 0.1(1+\cos(2\pi x))(\text{day}^{-1}), m(x) = 0. \qquad (37)
\end{aligned}
$$

and

$$
\begin{aligned}
& r = 0, \Omega = (0,1), d_S = 0.2, \lambda = 2 + 0.01\sin(2\pi x), \alpha = 0.2 + 0.01\sin(2\pi x),\\
& \beta(x) = 2(5+\sin(2\pi x)), \gamma = 1(\text{day}^{-1}), \qquad (38)\\
& \alpha = 0.1(1+\cos(2\pi x))(\text{day}^{-1}), m(x) = 0
\end{aligned}
$$

By Lemma 3, the basic reproduction number is decreasing in d_I, however Fig. 1 (a) and (b) shows that the epidemic size is increasing in d_I under $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$, respectively, which to some extent shows that the epidemic size may be a decreasing function of the basic reproduction number. We mention here that two numerical examples are special cases and the results are not general. Actually, for the SIS reaction diffusion model (1) proposed in Allen et al. (2008), larger $\mathcal{R}_0$ may not imply larger population size. One can see some theoretical results on this issue from epidemiology perspective in Gao (2020) and ecology perspective in Lou (2006). The epidemic model includes natural birth rate and the infection of newly increment population induces $H_i > 1$.

5.2 The effect of delayed media impact

In this part, we focus the simulations on system (17) for simplicity. We fixed the parameters as

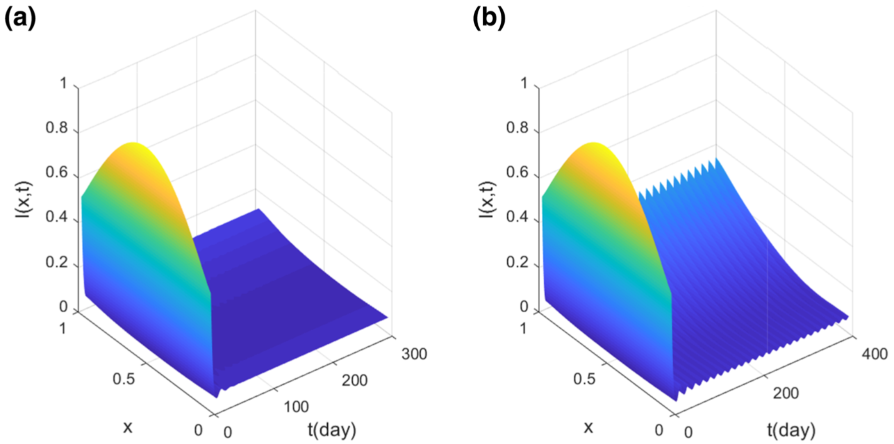

Fig. 2 Solutions of system (17) showing that **(A)** the endemic steady state is asymptotically stable for $r = 6.4 < r_0 \approx 6.4$, and **(B)** the bifurcated periodic solution is feasible for $r = 6.8 > r_0 \approx 6.4$. Parameters are fixed as (39)

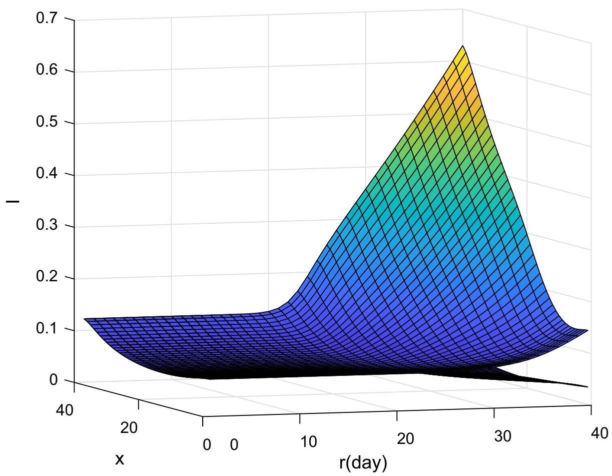

Fig. 3 Bifurcation diagram describing the dynamics of system (17) as r increases. Parameters are fixed as (39)

$$\begin{aligned}&\Omega = (0, 1), d = 0.001, \widetilde{N} = 1, \beta(x) = 0.4(x + 1), \gamma = 0.25(\text{day}^{-1}),\\&\alpha = 0.1(\text{day}^{-1}), m(x) = 10(1 + \sin(\pi x)).\end{aligned} \tag{39}$$

By direct calculation we obtain that $\mathcal{R}_0 = 1.1$ and the bifurcation point $r_0 \approx 6.4$. It can be observed from Fig. 2(A) that the endemic steady state is asymptotically stable for $r = 6 < r_0 \approx 6.4$ and from Fig. 2(A) that the bifurcated periodic solution is feasible for $r = 6.8 > r_0 \approx 6.4$. Further, we plotted the bifurcation diagram by using the delay r as the bifurcation parameter (shown in Fig. 3).

By direct calculation from (39), the local basic reproduction number of system (17) $\mathcal{R}(x) := \beta(x)/(\gamma(x)+\alpha(x)) < 1$ at $0 < x < 0.65$ and the basic reproduction number with respect to the spatial homogeneous environment ($\int_\Omega \beta(x)dx < \int_\Omega(\gamma(x) + \alpha(x))dx$) is smaller than one. However, Fig. 3 shows that for system (17), disease persists in any place and when the delay is large, disease oscillates at any place, which implies spatial heterogeneity enhances disease persistence and oscillation, making disease prediction and control much harder.

6 Conclusions

In this paper, we consider an SIS (suspected-infected-suspected) functional partial differential equation model cooperated with spatial heterogeneity and delayed media impact. The psychological impact of media coverage and rapid information flow on the public is depicted by the reduction in incidence rate at location x, which is expressed as a media function $e^{-m(x)I(x,t-r)}$ with x depending on location. r is the delay of mass media impact on infectious disease and it may directly means the mass media response time, or indirectly means individual response time to media reports, such as the time from symptom onset to hospitalization.

We first show the wellposedness of the model including the existence and uniqueness of the solution, and that the solution semiflow is point dissipative, so a global attractor follows. Then we define the basic reproduction number of the system, and prove that when the basic reproducing number $\mathcal{R}_0 < 1$, the disease-free steady state is globally asymptotically stable; when $\mathcal{R}_0 > 1$, the disease is uniformly persist. We find that the basic reproduction number has nothing to do with the media impact parameters ($m(x)$) and the lag effect (r), but only related to the diffusion rate. This shows that the delayed media impact does not affect the disease persistence, but the human mobility does. The asymptotic behaviors and monotonicity of the basic reproduction number with respect to the diffusion rate of the infected individuals are studied. The theoretical results show that the basic reproduction number in the spatial heterogeneous environment is larger than that in the spatial homogeneous environment, and when the diffusion rate of the infected individuals is small, even if the basic reproduction number in the space homogeneous environment is less than one (the disease is eliminated), the basic reproduction number is still greater than one under spatial heterogeneous environment.

We prove the existence of local Hopf bifurcation at the endemic steady state state with time delay as the bifurcation parameter and the global Hopf bifurcation theorem is used to prove the global continuation of periodic solutions. Here we mention that we can only obtain the existence of pure imaginary roots of eigenvalue equation in a small range of susceptibility diffusion coefficient by using implicit function theorem. Our theoretical and numerical results show that the lag effect of media impact may lead to the periodic oscillation of disease, which brings great challenges to the prevention and control of disease. Moreover, spatial heterogeneity not only makes the disease more likely to persist, but also makes the disease more prone to periodic oscillation and the oscillation places becomes larger, which makes disease harder to prevent and control. Human mobility in spatial heterogeneous environments makes the disease situation in different regions interact with each other. The prevention and control of this disease is no longer an independent matter of each region, but needs overall planning.

In epidemiology, for the spatially homogeneous ordinary differential epidemic model, a general conclusion is that the larger the basic reproduction number, the more people will eventually be infected. Therefore, the basic reproductive number can be used as an important indicator to control of infectious disease, assess the potential for disease invasion and persistence, to predict the extent of an epidemic, and to infer the impact of interventions and of relaxing control measures. However, the utility of $\mathcal{R}_0$ may be overstated. One misconception is that the reproductive number is enough to tell us how large an epidemic will be. For spatial epidemic model, the relationship between basic reproduction number and epidemic size is more subtle and complex. Smaller basic reproduction number may induce larger epidemic size. It may not be effective to control the basic reproduction number of the disease alone. This poses a greater challenge for more effective prediction and control of infectious diseases.

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Appendix

Proof of Theorem 2

We first prove (i) by constructing a Lyapunov functional and applying LaSalle's invariance principle ((Hale 1969, Theorem 1); (Zhao 2017, Theorem 1.1.1)) for infinite dimensional dynamical systems. Recall that system (2) defines a dynamical system on C_r^+, and $\Phi(t)$ is the solution semiflow of system (2) on C_r^+, i.e., $\Phi(t)\varphi = u_t(\varphi)$, $t \geq 0$, where $u(t,\varphi)$ is the unique solution of system (2) with $u_0 = \varphi$. By (Wu 1996, Theorem 2.2.6), $\Phi(t) = u_t(\cdot) : C_r^+ \to C_r^+$ is compact, and for each $\varphi \in C_r^+$ $t > r$, the orbit of φ under $\Phi(t)$ has compact closure in C_r^+.

For any $\psi \in C_r^+$, define the functional

$$L(\psi) = \int_\Omega \psi_2(0)\phi_I dx,$$

where ϕ_I is the positive eigenfunction corresponding to the principal eigenvalue λ_1 of the problem (13). For an arbitrary solution $u(t,\varphi)$ of (2), we obtain

$$\begin{aligned}
\frac{d}{dt}L(u_t(\varphi)) &= \int_\Omega I_t \phi_I dx \\
&= \int_\Omega \left(d_I \Delta I + \frac{\beta(x)e^{-m(x)I(x,t-r)}SI}{S+I} - \gamma(x)I - \alpha(x)I \right) \phi_I dx \\
&= \int_\Omega \left(\beta(x)I \left(\frac{e^{-m(x)I(x,t-r)}S}{S+I} - 1 \right) - \lambda_1 I \right) \phi_I dx. \qquad \text{(A.1)}
\end{aligned}$$

Therefore, $\frac{d}{dt}L(u_t(\varphi)) \leq 0$, which implies that $L(\psi)$ is a Lyapunov functional on C_r^+ relative to the system (2).

Next define

$$\dot{L}(\varphi) := \frac{d}{dt}L(u_t(\varphi))\Big|_{t=0} \quad \text{and} \quad \mathcal{S} = \{\varphi \in C_r^+ | \dot{L}(\varphi) = 0\},$$

where $u(t,\varphi)$ is the unique solution of (2) with initial condition $u_0 = \varphi \in C_r^+$. By (A.1), we have $\mathcal{S} = \{\varphi \in C_r^+ | \varphi_2 = 0\}$ and $\mathcal{S}$ is invariant under $\Phi(t)$. Thus by the LaSalle invariant principle ((Hale 1969, Theorem 1)), we obtain $\lim_{t\to\infty} I(\cdot,t) = 0$. By (Zhao 2017, Theorem 1.2.1 with Remark 1.3.2) (see also (Thieme 1992, Theorem 4.1)), we have $\lim_{t\to\infty} S(\cdot,t) = \widetilde{N}$.

For (ii). We appeal to the theory of uniform persistence theory developed in Zhao (2017); Magal and Zhao (2005). Denote

$$U_0 := \{(\varphi,\phi_2) \in C_r^+ | \phi_2(0) \neq 0\}; \quad \partial U_0 := C_r^+ \setminus U_0.$$

Then $C_r^+ = U_0 \cup \partial U_0$, U_0 and ∂U_0 are relatively open and closed subsets of U, respectively, and U_0 is convex. Let $\Phi(t)(s_0,i_0) = (S(\cdot,t), I(\cdot,t))$ be the unique solution of system (2) with the initial value $(s_0,i_0) \in C_r^+$ for any $t > 0$. By Theorem 1, $\Phi(t)$ has a global attractor.

Step 1. We have $\Phi(t)U_0 \subset U_0$ for all $t > 0$. This is a direct result of the strong maximum principle for parabolic equations.

Step 2. Let A_∂ be the maximal positively invariant set for $\Phi(t)$ in ∂U_0, i.e.

$$A_\partial := \{(s_0,i_0) \in C_r^+ | \Phi(t)(s_0,i_0) \in \partial U_0, t \geq 0\}.$$

It is easy to verify that $A_\partial = \{u_0 = (s_0,i_0) \in C_r^+ | i_0 = 0\}$. Denote $\omega((s_0,i_0))$ as the ω-limit set of (s_0,i_0) in C_r^+ (see Zhao (2017)) and

$$\hat{A}_\partial = \cup_{\{(s_0,i_0)\in A_\partial\}}\omega((s_0,i_0)).$$

It can be seen that $\hat{A}_\partial = \{E_0 = (\widetilde{N},0)\}$. Thus, $\{E_0\}$ is a compact and isolated invariant set for $\Phi(t)$ restricted in A_∂.

Step 3. We prove that there exists some constant $\epsilon_1 > 0$ independent of initial values such that

$$\limsup_{t\to\infty} \|\Phi(t)(s_0,i_0) - (\widetilde{N},0)\| > \epsilon_1.$$

Assume, on the contrary, that for any $\epsilon_2 > 0$, there exists some initial value (s_0^*, i_0^*) such that

$$\limsup_{t\to\infty} \|\Phi(t)(s_0^*,i_0^*) - (\widetilde{N},0)\| \leq \frac{\epsilon_2}{2}. \tag{A.2}$$

Given any small $\epsilon_3 > 0$ and let $\lambda_1(\epsilon_3)$ be the unique principal eigenvalue of the following eigenvalue problem with a positive eigenfunction ψ_I:

$$\begin{aligned}&-d_I \Delta\psi_I - \frac{\beta(x)e^{-m(x)\epsilon_3}(\widetilde{N}-\epsilon_3)}{\widetilde{N}+2\epsilon_3}\psi_I + (\gamma(x)+\alpha(x))\psi_I \\ &= \lambda\psi_I \text{ in } \Omega; \quad \frac{\partial\psi_I}{\partial n}\Big|\partial\Omega = 0.\end{aligned}$$

Note that $\lim_{\epsilon_3\to 0}\lambda_1(\epsilon_3) = \lambda_1 < 0$, where λ_1 is the principal eigenvalue of eigenvalue problem (13). Therefore, we can choose ϵ_3 such that $\lambda_1(\epsilon_3) < 0$. Since ϵ_2 is arbitrary, choose $\epsilon_2 = \epsilon_3$. By (A.2), there exists $T_0 > 0$ such that $|S^* - \widetilde{N}|, I^* \le \epsilon_3$ for any $x \in \overline{\Omega}, t \ge T_0$. By the strong maximum principal of parabolic equations, $S^*(\cdot, t), I^*(\cdot, t) > 0$ for all $t > 0$. Then we can find a small positive constant c_* such that $I^*(x, T) \ge c_*\psi_I$. It is easy to verify that $I^*(x, t)$ is a supersolution of the problem

$$\begin{cases} \dfrac{\partial \hat{I}}{\partial t} = d_I\Delta\hat{I} + \dfrac{\beta(x)e^{-m(x)\epsilon_3}(\widetilde{N}-\epsilon_3)}{\widetilde{N}+2\epsilon_3}\hat{I} - (\gamma(x)+\alpha(x))\hat{I}, & x \in \Omega, t > T, \\ \dfrac{\partial\hat{I}}{\partial n} = 0, & x \in \partial\Omega, t > T, \\ \hat{I}(x, T) = c_*\psi_I, \end{cases} \tag{A.3}$$

and $c_* e^{-\lambda_1(\epsilon_3)(t-T)}\phi_I$ is the unique solution to system (A.3). Note that $\lambda_1(\epsilon_3) < 0$, therefore $I^*(x, t) \ge c_* e^{-\lambda_1(\epsilon_3)(t-T)}\psi_I \to \infty$ uniformly in $\overline{\Omega}$ as $t \to \infty$. This contradiction finishes the proof of Step 3.

The result of Step 3 implies that $\{E_0\}$ is an isolated invariant set for $\Phi(t)$ in C_r^+, and $W^S(\{E_0\}) \cap U_0$ is an empty set, where $W^S(\{E_0\})$ is the stable set of $\{E_0\}$ for $\Phi(t)$.

Finally, by Steps 1-3 and (Zhao 2017, Theorem 1.3.1), Φ_t is uniformly persistent with respect to $(U, \partial U_0)$. Moreover, by (Zhao 2017, Theorem 1.3.7), (2) admits at least one endemic steady state.

Proof of Theorem 4

Before proving Theorem 4, we give three lemmas which will be used to conclude our assertion. To start with, we give estimates for solutions of (27).

Lemma A.1 *If $(w_\lambda, \theta_\lambda, \psi_\lambda)$ is a solution to (27) with $w_\lambda > 0, \theta_\lambda \in [0, 2\pi)$ and $\psi_\lambda \in X_{\mathbb{C}}\backslash\{0\}$, then*

$$\lambda\sin(\theta_\lambda)\int_\Omega N_\lambda|\psi_\lambda|^2dx - w_\lambda\int_\Omega |\psi_\lambda|^2dx = 0. \tag{A.4}$$

Moreover, $\frac{w_\lambda}{\lambda-\lambda_*}$ is bounded for $\lambda \in [\lambda_*, \lambda^*]$.

Proof It follows from substituting $(w_\lambda, \theta_\lambda, \psi_\lambda)$ into (27), multiplying (27) by $\overline{\psi}_\lambda$, and integrating the result by part over Ω that

$$-\int_\Omega |\nabla\psi_\lambda|^2 dx + \lambda\int_\Omega K_\lambda|\psi_\lambda|^2 dx - \lambda e^{-i\theta_\lambda}\int_\Omega N_\lambda|\psi_\lambda|^2 dx - i w_\lambda\int_\Omega |\psi_\lambda|^2 dx = 0,$$

which implies (A.4). Moreover, we obtain

$$\frac{w_\lambda}{\lambda-\lambda_*} = \frac{\lambda\sin(\theta_\lambda)\int_\Omega N_\lambda|\psi_\lambda|^2 dx}{(\lambda-\lambda_*)\int_\Omega |\psi_\lambda|^2 dx} \le \lambda_* \max(\beta m)(\|\ \phi\ \|_\infty + (\lambda-\lambda_*)\ \|\ \xi_\lambda\ \|_\infty).$$

Thus $\frac{w}{\lambda-\lambda_*}$ is bounded for $\lambda \in [\lambda_*, \lambda^*]$. □

By similar arguments as Lemma 2.3 in Busenberg and Huang (1996), we get the following result:

Lemma A.2 *If $z \in X_{\mathbb{C}}$ and $<\phi, z>= 0$, then $|<Lz, z>| \ge \mu_2 \ \|\ z\ \|^2_{Y_C}$, where μ_2 is the second eigenvalue of operator $-L$.*

We prove that $G(z, \kappa, h, \theta, \lambda) = 0$ is uniquely solvable for $\lambda = \lambda_*$.

Lemma A.3 *The following equation*

$$G(z, \kappa, h, \theta, \lambda_*) = 0, z \in (X_1)_{\mathbb{C}}, h > 0, \kappa \ge 0, \theta \in [0, 2\pi) \tag{A.5}$$

admits a unique solution $(z_{\lambda_*}, \kappa_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})$, where

$$\kappa_{\lambda_*} = 1, \theta_{\lambda_*} = \frac{\pi}{2}, h_{\lambda_*} = \frac{\int_\Omega \lambda_* m\beta A_{\lambda_*}\phi^3 dx}{\int_\Omega \phi^2 dx}$$

and $z_{\lambda_*} \in (X_1)_{\mathbb{C}}$ is the unique solution of

$$Lz_{\lambda^*} + (\beta-\gamma-\alpha-\lambda_*\beta A_{\lambda_*}(m+2/\widetilde{N})\phi)\phi + i\lambda_* m\beta A_{\lambda_*}\phi^2 - ih_{\lambda_*}\phi = 0, \tag{A.6}$$

where L is defined in (19).

Proof It follows from the second equation of (A.5) that $\kappa = \kappa_{\lambda_*} = 1$. Substituting $\kappa = 1$ and $\lambda = \lambda_*$ into $g_2 = 0$ yields

$$g_2 = Lz + (\beta-\gamma-\alpha-\lambda_*\beta A_{\lambda_*}(m+2/\widetilde{N})\phi)\phi - e^{-i\theta}\lambda_* m\beta A_{\lambda_*}\phi^2 - ih_{\lambda_*}\phi = 0,$$

which implies

$$\begin{cases} \lambda_*\int_\Omega m\beta A_{\lambda_*}\phi^3 dx\sin(\theta) = h\int_\Omega \phi^2 dx, \\ \lambda_*\int_\Omega m\beta A_{\lambda_*}\phi^3 dx\cos(\theta) = 0. \end{cases}$$

Therefore, $\theta = \theta_{\lambda_*} = \frac{\pi}{2}, h = h_{\lambda_*} = \frac{\int_\Omega \lambda_* m\beta A_{\lambda_*}\phi^3 dx}{\int_\Omega \phi^2 dx}$. Moreover, it is easy to verify that

$$(\beta - \gamma - \alpha - \lambda_*\beta A_{\lambda_*}(m + 2/\widetilde{N})\phi)\phi, -\lambda_* m\beta A_{\lambda_*}\phi^2 - h_{\lambda_*}\phi \in Y_1.$$

Thus, $g_1(z, \kappa_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*) = 0$ has a unique solution z_{λ_*} satisfies (A.6). □

Now we proceed to prove Theorem 4.

Proof let $T = (T_1, T_2) : (X_1)_{\mathbb{C}} \times \mathbb{R}^3 \to Y_{\mathbb{C}} \times \mathbb{R}$ be the Fréchet derivative of G with respect to (z, κ, h, θ) at $(z_{\lambda_*}, \kappa_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*)$, i.e.,

$$\begin{aligned} T(z, \kappa, h, \theta) &= Lz + \kappa\phi[\beta - \gamma - \alpha - \lambda_*\beta A_{\lambda_*}(m + \frac{2}{\widetilde{N}})\phi + i\lambda_* m\beta A_{\lambda_*}\phi - ih_{\lambda_*}] \\ &\quad - ih\phi + \theta\lambda_* m\beta A_{\lambda_*}\phi^2, \\ T_2(r) &= 2r \parallel \phi \parallel_{Y_C}^2 . \end{aligned}$$

Note that T is a bijection from $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$ to $Y_{\mathbb{C}} \times \mathbb{R}$. Thus it follows from the implicit function theorem that there exist $\tilde{\lambda}^* > \lambda_*$ and a continuously differentiable mapping $\lambda \to (z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda)$ from $[\lambda_*, \tilde{\lambda}^*]$ to $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$ such that $G(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$.

Now it remains to prove the uniqueness. We only need to verify that if $G(z, \kappa, h, \theta, \lambda) = 0$ with $z_\lambda \in (X_1)_{\mathbb{C}}, \kappa_\lambda, h_\lambda > 0, \theta_\lambda \in [0, 2\pi)$, then

$$(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda) \to (z_{\lambda_*}, \kappa_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})$$

as $\lambda \to \lambda_*$ in the norm of $X_{\mathbb{C}} \times \mathbb{R}^3$. To start with, we show z_λ is bounded in $(X_1)_{\mathbb{C}}$. Note that A_λ and $\|\xi_\lambda\|_\infty$ are bounded for $\lambda \in [\lambda_*, \tilde{\lambda}^*]$. By Lemma A.2 and similar arguments in (Busenberg and Huang 1996, Theorem 2.4), we can obtain that there exist $M_1, M_2 > 0$ such that

$$\lambda_2\|z_\lambda\|_{Y_{\mathbb{C}}}^2 \le | < Lz_\lambda, z_\lambda > | \le M_1\|\phi\|_{Y_{\mathbb{C}}}\|z_\lambda\|_{Y_{\mathbb{C}}} + M_2(\lambda - \lambda_*)\|z_\lambda\|_{Y_{\mathbb{C}}}^2.$$

Therefore, if $\tilde{\lambda}^*$ is sufficiently small, z_λ is bounded in $Y_{\mathbb{C}}$ for $\lambda \in [\lambda_*, \tilde{\lambda}^*]$. Then z_λ is also bounded in $(X_1)_{\mathbb{C}}$. Moreover, it follows from Lemma A.1 and (29) that $\kappa_\lambda, h_\lambda, \theta_\lambda$ are bounded for $\lambda \in [\lambda_*, \tilde{\lambda}^*]$, which together with the boundedness of z_λ in $(X_1)_{\mathbb{C}}$ implies that $\{(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda) : \lambda \in [\lambda_*, \tilde{\lambda}^*]\}$ is precompact in $Y_{\mathbb{C}} \times \mathbb{R}^3$. Then, there exists a subsequence $\{(z_{\lambda_n}, \kappa_{\lambda_n}, h_{\lambda_n}, \theta_{\lambda_n})\}$ is convergent in $Y_{\mathbb{C}} \times \mathbb{R}^3$ for $\lambda_n \to \lambda_*$ as $n \to \infty$. Taking the limit of the equation $L^{-1}g_1(z_{\lambda_n}, \kappa_{\lambda_n}, h_{\lambda_n}, \theta_{\lambda_n}, \lambda_n) = 0$ as $n \to \infty$, we can see from Lemma A.3 that

$$(z_{\lambda_n}, \kappa_{\lambda_n}, h_{\lambda_n}, \theta_{\lambda_n}) \to (z_{\lambda_*}, \kappa_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})$$

in $(X_1)_{\mathbb{C}} \times \mathbb{R}^3$ as $n \to \infty$. Thus we obtain the uniqueness and complete the proof. □

Proof of Lemma 7

It follows from Corollary 1 that $\mathcal{N}(\mathcal{A}_{\tau_n}(\lambda) - iw_\lambda) = \text{Span}\{e^{iw_\lambda\theta}\psi_\lambda\}$. If $\varphi \in \mathcal{N}(\mathcal{A}_{\tau_n}(\lambda) - iw_\lambda) \cap \mathcal{N}(\mathcal{A}_{\tau_n}(\lambda) - iw_\lambda)^2$, then

$$(\mathcal{A}_{\tau_n} - iw_\lambda)\varphi \in \mathcal{N}(\mathcal{A}_{\tau_n} - iw_\lambda) = \text{Span}\{e^{iw_\lambda\theta}\psi_\lambda\}$$

Therefore, there exists a constant a such that

$$(\mathcal{A}_{\tau_n} - iw_\lambda)\varphi = ae^{iw_\lambda\theta}\psi_\lambda,$$

which implies

$$\begin{cases} \dot{\varphi}(\theta) = iw_\lambda\varphi(\theta) + ae^{iw_\lambda\theta}\psi_\lambda, \theta \in [-\tau_n, 0], \\ \dot{\varphi}(0) = \Delta\varphi(0) + \lambda K_\lambda\varphi(0) - \lambda N_\lambda\varphi(-\tau_n). \end{cases} \tag{A.7}$$

We can obtain from the first equation of (A.7) that

$$\varphi(-\tau_n) = \varphi(0)e^{-iw_\lambda\tau_n} - a\tau_n e^{-iw_\lambda\tau_n}\psi_\lambda; \dot{\varphi}(0) = iw_\lambda\varphi(0) + a\psi_\lambda,$$

which together with the second equation of (A.7) yields

$$\Delta\varphi(0) + \lambda K_\lambda\varphi(0) - \lambda N_\lambda(\varphi(0)e^{-iw_\lambda\tau_n} - a\tau_n e^{-iw_\lambda\tau_n}\psi_\lambda) - iw_\lambda\varphi(0) - a\psi_\lambda = 0,$$

i.e.,

$$\Delta(\lambda, iw_\lambda, \tau_n)\varphi(0) = a\psi_\lambda(1 - \lambda N_\lambda\tau_n e^{-i\theta_\lambda}). \tag{A.8}$$

Multiplying (A.8) by $\overline{\psi}_\lambda$ and integrating over Ω by parts yield that

$$a\int_\Omega |\psi_\lambda|^2(1 - \lambda N_\lambda\tau_n e^{-i\theta_\lambda})dx = 0. \tag{A.9}$$

For (A.9), taking the limit $\lambda \to \lambda_*$, we have

$$\theta_\lambda \to \frac{\pi}{2}, \quad \tau_n(\lambda - \lambda_*) \to \frac{\pi/2 + 2n\pi}{h_{\lambda_*}},$$
$$\psi_\lambda \to \phi, \quad \frac{\lambda N_\lambda}{\lambda - \lambda_*} \to \lambda_* m\beta A_{\lambda_*}\phi \;\text{ in }\; X_{\mathbb{C}}.$$

Thus $a = 0$ can obtained from

$$\int_\Omega \psi_\lambda^2(1 - \lambda N_\lambda\tau_n e^{-iw_\lambda\tau_n})dx = \int_\Omega \phi^2\left(1 + i\frac{\pi/2 + 2n\pi}{h_{\lambda_*}}\lambda_* m\beta A_\lambda\phi\right)dx$$
$$+o(\lambda - \lambda_*) \neq 0.$$

Therefore,

$$\mathcal{N}(\mathcal{A}_{\tau_n} - iw_\lambda)^k = \mathcal{N}(\mathcal{A}_{\tau_n} - iw_\lambda), k = 2, 3, \cdots, n = 0, 1, 2, \cdots.$$

Thus $\mu = iw_\lambda$ is a simple eigenvalue of $\mathcal{A}_{\tau_n}(\lambda)$ for $n = 0, 1, 2, \cdots$.

Proof of Lemma 8

It follows from differentiating (32) with respect to τ at $\tau = \tau_n$ that

$$\Delta(\lambda, \mu, \tau)\psi' + \lambda N_\lambda \psi e^{-\mu\tau}\mu = \frac{d\mu}{d\tau}(1 - \lambda\tau N_\lambda e^{-\mu\tau})\psi.$$

Then multiplying by $\overline{\psi}$ and integrating over Ω by parts yield

$$\begin{aligned}\mathrm{Re}\left(\frac{d\mu}{d\tau}\right) &= \mathrm{Re}\left(\frac{\int_\Omega (1 - \lambda\tau N_\lambda e^{-\mu\tau})\psi^2 dx}{\int_\Omega \lambda N_\lambda \psi^2 e^{-\mu\tau} v dx}\right)\\ &= \mathrm{Re}\left(\frac{\int_\Omega \psi^2 dx}{\int_\Omega \lambda N_\lambda \psi^2 e^{-\mu\tau}\mu dx} - \frac{\tau}{\mu}\right).\end{aligned}$$

Let $\tau = \tau_n$. Recalling that $\mu(\tau_n) = iw_\lambda$, $\psi(\tau_n) = \psi_\lambda$, thus we obtain

$$\mathrm{Re}\left(\frac{d\mu}{d\tau}\Big|_{\tau=\tau_n}^{-1}\right) = \mathrm{Re}\left(\frac{\int_\Omega \psi_\lambda^2 dx}{\int_\Omega i\lambda N_\lambda \psi_\lambda^2 e^{-iw_\lambda\tau_n} w_\lambda dx}\right).$$

Moreover, taking the limit $\lambda \to \lambda_*$, we have

$$\begin{aligned}&\frac{w_\lambda}{\lambda - \lambda_*} \to h_{\lambda_*}, \quad \tau_n(\lambda - \lambda_*) \to \frac{\pi/2 + 2n\pi}{h_{\lambda_*}},\\ &\psi_\lambda \to \phi, \quad \frac{\lambda N_\lambda}{\lambda - \lambda_*} \to \lambda_* m\beta A_{\lambda_*}\phi \ \text{ in } \ X_{\mathbb{C}}.\end{aligned}$$

Therefore, we have

$$(\lambda - \lambda^*)^2 \mathrm{Re}\left(\frac{d\mu}{d\tau}\Big|_{\tau=\tau_n}^{-1}\right) = \frac{\int_\Omega \phi^2 dx}{\int_\Omega \lambda_* m\beta A_{\lambda^*}\phi^3 h_{\lambda_*} dx} + o(\lambda - \lambda_*) > 0.$$

This completes the proof.

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