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Temporal self-similarity of quantum dynamical maps as a concept of memorylessness

Shrikant Utagi[1,2], R. Srikanth[1] & Subhashish Banerjee[3✉]

The problem of defining quantum non-Markovianity has proven elusive, with various in-equivalent criteria put forth to address it. The concept of CP-indivisibility and the hierarchy of stronger divisibility criteria going up to P-indivisibility, capture a fundamental aspect of memory in quantum non-Markovianity. In practice, however, there can be a memory-like influence associated with divisible channels in the form of weakening, if not reversing, the effects of decoherence. Arguably, such a facet of memory relates to CP-indivisibility as quantum discord relates to entanglement. We concretize this weaker notion of non-Markovianity by identifying it with deviation from "temporal self-similarity", the property of a system dynamics whereby the propagator between two intermediate states is independent of the initial time t_0. We illustrate this idea through examples, and propose a geometric quantification of temporal self-similarity, and show how our approach complements the divisibility-based criterion of quantum non-Markovianity.

Practical quantum information processing must inevitably contend with quantum noise[1–4], and in particular, non-Markovian effects in the noise[5–10]. Classical Markovianity can be defined in terms of the divisibility of a process into intermediate transitions, or equivalently in terms of the fall in distinguishability of two states. The quantum adaptation of these ideas to define quantum non-Markovianity is not straightforward, and turns to lead to in-equivalent concepts, essentially because the Kolmogorov hierarchy of classical joint probability distributions cannot be transferred to the quantum case[11–15].

The classical identification of Markovianity with divisibility leads to a hierarchy of quantum divisibility criteria based on the positivity property of the intermediate map (the propagator of the dynamics between two arbitrary times) associated with a dynamical process being CP-divisible[16,17] or P-divisible[18], or an intermediate k-divisible[19,20]. In Ref.[21] quantum non-Markovian behavior was studied from the perspective of linear response theory. The effort to unify all such definitions into a single framework is important and remains studied by various authors; in this context, cf.[22]. It is fair to say that the question of how exactly to characterize memory effects in quantum processes remains a topic of active ongoing research[23–25].

As a rule of thumb, quantum non-Markovianity corresponds to the influence of memory of initial conditions that reverses the effect of quantum decoherence. However, in certain contexts channel memory may manifest itself simply by weakening the effect of quantum decoherence, without actually reversing it. To illustrate this idea, consider Fig. 1, which shows how quantum information gets degraded in three dephasing channels: the Holevo bound $B = S(\frac{\rho_1+\rho_2}{2}) - \frac{1}{2}[S(\rho_1) + S(\rho_2)]$ for a quantum dynamical semigroup channel (bottom-most plot), a CP-indivisible channel (oscillatory plot) and a non-QDS but CP-divisible channel. Although the last mentioned does not manifest recurrence, clearly it involves a memory effect in that it slows down the fall of distinguishability of the two initially orthogonal states ($|+\rangle\langle+|$ and $|-\rangle\langle-|$). Here, $S(\rho) \equiv -\text{Tr}[\rho \log(\rho)]$ denotes the von Neumann entropy.

In the present work, we identify the origin of this memory-like effect in such divisible channels with deviation from "temporal self-similarity" (TSS), which roughly refers to the idea of the intermediate map being oblivious of the initial time t_0. Furthermore, we prove that TSS may be identified with the quantum dynamical semigroup (QDS)[26], the 1-parameter semigroup of maps governing the system dynamics, generated by the time-independent linear map, namely, the Lindbladian $\mathcal{L}$, corresponding to the time-homogeneous master equation $\dot{\rho}(t) = \mathcal{L}[\rho(t)]$[27,28]. Now there has been a traditional background to identifying quantum Markovianity with

[1]Poornaprajna Institute of Scientific Research, Bangalore 562164, India. [2]Graduate Studies, Manipal Academy of Higher Education, Manipal 576104, India. [3]Interdisciplinary Research Platform - Quantum Information and Computation (IDRP-QIC), Indian Institute of Technology, Jodhpur 342037, India. ✉email: subhashish@iitj.ac.in
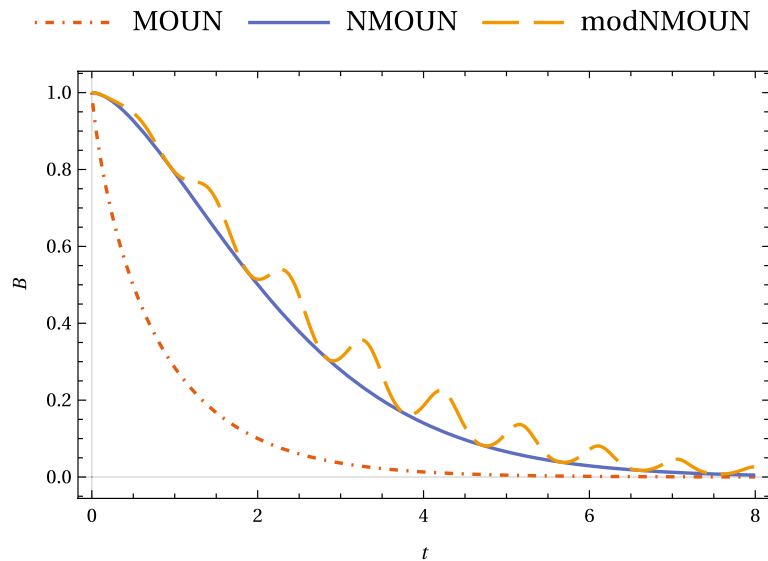
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Figure 1. (Color online) Distinguishability of two states initially $|+\rangle$ and $|-\rangle$ with respect to time under a Markovian OUN (QDS), non-Markovian OUN (non-QDS but CP-divisible) and modified OUN (CP-indivisible) dephasing channels, quantified by the Holevo bound B (these noisy channels are described below). In the first case, $G = 1$ (red, dot-dashed curve), for the second $G = 1$ and $g = 0.3$ (blue, bold curve), and for the third case, $a = 1$ and $r = 0.3$, and $k = 1.5$. (orange, dashed curve), cf. Eqs. (5) and (8). The figure illustrates how deviation from QDS, even when it doesn't reverse decoherence, can weaken it.

QDS[1,3,14,29]. This tradition was largely based on two broad physical considerations: (a) it could be justified on grounds that QDS is a reasonable quantum analogue of the classical Chapman–Kolmogorov equation, (b) it is also favored by considerations of the system–environment interaction, such as a strong coupling with the bath, the correlation times of the environment being very small in relation to the system's relaxation time or allowing for the Born–Markov approximation[1,3].

However, this historical picture was prior to the emergence of the divisibility-based criteria, and by our revisitation of it is show that it still holds relevance, but in light of a new perspective, which is not explicitly based on parameters related to the system-environment interaction, such as coupling strength, correlation time, etc. (though they play an indirect role, of course), but instead on memory-like effect manifested in the reduced dynamics . In quantum correlation theory, quantum discord is known to be strictly weaker than quantum entanglement[30], and yet of practical relevance[31]. The discovery of quantum discord was rather surprising in that it highlighted the existence of nonclassicality in the correlation among particles that are classically correlated. We think it is not amiss to consider that the broader concept of quantum non-Markovianity identified here with deviation from TSS stands in relation to the divisibility-based concept of quantum non-Markovanity as quantum discord stands with respect to quantum entanglement. Just as even classically correlated systems can manifest quantum discord, so too even divisible maps can manifest this weaker form of memory. We clarify this point later below.

Results

Temporal self-similarity. Suppose Alice obtains states $\rho(t_1)$ and $\rho(t_2)$ by applying CP evolution $\mathcal{E}(t_1, t_i)$ and $\mathcal{E}(t_2, t_i)$, respectively, to a system in initial state $\rho(t_i)$. Here $t_2 > t_1 > t_i$, and t_i is the initial time. She informs Bob the values t_1, t_2 and the *form* of the full map $\mathcal{E}(t_f, t_i)$, where t_f denotes the final time of the evolution. The form of the map could be represented as a set of Kraus operators, the Choi matrix, the dynamical map, etc. She asks Bob to compute the intermediate map $\mathcal{E}(t_2, t_1)$ that evolves $\rho(t_1)$ to $\rho(t_2)$. Quite generally $\mathcal{E}(t_2, t_i) = \mathcal{E}(t_2, t_1)\mathcal{E}(t_1, t_i)$. Assuming the invertibility of $\mathcal{E}(t_1, t_i)$, Bob's task is to compute $\mathcal{E}(t_2, t_1) = \mathcal{E}(t_2, t_i)\mathcal{E}(t_1, t_i)^{-1}$ (Fig. 2).

Clearly, this map is a function of t_i in general. This means that generally only if Alice supplies the value $t_i := t_0$ can Bob compute $\mathcal{E}(t_2, t_1)$. This is evidently a kind of memory effect. (Alternatively, Alice also supplies $\rho(t_1)$ in the beginning, and asks Bob to compute $\rho(t_2)$. In this case, she must reveal the initial state $\rho(t_0)$, reflecting the history dependence of the later state.) On the other hand, if the channel is *temporally self-similar*, i.e., the form of intermediate map is the same as that of the full map, then Bob can simply compute $\mathcal{E}(t_2, t_1)$ by $\mathcal{E}(t_f, t_i)|_{t_f := t_2, t_i := t_1}$. In this case, he doesn't require the t_0 information, which gives a notion of memorylessness, and which will be shown to be stronger than CP-divisibility. Thus, a channel $\mathcal{E}$ is *temporally self-similar* when its any intermediate map is oblivious of the initial time, and behaves like a full map in its own right[33]. We note that as a valid quantum channel, $\mathcal{E}$ should be completely positive (CP).

Assuming a constant system-bath Hamiltonian, $\mathcal{E}(t, t_i) = \mathcal{E}(t - t_i)$, which essentially follows from the fact that the CP map can be purified to a system-environment unitary $U(t - t_i)$ acting on a product state. Therefore, we have $\mathcal{E}(t_2, t_1) = \mathcal{E}(f(t_2 - t_i, t_1 - t_i))$ such that $f(t_2 - t_i, t_1 - t_i)$ has no dependence of t_i for arbitrary t_2, t_1 and t_i. Clearly, this holds only if $f(y, x) = f(y - x)$ for $y \geq x$. Setting $r \equiv t_1 - t_i$ and $s \equiv t_2 - t_1$, we find that
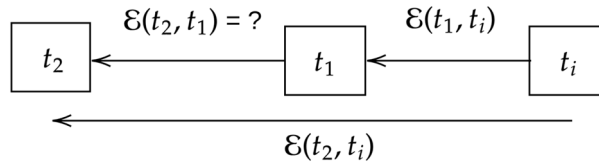
Figure 2. Deciding (non-)Markovianity: given *the form* $\mathcal{E}(t, t_i)$ (with free variables t, t_i) and values t_1 and t_2, the problem is to compute the intermediate map $\mathcal{E}(t_2, t_1)$. If this computation requires initial data $t_i := t_0$, then the map is non-Markovian, else it is not. (Equivalently, the original data given to Bob additionally includes $\rho(t_1)$, and his task would be to compute $\rho(t_2)$). Non-Markovianity in this case corresponds to this computation requiring the initial state ρ. Given this, t_0 can be computed by solving $\rho(t_1) = \mathcal{E}(t_1, t_i)\rho$ for t_i, and then $\rho(t_2)$ computed as $\mathcal{E}(t_2, t_0)\mathcal{E}(t_1, t_0)^{-1}\rho(t_1)$. Note that the non-invertibility of $\mathcal{E}(t, t_i)$ does not pose a problem to compute the initial time t_0 given $\rho(t_1)$ and ρ[32], Section IV).

$\mathcal{E}(r + s) = \mathcal{E}(s)\mathcal{E}(r)$, which is the defining composition rule for QDS. Conversely, QDS satisfies temporal self-similarity since the intermediate map under QDS is just $\mathcal{E}(t_2 - t_1)$. We thus identify temporal self-similarity with QDS.

It is worth comparing and contrasting our invocation of initial time memory with that proposed in Ref.[29]. There, a non-Markovian quantum evolution (obtained as the reduced dynamics of a time-independent Hamiltonian dynamics defined on the system and an ancilla) is identified with a time-local equation for the dynamical map $\mathcal{E}(t, t_0)$ with a time-homogeneous generator $\mathcal{L}(t - t_0)$ containing a memory of t_0. This is indeed analogous to our identification of memory with t_0 dependence of the intermediate map. On the other hand, the evolution described by a generalized Lindblad master equation with time-dependent generator $\mathcal{L}(t)$, deviating from QDS but satisfying inhomogeneous composition rule, is taken to be Markovian. Thus, this approach leads to a concept of non-Markovianity stronger than that proposed here. The reason, essentially, has to do with the fact that the initial time dependence in the case of non-Markovianity occurs in the generator $\mathcal{L}$ via a "$t - t_0$" term in the case of Ref.[29], whereas it occurs in the intermediate map $\mathcal{E}(t_2, t_1)$ in our case in more general functional forms of dependence on initial time (cf. herebelow Eq. (7) and the discussion below it).

We present a simple illustration of temporal self-similarity as below. Consider the amplitude damping channel (ADC), under which a quantum state ρ evolves to state ρ' via the map $\mathcal{E}^{AD}[\rho] \rightarrow \sum_j A_j(t)\rho A_j^\dagger(t)$, with the Kraus operators $A_j(t)$ given by $A_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \lambda(t)} \end{pmatrix}$ and $A_2(t) = \begin{pmatrix} 0 & \sqrt{\lambda(t)} \\ 0 & 0 \end{pmatrix}$, where

$$\lambda(t) = 1 - e^{-\gamma_0 t} \tag{1}$$

is the damping factor and γ_0 is the vacuum bath interaction parameter[34,35].

Now, suppose that the system evolves starting at t_i, going through t_1 to t_2. Let the damping factor associated with the full map $\mathcal{E}^{AD}(t_2, t_i)$ be λ, and that with the initial map $\mathcal{E}^{AD}(t_1, t_i)$ be μ. For $\mathcal{E}^{AD}(t_2, t_i)$ the Choi matrix[36] is found to be[17,37]:

$$\chi(t_2, t_i) = \begin{pmatrix} 1 & 0 & 0 & \sqrt{1 - \lambda} \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1 - \lambda} & 0 & 0 & 1 - \lambda \end{pmatrix}, \tag{2}$$

while that for the intermediate evolution map $\mathcal{E}^{AD}(t_2, t_1) = \mathcal{E}^{AD}(t_2, t_i)\mathcal{E}^{AD}(t_1, t_i)^{-1}$ is found to be[37]:

$$\chi(t_2, t_1) = \begin{pmatrix} 1 & 0 & 0 & \sqrt{\frac{1 - \lambda}{1 - \mu}} \\ 0 & \frac{\mu - \lambda}{\mu - 1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\frac{1 - \lambda}{1 - \mu}} & 0 & 0 & \frac{1 - \lambda}{1 - \mu} \end{pmatrix}. \tag{3}$$

Note that the matrices $\chi(t_2, t_i)$ and $\chi(t_2, t_1)$ are of the *same* form provided the functions $1 - \lambda$ and $1 - \mu$ have the exponential form e^{kt} for some k. In view of Eq. (1), this is indeed the case. We thus confirm that amplitude damping is indeed temporally self-similar.

If the Lindbladian $\mathcal{L}(t) = \mathcal{L}$ is a constant, then $\mathcal{E}(t_2, t_1) = e^{(t_2 - t_0)\mathcal{L}}e^{-(t_1 - t_0)\mathcal{L}} = e^{(t_2 - t_1)\mathcal{L}} = \mathcal{E}(t_2 - t_1)$, i.e., we obtain the self-similar form. But the converse is not true. This happens essentially when suitable continuity and limit requirements are not met. A simple example here would be the temporally self-similar map $\mathcal{E} = \mathbb{I}$ for $t = 0$, and $\mathcal{E}(\rho) = \sum_j \Pi_j \text{Tr}(\rho \Pi_j)$ for $t > 0$, where Π_j is any complete set of projectors[15]. However, in physically motivated scenarios, we can assume that the channel is continuous, satisfying the limit requirement $\lim_{t \to 0^+} \mathcal{E} = \mathbb{I}$. Given these assumptions, we can identify self-similarity at the master equation level with the time-independent Lindbladian. Consider the integro-differential time-nonlocal representation of the master equation of the system dynamics in terms of the linear memory kernel map $\mathcal{M}$, which makes this idea of memorylessness clearer. We have $\dot{\rho}(t) = \int_{t_0}^{t} ds \mathcal{M}_{t-s}\rho(s) = \int_{t_0}^{t} ds \mathcal{M}_{t-s}[\mathcal{E}(s - t_0)\mathcal{E}^{-1}(t - t_0)]\rho(t) = \int_{t_0}^{t} ds \mathcal{M}_{t-s}[e^{(s-t)\mathcal{L}}]\rho(t)$. This implies

that $\mathcal{M}_{t-s} = \delta(t-s)\mathcal{L}$, where $\delta(t-s)$ is the Dirac delta function, meaning that the dynamics remembers only the present time and has no influence from earlier times.

It is important to stress that this notion of memory as a dependence on the initial time (or, equivalently, break-down of temporal self-similarity of the map) shows up at the level of maps, and is not obvious at the level of generators. If the intermediate map $\mathcal{E}(t_2, t_1)$ is NCP, then it follows that $\mathcal{E}(t_2, t_i)$ is not QDS, and thus computing $\mathcal{E}(t_2, t_1)$ requires knowledge of t_i. Thus, TSS corresponds to a stronger concept of memorylessness than CP-divisibility.

We may expand on our analogy in the context of quantum correlations, where it is useful to invoke a result due to Sudarshan et al.[38]. It is known that (a subset of) entangled states turn negative under partial transpose, which can serve as a witness of entanglement[39]. Discordant states that are separable are necessarily positive under partial transpose. Now, CP-indivisible maps are associated with an intermediate matrix $A(t + \Delta t, t)$ that under an involution operation (cf "Methods") yields the intermediate B map or Choi matrix, which can be negative. That this negativity can be a witness to CP-indivisibility is precisely the RHP criterion[16]. By contrast, non-TSS states that are CP-divisible yield positive matrices under this involution of the intermediate map. Therefore, in the scheme of this analogy, CP-divisibility corresponds to separable states, whilst TSS to non-discordant states. On this basis, we may regard CP-indivisibility as representing a fundamental non-classical aspect of memory in quantum non-Markovianity, whilst deviation from TSS can include a "classical-like" aspect of memory in quantum non-Markovianity.

Examples: application to OUN and PLN.   As a particular instance of divisible noisy channels that manifest this sort of memory effect, we mention two quantum channels, namely Ornstein–Uhlenbeck noise (OUN) and power-law noise (PLN), that were traditionally introduced as non-Markovian based on certain physical arguments, but are Markovian in the CP-divisibility sense. The OUN model was developed in[40] in the context of the effect of non-Markovian evolution on the dynamics of entanglement. The model used was that of Gaussian noise with a colored auto-correlation function, modeling random frequency fluctuations and has its roots in the modern development of statistical mechanics[41]. In the limit of infinite noise bandwidth, this reduced to the well-known white noise which is Markovian in nature. PLN is a non-Markovian stationary noise process. The name *Power Law* points to the functional relationship between the spectral density and the frequency of the noise. It is a major source of decoherence in solid state quantum information processing devices such as superconducting qubits and has a well-defined Markovian limit[42].

The canonical Kraus representation for these channels has the form $\mathcal{E}(\rho) \equiv \sum_{j=I,Z} K_j \rho K_j^\dagger$ with $K_I(t) = \sqrt{\frac{1+p(t)}{2}} I$ and $K_Z(t) = \sqrt{\frac{1-p(t)}{2}} Z$, corresponding to the Choi matrix, $\chi \equiv (\mathcal{E} \otimes \mathbb{I})(|00\rangle + |11\rangle)$ given by:

$$\chi(t, 0) = \begin{pmatrix} 1 & 0 & 0 & p(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p(t) & 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

where I and Z are Pauli operators, and

$$p(t) = \begin{cases} \exp\left(\frac{-G}{2}\left(g^{-1}[\exp(-gt) - 1] + t\right)\right) & \text{case of OUN}, \\ \exp\left(-\frac{Gt(gt+2)}{2(gt+1)^2}\right) & \text{case of PLN}. \end{cases} \tag{5}$$

Here G is the inverse of the effective relaxation time, while g and $1/g$ are related to the noise band width, for the OUN and PLN noises, respectively.

The corresponding master equation in its canonical form[43] is $\dot{\rho} = \gamma(t)(-\rho + Z\rho Z)$ where $\gamma(t) = -\frac{\dot{p}(t)}{2p(t)}$[32] is the decoherence rate. It follows from Eq. (5)

$$\gamma(t) = \begin{cases} \frac{1}{4}G\left(1 - e^{-gt}\right) & \text{case of OUN}, \\ \frac{G}{2(gt+1)^3} & \text{case of PLN}, \end{cases} \tag{6}$$

showing that $\gamma(t)$ remains positive for all t in both these cases. Thus, neither of them is CP-indivisibe[44].

According to the system-environment criterion, OUN (resp., PLN) have their Markovian limit by setting g (resp., $1/g$) $\rightarrow \infty$, in Eq. (5). In this limit, we may replace Eq. (5) by $p^*(t) = e^{-Gt/2}$ (for OUN) and $p^*(t) = e^{-Gt}$ (for PLN). The corresponding rate $\gamma^* \equiv -\dot{p}^*/(2p^*)$ becomes a positive constant, $\frac{G}{4}$ and $\frac{G}{2}$, respectively, in which case the master equation corresponds to the strict (i.e., time-homogeneous) Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) equation[26,45]. Hence this would correspond to the QDS-limit. For the general case, satisfying Eq. (5), the decoherence rate is time-dependent, implying that the dynamics corresponds to the time-inhomogeneous GKSL master equation.

The Choi matrix of the intermediate map that evolves the system from time t_1 to t_2 is given by (cf. the "Methods" Part, where it is derived via the formalism of dynamical maps A and B[38]):

$$\chi(t_2, t_1) = \begin{pmatrix} 1 & 0 & 0 & \frac{p(t_2)}{p(t_1)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{p(t_2)}{p(t_1)} & 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

From Eq. (5), we have for PLN that $p(t_j) = \exp\left(-\frac{G[t_j - t_0](g[t_j - t_0] + 2)}{2(g[t_j - t_0] + 1)^2}\right)$. From this, we readily find that the decoherence term $\frac{p(t_2)}{p(t_1)}$ in Eq. (7) does not simplify to a form that is independent of t_0. A similar argument holds for OUN. Thus, even though these two channels are CP-divisible, they carry a memory of the initial time t_0, which is required to construct the propagator between any two arbitrary instances.

On the other hand, for the QDS-limit rates, we find $\frac{p(t_2)}{p(t_1)} = e^{-\frac{G}{2}(t_2 - t_1)}$ and $e^{-G(t_2 - t_1)}$, respectively, for OUN and PLN, showing that the intermediate map is oblivious of t_0.

This brings us to the important issue of empirical or practical implication of such weaker-than-CP-indivisible memory-like effect. Two examples may be pointed out. In the OUN model, this effect was made use of[40] to study its impact on the prolongation of the time to entanglement sudden death (ESD); while in[37], the analogous nature in the OUN and PLN models is shown to counteract decoherence for quantum walks. In both these examples, the weaker-than-CP-indivisible effect acts as a memory resource.

In the context of Fig. 1, we introduce a non-Markovian noise, which is a modification of non-Markovian OUN inspired by the random-telegraph noise (RTN), by introducing the time-dependent mixing parameter given by

$$\tilde{p}(t) = \exp\left[\frac{-a}{2}\left(\frac{e^{-rt} - 1}{r} + t\right)\left(\frac{\sin^2(\frac{t}{r})}{k} + \cos^2(\frac{t}{r})\right)\right],\tag{8}$$

and call it modified NMOUN. Here k is some real number. Modified OUN (or, modOUN) has a QDS limit when $r \to \infty$. It is CP-indivisible if $k \neq 1$.

Quantifying deviation from temporal self-similarity.

The above considerations suggest that the non-Markovianity in this weaker sense may be geometrically quantified by the minimum distance of an evolution $\mathcal{E}$ from a QDS form either at the level of maps or of generators. In the discrete-time case, only the former is possible. In practice, this approach can be computationally complicated to realize, given the non-convex nature of set of CP-divisible (including QDS) maps[29]. Typically, for physically well motivated channels, we may assume that the channel satisfies time-continuity and the above-mentioned limit assumption. Accordingly, as one possibility to quantify non-Markovianity in this sense, one may consider minimizing the distance $\|\mathcal{E}(t) - e^{\mathcal{L}^* t}\|$ for arbitrary maps, where $\mathcal{L}^*$ is a time-independent Lindblad generator. But here again one must contend with the non-convexity issue, noted above.

To circumvent this difficulty, we propose here to realize this geometric measure at the level of generators, instead of maps. We require to define a suitable measure that quantifies $|\mathcal{L}(t) - \mathcal{L}^*|$. To this end, we consider the infinitesimal intermediate map $\delta\mathcal{E}$ of the given channel $\mathcal{E}$, evolving the system of Hilbert space dimension d from time t to time $t + dt$. We have $(\delta\mathcal{E})\rho(t) = \mathcal{T}\exp\left(\int_t^{t+dt}\mathcal{L}(s)ds\right)\rho(t) = (1 + \mathcal{L}(t)dt)\rho(t)$. By the Choi-Jamiolkowski isomorphism, for any map $\mathcal{E}$ acting on a d-dimensional system, we can associate the unique $d^2 \times d^2$ Choi matrix $\hat{\mathcal{E}}(t) \equiv d(\mathcal{E} \otimes \mathbb{I})|\Phi^+\rangle\langle\Phi^+|$, where $|\Phi^+\rangle \equiv d^{-1/2}\sum_j |j, j\rangle$ is the maximally entangled state. The Choi matrix of the infinitesimal intermediate map defined above is thus $(d|\Phi^+\rangle\langle\Phi^+| + \hat{\mathcal{L}}(t)dt)$. Here, the uniqueness means that the Choi matrix is insensitive to the unitary freedom in the representation of the generator, and in particular, is independent of whether the generator has been represented in its canonical form. Let $\delta\mathcal{E}^*(t)$ be the infinitesimal intermediate map of a QDS channel, and $\mathcal{L}^*$ the corresponding generator. Then the difference between the two infinitesimal intermediate maps is $\Delta L \equiv \delta\mathcal{E}(t) - \delta\mathcal{E}^*(t) = (\mathcal{L}(t) - \mathcal{L}^*)dt$, from which we have:

$$\zeta = \min_{\mathcal{L}^*} \frac{1}{T}\int_0^T \|\hat{\mathcal{L}}(t) - \hat{\mathcal{L}}^*\|dt,\tag{9}$$

where $\|A\| = \text{Tr}\sqrt{AA^\dagger}$ is the trace norm of a matrix A.

This measure of non-Markovianity has the following desirable properties. There is an inherent normalization, whereby $\zeta = 0$ iff the channel is QDS and $\zeta > 0$ otherwise. The measure is easily computable since the general numerical optimization problem of Eq. (9) can be implemented in $d^2 - 1$ dimensions efficiently. The Lindbladians can be represented in their canonical forms, and those with positive Lindblad terms will form a convex set. Furthermore, for specific examples we consider, the minimization over the Lindbladian reduces to the problem of minimizing over a single parameter. The measure respects continuity, which we illustrate with a few examples. It is basis-independent, and has the operational meaning of non-Markovianity as deviating from the property of self-similarity and thus requiring memory of the initial time t_0.

A similar geometric approach to quantifying non-Markovianity is considered in[46], namely minimizing the distance $\|\mathcal{E}(t = 1) - e^{\mathcal{L}^*}\|$ for arbitrary maps. They consider the distance of a map (snapshot of an evolution), whereas we consider the evolution over a finite period of time. The problem with considering a snapshot is that there can be infinite number of evolutions passing through the map at that instant. For example, the map Eq. (4), with a fixed $p(t = T)$, has an infinite number of ways to assign values to the pair (g, G).

The measure Eq. (9) is now applied to a number of well-known channels. Immediately below, we consider the CP-divisible channels of OUN and PLN to quantify their non-Markovianity in the present approach. For dephasing channels, we find $\mathcal{L}\rho = \gamma(-\rho + Z\rho Z)$. Thus, $\hat{\mathcal{L}}(t) - \hat{\mathcal{L}}^* = (\gamma^* - \gamma)(|\Phi^+\rangle\langle\Phi^+| - |\Phi^-\rangle\langle\Phi^-|)$, where $|\Phi^-\rangle$ is the Bell state with even parity and negative phase.

For example, for OUN, it follows from their canonical master equation that $\zeta = \min_c \frac{1}{T}\int_0^T \left|\frac{1}{4}G(1 - e^{-gt}) - c\right|dt$ where c is the rate of a QDS dephasing channel. The minimization may be determined numerically and would in general depend on T. For OUN and PLN, it is simpler to consider an estimate of c to be the family's natural QDS limit, as may be found from the decoherence rates and p^*, given
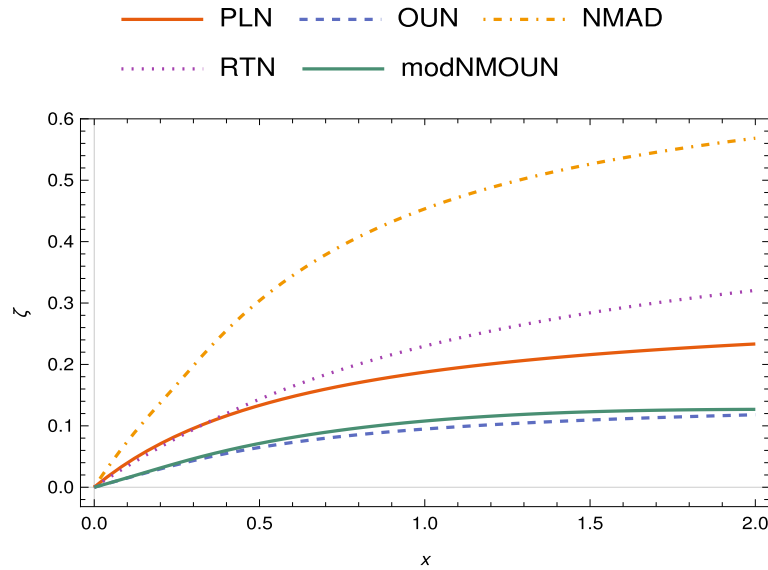
Figure 3. (Color online) Upper bound on the non-Markovianity measure ζ for CP-divisible and CP-indivisible channels, with $T \equiv 1$ [Eq. (9)]. Here, $x \equiv g^{-1}$ for OUN, NMAD and RTN and $x \equiv g$ for PLN. The value of parameters used are: $G = 0.6$ for both PLN (bold, red curve) and OUN (dashed, blue), and $\gamma_0 = 0.3$ for NM-AD (dot-dashed, orange) and $\gamma_0 = 0.6$ for RTN (dotted, purple). The curve (bold, dark green) is an upper bound on the non-Markovianity of modified NMOUN with $k = 1.5$ and $a = 0.6$, and $x \equiv r^{-1}$.

below Eq. (6). The corresponding plot (Fig. 3) may be considered as giving an upper bound on the weak memory effect. Similarly for the modified NMOUN case, Eq. (8), for which the corresponding plot appears in Fig. (3).

Application to stronger manifestations of non-Markovianity.   Although the quantity ζ has been motivated to explore the idea of memory weaker than CP-indivisibility, one would expect it to be applicable to such stronger manifestations of non-Markovianity, since in those cases, the deviation from the QDS form would be greater. For completeness, we shall consider a few representative examples, one that is P-divisible but CP-indivisible, another that is P-indivisible, involving both unital and non-unital maps. We present detailed analyses of these in the following.

Eternal non-Markovian channels.   An example of a model noise which is non-Markovian in the sense of CP-indivisibility but BLP Markovian[18] (i.e., P-divisible), is the interesting model called "Eternally non-Markovian" (ENM) Pauli channel, proposed in[44], with the decay rate $\gamma_3(t)$ being negative for all $t > 0$, whence the name 'eternal'. The canonical form of master equation for the evolution of qubit under this noise is the dephasing channel $\dot{\rho} = \sum_{j=1}^{3} \gamma_j(p)(\sigma_j \rho \sigma_j^\dagger - \rho)$, with $\gamma_1 = \gamma_2 = 1$ and $\gamma_3(t) = -\tanh(t)$.

The measure Eq. (9), in this case is found to be $\zeta = \min_c \frac{1}{T} \int_0^T |-\tanh(t) - c| dt$, for which optimal QDS channel is clearly the dephasing channel with $\gamma_1 = \gamma_2 = 1$ and $\gamma_3 = c$, and $c = 0$. Setting $T \equiv 1$, the degree of non-Markovianity here is $\int_0^1 \tanh(t) dt = \log[\cosh(1)] \approx 0.433$.

Random telegraph noise (RTN): P-indivisible dephasing.   As our next example, we consider random telegraph noise (RTN), which is a very well studied pure dephasing process, known to be non-Markovian according to information back-flow and CP-divisibility criteria[37,47]. The Kraus operators characterizing this process has a functional form similar to that of PLN and OUN with the function $p(t)$ in Eq. (5) having the form $p(t) = \exp\{-gt\}\left(\cos(g\omega t) + \frac{\sin(g\omega t)}{\omega}\right)$, with $\omega = \sqrt{(\frac{2\gamma_0}{g})^2 - 1}$. Here g is the spectral band width, which is the inverse of environmental correlation time scale τ, and γ_0 defines the coupling strength between the system and the environment. The decay rate is found to be $\gamma(t) = \frac{2\gamma_0^2}{g\left(1 + \sqrt{\frac{4\gamma_0^2}{g^2} - 1} \cot\left(gt\sqrt{\frac{4\gamma_0^2}{g^2} - 1}\right)\right)}$, which vanishes in the limit $g \gg 2\gamma_0$ and represents the QDS limit of the family. In the limit $g > 2\gamma_0$, the process is described by a master equation with time-dependent all-time-positive decay rate $\gamma(t)$, and hence CP- and P-divisible. This would correspond to Markovian behavior from the divisibility perspective, but from the present point of view would be non-Markovian due to deviation from QDS. In the limit $g < 2\gamma_0$, the process becomes non-Markovian in the sense of CP-divisibility and $\gamma(t)$ oscillates between negative and positive values, giving rise to intervals of breakdown of CP-divisibility and even P-divisibility. As $g \to 0$, the noise becomes more colored, hence non-Markovian. Setting $\gamma^* = 0$ for this noise, an upper bound for the measure of non-Markovianity Eq. (9), for channels in this family parametrized by $x \equiv \frac{1}{g}$ is depicted in Fig. (3).

Non-Markovian amplitude damping. The above method, considered so far for unital channels, can straightforwardly be extended to the non-unital case. As a specific example, we may consider the non-Markovian (P-indivisible) amplitude damping (AD) as an example of non-unital channel, the NM-AD channel.

Consider a qubit interacting dissipatively with a bath of harmonic oscillators, whose spectral density is given by the Lorentzian

$$I(\omega) = \gamma_0 g^2 (2\pi(\omega_0 + \Delta - \omega)^2 + g^2)^{-1},$$

where g is the width of the spectral density function, centered at a frequency detuned from the atomic frequency ω_0 by an amount Δ, and the rate γ_0 quantifies the strength of the system-environment coupling. If we assume $\Delta = 0$ (no detuning) i.e., when the qubit is in *resonance* with the central frequency of the bath, then the GKSL-like time-dependent master equation, with the rotating wave approximation, is given by

$$\frac{d\rho_s(t)}{dt} = \gamma(t)[\sigma_-\rho_s(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho_s(t)\}], \qquad (10)$$

where $\gamma(t) = -2\Re[\frac{\dot{G}(t)}{G(t)}]$ is the time-dependent decoherence rate, and G is the decoherence function given by

$$G(t) = e^{-\frac{gt}{2}}\left(\frac{g}{l}\sinh\left[\frac{lt}{2}\right] + \cosh\left[\frac{lt}{2}\right]\right), \qquad (11)$$

with $l = \sqrt{g^2 - 2\gamma_0 g}$ and $\sigma_\pm$ are the standard atomic raising (lowering) operators. This is the time-dependent AD channel and the decay rate is given by

$$\begin{aligned}\gamma(t) &= -\frac{2}{|G(t)|}\frac{d|G(t)|}{dt} \\ &= 2\Re\left(\frac{\gamma_0}{\sqrt{1 - \frac{2\gamma_0}{g}}\coth\left(\frac{1}{2}gt\sqrt{1 - \frac{2\gamma_0}{g}}\right) + 1}\right).\end{aligned} \qquad (12)$$

Now, the expression Eq. (1) for the damping factor of the ADC is replaced by $\lambda(t) = 1 - |G(t)|^2$[14]. In the limit $g < 2\gamma_0$, the decay rate (12) oscillates, and becomes negative for certain intervals giving rise to non-Markovian evolution. In the limit $g > 2\gamma_0$, the dynamics is time-dependent Markovian. (The point $g = 2\gamma_0$, however, corresponds to a point at which the time-local master equation lacks a perturbation expansion.) One readily sees that in the limit $g \gg 2\gamma_0$, the decay rate $\gamma(t) = \gamma_0$, i.e., it becomes time-*in*dependent, corresponding to a QDS evolution, the standard AD channel.

The measure Eq. (9) is found to be $\zeta = \min_{\gamma^*}\frac{1}{T}\int_0^T |\gamma(t) - \gamma^*|(1 + \sqrt{2})dt$. Once again, an upper bound of non-Markovianity parameter ζ, can be obtained by choosing γ^* to the QDS limit of the family of non-Markovian AD channels. As in the dephasing case, we find that the general optimization of the measure reduces to minimizing over a single parameter γ^*, which can be done numerically and is depicted in Fig. (3). The measure Eq. (9) can be suitably adapted through a renormalization procedure to handle the scenario wherein the generator has singularities, cf.[48].

Discussion

Over the last decade, the concept of what constitutes quantum non-Markovianity has been debated and studied in depth. Here we pointed out that even when the system's dynamical map is divisible, there can be a kind of memory of the initial time t_0 encoded into the form of the intermediate map, causing the dynamics to deviate from "temporal self-similarity" (TSS). Operationally, this memory effect may simply show up as a mitigation of decoherence, rather than a reversal of decoherence. We identify this concept of memorylessness with the quantum dynamical semigroup (QDS), and quantify it, providing a number of examples. We argue that this weaker concept of quantum non-Markovianity stands in relation to the standard divisibility based definition of quantum Markovianity, as quantum discord stands with respect to entanglement. Of practical importance is that this measure would be applicable in scenarios where the map is divisible, and yet a memory-like effect is seen to counter decoherence caused, for e.g., by modifying the frequency spectrum of the interacting bath or broadening the memory kernel of the dynamics. To illustrate this, we studied various examples, among them the well known channels of OUN and PLN.

A number of new directions are opened up by the present work. In[49] it is shown how maximally non-Markovian system dynamics can arise without backflow of information from the system to the environment modelled via classical degrees of freedom. While QDS was historically motivated, but not defined, by considerations of the system-bath interaction, another concept of Markovianity that explicitly relies on such considerations is the quantum regression formula (QRF), which is based on the system's two-point or higher-order correlation functions[50]. Its relation to divisibility-based criteria for non-Markovianity have been studied by various authors[12,15]. In light of these observations, one might ask what the most general system-environment correlations are that lead to deviation from TSS, and also how to reconcile the two different (but related) viewpoints on non-Markovianity, based on the system dynamics vs that based on knowledge of the system-environment interaction and correlations.

In[51], the authors consider a resource theory of non-Markovianity wherein the Choi matrices corresponding to small-time divisible maps constitute the free states. Our work suggests the possibility of constructing a more relaxed resource theory of small-time maps, wherein resourceful states would correspond to maps of processes deviating from temporal self-similarity. Earlier, we noted the instances where the memory-like signature of the

OUN model, as characterized here, helps to prolong the time to ESD[40] and also counteracts decoherence in quantum walk[37]. This should pave the way for more such examples which could have an impact on broadening the scope of memory in quantum phenomena.

In[19], studying the positivity of the propagator between two arbitrary times in an extended Hilbert space (divisibility), the authors defined a non-Markovianity degree, as the analogue of Schmidt number in quantum entanglement, such that the analogue of maximally entangled states are maximally non-Markovian quantum dynamics. It would be interesting to study how channels weaker than CP-indivisible fit into this hierarchy, in particular, whether they could be considered as the analogues of separable states with non-vanishing quantum discord.

Methods

Sudarshan A and B matrices for PLN-OUN. The intermediate dynamics can be studied by way of the dynamical map A introduced by Sudarshan et al.[38]. The map A represents the noise superoperator as a $d^2 \times d^2$ matrix on a vector, obtained by vectorizing the density operator, i.e., $\rho' = A(t_1, t_0) \cdot \rho$. Thus, $\rho'_{j'k'} = \sum_{j,k} A_{j'k';jk}\rho_{jk}$.

Here d is the system's Hilbert space dimension. Given a qubit density operator $\begin{bmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{bmatrix}$, the channel (4) can be represented as:

$$
\begin{pmatrix} \rho'_{00} \\ \rho'_{01} \\ \rho'_{10} \\ \rho'_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p(t) & 0 & 0 \\ 0 & 0 & p(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_{00} \\ \rho_{01} \\ \rho_{10} \\ \rho_{11} \end{pmatrix},
\tag{13}
$$

where $p(t)$ is given by the corresponding probabilities $p^*(t)$ when temporal self-similarity holds, and by Eq. (5) in the general case.

Unlike the Choi matrix[36], the map A can be composed directly by matrix multiplication: thus $\rho(t_2) = A(t_2, t_1) \cdot A(t_1, t_0)\rho(t_0)$. The intermediate dynamical map $A(t_2, t_1)$ can thus be directly computed as $A(t_2, t_1) = A(t_2, t_0) \cdot A(t_1, t_0)^{-1}$, where the inverse is the matrix inverse and assumed to be non-singular. For Eq. (13), one readily finds that

$$
A(t_2, t_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{p(t_2)}{p(t_1)} & 0 & 0 \\ 0 & 0 & \frac{p(t_2)}{p(t_1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\tag{14}
$$

By re-shuffling the terms of the A-matrix according to an "involution map" given by $B_{j'j;k'k} = A_{j'k';jk}$[38], from Eq. (14), one obtains the corresponding B-matrix, which is just the Choi matrix.

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Author contributions

R.S. and S.B. developed the basic theoretical ideas. S.U. made further contributions to this and also performed all the calculations.

Competing interests

The authors declare no competing interests.

Additional information

Correspondence and requests for materials should be addressed to S.B.

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