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Research article

A unified exponential-H family for modeling real-life data: Properties and inference

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ABSTRACT

The exponential distribution is one of the most widely used statistical distribution for reliability issues. In this paper, we introduce a novel family based on the exponential model, called the new exponential-H (NEx-H) family. The sub-models of the NEx-H family are capable of accommodating variable failure rates, as well as unimodal, bimodal, left-skewed, symmetric, right-skewed, and J-shape densities. The mathematical features of the NEx-H family are derived. The parameters of the NEx-Weibull distribution are estimated by using seven estimation methods. Detailed numerical simulations are presented. Based on our study, the maximum likelihood is the best estimation method for estimating the NEx-Weibull parameters. Three real-life data sets are fitted using the NEx-Weibull distribution. The NEx-Weibull model provides better fit as compared to some competing Weibull models.

1. Introduction

Recently, adding extra parameters to baseline distributions have received an increased interest among statisticians. The extra parameters provide new insights for flexible modeling of real-life data in several applied areas such as biology, public health, medicine, engineering, computer science, industry, life-testing, and insurance, among others. Hence, flexible lifetime models are very important to provide adequate fits to real-life data in these fields due to the drawbacks of classical distributions which are unable to show wide flexibility as well as very limited in their properties, and can not present adequate fits to real-life data.

The new generators provide more flexible distributions for fitting real-life data in several applied fields. Hence, several families or generators having one or more extra shape parameters have been introduced in the literature to generate more flexible models. For example, the exponentiated-G [1], Marshall–Olkin-G [2], beta-G [3], transmuted-G [4], Kumaraswamy-G [5], Kumaraswamy odd log-logistic-G [6], type-I half-logistic-G [7], Weibull Marshall–Olkin-G [8], Kumaraswamy alpha-power-G [9], Marshall–Olkin Burr-III-G [10], Marshall–Olkin Weibull-H [11], and odd JCA-G [12] families, among others. The above mentioned families provide induction of one, two or three extra shape parameters. More information can be explored in [13].

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In this article, a new class called the new exponential-H (NEx-H) family is introduced. The NEx-H family is formulated based on the exponential (Ex) distribution and the T-X family [14]. The most important feature of the new family, with only one extra shape parameter, represents in its ability to provide bimodal density shapes for its special sub-models.

The NEx-H family has the following desirable properties. (i) The probability density function (pdf) of the NEx-H family has a simple closed form. Then, the special sub-models of the NEx-H class can be used for modeling and analyzing real-life data in different applied areas; (ii) The extra parameter of the NEx-H family makes its special sub-models very flexible to exhibit all important failure rate shapes; (iii) Additionally, the densities of the NEx-H sub-models can also provide more flexible shapes; (iv) The NEx-Weibull (NExW) distribution, as a special sub-model of NEx-H family, provides a close fit to unimodal and bimodal real-life data sets.

Moreover, the parameters of the NExW distribution are estimated by using seven classical estimation approaches. The performance of these methods is addressed based on detailed simulation studies.

The paper is organized in seven sections. The NEx-H family is defined in Section 2. Five special sub-models are introduced in Section 3. In Section 4, important properties of the NEx-H family are derived. Inference about the NExW parameters is presented In Section 5. Section 6 explores simulation Section 7 presents three real-life applications. Final remarks are presented in Section 8.

2. The NEx-H family

Consider the Ex distribution with pdf, $w(t;a) = a\exp(-at)$, $(t > 0)$ and cdf, $W(t;a) = 1 - \exp(-at)$, $(t > 0)$, $a > 0$. Based on the Ex density and by replacing t with the quantity $H(t;\boldsymbol{\varpi})(2 - H(t;\boldsymbol{\varpi}))/\bar{H}(t;\boldsymbol{\varpi})$, the cumulative distribution function (cdf) of the NEx-H family follows (for $t > 0$) as

$$K(t;a,\boldsymbol{\varpi}) = \int_0^{H(t;\boldsymbol{\varpi})(2-H(t;\boldsymbol{\varpi}))/\bar{H}(t;\boldsymbol{\varpi})} a\exp(-a\,t)\,dt$$

$$= 1 - \exp\left\{-a\left[\frac{H(t;\boldsymbol{\varpi})(2 - H(t;\boldsymbol{\varpi}))}{1 - H(t;\boldsymbol{\varpi})}\right]\right\}, a > 0, \quad (2.1)$$

where $H(t;\boldsymbol{\varpi})$ denotes the baseline cdf which depends on a vector of parameters $\boldsymbol{\varpi}$ and $\bar{H}(t;\boldsymbol{\varpi}) = 1 - H(t;\boldsymbol{\varpi})$ is the survival function (sf).

The pdf of the NEx-H family reduces (for $t > 0$ and $a > 0$) to

$$k(t;a,\boldsymbol{\varpi}) = \frac{a\,h(t;\boldsymbol{\varpi})}{[1 - H(t;\boldsymbol{\varpi})]^2}\left\{1 + [1 - H(t;\boldsymbol{\varpi})]^2\right\}\exp\left\{-a\left[\frac{H(t;\varpi)(2 - H(t;\boldsymbol{\varpi}))}{1 - H(t;\boldsymbol{\varpi})}\right]\right\}, \quad (2.2)$$

where $h(t;\boldsymbol{\varpi})$ denotes the baseline density. Hereafter, any random variable (*rv*) having the pdf (2.2) will be referred as $T \sim$NEx-H$(a,\boldsymbol{\varpi})$.

The hazard rate function (hrf) of the NEx-H family takes the form

$$w(t;a,\boldsymbol{\varpi}) = \frac{a\,h(t;\boldsymbol{\varpi})\left\{1 + [1 - H(t;\boldsymbol{\varpi})]^2\right\}}{[1 - H(t;\boldsymbol{\varpi})]^2}.$$

The reversed-hrf (rhrf) and cumulative-hrf of the NEx-H family are

$$r(t;a,\boldsymbol{\varpi}) = \frac{a\,h(t;\boldsymbol{\varpi})\left\{1 + [1 - H(t;\boldsymbol{\varpi})]^2\right\}\exp\left\{-a\left[\frac{H(t;\boldsymbol{\varpi})(2-H(t;\boldsymbol{\varpi}))}{1-H(t;\boldsymbol{\varpi})}\right]\right\}}{[1 - H(t;\boldsymbol{\varpi})]^2\left(1 - \exp\left\{-a\left[\frac{H(t;\boldsymbol{\varpi})(2-H(t;\boldsymbol{\varpi}))}{1-H(t;\boldsymbol{\varpi})}\right]\right\}\right)}$$

and

$$R(t;a,\boldsymbol{\varpi}) = -\log S(t;a,\varpi) = \frac{a\,H(t;\boldsymbol{\varpi})\,[2 - H(t;\boldsymbol{\varpi})]}{1 - H(t;\boldsymbol{\varpi})},$$

where $S(t;a,\boldsymbol{\varpi}) = 1 - K(T;a,\varpi)$ is the sf of the NEx-H class.

The quantile function (qf) of the NEx-H family has the following explicit form

$$T_u = H^{-1}\left[\frac{-(\log(1-u) - 2a) \pm \sqrt{\log^2(1-u) + 4a^2}}{2a}\right], \qquad u \in (0,1).$$

In the following subsequent sections, we provide some special sub-models of the NEx-H family and derive some of its key mathematical features.

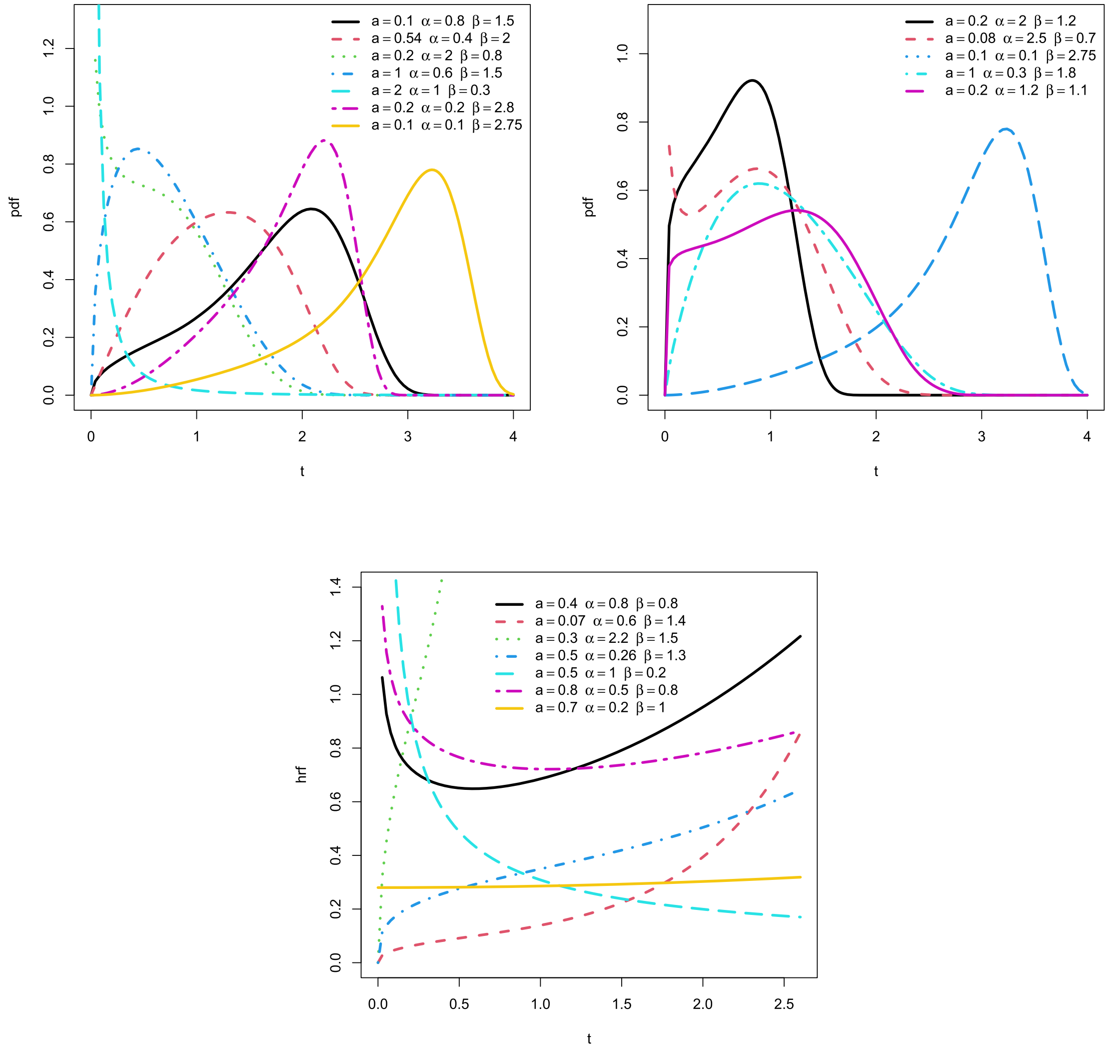

Fig. 1. Plots of the pdf and hrf of the NExW distribution with different parameter values.

3. Five special sub-models

This section provides five special sub-models of the NEx-H family. The extra shape parameter a of the NEx-H family makes the baseline hrf more flexible to exhibit all important hrf shapes including constant, bathtub, increasing, J-shape, unimodal, reversed J-shape, and decreasing shapes as shown in Figs. 1–5. Furthermore, the densities of these sub-models provide great flexibility in their shapes. They can exhibit left-skewed, bimodal, right-skewed, unimodal, symmetric, J and reversed-J shapes as illustrated in Figs. 1–5.

3.1. NEx-Weibull distribution

Consider the Weibull distribution with pdf $h(t;\alpha,\beta)=\alpha\beta t^{\beta-1}\exp(-\alpha t^{\beta})$, $t>0$, $\alpha,\ \beta>0$. Then, the pdf and hrf of the NExW distribution take the forms (for $t>0$)

$$k(t;a,\alpha,\beta)=a\alpha\beta t^{\beta-1}\exp\left(\alpha t^{\beta}\right)\left[1-\exp\left(-2\alpha t^{\beta}\right)\right]\exp\left\{-a\left[\frac{1-\exp\left(-2\alpha t^{\beta}\right)}{\exp\left(-\alpha t^{\beta}\right)}\right]\right\},\ \alpha,\ \beta,\ a>0$$

and

$$w(t;a,\alpha,\beta)=a\alpha\beta t^{\beta-1}\exp\left(\alpha t^{\beta}\right)\left[1-\exp\left(-2\alpha t^{\beta}\right)\right].$$

Fig. 1 shows plots of density and hrf of the NExW distribution for different parametric values.

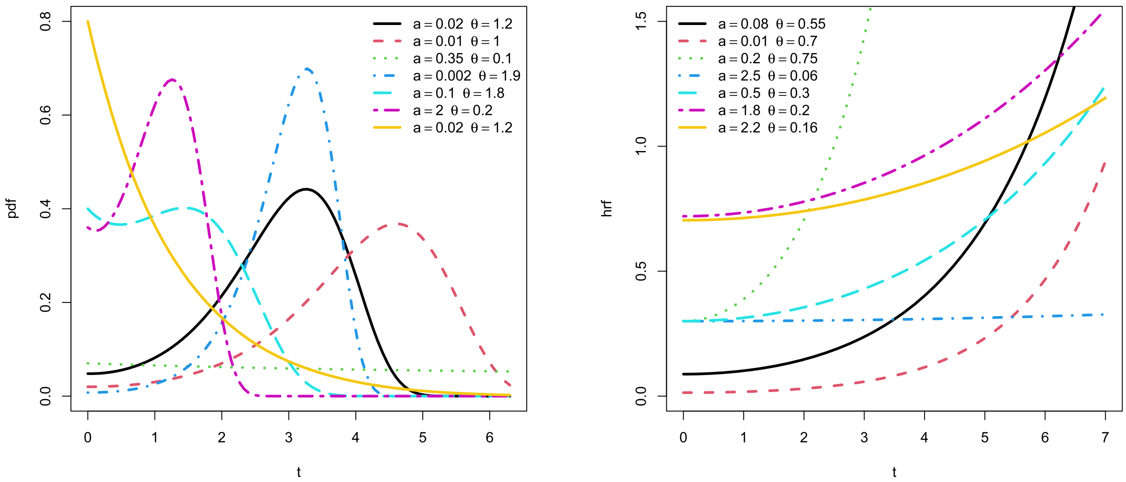

Fig. 2. Plots of the pdf and hrf of the NExEx distribution with different parameter values.

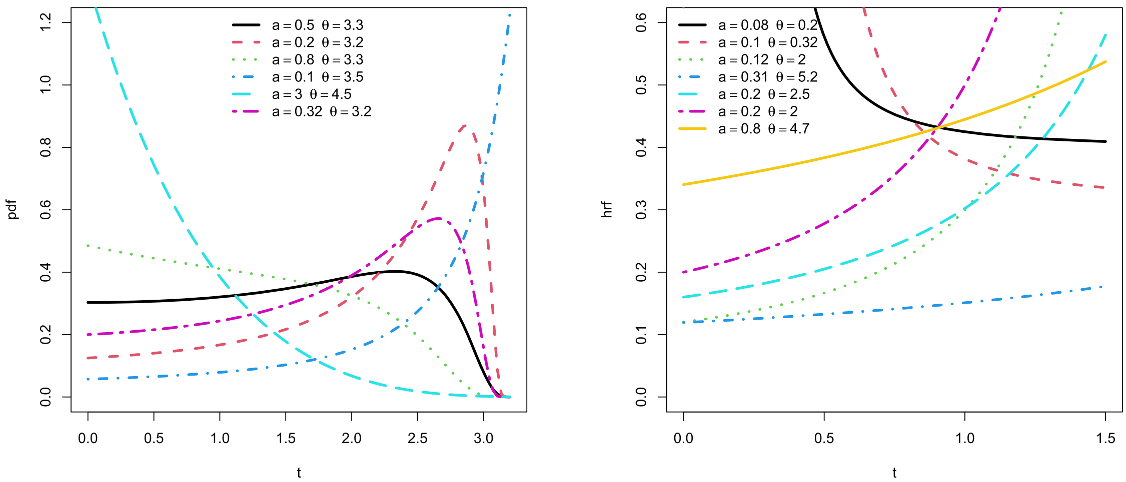

Fig. 3. Plots of the pdf and hrf of the NExU distribution with different parameter values.

3.2. NEx-exponential distribution

Consider the Ex distribution with pdf $h(t;\theta) = \theta\,\exp(-\theta\,t),\ t > 0,\ \theta > 0$. The pdf of the NEx-exponential (NExEx) model takes the form

$$k(t;a,\theta) = a\,\theta\left[1+\exp\left(-2\theta t\right)\right]\exp\left\{\theta\,t - a\exp\left(\theta t\right)\left[1-\exp\left(-2\theta t\right)\right]\right\},\ a,\ \theta > 0.$$

Fig. 2 gives some plots of the NExEx pdf and hrf for several parametric values.

3.3. NEx-uniform distribution

The uniform distribution has the pdf $h(t;\theta) = \frac{1}{\theta}$, $0 < t < \theta < \infty$, $\theta > 0$. The pdf of the NEx-uniform (NExU) model follows as

$$k(t;a,\theta) = \frac{a}{\theta}\left[\frac{2\theta(\theta-t)+t^2}{(\theta-t)^2}\right]\exp\left[-\frac{a\,t}{\theta}\left(\frac{2\theta-t}{\theta-t}\right)\right], \qquad 0 < t < \theta, a < \infty.$$

Fig. 3 illustrates the pdf and hrf plots of the NExU distribution.

3.4. NEx-Burr-XII distribution

Consider the Burr XII (BXII) distribution with pdf $h(t;\alpha,\beta,\gamma) = \alpha\beta\gamma^{-\alpha}t^{\alpha-1}[1+(t/\gamma)^{\alpha}]^{-\beta-1}$, $t > 0$, $\alpha,\beta,\gamma > 0$. Then, the pdf of the NExBXII distribution takes the form

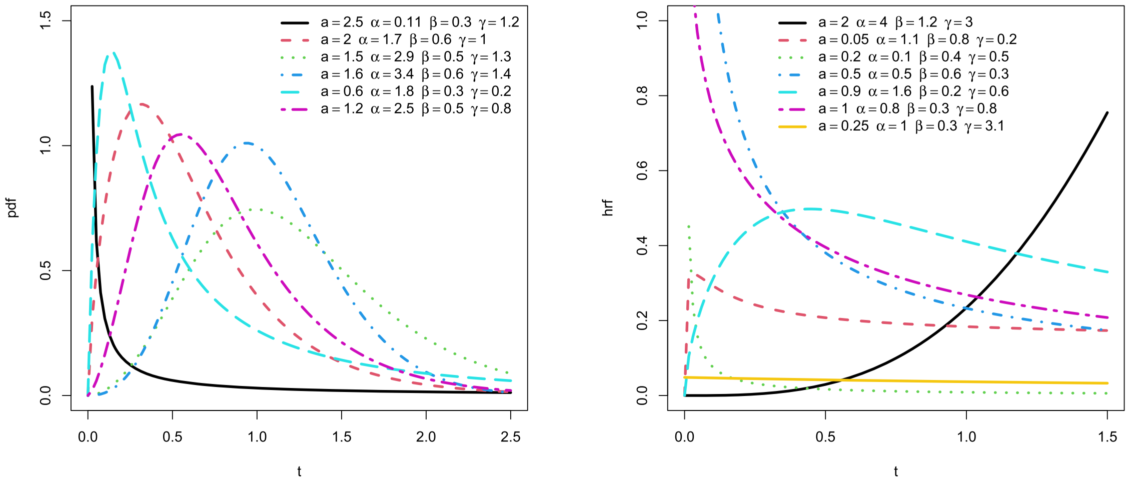

Fig. 4. Plots of the pdf and hrf of the NExBXII distribution with different parameter values.

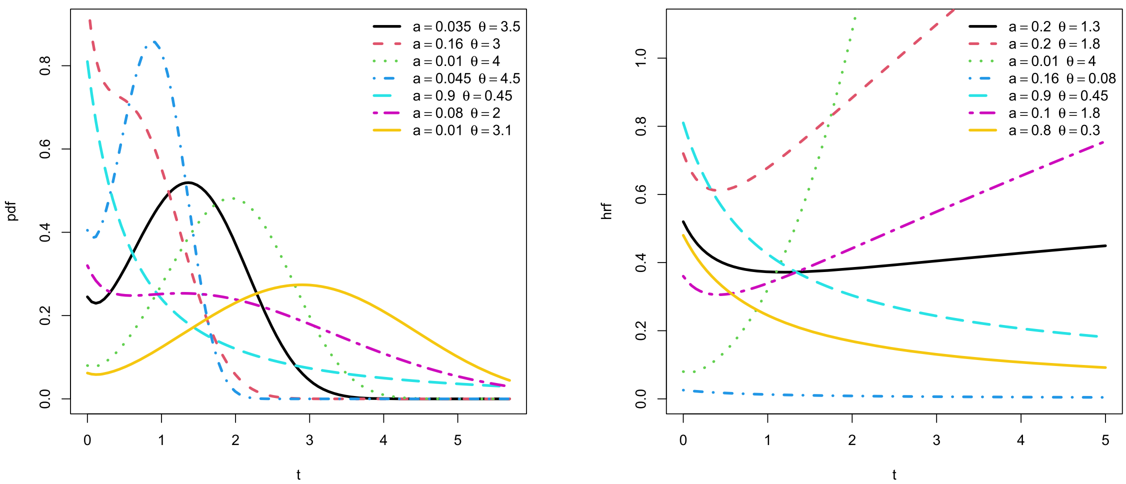

Fig. 5. Plots of the pdf and hrf of the NExL distribution with different parameter values.

$$k(t;a,\alpha,\beta,\gamma)=\frac{a\alpha\beta\left\{1+[1+(t/\gamma)^{\alpha}]^{-2\beta}\right\}}{\gamma\,(t/\gamma)^{1-\alpha}\left[1+(t/\gamma)^{\alpha}\right]^{1-\beta}}\exp\left(-a\left\{\frac{1-[1+(t/\gamma)^{\alpha}]^{-2\beta}}{[1+(t/\gamma)^{\alpha}]^{-\beta}}\right\}\right),t>0,\alpha,\beta,\gamma,a>0.$$

Fig. 4 displays some plots of the NExBXII pdf and hrf for several parameter values.

3.5. *NEx-Lomax distribution*

Consider the Lomax distribution with pdf $h(t;\theta)=\theta(1+t)^{-\theta-1}$, $t>0$, $\theta>0$. The NEx-Lomax (NExL) pdf reduces to

$$k(t;a,\theta)=a\,\theta(1+t)^{\theta-1}\left[1+(1+t)^{-2\theta}\right]\exp\left\{-a\left[\frac{1-(1+t)^{-2\theta}}{(1+t)^{-\theta}}\right]\right\},\qquad t>0,\ a,\theta>0.$$

The graphs of the density and hazard rate functions of the NExL distribution are displayed in Fig. 5.

4. Key properties of the NEx-H family

In this section, we discuss some key mathematical features of the family including mixture representation, moments, stress-strength analysis, entropy, order statistics, residual and reversed residual lives.

4.1. Linear representation

A mixture representation of the cdf (2.1) can be derived using the following series expansions:
$(1-z)^b = \sum_{i=0}^{\infty} \binom{b}{i}(-1)^i z^i$, $(1+z)^b = \sum_{i=0}^{\infty} \binom{b}{i} z^i$ and $\exp(-ax) = \sum_{i=0}^{\infty} \frac{(-a)^i x^i}{i!}$.
Hence, the cdf (2.1) can be express as

$$K(t;a,\boldsymbol{\varpi}) = 1 - \sum_{i=0}^{\infty} \frac{(-a)^i}{i!}\left[\frac{H(t;\boldsymbol{\varpi})(2-H(t;\boldsymbol{\varpi}))}{1-H(t;\boldsymbol{\varpi})}\right]^i = 1 - \sum_{i=0}^{\infty} \frac{(-a)^i H(t;\boldsymbol{\varpi})^i}{i!\,[1-H(t;\boldsymbol{\varpi})]^i}\,[1+(1-H(t;\boldsymbol{\varpi}))]^i.$$

The last term can be expressed as $[1+(1-H(t;\boldsymbol{\varpi}))]^i = \sum_{j=0}^{\infty}\binom{i}{j}(1-H(t;\boldsymbol{\varpi}))^j$. Hence, the above equation reduces to

$$\begin{aligned} K(t;\boldsymbol{\varpi}) &= 1 - \sum_{i,j=0}^{\infty} \frac{(-a)^i H(t;\boldsymbol{\varpi})^i}{i!}\binom{i}{j}(1-H(t;\boldsymbol{\varpi}))^{j-i} \\ &= 1 - \sum_{i,j,k=0}^{\infty} \frac{(-a)^i(-1)^k}{i!}\binom{i}{j}\binom{j-i}{k} H(t;\boldsymbol{\varpi})^{i+k} \\ &= 1 + \sum_{i,j,k=0}^{\infty} \frac{(-a)^i(-1)^{k+1}}{i!}\binom{i}{j}\binom{j-i}{k} H(t;\boldsymbol{\varpi})^{i+k}. \end{aligned}$$

Hence, the cdf of the NEx-H family reduces to

$$K(t;\boldsymbol{\varpi}) = 1 + \sum_{i,k=0}^{\infty} \varphi_{i,k} H(t;\boldsymbol{\varpi})^{i+k}, \tag{4.1}$$

where $\varphi_{i,k} = \sum_{j=0}^{\infty} \frac{(-a)^i(-1)^{k+1}}{i!}\binom{i}{j}\binom{j-i}{k}$.

Taking the derivative of Equation (4.1), the pdf of the NEx-H family takes the form

$$k(t;\boldsymbol{\varpi}) = \sum_{i,k=0}^{\infty} \varphi_{i,k}\pi_{i+k}(t;\boldsymbol{\varpi}), \tag{4.2}$$

where $\pi_{i+k}(t;\boldsymbol{\varpi}) = (i+k)\,h(t;\boldsymbol{\varpi})\,H(t;\boldsymbol{\varpi})^{i+k-1}$ is the exponentiated-H (exp-H) density with power parameter $i+k$.

4.2. Moments

Equation (4.2) can be used to obtain the r^{th} moments of the NEx-H family as follows

$$\mu_r' = E(t^r) = \int_{-\infty}^{\infty} t^r\, k(t;a,\boldsymbol{\varpi})dt = \sum_{i,k=0}^{\infty} \varphi_{i,k} \int_0^{\infty} t^r \pi_{i+k}(t;\boldsymbol{\varpi})dt.$$

Therefore

$$\mu_r' = \sum_{i,k=0}^{\infty} \varphi_{i,k}\, Q_{i+k,r}(t;\boldsymbol{\varpi}),$$

where $Q_{i+k,r}(t;\boldsymbol{\varpi}) = \int_0^{\infty} t^r\, \pi_{i+k}(t;\varpi)dt$.

Some numerical values of the NExW moments are provided for different values of its parameters. The first four moments, μ_1', μ_2', μ_3' and μ_4', variance (σ^2), skewness (η_1), kurtosis (η_2) and coefficient of variation (C_V) of the NExW distribution are reported in Table 1. Plots for η_1 and η_2 of the NExW distribution are shown in Fig. 6 for $a=2$. The two measures are increasing functions in α and β.

According to Equation (4.2), the m^{th} incomplete moment of the NEx-H family reduces to

$$\mu^m(t) = \int_{-\infty}^{t} t^m k(t;a,\boldsymbol{\varpi})dt = \sum_{i,k=0}^{\infty} \varphi_{i,k} \int_0^{t} t^m \pi_{i+k}(t;\boldsymbol{\varpi})dt.$$

Hence

$$\mu^m(t) = \sum_{i,k=0}^{\infty} \varphi_{i,k}\, L_{i+k,m}(t;\boldsymbol{\varpi}),$$

Table 1
The numerical values of μ_1', μ_2', μ_3' and μ_4', σ^2, η_1, η_2 and C_V of the NExW distribution.

(a, α, β)	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	η_1	η_2	C_V
(0.3, 0.2, 0.2)	0.01712	0.00986	0.00681	0.00518	0.00957	6.73785	51.66531	5.71621
(0.5, 0.2, 0.5)	0.05605	0.03420	0.02433	0.01885	0.03106	3.45794	14.57896	3.14437
(1, 0.6, 0.8)	0.24715	0.14704	0.10250	0.07821	0.08596	0.93903	5.67902	1.18627
(1, 1.5, 1.2)	0.31773	0.16175	0.09836	0.06707	0.06079	0.55573	19.10565	0.77603
(2, 1.8, 1.5)	0.22452	0.07848	0.03272	0.01562	0.02807	0.53025	22.33164	0.74621
(0.5, 2, 1)	0.33896	0.18758	0.12175	0.08720	0.07269	0.45343	17.22896	0.79542
(1, 2, 2)	0.42045	0.21934	0.12908	0.08271	0.04256	0.12170	106.0185	0.49065
(2, 2, 1.5)	0.20737	0.06801	0.02648	0.01179	0.02501	0.50885	20.65634	0.76263
(3, 5, 1)	0.00559	0.00074	0.00010	1.50878	0.00071	4.80220	25.61941	4.76554
(1.5, 3, 2)	0.28257	0.10296	0.04256	0.01935	0.02312	0.11561	74.27237	0.53808
(4, 1, 3)	0.44468	0.22396	0.12316	0.07256	0.02621	0.05935	344.1554	0.36410
(0.7, 3, 0.5)	0.04333	0.01072	0.00327	0.00121	0.00884	2.45743	9.85392	2.17064
(0.5, 4, 3)	0.52821	0.30621	0.18865	0.12145	0.02721	-0.41033	633.4038	0.31229
(2, 0.6, 4)	0.64032	0.47397	0.36736	0.29488	0.06395	-1.11505	250.4555	0.39493
(0.8, 2, 0.3)	0.04034	0.01570	0.00821	0.00520	0.01408	3.85576	20.38106	2.94115
(4, 5, 3)	0.25752	0.07640	0.02463	0.00849	0.01008	-0.23791	262.5072	0.39004
(5, 3, 6)	0.52593	0.28691	0.16132	0.09306	0.01030	-0.39447	4328.402	0.19298
(3, 0.4, 2)	0.46759	0.29116	0.20241	0.15142	0.07252	-0.07971	56.70898	0.57590
(3, 2, 0.8)	0.02372	0.00448	0.00099	0.00026	0.00391	2.87439	12.04306	2.63770
(5, 5, 4)	0.34008	0.12492	0.04860	0.01981	0.00927	-0.21494	936.6195	0.28314
(0.3, 3, 2)	0.54961	0.35099	0.24284	0.17731	0.04892	-0.35444	231.2084	0.40242
(2, 4, 3)	0.35012	0.13872	0.05966	0.02735	0.01614	-0.10286	348.8546	0.36288
(1, 1, 1)	0.31125	0.17629	0.11807	0.08735	0.07941	0.61494	11.25464	0.90541
(2, 1, 1)	0.21286	0.09317	0.05131	0.03292	0.04786	1.06021	9.04297	1.02781
(1, 2, 1)	0.20402	0.08171	0.04034	0.02309	0.04008	0.91139	9.81712	0.98136
(1, 1, 2)	0.49119	0.31993	0.23023	0.17687	0.07865	-0.18994	58.55786	0.57097

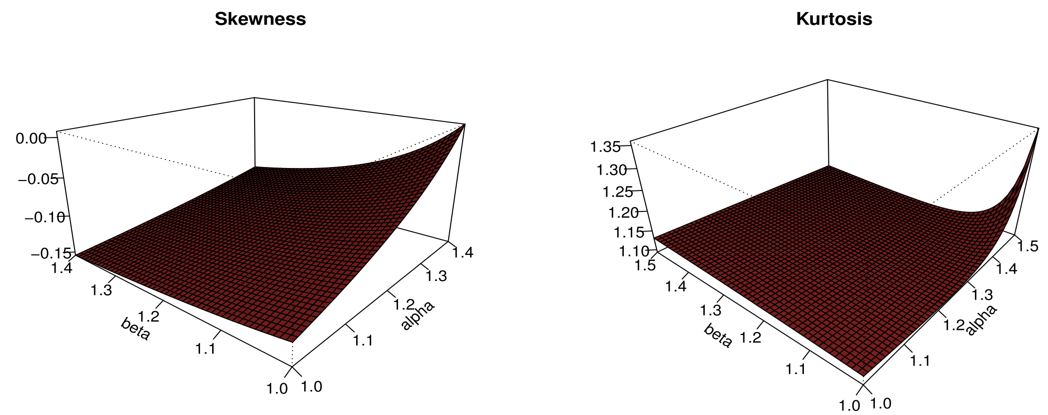

Fig. 6. Plots of η_1 and η_2 for the NExW distribution.

where $L_{i+k,m}(t;\boldsymbol{\varpi}) = \int_0^t t^m \,\pi_{i+k}(t;\boldsymbol{\varpi})dt$.

The negative moments of the NEx-H family are defined as

$$\mu'_{-r} = E(t^{-r}) = \int_{-\infty}^{\infty} t^{-r} k(t;a,\varpi)dt = \sum_{i,k=0}^{\infty} \varphi_{i,k} \int_0^{\infty} t^{-r}\,\pi_{i+k}(t;\boldsymbol{\varpi})dt.$$

Then, we can write

$$\mu'_{-r} = \sum_{i,k=0}^{\infty} \varphi_{i,k}\, R(t;\boldsymbol{\varpi}),$$

where $R(t;\boldsymbol{\varpi}) = \int_0^{\infty} t^{-r}\,\pi_{i+k}(t;\varpi)dt$.

The moment generating function (mgf) of the NEx-H family can be expressed, from (4.2), as

$$M_t(y) = E(e^{ty}) = \int_0^\infty e^{ty} k(t; a, \boldsymbol{\varpi}) dt = \sum_{i,k=0}^\infty \varphi_{i,k} \int_0^\infty e^{ty} \pi_{i+k}(t; \boldsymbol{\varpi}) dt$$

$$= \sum_{i,k=0}^\infty \varphi_{i,k} \sum_{\ell=0}^\infty \frac{y^\ell}{\ell!} \int_0^\infty t^\ell \, \pi_{i+k}(t; \boldsymbol{\varpi}) dt = \sum_{i,k=\ell=0}^\infty \varphi_{i,k} \frac{y^\ell}{\ell!} J(t; \boldsymbol{\varpi}),$$

where $J(t; \boldsymbol{\varpi}) = \int_0^\infty t^\ell \, \pi_{i+k}(t; \boldsymbol{\varpi}) dt$.

The characteristics function, say $\phi_t(y)$, of the NEx-H family is defined by

$$\phi_t(y) = E(e^{\imath t y}) = \int_0^\infty e^{\imath t y} k(t; a, \boldsymbol{\varpi}) dt.$$

Using the exponential series $\exp(a\,y) = \sum_{j=0}^\infty \frac{a^j}{j!} y^j$ and Equation (4.2), we have

$$\phi_t(y) = \sum_{i,k=0}^\infty \varphi_{i,k} \sum_{n=0}^\infty \frac{(\imath\, y)^n}{n!} \int_0^\infty t^n \pi_{i+k}(t; \boldsymbol{\varpi}) dt = \sum_{i,k,n=0}^\infty \varphi_{i,k} \frac{(\imath\, y)^n}{n!} D(t; \boldsymbol{\varpi}),$$

where $D(t; \boldsymbol{\varpi}) = \int_0^\infty t^n \, \pi_{i+k}(t; \boldsymbol{\varpi}) dt$.

4.3. Rényi entropy

The entropy measure represents the variability of the uncertainty. The Rényi entropy is defined as

$$I_R = \frac{1}{1-r} \log \left[I(\delta) \right], \qquad r > 0, r \neq 1,$$

where $I(\delta) = \int_0^\infty k^r(t; \boldsymbol{\varpi}) dt$. Based on Equation (4.2), we can write

$$I(\delta) = \left(\sum_{i,k=0}^\infty \varphi_{i,k} \right)^r \int_0^\infty \pi_{i+k}(t; \boldsymbol{\varpi})^r dt.$$

Using the last two equations, the Rényi entropy of the NEx-H family follows as

$$I_R = \frac{1}{1-r} \log \left[\left(\sum_{i=k=0}^\infty \varphi_{i,k} \right)^r \int_0^\infty \pi_{i+k}(t; \boldsymbol{\varpi})^r dt \right].$$

Note that Shannon entropy follows from the Rényi entropy if $r \to 1$.

4.4. Stress-strength model

The stress-strength model, is also known as reliability (R), defines the life of a part that contains a random strength, say T_1, that is subjected to an accidental stress, say T_2. The element fails at the moment that the stress applied to it exceeds the strength, and therefore the component can function suitably for $T_1 > T_2$. Hence, $R = P(T_2 < T_1)$ could be a life of component reliability. It has several applications particularly within the area of engineering. Now, we derive the reliability R when T_1 and T_2 have independent NEx-H$(a_1, \boldsymbol{\varpi})$ and NEx-H$(a_2, \boldsymbol{\varpi})$ distributions. From Equations (4.1) and (4.2) the reliability R reduces to

$$R = P(T_1 < T_2) = \int_0^\infty k_1(t; a_1, \boldsymbol{\varpi}) \, K_2(t; a_2, \boldsymbol{\varpi}) dt.$$

Hence

$$R = P(T_1 < T_2) = \sum_{i,k=0}^\infty \varphi_{i,k} \int_0^\infty \pi_{i+k}(t; \boldsymbol{\varpi}) dt + \left(\sum_{i,k=0}^\infty \varphi_{i,k} \right)^2 \int_0^\infty \pi_{i+k}(t; \boldsymbol{\varpi}) H(t; \boldsymbol{\varpi})^{i+k} dt,$$

where $H(t;\boldsymbol{\varpi})$ is the baseline cdf and $\pi_{i+k}(t;\boldsymbol{\varpi})$ is the Exp-G pdf with power parameter $i+k$.

4.5. Order statistics

Let $T_1, T_2, \ldots, T_n$ be a random sample from the NEx-H family with respective cdf (4.1) and pdf (4.2). Then, the density of i^{th} order statistic is defined by

$$k_{i:n}(t;a,\varpi) = \frac{n!}{(i-1)!(n-i)!} k(t;a,\boldsymbol{\varpi}) K(t;a,\boldsymbol{\varpi})^{i-1} \bar{K}(t;a,\boldsymbol{\varpi})^{n-i},$$

where

$$K(t;a,\boldsymbol{\varpi})^{i-1} = \left[1 - \sum_{i,k=0}^{\infty} \varphi_{i,k} H(t;\boldsymbol{\varpi})^{i+k}\right]^{i-1} = \sum_{\ell=0}^{\infty} \binom{i-1}{\ell} (-1)^{\ell} \left[\sum_{i,k=0}^{\infty} \varphi_{i,k} H(t;\boldsymbol{\varpi})^{i+k}\right]^{\ell}$$

and

$$\bar{K}(t;a,\boldsymbol{\varpi})^{n-i} = [1 - K(t;a,\boldsymbol{\varpi})]^{n-i} = \left[\sum_{i,k,\ell=0}^{\infty} \varphi_{i,k} H(t;\boldsymbol{\varpi})^{i+k}\right]^{n-i}.$$

Hence, we have

$$k(t;a,\boldsymbol{\varpi}) K(t;a,\boldsymbol{\varpi})^{i-1} \bar{K}(t;a,\boldsymbol{\varpi})^{n-i} = \left[\sum_{i,k=0}^{\infty} \varphi_{i,k}\right]^{\ell+n-i+1} \sum_{\ell=0}^{\infty} \binom{i-1}{\ell} (-1)^{\ell} \pi_{i+k}(t;\boldsymbol{\varpi}) \left[H(t;\boldsymbol{\varpi})^{i+k}\right]^{(\ell+n-i)}.$$

Then, the pdf of the i^{th} order statistic for the NEx-H family takes the form

$$k_{i:n}(t;a,\boldsymbol{\varpi}) = \frac{n!}{(i-1)!(n-i)!} \left[\sum_{i,k=0}^{\infty} \varphi_{i,k}\right]^{\ell+n-i+1} \sum_{\ell=0}^{\infty} \binom{i-1}{\ell} (-1)^{\ell} \pi_{i+k}(t;\boldsymbol{\varpi}) \left[H(t;\boldsymbol{\varpi})^{i+k}\right]^{(\ell+n-i)},$$

where $H(t;\boldsymbol{\varpi})$ and $\pi_{i+k}(t;\boldsymbol{\varpi})$ are defined in the above section.

4.6. Residual and reversed-residual lives

The n^{th} moment of the residual life is determined by

$$m_n(v) = \frac{1}{S(v)} \int_{v}^{\infty} (t-v)^n \, k(t;a,\boldsymbol{\varpi})\, dt.$$

Using Equation (4.2), we obtain $m_n(v)$ for the NEx-H family as

$$m_n(v) = \frac{1}{S(v)} \sum_{i,k=0}^{\infty} \varphi_{i,k} \int_{v}^{\infty} (t-v)^n \pi_{i+k}(t;\boldsymbol{\varpi})\, dt.$$

By using the binomial expansion given by $(a-b)^n = \sum_{r=0}^{n} \binom{n}{r} b^{n-r} a^r (-1)^{n-r}$, we have

$$m_n(v) = \frac{1}{S(v)} \sum_{i=k=0}^{\infty} \sum_{\ell=0}^{n} \varphi_{i,k} \binom{n}{\ell} (v)^{n-\ell} (-1)^{n-\ell} \int_{v}^{\infty} t^{\ell} \, \pi_{i+k}(t;\varpi)\, dt.$$

The n^{th} moment of reversed residual life of T is defined by

$$M_n(v) = \frac{1}{K(v)} \int_{0}^{v} (v-t)^n \, k(t;a,\boldsymbol{\varpi})\, dt.$$

Hence, $M_n(v)$ of the NEx-H family follows as

$$M_n(v) = \frac{1}{K(v)} \sum_{i=k=0}^{\infty} \sum_{\ell=0}^{n} \varphi_{i,k} \binom{n}{\ell} v^{\ell} (-1)^{n-\ell} \int_{v}^{\infty} t^{n-\ell} \pi_{i+k}(t;\boldsymbol{\varpi})\, dt.$$

The mean inactivity time, $M_1(v)$, (also known as mean waiting time) of the NEx-H family follows by setting $n=1$ in above equation.

5. Estimation methods for the NExW distribution

This section describes the estimation of the NExW parameters using different methods of estimation including the maximum-likelihood (ML), least-squares (LS), Cramér–von Mises (CVM), weighted-least squares (WLS), maximum-product of spacing (MPS), Anderson–Darling (AD), and right-tail Anderson–Darling (RTAD). Consider a random sample from the NExW distribution denoted by $t_1, ...t_n$ and its associated order statistics denoted by $t_{(1)} < t_{(2)} < .... < t_{(n)}$.

The log-likelihood function of the NExW model takes the form

$$\ell = n\ln a + n\ln\alpha + n\ln\beta + (\beta-1)\sum_{i=1}^{n}\ln t_i + \sum_{i=1}^{n}\ln\left[1-\exp\left(-2\alpha t_i^{\beta}\right)\right] + \sum_{i=1}^{n}\alpha t_i^{\beta} - a\sum_{i=1}^{n}\left[\frac{1-\exp\left(-2\alpha t_i^{\beta}\right)}{\exp\left(-\alpha t_i^{\beta}\right)}\right]. \tag{5.1}$$

The ML estimates (MLEs) of a, α and β can be determined by either solving the following nonlinear equations or by maximizing (5.1) with regard to a, α and β.

$$\frac{\partial\ell}{\partial a} = \frac{n}{a} - \sum_{i=1}^{n}\left[\frac{1-\exp\left(-2\alpha t_i^{\beta}\right)}{\exp\left(-\alpha t_i^{\beta}\right)}\right] = 0,$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} t_i^{\beta} + \sum_{i=1}^{n}\frac{2t_i^{\beta}}{\exp\left(2\alpha t_i^{\beta}\right)-1} - \sum_{i=1}^{n} a t_i^{\beta}\exp\left(-\alpha t_i^{\beta}\right)\left[1+\exp\left(2\alpha t_i^{\beta}\right)\right] = 0$$

and

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln(t_i) + \sum_{i=1}^{n}\alpha t_i^{\beta}\ln(t_i) + \sum_{i=1}^{n}\frac{2\alpha t_i^{\beta}\ln(t_i)}{\exp\left(2\alpha t_i^{\beta}\right)-1} - \sum_{i=1}^{n} a\alpha t_i^{\beta}\ln(t_i)\exp\left(-\alpha t_i^{\beta}\right)\left[1+\exp\left(2\alpha t_i^{\beta}\right)\right] = 0.$$

The likelihood function can be maximized using the R (optimal function), Ox program (sun procedure MaxBFGS), or Newton Rapshon method to get the MLEs.

The LS estimates (LSEs) of NExW parameters follow by minimizing

$$T(a,\alpha,\beta) = \sum_{i=1}^{n}\left\{K(t_{(i)};a,\alpha,\beta) - \frac{i}{n+1}\right\}^2 = \sum_{i=1}^{n}\left[1-\exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right) - \frac{i}{n+1}\right]^2,$$

where $\delta(t_{(i)};\alpha,\beta) = \exp\left(\alpha t_{(i)}^{\beta}\right)\left[1-\exp\left(-2\alpha t_{(i)}^{\beta}\right)\right]$.

Also, the LSEs are obtained by solving the following equations simultaneously

$$\frac{\partial T(a,\alpha,\beta)}{\partial a} = \sum_{i=1}^{n} 2\left\{1-\exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right) - \frac{i}{n+1}\right\} T_1'(t_{(i)};a,\alpha,\beta) = 0,$$

$$\frac{\partial T(a,\alpha,\beta)}{\partial\alpha} = \sum_{i=1}^{n} 2\left\{1-\exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right) - \frac{i}{n+1}\right\} T_2'(t_{(i)};a,\alpha,\beta) = 0$$

and

$$\frac{\partial T(a,\alpha,\beta)}{\partial\beta} = \sum_{i=1}^{n} 2\left\{1-\exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right) - \frac{i}{n+1}\right\} T_3'(t_{(i)};a,\alpha,\beta) = 0,$$

where

$$T_1'(t_{(i)};a,\alpha,\beta) = \delta(t_{(i)};\alpha,\beta)\exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right), \tag{5.2}$$

$$T_2'(t_{(i)};a,\alpha,\beta) = a t_{(i)}^{\beta}\exp\left(\alpha t_{(i)}^{\beta}\right)\left[1+\exp\left(-2\alpha t_{(i)}^{\beta}\right)\right]\exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right) \tag{5.3}$$

and

$$T_3'(t_{(i)};a,\alpha,\beta) = a\alpha t_{(i)}^{\beta}\ln(t_{(i)})\exp\left(\alpha t_{(i)}^{\beta}\right)\left[1+\exp\left(-2\alpha t_{(i)}^{\beta}\right)\right]\exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right). \tag{5.4}$$

The weighted-LSEs (WLSEs) of the NExW parameters are found by minimizing

$$\begin{aligned}TS(a,\alpha,\beta) &= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left\{ K(t_{(i)};a,\alpha,\beta) - \frac{i}{n+1} \right\}^2 \\ &= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left\{ 1 - \exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right) - \frac{i}{n+1} \right\}^2.\end{aligned}$$

The WLSEs are also determined by solving the equation

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left\{ 1 - \exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right) - \frac{i}{n+1} \right\} T_j'(t_{(i)};a,\alpha,\beta) = 0,\ j = 1,2,3,$$

where $T_1'(t_{(i)};a,\alpha,\beta)$, $T_2'(t_{(i)};a,\alpha,\beta)$ and $T_3'(t_{(i)};a,\alpha,\beta)$ are defined in (5.2)–(5.4).

The CVM estimates (CVMEs) of the NExW parameters follow by minimizing the following function

$$CV(a,\alpha,\beta) = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ 1 - \exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right) - \frac{2i-1}{2n} \right\}^2,$$

where $\delta(t_i;\alpha,\beta) = \exp\left(\alpha t_{(i)}^{\beta}\right)\left[1 - \exp\left(-2\alpha t_{(i)}^{\beta}\right)\right]$.

Additionally, the CVMEs can be determined by evaluating the following nonlinear questions

$$\sum_{i=1}^{n} \left\{ 1 - \exp\left(-a\,\delta(t_(i);\alpha,\beta)\right) - \frac{2i-1}{2n} \right\} T_j'(t_{(i)};a,\alpha,\beta) = 0,\ j = 1,2,3,$$

where $T_1'(t_{(i)};a,\alpha,\beta)$, $T_2'(t_{(i)};a,\alpha,\beta)$ and $T_3'(t_{(i)};a,\alpha,\beta)$ are defined in (5.2)–(5.4).

The uniform spacings of a random sample of size n from the NExW distribution are defined by

$$\begin{aligned}D_i(a,\alpha,\beta) &= K(t_{(i)};a,\alpha,\beta) - K(t_{(i-1)};a,\alpha,\beta) \\ &= \exp\left(-a\,\delta(t_{(i-1)};\alpha,\beta)\right) - \exp\left(-a\,\delta(t_{(i)};\alpha,\beta)\right),\end{aligned}$$

where $\delta(t_{(i)};\alpha,\beta) = \exp\left(\alpha t_{(i)}^{\beta}\right)\left[1 - \exp\left(-2\alpha t_{(i)}^{\beta}\right)\right]$ and $K(t_{(0)};a,\alpha,\beta) = 0$, $K(t_{(n+1)};a,\alpha,\beta) = 1$ and $\sum_{i=0}^{n+1} D_i(a,\alpha,\beta) = 1$.

Hence, the MPS estimates (MPSEs) of the NExW parameters are obtained by maximizing the function

$$H(a,\alpha,\beta) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log D_i(a,\alpha,\beta).$$

Additionally, the MPSEs of the NExW parameters are also be calculated by solving

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(a,\alpha,\beta)} \left[T_j'(t_{(i)};a,\alpha,\beta) - T_j'(t_{(i-1)};a,\alpha,\beta) \right] = 0, \qquad j = 1,2,3,$$

where $T_1'(t_{(i)};a,\alpha,\beta)$, $T_2'(t_{(i)};a,\alpha,\beta)$ and $T_3'(t_{(i)};a,\alpha,\beta)$ are defined in (5.2)–(5.4).

Another kind of minimal distance estimators is the AD estimates (ADEs). We obtain the ADEs of the NExW parameters by minimizing the following function

$$A(a,\alpha,\beta) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left[\log\{K(t_{(i)};a,\alpha,\beta)\} - \log\{S(t_{(n+1-i)};a,\alpha,\beta)\} \right],$$

with respect to a, α, and β.

The ADEs are also determined by solving the following nonlinear equations

$$\sum_{i=1}^{n} (2i-1) \left[\frac{T_j'(t_{(i)};a,\alpha,\beta))}{K(t_{(i)};a,\alpha,\beta)} - \frac{T_j'(t_{(n+1-i)};a,\alpha,\beta)}{S(t_{(n+1-i)};a,\alpha,\beta)} \right] = 0,\ j = 1,2,3,$$

$T_1'(t_{(i)};a,\alpha,\beta)$, $T_2'(t_{(i)};a,\alpha,\beta)$ and $T_3'(t_{(i)};a,\alpha,\beta)$ are defined in (5.2)–(5.4).

Table 2
The MEs and MSEs (in parentheses) of the NExW parameters for $n = 20$.

Parameters	MLEs	LSEs	LSEs	CVMEs	MPSEs	ADEs	RTADEs
$a = 0.5$	0.748(0.161)	0.810(0.275)	0.823(0.280)	0.811(0.276)	0.820(0.273)	0.847(0.278)	0.796(0.269)
$\alpha = 2$	3.028(0.286)	4.049(0.417)	4.167(0.432)	3.937(0.409)	3.846(0.396)	3.485(0.426)	3.157(0.388)
$\beta = 1.8$	2.831(0.184)	3.387(0.257)	3.576(0.289)	3.145(0.286)	3.032(0.282)	3.108(0.278)	2.956(0.201)
$a = 1$	2.051(0.176)	2.725(0.238)	2.801(0.227)	2.719(0.214)	2.812(0.241)	2.776(0.247)	2.710(0.185)
$\alpha = 0.3$	0.917(0.184)	1.413(0.262)	1.454(0.247)	1.503(0.261)	1.821(0.227)	1.515(0.243)	1.071(0.196)
$\beta = 3$	4.137(0.189)	4.864(0.315)	4.747(0.357)	4.486(0.374)	4.571(0.363)	4.554(0.346)	4.251(0.277)
$a = 2$	3.135(0.175)	3.454(0.267)	3.536(0.273)	3.493(0.276)	3.575(0.270)	3.486(0.260)	3.244(0.201)
$\alpha = 2$	3.182(0.231)	3.470(0.318)	3.577(0.362)	3.647(0.343)	3.545(0.306)	3.486(0.313)	3.274(0.296)
$\beta = 0.3$	0.984(0.164)	1.036(0.251)	1.143(0.235)	1.084(0.215)	1.316(0.256)	1.415(0.224)	1.218(0.201)
$a = 1$	2.034(0.142)	2.672(0.247)	2.686(0.238)	2.726(0.253)	2.628(0.247)	2.558(0.216)	2.227(0.168)
$\alpha = 1.5$	2.472(0.294)	2.982(0.378)	3.156(0.383)	3.213(0.352)	3.242(0.337)	3.219(0.352)	2.975(0.310)
$\beta = 1.8$	3.029(0.252)	3.647(0.339)	3.532(0.345)	3.264(0.325)	3.450(0.338)	3.218(0.314)	3.126(0.278)
$a = 1$	2.135(0.184)	2.642(0.237)	2.447(0.252)	2.500(0.231)	2.418(0.253)	2.387(0.247)	2.297(0.218)
$\alpha = 0.3$	1.136(0.184)	1.564(0.238)	1.432(0.207)	1.342(0.219)	1.288(0.276)	1.311(0.253)	1.254(0.203)
$\beta = 3$	4.208(0.259)	4.878(0.374)	4.753(0.383)	5.012(0.358)	4.675(0.337)	4.562(0.380)	4.456(0.293)
$a = 2$	3.057(0.252)	3.653(0.381)	3.684(0.376)	3.774(0.372)	4.164(0.354)	3.583(0.326)	3.136(0.286)
$\alpha = 2$	3.162(0.270)	3.742(0.354)	3.761(0.372)	3.628(0.369)	3.580(0.317)	3.432(0.334)	3.231(0.295)
$\beta = 0.3$	1.234(0.169)	1.542(0.260)	1.478(0.235)	1.506(0.252)	1.481(0.274)	1.517(0.262)	1.384(0.188)
$a = 0.5$	1.135(0.179)	1.564(0.285)	1.484(0.276)	1.472(0.259)	1.668(0.209)	1.474(0.242)	1.273(0.201)
$\alpha = 1.5$	2.263(0.214)	2.547(0.314)	2.637(0.319)	2.571(0.336)	2.479(0.361)	2.576(0.323)	2.368(0.266)
$\beta = 3$	4.136(0.283)	4.375(0.364)	4.428(0.354)	4.472(0.329)	4.581(0.338)	4.495(0.367)	4.293(0.301)
$a = 1$	2.015(0.185)	2.248(0.237)	2.346(0.226)	2.472(0.246)	2.417(0.260)	2.503(0.239)	2.184(0.208)
$\alpha = 0.3$	0.962(0.157)	1.261(0.183)	1.362(0.402)	1.348(0.323)	1.326(0.290)	1.361(0.276)	1.027(0.214)
$\beta = 1.8$	2.863(0.273)	3.138(0.347)	3.161(0.369)	3.182(0.373)	3.319(0.362)	3.159(0.338)	2.910(0.286)

By minimizing the following function with respect to a, α, and β

$$R(a,\alpha,\beta) = \frac{n}{2} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \bar{K}(t_{(n+1-i)};a,\alpha,\beta) - 2\sum_{i=1}^{n} K(t_{(i)};a,\alpha,\beta),$$

we obtain the right-tail ADEs (RTADEs) of the NExW parameters.

6. Simulation study

In this section, a simulation study is conducted to evaluate the effectiveness of seven different estimators of the NExW parameters. Various parameter values of $a = (0.5, 2, 1)$, $\alpha = (2, 0.3, 1.5)$, and $\beta = (0.3, 3, 1.8)$, and different sample sizes $n = (20, 50, 100, 200, 350, 500, 600, 750)$. We calculate the mean estimates (MEs) and associated mean-square errors (MSEs) using the seven estimation approaches for each combination of the parameters. The simulation results are carried out by using the R software, and the number of Monte Carlo replications is $N = 2000$. The samples are generated from the NExW model using its qf which is defined by

$$T_u = \left(\frac{-1}{\alpha}\log\left\{1 - \left[\frac{-(\log(1-u) - 2a) \pm \sqrt{\log^2(1-u) + 4a^2}}{2a}\right]\right\}\right)^{1/\beta}, u \in (0,1).$$

The MLEs, LSEs, MPSEs, WLSEs, ADEs, CVMEs, and RTADEs are evaluated based on the MEs and MSEs. Tables 2–9 list the values of MEs and MSEs for the seven methods. It is noted that the behavior of different estimates of the NExW parameters is entirely good, illustrating small MSEs in all studied cases. Hence, the calculated estimates are quite reliable and very close to the actual values of the parameters.

Moreover, as the sample size increases, the MSEs decrease, indicating that these estimates are consistent for the NExW parameters. Furthermore, by comparing different estimation methods, the performance ordering of the estimates based on the MSEs (from best to worst) for all NExW parameters is MLEs, LSEs, MPSEs, RTADEs, CMEs, WLSEs, and ADEs in most of the studied cases.

7. Data analysis

In this section, the importance and applicability of the NExW distribution are addressed using three real-life data sets. We compare the fit of the NExW distribution with its competing Weibull extensions including the alpha power-Weibull (APW) [15],

Table 3
The MEs and MSEs (in parentheses) of the NExW parameters for $n = 50$.

Parameters	MLEs	LSEs	LSEs	CVMEs	MPSEs	ADEs	RTADEs
$a = 0.5$	0.720(0.132)	0.735(0.243)	0.716(0.253)	0.752(0.253)	0.701(0.243)	0.677(0.254)	0.733(0.252)
$\alpha = 2$	2.816(0.254)	3.479(0.325)	3.667(0.326)	3.463(0.316)	3.628(0.347)	3.231(0.352)	3.016(0.337)
$\beta = 1.8$	2.542(0.162)	3.154(0.224)	3.241(0.253)	3.021(0.264)	2.901(0.237)	2.872(0.245)	2.613(0.186)
$a = 1$	1.857(0.153)	2.431(0.214)	2.474(0.215)	2.403(0.201)	2.526(0.230)	2.462(0.215)	2.502(0.162)
$\alpha = 0.3$	0.702(0.152)	1.106(0.241)	1.231(0.232)	1.314(0.230)	1.416(0.214)	1.312(0.231)	0.862(0.173)
$\beta = 3$	4.012(0.157)	4.427(0.301)	4.231(0.332)	4.142(0.351)	4.240(0.324)	4.226(0.321)	4.102(0.251)
$a = 2$	2.896(0.142)	3.213(0.241)	3.302(0.231)	3.261(0.242)	3.242(0.237)	3.261(0.242)	3.102(0.196)
$\alpha = 2$	3.013(0.215)	3.152(0.303)	3.216(0.340)	3.214(0.321)	3.314(0.287)	3.152(0.285)	3.122(0.262)
$\beta = 0.3$	0.756(0.143)	0.813(0.230)	1.101(0.212)	0.974(0.202)	1.113(0.233)	1.201(0.212)	1.105(0.186)
$a = 1$	1.894(0.123)	2.321(0.223)	2.346(0.214)	2.314(0.231)	2.215(0.214)	2.327(0.202)	2.011(0.124)
$\alpha = 1.5$	2.241(0.252)	2.537(0.345)	3.112(0.351)	3.103(0.322)	3.121(0.313)	3.204(0.321)	2.426(0.268)
$\beta = 1.8$	2.867(0.223)	3.214(0.315)	3.311(0.324)	3.118(0.301)	3.126(0.314)	3.104(0.285)	2.914(0.235)
$a = 1$	2.101(0.157)	2.328(0.221)	2.315(0.231)	2.263(0.210)	2.233(0.241)	2.163(0.225)	2.042(0.201)
$\alpha = 0.3$	1.012(0.151)	1.337(0.214)	1.271(0.181)	1.310(0.203)	1.224(0.243)	1.203(0.224)	1.212(0.176)
$\beta = 3$	4.116(0.232)	4.352(0.337)	4.204(0.336)	4.547(0.316)	4.218(0.314)	4.115(0.323)	4.127(0.252)
$a = 2$	3.011(0.213)	3.214(0.336)	3.432(0.325)	3.245(0.341)	4.013(0.220)	3.237(0.218)	3.026(0.253)
$\alpha = 2$	2.642(0.241)	3.318(0.312)	3.237(0.345)	3.214(0.332)	3.178(0.303)	3.212(0.311)	3.102(0.253)
$\beta = 0.3$	1.101(0.145)	1.254(0.223)	1.239(0.212)	1.214(0.231)	1.236(0.225)	1.205(0.234)	1.163(0.162)
$a = 0.5$	1.007(0.154)	1.241(0.263)	1.251(0.253)	1.246(0.236)	1.417(0.201)	1.144(0.215)	1.084(0.172)
$\alpha = 1.5$	2.132(0.202)	2.315(0.302)	2.341(0.313)	2.316(0.314)	2.176(0.320)	2.341(0.311)	2.147(0.224)
$\beta = 3$	4.110(0.241)	4.144(0.332)	4.214(0.317)	4.153(0.315)	4.216(0.314)	4.156(0.315)	4.037(0.285)
$a = 1$	1.876(0.143)	2.123(0.214)	2.123(0.213)	2.231(0.223)	2.142(0.214)	2.214(0.216)	2.041(0.202)
$\alpha = 0.3$	0.911(0.133)	1.120(0.151)	1.231(0.361)	1.314(0.311)	1.224(0.241)	1.240(0.242)	0.834(0.201)
$\beta = 1.8$	2.316(0.241)	3.014(0.315)	3.012(0.335)	3.031(0.324)	3.134(0.321)	3.112(0.314)	2.401(0.244)

Table 4
The MEs and MSEs (in parentheses) of the NExW parameters for $n = 100$.

Parameters	MLEs	LSEs	LSEs	CVMEs	MPSEs	ADEs	RTADEs
$a = 0.5$	0.675(0.121)	0.713(0.231)	0.711(0.241)	0.727(0.238)	0.689(0.231)	0.645(0.242)	0.712(0.232)
$\alpha = 2$	2.603(0.242)	3.463(0.313)	3.425(0.312)	3.431(0.304)	3.514(0.325)	3.122(0.331)	2.703(0.314)
$\beta = 1.8$	2.481(0.141)	3.112(0.212)	3.140(0.232)	2.764(0.247)	2.742(0.224)	2.712(0.233)	2.501(0.167)
$a = 1$	1.597(0.137)	2.252(0.204)	2.149(0.202)	2.216(0.175)	2.264(0.218)	2.251(0.204)	2.463(0.141)
$\alpha = 0.3$	0.595(0.140)	1.054(0.226)	1.118(0.221)	1.148(0.216)	1.213(0.202)	1.128(0.219)	0.676(0.142)
$\beta = 3$	3.968(0.142)	4.275(0.281)	4.119(0.320)	4.142(0.341)	4.110(0.313)	4.104(0.311)	4.022(0.236)
$a = 2$	2.764(0.132)	3.154(0.232)	3.143(0.225)	3.145(0.228)	3.117(0.216)	3.174(0.221)	2.892(0.168)
$\alpha = 2$	2.852(0.202)	3.036(0.278)	3.168(0.337)	3.107(0.311)	3.183(0.263)	3.073(0.263)	3.100(0.241)
$\beta = 0.3$	0.742(0.130)	0.804(0.221)	0.874(0.204)	0.944(0.187)	1.006(0.217)	1.101(0.208)	0.814(0.166)
$a = 1$	1.851(0.117)	2.152(0.211)	2.165(0.205)	2.174(0.224)	2.182(0.207)	2.246(0.179)	1.914(0.112)
$\alpha = 1.5$	2.126(0.241)	2.356(0.321)	3.104(0.327)	3.011(0.300)	3.014(0.286)	3.111(0.310)	2.251(0.245)
$\beta = 1.8$	2.653(0.212)	3.102(0.301)	3.102(0.311)	3.085(0.286)	3.110(0.302)	3.010(0.253)	2.743(0.221)
$a = 1$	1.869(0.142)	2.148(0.210)	2.103(0.211)	2.137(0.186)	2.116(0.225)	2.034(0.213)	1.949(0.188)
$\alpha = 0.3$	0.765(0.134)	1.125(0.204)	1.016(0.153)	1.162(0.192)	1.121(0.232)	1.061(0.210)	0.932(0.161)
$\beta = 3$	4.042(0.224)	4.128(0.314)	4.013(0.321)	4.232(0.300)	4.101(0.287)	4.058(0.311)	4.073(0.231)
$a = 2$	2.851(0.201)	3.144(0.317)	3.328(0.312)	3.131(0.325)	3.673(0.218)	3.214(0.205)	2.924(0.229)
$\alpha = 2$	2.513(0.225)	3.142(0.279)	3.076(0.311)	3.071(0.303)	3.016(0.262)	3.056(0.273)	2.842(0.213)
$\beta = 0.3$	0.902(0.122)	1.138(0.202)	1.157(0.171)	1.122(0.203)	1.128(0.201)	1.135(0.203)	1.112(0.125)
$a = 0.5$	0.854(0.133)	1.1026(0.227)	1.123(0.214)	1.113(0.201)	1.155(0.164)	1.012(0.186)	1.001(0.115)
$\alpha = 1.5$	2.108(0.154)	2.134(0.264)	2.178(0.249)	2.139(0.286)	2.118(0.289)	2.303(0.244)	2.118(0.204)
$\beta = 3$	3.874(0.185)	4.025(0.269)	4.001(0.247)	4.011(0.287)	4.082(0.257)	4.064(0.284)	4.003(0.238)
$a = 1$	1.817(0.119)	2.024(0.194)	2.063(0.168)	2.162(0.202)	2.111(0.174)	2.201(0.187)	2.002(0.153)
$\alpha = 0.3$	0.738(0.116)	1.045(0.124)	1.186(0.321)	1.247(0.254)	1.205(0.229)	1.213(0.214)	0.802(0.171)
$\beta = 1.8$	2.268(0.217)	2.843(0.286)	2.845(0.307)	3.002(0.301)	3.012(0.298)	3.013(0.284)	2.178(0.227)

Table 5
The MEs and MSEs (in parentheses) of the NExW parameters for $n = 200$.

Parameters	MLEs	LSEs	LSEs	CVMEs	MPSEs	ADEs	RTADEs
$a = 0.5$	0.648(0.116)	0.701(0.218)	0.704(0.226)	0.717(0.224)	0.653(0.218)	0.639(0.229)	0.703(0.228)
$\alpha = 2$	2.537(0.227)	3.421(0.304)	3.409(0.300)	3.418(0.296)	3.461(0.307)	3.082(0.319)	2.611(0.301)
$\beta = 1.8$	2.457(0.125)	3.045(0.201)	3.121(0.217)	2.715(0.218)	2.725(0.213)	2.607(0.214)	2.442(0.129)
$a = 1$	1.574(0.114)	2.228(0.189)	2.132(0.178)	2.183(0.156)	2.218(0.202)	2.226(0.176)	2.421(0.117)
$\alpha = 0.3$	0.563(0.129)	1.013(0.213)	1.101(0.214)	1.033(0.202)	1.201(0.187)	1.116(0.207)	0.643(0.128)
$\beta = 3$	3.754(0.128)	4.253(0.265)	4.017(0.308)	4.026(0.327)	4.001(0.304)	4.012(0.294)	3.873(0.215)
$a = 2$	2.746(0.123)	3.131(0.228)	3.137(0.211)	3.109(0.213)	3.071(0.201)	3.044(0.210)	2.841(0.145)
$\alpha = 2$	2.826(0.187)	3.012(0.253)	3.132(0.314)	3.014(0.303)	3.141(0.237)	3.036(0.241)	2.941(0.233)
$\beta = 0.3$	0.723(0.115)	0.712(0.218)	0.843(0.182)	0.917(0.170)	0.894(0.202)	0.889(0.187)	0.807(0.135)
$a = 1$	1.813(0.102)	2.134(0.200)	2.142(0.187)	2.148(0.212)	2.154(0.201)	2.214(0.165)	1.901(0.101)
$\alpha = 1.5$	2.112(0.219)	2.312(0.317)	2.817(0.314)	2.793(0.252)	2.886(0.265)	3.032(0.314)	2.211(0.213)
$\beta = 1.8$	2.637(0.204)	2.943(0.287)	2.876(0.285)	3.013(0.263)	2.859(0.278)	2.865(0.238)	2.539(0.210)
$a = 1$	1.852(0.128)	2.115(0.202)	2.038(0.203)	2.103(0.166)	2.019(0.212)	2.011(0.204)	1.871(0.164)
$\alpha = 0.3$	0.747(0.121)	1.110(0.185)	1.010(0.132)	1.128(0.154)	1.111(0.220)	1.017(0.203)	0.915(0.132)
$\beta = 3$	3.965(0.210)	4.115(0.302)	4.001(0.316)	4.218(0.285)	4.000(0.287)	4.014(0.301)	4.031(0.219)
$a = 2$	2.814(0.187)	3.118(0.302)	3.286(0.301)	3.073(0.313)	3.633(0.201)	3.152(0.200)	2.962(0.242)
$\alpha = 2$	2.528(0.232)	3.164(0.303)	3.112(0.321)	3.117(0.318)	3.124(0.286)	3.112(0.301)	2.874(0.237)
$\beta = 0.3$	0.916(0.136)	1.173(0.211)	1.195(0.187)	1.146(0.217)	1.152(0.213)	1.156(0.218)	1.137(0.145)
$a = 0.5$	0.879(0.142)	1.116(0.242)	1.137(0.238)	1.129(0.204)	1.185(0.186)	1.117(0.202)	1.013(0.146)
$\alpha = 1.5$	2.120(0.185)	2.178(0.281)	2.217(0.304)	2.183(0.302)	2.134(0.311)	2.318(0.263)	2.121(0.217)
$\beta = 3$	4.014(0.201)	4.072(0.311)	4.029(0.284)	4.015(0.300)	4.202(0.285)	4.122(0.303)	4.015(0.253)
$a = 1$	1.831(0.131)	2.110(0.204)	2.111(0.201)	2.218(0.217)	2.123(0.197)	2.207(0.203)	2.018(0.185)
$\alpha = 0.3$	0.753(0.121)	1.074(0.135)	1.218(0.335)	1.301(0.279)	1.213(0.232)	1.227(0.226)	0.819(0.187)
$\beta = 1.8$	2.113(0.203)	2.831(0.262)	2.813(0.302)	2.874(0.287)	2.783(0.264)	2.693(0.256)	2.135(0.215)

Table 6
The MEs and MSEs (in parentheses) of the NExW parameters for $n = 350$.

Parameters	MLEs	LSEs	LSEs	CVMEs	MPSEs	ADEs	RTADEs
$a = 0.5$	0.621(0.104)	0.674(0.213)	0.682(0.210)	0.701(0.216)	0.637(0.205)	0.626(0.214)	0.677(0.207)
$\alpha = 2$	2.336(0.214)	3.168(0.275)	3.214(0.265)	3.207(0.272)	3.153(0.245)	2.372(0.302)	2.471(0.275)
$\beta = 1.8$	2.217(0.112)	2.485(0.183)	2.542(0.202)	2.248(0.211)	2.316(0.201)	2.413(0.200)	2.246(0.120)
$a = 1$	1.341(0.101)	1.682(0.153)	1.764(0.136)	1.659(0.124)	1.498(0.186)	1.944(0.142)	1.876(0.103)
$\alpha = 0.3$	0.438(0.117)	0.542(0.201)	0.724(0.203)	0.813(0.178)	0.658(0.153)	0.765(0.159)	0.472(0.115)
$\beta = 3$	3.261(0.122)	3.762(0.236)	3.537(0.258)	3.662(0.282)	3.817(0.255)	3.552(0.245)	3.436(0.201)
$a = 2$	2.436(0.118)	2.746(0.215)	2.615(0.201)	2.819(0.189)	2.761(0.178)	2.547(0.185)	2.564(0.124)
$\alpha = 2$	2.518(0.143)	2.703(0.232)	2.824(0.247)	2.581(0.272)	2.655(0.214)	2.714(0.228)	2.627(0.216)
$\beta = 0.3$	0.531(0.101)	0.625(0.203)	0.733(0.161)	0.804(0.153)	0.772(0.175)	0.737(0.173)	0.675(0.123)
$a = 1$	1.432(0.086)	1.748(0.175)	1.806(0.156)	1.704(0.201)	1.779(0.186)	1.843(0.142)	1.622(0.095)
$\alpha = 1.5$	1.852(0.187)	2.161(0.302)	2.314(0.242)	2.543(0.231)	2.465(0.253)	2.615(0.256)	1.861(0.201)
$\beta = 1.8$	2.034(0.175)	2.638(0.243)	2.664(0.255)	2.817(0.241)	2.436(0.253)	2.655(0.214)	2.235(0.185)
$a = 1$	1.571(0.114)	1.854(0.159)	1.836(0.174)	1.764(0.143)	1.814(0.183)	1.755(0.168)	1.654(0.147)
$\alpha = 0.3$	0.535(0.113)	0.731(0.154)	0.752(0.125)	0.815(0.132)	0.563(0.215)	0.672(0.182)	0.711(0.125)
$\beta = 3$	3.423(0.142)	3.556(0.264)	3.718(0.302)	3.904(0.257)	3.676(0.264)	3.748(0.276)	3.761(0.203)
$a = 2$	2.463(0.146)	2.725(0.273)	2.861(0.265)	2.742(0.296)	2.810(0.178)	2.754(0.178)	2.641(0.214)
$\alpha = 2$	2.315(0.221)	2.641(0.311)	2.749(0.314)	2.682(0.302)	2.743(0.252)	2.825(0.265)	2.544(0.221)
$\beta = 0.3$	0.624(0.124)	0.763(0.185)	0.841(0.153)	0.668(0.204)	0.682(0.201)	0.675(0.211)	0.714(0.131)
$a = 0.5$	0.664(0.130)	0.835(0.231)	0.774(0.215)	0.679(0.164)	0.824(0.172)	0.756(0.174)	0.688(0.127)
$\alpha = 1.5$	1.671(0.154)	1.853(0.253)	1.914(0.275)	1.782(0.263)	1.835(0.294)	1.942(0.241)	1.766(0.202)
$\beta = 3$	3.545(0.134)	3.762(0.262)	3.615(0.253)	3.755(0.211)	3.622(0.257)	3.714(0.268)	3.616(0.236)
$a = 1$	1.463(0.116)	1.723(0.174)	1.810(0.168)	1.873(0.203)	1.644(0.175)	1.880(0.148)	1.638(0.155)
$\alpha = 0.3$	0.554(0.110)	0.726(0.124)	0.801(0.314)	0.783(0.254)	0.775(0.221)	0.792(0.215)	0.657(0.174)
$\beta = 1.8$	2.011(0.138)	2.543(0.241)	2.601(0.274)	2.558(0.265)	2.534(0.241)	2.541(0.232)	2.058(0.201)

Table 7
The MEs and MSEs (in parentheses) of the NExW parameters for $n = 500$.

Parameters	MLEs	LSEs	LSEs	CVMEs	MPSEs	ADEs	RTADEs
$a = 0.5$	0.576(0.085)	0.632(0.201)	0.641(0.177)	0.673(0.203)	0.616(0.182)	0.612(0.178)	0.574(0.176)
$\alpha = 2$	2.183(0.163)	2.544(0.253)	2.618(0.242)	2.436(0.250)	2.461(0.223)	2.175(0.274)	2.233(0.252)
$\beta = 1.8$	2.104(0.076)	2.263(0.142)	2.325(0.185)	2.115(0.196)	2.128(0.184)	2.301(0.188)	2.134(0.116)
$a = 1$	1.203(0.077)	1.425(0.134)	1.537(0.124)	1.417(0.113)	1.254(0.163)	1.531(0.128)	1.467(0.098)
$\alpha = 0.3$	0.366(0.076)	0.431(0.184)	0.468(0.189)	0.512(0.153)	0.434(0.137)	0.458(0.145)	0.348(0.102)
$\beta = 3$	3.176(0.117)	3.453(0.218)	3.344(0.235)	3.372(0.251)	3.576(0.234)	3.341(0.223)	3.276(0.177)
$a = 2$	2.268(0.095)	2.535(0.202)	2.457(0.185)	2.516(0.147)	2.449(0.153)	2.381(0.157)	2.367(0.112)
$\alpha = 2$	2.327(0.079)	2.452(0.226)	2.570(0.231)	2.358(0.241)	2.345(0.184)	2.471(0.214)	2.328(0.201)
$\beta = 0.3$	0.423(0.078)	0.476(0.194)	0.525(0.146)	0.574(0.134)	0.485(0.156)	0.511(0.144)	0.457(0.114)
$a = 1$	1.213(0.067)	1.385(0.137)	1.573(0.132)	1.342(0.178)	1.451(0.164)	1.448(0.123)	1.353(0.067)
$\alpha = 1.5$	1.665(0.153)	1.864(0.246)	1.964(0.217)	1.797(0.215)	1.712(0.237)	1.988(0.232)	1.667(0.167)
$\beta = 1.8$	1.972(0.147)	2.415(0.221)	2.456(0.234)	2.510(0.219)	2.214(0.235)	2.328(0.202)	1.987(0.153)
$a = 1$	1.316(0.072)	1.546(0.136)	1.415(0.144)	1.343(0.131)	1.476(0.152)	1.352(0.142)	1.265(0.124)
$\alpha = 0.3$	0.415(0.081)	0.518(0.132)	0.473(0.113)	0.530(0.124)	0.461(0.184)	0.459(0.161)	0.440(0.113)
$\beta = 3$	3.264(0.076)	3.468(0.243)	3.353(0.277)	3.448(0.233)	3.345(0.245)	3.349(0.242)	3.278(0.192)
$a = 2$	2.324(0.121)	2.477(0.234)	2.528(0.231)	2.455(0.266)	2.516(0.145)	2.387(0.152)	2.344(0.201)
$\alpha = 2$	2.241(0.196)	2.365(0.286)	2.342(0.254)	2.351(0.273)	2.338(0.230)	2.455(0.241)	2.342(0.213)
$\beta = 0.3$	0.418(0.067)	0.537(0.154)	0.556(0.138)	0.480(0.194)	0.474(0.169)	0.426(0.171)	0.547(0.124)
$a = 0.5$	0.543(0.121)	0.674(0.219)	0.628(0.187)	0.587(0.145)	0.653(0.148)	0.668(0.156)	0.575(0.114)
$\alpha = 1.5$	1.558(0.136)	1.675(0.236)	1.748(0.255)	1.635(0.241)	1.654(0.253)	1.714(0.225)	1.647(0.189)
$\beta = 3$	3.321(0.118)	3.464(0.241)	3.376(0.237)	3.364(0.184)	3.387(0.235)	3.422(0.243)	3.347(0.214)
$a = 1$	1.236(0.075)	1.437(0.148)	1.412(0.146)	1.472(0.185)	1.347(0.155)	1.463(0.125)	1.388(0.132)
$\alpha = 0.3$	0.339(0.069)	0.490(0.117)	0.553(0.279)	0.426(0.232)	0.418(0.215)	0.435(0.185)	0.388(0.148)
$\beta = 1.8$	1.964(0.125)	2.327(0.225)	2.336(0.248)	2.384(0.232)	2.374(0.229)	2.311(0.220)	1.986(0.193)

Table 8
The MEs and MSEs (in parentheses) of the NExW parameters for $n = 600$.

Parameters	MLEs	LSEs	LSEs	CVMEs	MPSEs	ADEs	RTADEs
$a = 0.5$	0.512(0.053)	0.536(0.186)	0.610(0.143)	0.632(0.169)	0.543(0.153)	0.562(0.145)	0.543(0.142)
$\alpha = 2$	2.042(0.124)	2.313(0.221)	2.304(0.225)	2.224(0.231)	2.234(0.211)	2.132(0.245)	2.121(0.223)
$\beta = 1.8$	1.812(0.042)	2.134(0.123)	2.215(0.153)	2.012(0.163)	2.016(0.145)	2.140(0.146)	1.916(0.102)
$a = 1$	1.014(0.042)	1.213(0.112)	1.342(0.109)	1.203(0.101)	1.216(0.122)	1.362(0.105)	1.237(0.045)
$\alpha = 0.3$	0.314(0.034)	0.415(0.152)	0.423(0.146)	0.401(0.132)	0.371(0.124)	0.423(0.121)	0.326(0.084)
$\beta = 3$	3.037(0.075)	3.231(0.202)	3.125(0.212)	3.228(0.232)	3.319(0.213)	3.118(0.201)	3.127(0.142)
$a = 2$	2.024(0.053)	2.216(0.169)	2.214(0.153)	2.265(0.125)	2.217(0.133)	2.140(0.135)	2.143(0.087)
$\alpha = 2$	2.104(0.045)	2.226(0.201)	2.241(0.211)	2.136(0.225)	2.132(0.147)	2.238(0.200)	2.116(0.177)
$\beta = 0.3$	0.308(0.034)	0.423(0.143)	0.410(0.123)	0.434(0.102)	0.351(0.123)	0.400(0.121)	0.411(0.073)
$a = 1$	1.101(0.032)	1.256(0.124)	1.316(0.111)	1.207(0.145)	1.238(0.142)	1.214(0.112)	1.175(0.051)
$\alpha = 1.5$	1.510(0.134)	1.636(0.223)	1.649(0.202)	1.596(0.178)	1.562(0.215)	1.677(0.219)	1.544(0.149)
$\beta = 1.8$	1.822(0.125)	2.202(0.216)	2.238(0.213)	2.301(0.204)	2.119(0.214)	2.116(0.178)	1.878(0.137)
$a = 1$	1.104(0.031)	1.232(0.124)	1.210(0.123)	1.187(0.120)	1.241(0.131)	1.232(0.122)	1.175(0.112)
$\alpha = 0.3$	0.302(0.034)	0.412(0.120)	0.431(0.103)	0.417(0.112)	0.424(0.153)	0.421(0.136)	0.427(0.100)
$\beta = 3$	3.046(0.021)	3.235(0.221)	3.165(0.246)	3.217(0.212)	3.128(0.224)	3.139(0.222)	3.087(0.151)
$a = 2$	2.101(0.064)	2.245(0.218)	2.252(0.216)	2.241(0.234)	2.254(0.124)	2.186(0.122)	2.177(0.169)
$\alpha = 2$	2.021(0.133)	2.157(0.261)	2.148(0.232)	2.136(0.235)	2.179(0.214)	2.210(0.225)	2.137(0.174)
$\beta = 0.3$	0.302(0.042)	0.424(0.132)	0.431(0.119)	0.376(0.145)	0.368(0.136)	0.359(0.143)	0.343(0.112)
$a = 0.5$	0.503(0.078)	0.641(0.204)	0.611(0.132)	0.546(0.121)	0.632(0.125)	0.624(0.132)	0.534(0.101)
$\alpha = 1.5$	1.511(0.121)	1.632(0.218)	1.646(0.234)	1.602(0.226)	1.613(0.231)	1.642(0.211)	1.574(0.155)
$\beta = 3$	3.102(0.103)	3.245(0.227)	3.263(0.215)	3.195(0.147)	3.253(0.213)	3.278(0.221)	3.176(0.172)
$a = 1$	1.021(0.032)	1.235(0.125)	1.254(0.123)	1.228(0.153)	1.159(0.127)	1.236(0.109)	1.161(0.124)
$\alpha = 0.3$	0.308(0.035)	0.423(0.102)	0.438(0.227)	0.401(0.219)	0.385(0.185)	0.317(0.157)	0.356(0.126)
$\beta = 1.8$	1.802(0.078)	2.107(0.213)	2.115(0.225)	2.144(0.218)	2.146(0.196)	2.102(0.207)	1.925(0.139)

Table 9
The MEs and MSEs (in parentheses) of the NExW parameters for $n = 750$.

Parameters	MLEs	LSEs	LSEs	CVMEs	MPSEs	ADEs	RTADEs
$a = 0.5$	0.501(0.042)	0.512(0.164)	0.553(0.131)	0.541(0.135)	0.537(0.139)	0.541(0.132)	0.530(0.121)
$\alpha = 2$	2.001(0.121)	2.164(0.218)	2.216(0.213)	2.129(0.227)	2.151(0.200)	2.061(0.213)	2.105(0.217)
$\beta = 1.8$	1.801(0.027)	1.876(0.112)	1.854(0.138)	1.912(0.163)	1.913(0.132)	1.887(0.131)	1.864(0.093)
$a = 1$	1.002(0.026)	1.117(0.063)	1.228(0.077)	1.131(0.080)	1.134(0.117)	1.145(0.074)	1.126(0.023)
$\alpha = 0.3$	0.302(0.013)	0.351(0.138)	0.352(0.132)	0.342(0.122)	0.334(0.116)	0.328(0.119)	0.313(0.046)
$\beta = 3$	3.012(0.053)	3.125(0.168)	3.113(0.173)	3.185(0.216)	3.203(0.202)	3.042(0.201)	3.114(0.128)
$a = 2$	2.013(0.036)	2.165(0.135)	2.148(0.136)	2.127(0.118)	2.142(0.121)	2.124(0.121)	2.125(0.052)
$\alpha = 2$	2.014(0.023)	2.148(0.185)	2.173(0.200)	2.114(0.213)	2.083(0.125)	2.153(0.158)	2.069(0.152)
$\beta = 0.3$	0.301(0.022)	0.414(0.137)	0.305(0.112)	0.351(0.085)	0.338(0.111)	0.315(0.115)	0.392(0.042)
$a = 1$	1.013(0.010)	1.138(0.117)	1.204(0.110)	1.142(0.131)	1.125(0.117)	1.148(0.101)	1.133(0.034)
$\alpha = 1.5$	1.501(0.122)	1.614(0.211)	1.625(0.178)	1.553(0.146)	1.532(0.202)	1.635(0.204)	1.528(0.135)
$\beta = 1.8$	1.802(0.113)	2.029(0.202)	2.116(0.167)	2.150(0.183)	1.963(0.200)	1.952(0.136)	1.852(0.123)
$a = 1$	1.013(0.016)	1.128(0.112)	1.108(0.116)	1.123(0.113)	1.128(0.124)	1.129(0.110)	1.152(0.101)
$\alpha = 0.3$	0.300(0.013)	0.401(0.105)	0.416(0.095)	0.403(0.101)	0.413(0.138)	0.411(0.124)	0.416(0.084)
$\beta = 3$	3.013(0.018)	3.157(0.183)	3.126(0.228)	3.178(0.168)	3.113(0.213)	3.124(0.211)	3.025(0.130)
$a = 2$	2.015(0.041)	2.124(0.202)	2.127(0.200)	2.136(0.215)	2.143(0.118)	2.166(0.101)	2.145(0.135)
$\alpha = 2$	2.011(0.121)	2.125(0.250)	2.124(0.215)	2.113(0.214)	2.136(0.202)	2.183(0.213)	2.125(0.148)
$\beta = 0.3$	0.301(0.021)	0.412(0.127)	0.416(0.104)	0.342(0.126)	0.337(0.124)	0.336(0.139)	0.324(0.101)
$a = 0.5$	0.501(0.035)	0.610(0.163)	0.542(0.120)	0.533(0.119)	0.601(0.113)	0.610(0.124)	0.512(0.094)
$\alpha = 1.5$	1.500(0.081)	1.616(0.203)	1.624(0.221)	1.582(0.213)	1.600(0.218)	1.618(0.183)	1.546(0.132)
$\beta = 3$	3.014(0.043)	3.126(0.215)	3.234(0.202)	3.157(0.125)	3.138(0.201)	3.135(0.211)	3.135(0.142)
$a = 1$	1.011(0.021)	1.213(0.114)	1.172(0.114)	1.116(0.138)	1.126(0.116)	1.215(0.098)	1.125(0.112)
$\alpha = 0.3$	0.301(0.024)	0.412(0.098)	0.414(0.215)	0.398(0.203)	0.347(0.156)	0.311(0.135)	0.327(0.115)
$\beta = 1.8$	1.800(0.035)	1.913(0.178)	2.102(0.213)	2.122(0.186)	2.123(0.164)	1.831(0.178)	1.818(0.124)

exponentiated-Weibull (EW) [16], inverse-Weibull (IW) [17], odd Burr-III Weibull (OBIIIW) [18], Topp–Leone inverse-Weibull (TLIW) [19] distributions.

We fitted the NExW model and other models using the following three real-life data sets.

Data I: The first data about breaking stress of carbon fibers (in Gba) is given in [20]. This data is previously studied by [21], [22], and [23], [24]. It consists of 100 observations:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

Data II: The second data about strengths of glass fibers 1.5 cm is made at the National Physical Laboratory in England [25]. The data is studied by [26] and [27]. It consists of 63 observations:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Data III: The third data about petal width (in cm) samples of three species of Iris. The data is introduced by Fisher (1936) and it available in the R software [28]. This data contains 150 observations:

0.2, 0.2, 0.2, 0.2, 0.2, 0.4, 0.3, 0.2, 0.2, 0.1, 0.2, 0.2, 0.1, 0.1, 0.2, 0.4, 0.4, 0.3, 0.3, 0.3, 0.2, 0.4, 0.2, 0.5, 0.2, 0.2, 0.4, 0.2, 0.2, 0.2, 0.2, 0.4, 0.1, 0.2, 0.2, 0.2, 0.2, 0.1, 0.2, 0.2, 0.3, 0.3, 0.2, 0.6, 0.4, 0.3, 0.2, 0.2, 0.2, 0.2, 1.4, 1.5, 1.5, 1.3, 1.5, 1.3, 1.6, 1.0, 1.3, 1.4, 1.0, 1.5, 1.0, 1.4, 1.3, 1.4, 1.5, 1.0, 1.5, 1.1, 1.8, 1.3, 1.5, 1.2, 1.3, 1.4, 1.4, 1.7, 1.5, 1.0, 1.1, 1.0, 1.2, 1.6, 1.5, 1.6, 1.5, 1.3, 1.3, 1.3, 1.2, 1.4, 1.2, 1.0, 1.3, 1.2, 1.3, 1.3, 1.1, 1.3, 2.5, 1.9, 2.1, 1.8, 2.2, 2.1, 1.7, 1.8, 1.8, 2.5, 2.0, 1.9, 2.1, 2.0, 2.4, 2.3, 1.8, 2.2, 2.3, 1.5, 2.3, 2.0, 2.0, 1.8, 2.1, 1.8, 1.8, 1.8, 2.1, 1.6, 1.9, 2.0, 2.2, 1.5, 1.4, 2.3, 2.4, 1.8, 1.8, 2.1, 2.4, 2.3, 1.9, 2.3, 2.5, 2.3, 1.9, 2.0, 2.3, 1.8.

The MLEs are calculated using the quasi-Newton procedure for bound numerical optimization (L-BFGS-B). The goodness-of-fit indicators include the Akaike information criterion (C_1), and corrected-Akaike information criterion (C_2), log-likelihood $C_{-\hat{\ell}}$, Bayesian information criterion (C_3), Hannan–Quinn criterion (C_4), CVM (C_5), AD (C_6), and Kolmogorov–Smirnov (KS) statistics, and KS p-value are computed for all considered distributions.

The MLEs and corresponding standard errors (SEs) for all fitted distributions are reported in Tables 10, 13 and 16 for the three data sets, respectively. Moreover, the goodness-of-fit measures are listed in Tables 11, 14 and 17 for the three data sets. The figures

Table 10
The MLEs and corresponding SEs (in parentheses) for data I.

Distribution	α	β	a	γ	c	k
NExW	0.7701	2.1971	0.2563	-	-	-
	(0.5042)	(0.3542)	(0.1051)	-	-	
APW	15.0130	-1.2473	4.9166	-	-	-
	(7.8999)	(0.1092)	(0.5506)	-	-	-
EW	1.0556	3.0972	0.3332	-	-	-
	(0.3327)	(0.5931)	(0.0309)	-	-	-
IW	-	3.2248	-	1.6496	-	-
	-	(0.4189)	-	(0.1226)	-	-
OBIIIW	0.0040	1.9993	-	-	12.6840	4.9609
	(0.0008)	(0.1897)	-	-	(3.1202)	(1.2320)
TLIW	0.9601	4.4207	-	1.3638	-	-
	(0.5171)	(1.9441)	-	(0.1278)	-	-

Table 11
The values of goodness-of-fit indicators for data I.

Distribution	$C_{-\hat{\ell}}$	C_1	C_2	C_3	C_4	C_5	C_6	KS	KS p-value
NExW	84.8213	177.6426	178.2983	186.4012	181.1035	0.0430	0.2836	0.0693	0.9087
APW	115.2805	236.5610	236.9481	243.13	239.1568	0.8162	4.5785	0.2269	0.0022
EW	86.4241	178.8482	179.2353	185.4172	182.4439	0.1027	0.5667	0.1049	0.4610
IW	121.195	246.39	246.5805	250.7693	248.1205	0.9594	5.3886	0.2296	0.0019
OBIIIW	98.9092	203.8184	204.2055	210.3874	206.4141	0.1601	0.8561	0.1757	0.0339
TLIW	112.2375	230.4750	230.8621	237.0439	233.0707	0.7482	4.2250	0.200	0.0102

Table 12
Confidence limits of the NExW parameters for data I.

C-I	α	β	a
95%	[0 1.7583]	[1.5028 2.8913]	[0.0503 0.4623]
99%	[0 2.0709]	[1.2832 3.1109]	[0 0.5274]

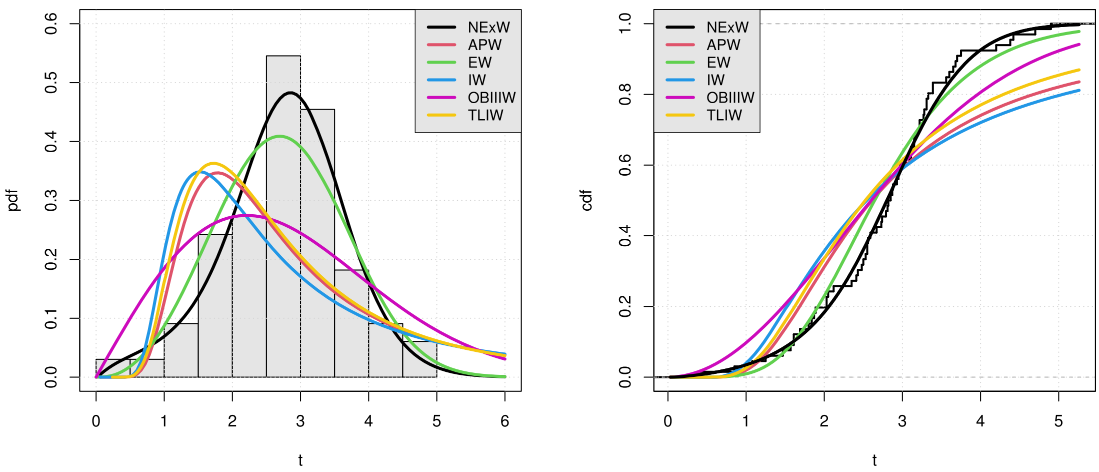

Fig. 7. Plots of the fitted pdf and cdf of the NExW model for data I.

in Tables 10, 13 and 16 reveal that the NExW distribution yields the lowest values of these statistics. Then, the NExW distribution provides the best fit to these three data sets. It is also shown that the NExW model has the largest KS p–value among all fitted models.

Tables 12, 15 and 18 list the confidence intervals of the NExW parameters for the three data sets. For visual comparisons, the fitted pdfs and estimated cdfs of the NExW distribution and other distributions are shown in Figs. 7, 9 and 11.

The TTT, box, and QQ plots are shown in Figs. 8, 10 and 12 for the three data sets, respectively. The numerical findings and charts show that the newly developed NExW distribution fits the three studied data sets better than other competing models.

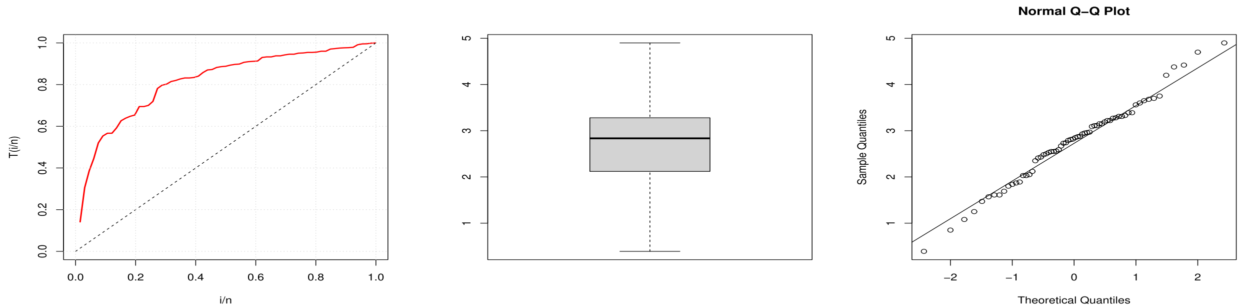

Fig. 8. The TTT, box, and QQ plots of the NExW model for data I.

Table 13
The MLEs and corresponding SEs (in parentheses) for data II.

Distribution	α	β	a	γ	c	k
NExW	0.2011	2.1775	0.2743	-	-	-
	(0.3187)	(2.835)	(0.1005)	-	-	
APW	15.9295	-2.1408	3.4831	-	-	-
	(9.0403)	(0.2014)	(0.3878)	-	-	-
EW	0.7219	7.1702	0.5858	-	-	-
	(0.3612)	(2.2189)	(0.0388)	-	-	-
IW	-	1.9695	-	2.8857	-	-
	-	(0.2486)	-	(0.2343)	-	-
OBIIIW	0.0145	5.3505	-	-	6.4685	2.5220
	(0.0223)	(0.7807)	-	-	(4.0106)	(1.1742)
TLIW	0.6819	3.8264	-	2.4243	-	-
	(0.4159)	(1.8774)	-	(0.2574)	-	-

Table 14
The values of goodness-of-fit indicators for data II.

Distribution	$C_{-\hat{\ell}}$	C_1	C_2	C_3	C_4	C_5	C_6	KS	KS p-value
NExW	11.5352	31.0705	31.7601	39.6430	34.4421	0.0959	0.5357	0.1182	0.3415
APW	43.2036	92.4073	92.8141	98.8367	94.9360	1.1276	6.0331	0.2330	0.0021
EW	14.7319	35.4638	35.8705	41.8932	37.9925	0.2018	1.1206	0.1357	0.1961
IW	46.8533	97.7067	97.9067	101.9930	99.3925	1.2251	6.4851	0.2440	0.0011
OBIIIW	15.6037	37.2074	37.6141	43.6367	39.7361	0.2458	1.3572	0.1653	0.0638
TLIW	39.5849	85.1699	85.5767	91.5993	87.6986	1.0370	5.5574	0.2530	0.0006

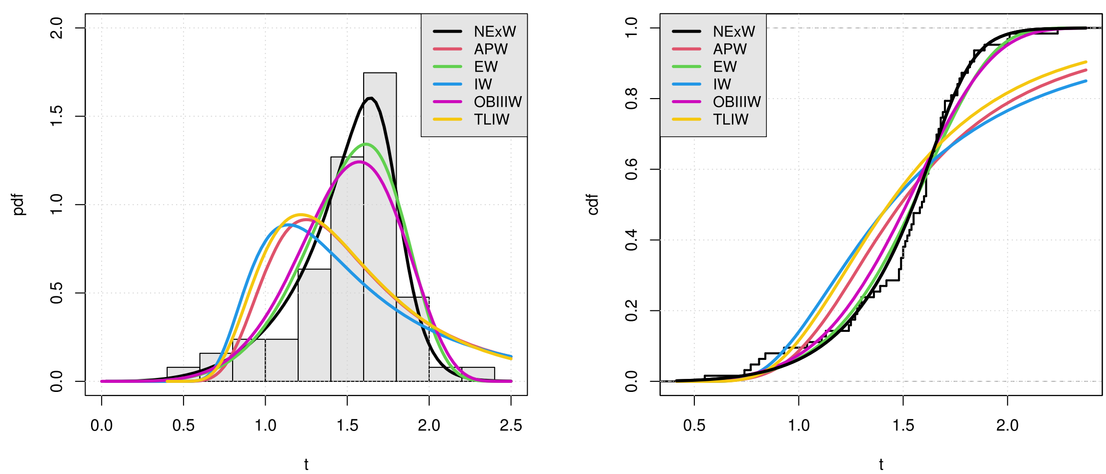

Fig. 9. Plots of the fitted pdf and cdf of the NExW model for data II.

Table 15
Confidence limits of the NExW parameters for data II.

C-I	α	β	a
95%	[0 0.8257]	[0 7.7341]	[0.0773 0.4713]
99%	[0 1.0233]	[0 9.4918]	[0.0150 0.5336]

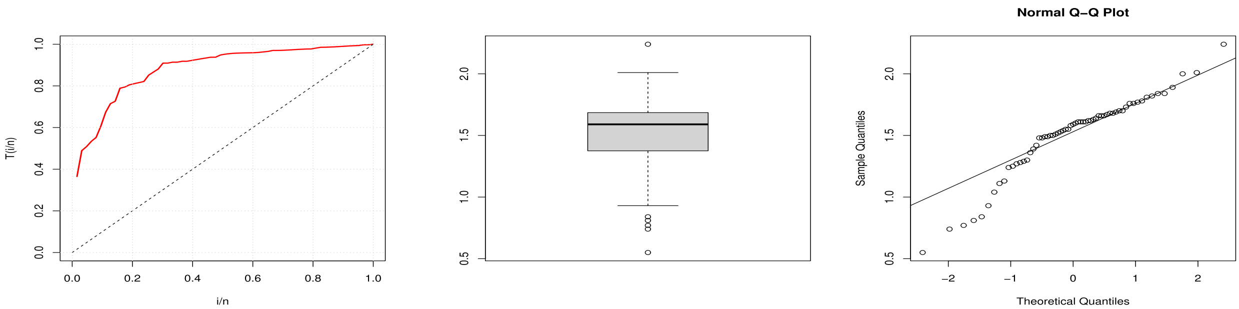

Fig. 10. The TTT, box, and QQ plots of the NExW model for data II.

Table 16
The MLEs and corresponding SEs (in parentheses) for data III.

Distribution	α	β	a	γ	c	k
NExW	0.6764	3.6006	0.2356	-	-	-
	(0.2541)	(0.4542)	(0.0365)	-	-	
APW	0.0948	-1.1538	0.2317	-	-	-
	(0.0658)	(0.0692)	(0.0563)	-	-	-
EW	0.1395	7.0766	0.4411	-	-	-
	(0.0250)	(1.0507)	(0.0149)	-	-	-
IW	-	0.5073	-	0.9764	-	-
	-	(0.0546)	-	(0.0589)	-	-
OBIIIW	0.0529	1.4409	-	-	6.2985	1.3281
	(0.0306)	(0.1016)	-	-	(3.6657)	(0.2002)
TLIW	0.2535	2.6773	-	0.8510	-	-
	(0.1153)	(0.92803)	-	(0.0672)	-	-

Table 17
The values of goodness-of-fit indicators for data III.

Distribution	$C_{-\hat{\ell}}$	C_1	C_2	C_3	C_4	C_5	C_6	KS	KS p-value
NExW	134.352	276.7039	276.9798	288.7464	281.5964	0.3478	2.6038	0.1215	0.0238
APW	198.492	402.9839	403.1483	412.0158	406.6533	2.4429	13.3283	0.2405	0.0000
EW	143.029	292.0585	292.2229	301.0904	295.7279	0.6006	3.9404	0.1434	0.0042
IW	203.812	411.6244	411.7061	417.6457	414.0707	2.5624	13.8997	0.2687	0.0000
OBIIIW	164.918	335.8375	336.0019	344.8694	339.5069	1.3737	8.0702	0.1892	0.0000
TLIW	194.289	394.5791	394.7435	403.611	398.2485	2.3781	12.9763	0.266	0.0000

Table 18
Confidence limits of the NExW parameters for data III.

C-I	α	β	a
95%	[0.1783 1.1744]	[2.7103 4.4908]	[0.1640 0.3071]
99%	[0.0208 1.3319]	[2.4287 4.7724]	[0.1414 0.3297]

8. Conclusions

We introduce a new family called the new exponential-H (NEx-H) family. The extra parameter of the NEx-H family provides greater flexibility to the baseline distributions such as Weibull, exponential, uniform, Lomax and Burr XII models. We provide five special sub-models of the NEx-H family. These special sub-models can provide constant, bathtub, increasing, reversed J-shape, unimodal, J-shape, and decreasing failure rate shapes, as well as left-skewed, bimodal, right-skewed, symmetric, J, unimodal, and reversed-J densities. The key mathematical characteristics of the NEx-H family are explored. Parameter estimation of a special sub-model called NEx-Weibull (NExW) distribution is addressed by using seven estimation approaches. The simulation results show that

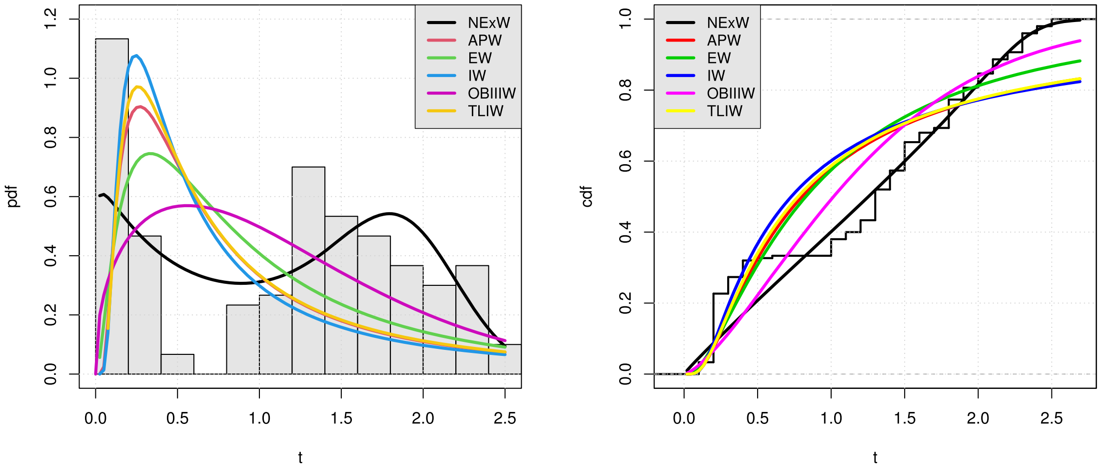

Fig. 11. Plots of the fitted pdf and cdf of the NExW model for data III.

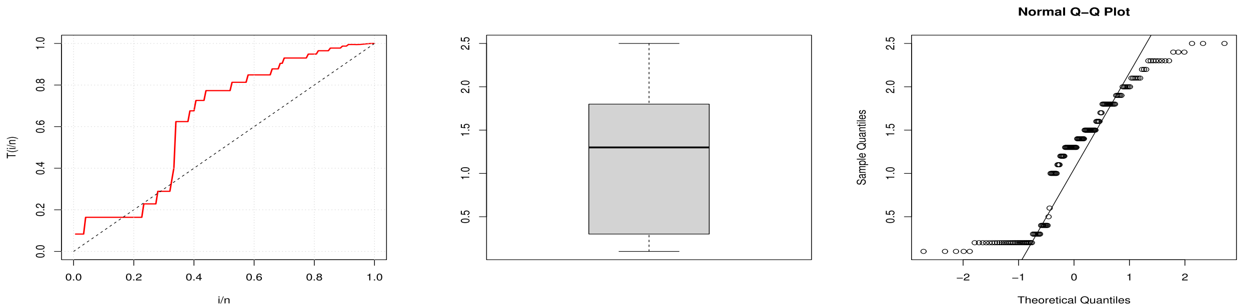

Fig. 12. The TTT, box, and QQ plots of the NExW model for data III.

the maximum likelihood is recommended for estimating the NExW parameters. Finally, we show the empirical importance of the NEx-H family using three real-life unimodal and bimodal data sets. The NExW distribution fits the three real-life data very well as compared to some extensions of the Weibull distribution.

CRediT authorship contribution statement

Farrukh Jamal: Writing – review & editing, Writing – original draft, Resources, Formal analysis, Conceptualization. **Mohammed Alqawba:** Writing – review & editing, Visualization, Software, Investigation, Conceptualization. **Yasser Altayab:** Writing – review & editing, Visualization, Methodology, Formal analysis. **Tariq Iqbal:** Writing – original draft, Validation, Software, Resources, Conceptualization. **Ahmed Z. Afify:** Writing – review & editing, Visualization, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

This work is mainly a methodological development and has been applied on secondary data which are provided in the manuscript.

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