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Linear algebraic structure of zero-determinant strategies in repeated games

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Abstract

Zero-determinant (ZD) strategies, a recently found novel class of strategies in repeated games, has attracted much attention in evolutionary game theory. A ZD strategy unilaterally enforces a linear relation between average payoffs of players. Although existence and evolutional stability of ZD strategies have been studied in simple games, their mathematical properties have not been well-known yet. For example, what happens when more than one players employ ZD strategies have not been clarified. In this paper, we provide a general framework for investigating situations where more than one players employ ZD strategies in terms of linear algebra. First, we theoretically prove that a set of linear relations of average payoffs enforced by ZD strategies always has solutions, which implies that incompatible linear relations are impossible. Second, we prove that linear payoff relations are independent of each other under some conditions. These results hold for general games with public monitoring including perfect-monitoring games. Furthermore, we provide a simple example of a two-player game in which one player can simultaneously enforce two linear relations, that is, simultaneously control her and her opponent's average payoffs. All of these results elucidate general mathematical properties of ZD strategies.

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Introduction

Game theory is a powerful framework explaining rational behaviors of human beings [1] and evolutionary behaviors of biological systems [2, 3]. In a simple example of prisoner's dilemma game, mutual defection is realized as a result of rational thought, even if mutual cooperation is more favorable. On the other hand, when the game is repeated infinite times, cooperation can be realized if players are far-sighted, which is confirmed as folk theorem. Axelrod's famous tournaments on infinitely repeated prisoner's dilemma game [4, 5] also showed that cooperative but retaliating strategy, called the tit-for-tat strategy, is successful in the setting of infinitely repeated game.

Recently, in repeated games with perfect monitoring, a novel class of strategies, called zero-determinant (ZD) strategy, was discovered [6]. Surprisingly, ZD strategy unilaterally enforces a linear relation between average payoffs of players. A strategy which unilaterally sets her opponent's average payoff (equalizer strategy) is one example. Another example is extortionate strategy in which the player can earn more average payoff than her opponent. ZD strategies contain the well-known tit-for-tat strategy as a special example. After the pioneering work of Press and Dyson, stability of ZD strategies has been studied in the context of evolutionary game theory [7–12], and it was found that some kind of ZD strategies, called generous ZD strategies, can stably exist. Performance of ZD strategies has also been studied in human experiments [13, 14]. Although ZD strategy was originally formulated in two-player two-action (iterated prisoner's dilemma) games, ZD strategy was extended to multi-player two-action (iterated social dilemma) games [15, 16], two-player multi-action games [17, 18], and multi-player multi-action games [19]. In addition, ZD strategy was extended to two-player two-action noisy games [20, 21], which is one example of the repeated games with imperfect monitoring. Furthermore, besides these fundamental theoretical studies, ZD strategies are also applied to resource sharing in wireless networks [22, 23]. See Ref. [24] for a review of ZD strategies in the context of direct reciprocity.

The contributions of this paper are four-fold. First, we extend ZD strategy for general multi-player multi-action repeated games with public monitoring, where players know the structure of games (players, sets of actions of all players, and payoffs of all players) but cannot observe actions of other players. A typical example of such situation is auction. In a sealed-bid auction, a player cannot know actions (bids) of other players, but only knows the result of the game (whether she is the winner or not). Second, we prove, in terms of a linear-algebraic argument, that linear payoff relations enforced by players with ZD strategies are consistent, that is, always have solutions. Third, we introduce the notion of independence of ZD strategies, and prove, again in terms of a linear-algebraic argument, that linear payoff relations enforced by players with ZD strategies are independent under a general condition. Fourth, as an application of linear algebraic formulation, we provide a simple example of a two-player game in which one player can simultaneously enforce two linear relations. This means that she can simultaneously control her and her opponent's average payoffs, which has never been reported in the context of ZD strategies. All of these results develop deeper understanding of mathematical properties of ZD strategies in general games.

We remark on discounting. In standard repeated games, discounting of future payoffs is considered by introducing a discounting factor $\delta \leq 1$ [1]. In the original work on ZD strategy by Press and Dyson, only the case without discounting (i.e., $\delta = 1$) was investigated [6]. After their work, ZD strategy was extended to $\delta < 1$ case [18, 25, 26]. In this paper, we consider only the non-discounting case $\delta = 1$.

Setup

We consider an N-player multi-action repeated game, in which player $n \in \{1, \cdots, N\}$ has M_n possible actions, where M_n is a positive integer. Let $\sigma \equiv (\sigma_1, \cdots, \sigma_N) \in \Sigma \equiv \prod_{n=1}^{N} \{1, \cdots, M_n\}$ denote a state of the game, which is the combination of the actions taken by the N players. Let $M \equiv \prod_{n=1}^{N} M_n$ be the size of the state space Σ. We assume that player n decides the next action stochastically according to her own previous action $\boldsymbol{\sigma}'_n$ and common information $\tau \in B$ with the conditional probability $\hat{T}_n(\sigma_n|\boldsymbol{\sigma}'_n, \tau)$, where B is some set. We also define the conditional probability that common information τ arises when actions of players in the preceding round are $\boldsymbol{\sigma}'$ by $W(\tau|\boldsymbol{\sigma}')$. (An example of τ is the winner in each round; see S1 Text) Then the sequence of states of the repeated game forms a Markov chain

$$P(\boldsymbol{\sigma}, t+1) = \sum_{\boldsymbol{\sigma}'} T(\boldsymbol{\sigma}|\boldsymbol{\sigma}') P(\boldsymbol{\sigma}', t) \tag{1}$$

with the transition probability

$$T(\boldsymbol{\sigma}|\boldsymbol{\sigma}') \equiv \sum_{\tau} W(\tau|\boldsymbol{\sigma}') \prod_{n=1}^{N} \hat{T}_n(\sigma_n|\boldsymbol{\sigma}'_n, \tau), \tag{2}$$

where $P(\boldsymbol{\sigma}, t)$ denotes the state distribution at time t. We assume that all players know the function $W(\tau|\boldsymbol{\sigma}')$ but cannot directly observe $\boldsymbol{\sigma}'$. When $B = \Sigma$ and $W(\tau|\boldsymbol{\sigma}') = \delta_{\tau,\boldsymbol{\sigma}'}$, the above formulation reduces to that of perfect monitoring games. Otherwise, it represents games with public monitoring, where players cannot directly observe actions of other players. The model treated here can therefore be regarded as an extension of repeated games with perfect monitoring to those with imperfect monitoring, and the extension includes the former as a special case.

For each state $\boldsymbol{\sigma}$, a payoff of player n is defined as $s_n(\boldsymbol{\sigma})$. Let $\mathbf{s}_n \equiv (s_n(\boldsymbol{\sigma}'))_{\boldsymbol{\sigma}' \in \Sigma}$ be the M-dimensional vector representing the payoffs of player n, which we call the payoff vector of player n. It should be noted that in the following analysis we do not assume the payoffs to be symmetric, unless otherwise stated.

Results

Zero-determinant strategies

Because a discounting factor δ is one, the payoffs of players are the average payoffs with respect to the stationary distribution of the Markov chain. Let $P^{(s)}(\boldsymbol{\sigma})$ denote the stationary distribution, which may depend on the initial condition when the Markov chain is not irreducible. It satisfies

$$P^{(\mathrm{s})}(\sigma) = \sum_{\boldsymbol{\sigma}'} T(\boldsymbol{\sigma}|\boldsymbol{\sigma}') P^{(\mathrm{s})}(\boldsymbol{\sigma}'). \tag{3}$$

Taking summation of both sides of Eq (3) with respect to $\boldsymbol{\sigma}_{-n} \equiv \boldsymbol{\sigma} \backslash \sigma_n$ with an arbitrary n, we obtain

$$0 = \sum_{\boldsymbol{\sigma}'} [T_n(\sigma_n|\boldsymbol{\sigma}') - \delta_{\sigma_n,\boldsymbol{\sigma}'_n}] P^{(\mathrm{s})}(\boldsymbol{\sigma}'), \tag{4}$$

where we have defined

$$T_n(\sigma_n|\boldsymbol{\sigma}') \equiv \sum_{\tau} W(\tau|\boldsymbol{\sigma}') \hat{T}_n(\sigma_n|\boldsymbol{\sigma}'_n, \tau). \tag{5}$$

Regarding $\delta_{\sigma_n,\boldsymbol{\sigma}'_n}$ as representing the strategy "Repeat", where player n repeats the previous action with probability one, one can readily see that Eq (4) is an extension of Akin's lemma [15, 18, 27, 28], relating a player's strategy with the stationary distribution, to the multi-player multi-action public-monitoring case. Letting

$$\tilde{T}_n(\sigma_n|\boldsymbol{\sigma}') \equiv T_n(\sigma_n|\boldsymbol{\sigma}') - \delta_{\sigma_n,\boldsymbol{\sigma}'_n}, \tag{6}$$

Eq (4) means that the average of $\tilde{T}_n(\sigma_n|\sigma')$ with respect to the stationary distribution is zero for any n and σ_n. We remark that all players are assumed to know the functional form of $W(\tau|\boldsymbol{\sigma}')$, and that $\hat{T}_n(\sigma_n|\boldsymbol{\sigma}'_n, \tau)$, and thus $T_n(\sigma_n|\boldsymbol{\sigma}')$ as well, are solely under control of player n.

Because of the normalization condition $\sum_{\sigma_n=1}^{M_n} T_n(\sigma_n|\sigma') = 1$, the relation

$$\sum_{\sigma_n=1}^{M_n} \tilde{T}_n(\sigma_n|\boldsymbol{\sigma}') = 0 \tag{7}$$

holds.

Let $\tilde{\boldsymbol{T}}_n(\sigma_n) \equiv (\tilde{T}_n(\sigma_n|\boldsymbol{\sigma}'))_{\boldsymbol{\sigma}'\in\Sigma}$, which we call the strategy vector of player n associated with action σ_n. (Another name for $\tilde{\boldsymbol{T}}_n(\sigma_n)$ is the Press-Dyson vector [27].) A strategy of player n is represented as an $M \times M_n$ matrix $T_n \equiv (\tilde{\boldsymbol{T}}_n(1), \cdots, \tilde{\boldsymbol{T}}_n(M_n))$ composed of the strategy vectors for her actions $\sigma_n \in \{1, \ldots, M_n\}$. For a matrix A, let span A be the subspace spanned by the column vectors of A. Let $\mathbf{0}_m$ and $\mathbf{1}_m$ denote the m-dimensional zero vector and the m-dimensional vector of all ones, respectively. From Eq (7), one has

$$\mathcal{T}_n \mathbf{1}_{M_n} = \sum_{\sigma_n=1}^{M_n} \tilde{\boldsymbol{T}}_n(\sigma_n) = \mathbf{0}_M \tag{8}$$

for any player n, implying that the dimension of span T_n is at most $(M_n - 1)$.

Let $\boldsymbol{\rho} \equiv (P^{(s)}(\boldsymbol{\sigma}))_{\boldsymbol{\sigma}\in\Sigma}$ be the vector representation of the stationary distribution $P^{(s)}(\boldsymbol{\sigma})$. When player n chooses a strategy T_n, for any vector $\upsilon \in$ span T_n, one has $\rho^\top \upsilon = 0$ due to Eq (4). In other words, the expectation of $\boldsymbol{v}$ with respect to the stationary distribution $\boldsymbol{\rho}$ vanishes.

Let $\rho \equiv (\mathbf{1}_M, \boldsymbol{s}_1, \cdots, \boldsymbol{s}_N)$ and $V_n \equiv$ span $T_n \cap$ span S. The following definition is an extension of the notion of the ZD strategy [6, 27] to multi-player multi-action public-monitoring games.

Definition 1. *A zero-determinant* (ZD) strategy *is defined as a strategy T_n for which* dim $V_n \geq 1$ *holds*.

To see that this is indeed an extended definition of the ZD strategy, note that any vector $\boldsymbol{u} \in$ span S is represented as $\boldsymbol{u} = S\alpha$, where $\alpha \equiv (\alpha_0, \alpha_1, \cdots, \alpha_N)^\top$ is the coefficient vector. Let $\boldsymbol{e} \equiv (1, e_1, \cdots, e_N)^\top = S^\top \rho$ be the vector with element e_n equal to the expected payoff $e_n \equiv \langle s_n(\boldsymbol{\sigma})\rangle_s$ of player n in the steady state. When player n employs a ZD strategy, it amounts to enforcing linear relations $\boldsymbol{e}^\top \alpha = \rho^\top S\alpha = 0$ on $\mathbf{e}$ with $\boldsymbol{\alpha}$ satisfying $S\alpha \in V_n$.

Consistency

A question naturally arises: When more than one of the players employ ZD strategies, are they "consistent", that is, do linear payoff relations enforced by the players always have solutions? For example, in a two-player game, when player 1 enforces $\sum_{n=1}^{2} \alpha_n e_n = \gamma$ by a ZD strategy and player 2 enforces $\sum_{n=1}^{2} \alpha'_n e_n = \gamma'$ by a ZD strategy, do the simultaneous equations of (e_1, e_2) have a solution? Let N' be the set of players who employ ZD strategies. The set $E \equiv \{\boldsymbol{e} \in \{1\} \times \mathbb{R}^N : \boldsymbol{e}^\top \alpha = 0, \forall \alpha, S\alpha \in \text{span}(V_n)_{n\in N'}\}$ consists of all combinations of the expected payoffs that satisfy the enforced linear relations by the players in N'. If E is empty, then it implies that the set of ZD strategies is inconsistent in the sense that there is no valid solution of the linear relations enforced by the players.

Definition 2. *ZD strategies are said to be consistent when E is not empty*.

In the multi-player setting, one may regard N' as a variant of a ZD strategy alliance [15], where the players in N' agree to coordinate on the linear relations to be enforced on the expected payoffs. The above question then amounts to asking whether it is possible for a player to serve as a counteracting agent who participates in the ZD strategy alliance with a hidden intention to invalidate it by adopting a ZD strategy that is inconsistent with others.

The following proposition is the first main result of this paper.

Proposition 1. *Any set of ZD strategies is consistent.*

Proof. We first note that the following property holds for strategy vectors, whose proof is given in Methods.

Lemma 1. *Let* $T = (T_1, \cdots, T_N)$. *Then* $\mathbf{1}_M \notin \text{span } T$.

For any set $\text{span}(V_n)_{n\in N'}$ of ZD strategies, let K be the dimension of $\text{span}(V_n)_{n\in N'}$, and let $\boldsymbol{u}_1 = S\alpha_1, \cdots, \boldsymbol{u}_K = S\alpha_K$ be a basis of $\text{span}(V_n)_{n\in N'}$. The expected payoff vector $\boldsymbol{e} = (1, \bar{\boldsymbol{e}}^\top)^\top$ should be given by a non-zero solution of the linear equation $\bar{\boldsymbol{e}}^\top \bar{A} + \boldsymbol{b}^\top = \mathbf{0}_K^\top$ in $\bar{\boldsymbol{e}}$, where we define A, $\boldsymbol{b}$, and $\bar{A}$ as

$$A = \begin{pmatrix} \boldsymbol{b}^\mathsf{T} \\ \\ \bar{A} \end{pmatrix} \equiv (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_K). \tag{9}$$

One has

$$\mathcal{S}A = (\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_K) = \mathbf{1}_M \boldsymbol{b}^\mathsf{T} + \bar{\mathcal{S}}\bar{A}, \tag{10}$$

where $\bar{S} \equiv (\boldsymbol{s}_1, \cdots, \boldsymbol{s}_N)$.

The Rouché-Capelli theorem [29] tells us that rank $\bar{A}$ = rank A is a necessary and sufficient condition for the linear equation $\bar{\boldsymbol{e}}^\top \bar{A} + \boldsymbol{b}^\top = \mathbf{0}_K^\top$ in $\bar{\boldsymbol{e}}$ to have a solution, that is, for span $(V_n)_{n\in N'}$ to be consistent (because A is augmented matrix). An equivalent expression of this condition is that there is no vector $\boldsymbol{c} \in \mathbb{R}^K$ such that $\bar{A}\boldsymbol{c} = \mathbf{0}_N$ and $\boldsymbol{b}^\top \boldsymbol{c} \neq 0$ hold (which ensures that there is no elementary operations which make the rank of augmented matrix larger than that of the original matrix). Assume to the contrary that there exist $\boldsymbol{c} \in \mathbb{R}^K$ such that $\bar{A}\boldsymbol{c} = \mathbf{0}_N$ and $\boldsymbol{b}^\top \boldsymbol{c} \neq 0$ hold. One would then have

$$\mathcal{S}A\mathbf{c} = \mathbf{1}_M \boldsymbol{b}^\mathsf{T}\mathbf{c} + \bar{\mathcal{S}}\bar{A}\mathbf{c} = (\boldsymbol{b}^\mathsf{T}\mathbf{c})\mathbf{1}_M. \tag{11}$$

On the other hand, $SA\boldsymbol{c} = \sum_{k=1}^K c_k \boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_K \in \text{span}(V_n)_{n\in N'} \subset \text{span } T$, so that Lemma 1 states that it should be zero if it is proportional to $\mathbf{1}_M$, leading to contradiction.

Proposition 1 states that it is impossible for any player to serve as a counteracting agent to invalidate ZD strategy alliances. This statement is quite general in that it applies to any instance of repeated games covered by our formulation.

In Ref. [19], it was shown that every player can have at most one master player, who can play an equalizer strategy on the given player (that is, controlling the expected payoff of the given player), in multi-player multi-action games. Indeed, our general result on the absence of inconsistent ZD strategies (Proposition 1) immediately implies that more than one ZD players cannot simultaneously control the expected payoff of a player to different values. Therefore, our result generalizes their result on equalizer strategy to arbitrary ZD strategies.

Since the dimension of span T_n is at most $(M_n - 1)$, depending on S, it should be possible for player n with $M_n \geq 3$ to adopt a ZD strategy for which dim $V_n \geq 2$ holds. The dimension of V_n corresponds to the number of independent linear relations to be enforced on the expected payoffs of the players, so that it implies that one player may be able to enforce multiple independent linear relations. On the other hand, our result on the absence of inconsistent ZD strategies implies that for any set N' of ZD players the dimension of $\text{span}(V_n)_{n\in N'}$ should be at most N, the number of players, since any set of ZD strategies should contain at most N independent linear relations if it is consistent. This in turn implies that if the dimension of $\text{span}(V_n)_{n\in N'}$ is equal to N for a subset N' of players then players not in N' cannot employ independent ZD strategy any more.

Independence

Another naturally-arising question would be regarding independence for a set of ZD strategies, which we define as follows:

Definition 3. *A set* $\{T_n\}_{n\in N'}$ *of ZD strategies is independent if any set* $\{\mathbf{v}_n\}_{n\in N'}$ *of non-zero vectors* $\boldsymbol{v}_n$ *in* V_n *is linearly independent. Otherwise,* $\{T_n\}_{n\in N'}$ *is said to be* dependent.

If a set of ZD strategies is dependent, then there exists a ZD player whose ZD strategy adds no linear constraints other than those already imposed by other ZD players. One of the simplest example of a dependent set of ZD strategies is the case where two players enforce exactly the same linear relation to the expected payoffs. Our second main result is to show that any set of ZD strategies is independent under a general condition.

Proposition 2. *Let* N' *be a subset of players. Assume that* $\tilde{\boldsymbol{T}}_n(\sigma_n)$ *does not have zero elements for any* $n \in N'$ *and any* $\sigma_n \in \{1, \ldots, M_n\}$*. Then, any set* $\{T_n\}_{n\in N'}$ *of ZD strategies of players in* N' *is independent*.

See Methods for the proof.

It should be noted that when $\tilde{\boldsymbol{T}}_n(\sigma_n)$ has zero elements then one might have dependent ZD strategies. A simple example can be found in a two-player two-action perfect-monitoring (iterated prisoner's dilemma) game: Let the payoff vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ for players 1 and 2 be $\boldsymbol{s}_1 = (R, S, T, P)^\top$ and $\boldsymbol{s}_2 = (R, T, S, P)^\top$, with $T \neq S$. If player 1 adopts the strategy

$$\tilde{\boldsymbol{T}}_1(1) = (0, -1, 1, 0)^\mathsf{T} = \frac{1}{T-S}\boldsymbol{s}_1 - \frac{1}{T-S}\boldsymbol{s}_2, \tag{12}$$

then it enforces the linear payoff relation $e_1 = e_2$. This strategy is a well-known tit-for-tat strategy [6]. By symmetry, player 2 can also adopt the same strategy $\tilde{\boldsymbol{T}}_2(1) = -\tilde{\boldsymbol{T}}_1(1)$, implying that these two strategies are indeed dependent.

Simultaneous multiple linear relations by one player

As mentioned above, when the number M_n of possible actions for player n is more than two, player n may be able to employ a ZD strategy with dim $V_n \geq 2$ to simultaneously enforce more than one linear relations. (We note that this is impossible for public goods game [15, 16] because the number of action for each player is two.) Such a possibility has never been reported in the context of ZD strategies. Here, we provide a simple example of such a situation in a two-player three-action symmetric game.

We consider the 3×3 symmetric game

$$\begin{aligned} \boldsymbol{s}_1 &= (0, r_1, 0, r_2, 0, 0, 0, 0, 0)^\mathsf{T} \\ \boldsymbol{s}_2 &= (0, r_2, 0, r_1, 0, 0, 0, 0, 0)^\mathsf{T}. \end{aligned} \tag{13}$$

We remark that $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{1}_9$ are linearly independent when $r_1 \neq r_2$ and $r_1 \neq -r_2$. We choose strategies of player 1 as

$$\begin{aligned} \boldsymbol{T}_1(1) &= (1, 1-p, 1, p', 0, 0, 0, 0, 0)^\mathsf{T} \\ \boldsymbol{T}_1(2) &= (0, q, 0, 1-q', 1, 1, 0, 0, 0)^\mathsf{T} \\ \boldsymbol{T}_1(3) &= (0, p-q, 0, q'-p', 0, 0, 1, 1, 1)^\mathsf{T} \end{aligned} \tag{14}$$

with $0 \leq p \leq 1$, $0 \leq q \leq 1$, $0 \leq p' \leq 1$, $0 \leq q' \leq 1$, $q \leq p$, and $p' \leq q'$. Then we obtain

$$\frac{q'r_1 + qr_2}{p'q - pq'}\tilde{\boldsymbol{T}}_1(1) + \frac{p'r_1 + pr_2}{p'q - pq'}\tilde{\boldsymbol{T}}_1(2) \quad = \boldsymbol{s}_1 \tag{15}$$

$$\frac{q'r_2 + qr_1}{p'q - pq'}\tilde{\boldsymbol{T}}_1(1) + \frac{p'r_2 + pr_1}{p'q - pq'}\tilde{\boldsymbol{T}}_1(2) \quad = \boldsymbol{s}_2. \tag{16}$$

Therefore, player 1 can simultaneously control average payoffs of both players, e_1 and e_2, as $e_1 = e_2 = 0$. Note that $\boldsymbol{\sigma}$ with $s_1(\boldsymbol{\sigma}) = 0$ is an absorbing state regardless of the strategy of player 2 in this case.

In general, when one player simultaneously enforces two linear relations in two-player multi-action symmetric games, only $e_1 = e_2 = C$ is allowed with some C. This is explained as follows: Assume that player 1 can simultaneously enforce $e_1 = C_1$ and $e_2 = C_2$ with $C_1 \neq C_2$ by one ZD strategy. Because the game is symmetric, player 2 can also simultaneously enforce $e_1 = C_2$ and $e_2 = C_1$ independently by one ZD strategy. This contradicts the consistency of ZD strategies (Proposition 1). Therefore, the only possibility is $e_1 = e_2 = C$.

The above argument can be extended straightforwardly to the multi-player case. For that purpose, we introduce some notions of symmetric multi-player games. The following definition of a symmetric multi-player game is due to von Neumann and Morgenstern [30, Section 28].

Definition 4. *A game is* symmetric *with respect to a permutation* π *on* $\{1, \ldots, N\}$ *if* $M_n = M_{\pi(n)}$ *holds for any* $n \in \{1, \ldots, N\}$ *and if* π *preserves the payoff structure of the game, that is,*

$$s_{\pi(n)}(\boldsymbol{\sigma}) = s_n(\boldsymbol{\sigma}_\pi) \tag{17}$$

holds for any $\boldsymbol{\sigma} \in \Sigma$ *and for any* $n \in \{1, \ldots, N\}$, *where* $\boldsymbol{\sigma}_\pi \equiv (\sigma_{\pi(1)}, \ldots, \sigma_{\pi(N)})$.

The following definition is due to Ref. [31].

Definition 5. *A game is* weakly symmetric *if for any pair of players n and* $\bar{n}$ *there exists some permutation* π *on* $\{1, \ldots, N\}$ *satisfying* $\pi(n) = \bar{n}$ *such that the game is symmetric with respect to* π.

Consider an N-player weakly symmetric game. Assume that one player simultaneously enforces N independent linear relations on the average payoffs $\{e_n\}_{n \in \{1, \ldots, N\}}$ of N players via adopting an N-dimensional ZD strategy. (Note that for this to be possible the number M_n of actions should satisfy $M_n \geq N + 1$). Then, the average payoffs $\{e_n\}_{n\in\{1,\ldots,N\}}$ should be simultaneously controlled, but they should satisfy $e_1 = e_2 = \cdots = e_n$ due to the consistency of ZD strategies.

The difficulty of construction of a ZD strategy of one player with dimension N in weakly symmetric N-player games can be seen in the following two propositions, whose proofs are given in Methods.

Proposition 3. *In a weakly symmetric N-player game, if the strategy vectors of one player contain no zero element, then a ZD strategy of the player with dimension N is impossible.*

Proposition 4. *In a weakly symmetric N-player game, if payoffs* $s_n(\boldsymbol{\sigma})$ *of player n are different from each other for all* $\boldsymbol{\sigma}$, *then a ZD strategy with dimension N is impossible.*

Discussion

In this paper, we have derived ZD strategies for general multi-player multi-action public-monitoring games, in which players cannot observe actions of other players. By formulating ZD strategy in terms of linear algebra, we have proved that linear payoff relations enforced by ZD players are consistent. Furthermore, we have proved that linear payoff relations enforced by players with ZD strategies are independent under a general condition. We emphasize that these results hold not only for imperfect-monitoring games but also for perfect-monitoring games. We have also provided a simple example in which one player can simultaneously enforce more than one linear constraints on the expected payoffs. These results elucidate constraints on ZD strategies in terms of linear algebra.

Although we have discussed mathematical properties of ZD strategies if exist, we do not know the criterion for whether ZD strategies exist or not when a game is given. For example, we can easily show that ZD strategy does not exist for the rock-paper-scissors game, which is the simplest two-player three-action symmetric zero-sum game. (See S1 Text for the proof.) Whereas, we can also show that there is a two-player three-action symmetric zero-sum game for which ZD strategy exists, which is also provided in S1 Text. Generally, the dimension of span S is smaller than $N + 1$ for zero-sum games, and construction of ZD strategies for zero-sum games is expected to be more difficult compared to non-zero-sum games. Consistency together with constraints on payoffs such as symmetry and linear dependence may be useful to specify the space of ZD strategies which can exist. Specifying a general criterion for the existence of ZD strategies is an important future problem.

In addition, it should be noted that ZD strategies are not always "rational" strategies, which have been a main subject of game theory. Therefore, investigation of ZD strategies in terms of bounded rationality [32] may be needed. Specifying the situation where ZD strategies are adopted is another important problem.

Another remark is related to memory of strategies. In this work, we considered only memory-one strategies. In Ref. [6], it has been proved that a player with longer memory does not have advantage over a player with short memory in terms of average payoff in two-player games. In Ref. [16, 19], it has been shown that this statement also holds for multi-player games. Therefore, considering only memory-one strategies should be sufficient even in our public-monitoring situation. Longer memory strategies attract much attentions in repeated games with implementation errors [33, 34]. Extension of ZD strategies to longer memory case may lead to different evolutionary behavior compared to memory-one strategies.

We remark on the effect of imperfect monitoring. In perfect monitoring case, the strategy vectors are arbitrary as long as they satisfy the conditions for probability distributions. In contrast, in imperfect monitoring case, forms of the strategy vectors are constrained by Eq (5). Therefore, the space of ZD strategies for imperfect-monitoring games is generally smaller than that for perfect-monitoring games. In S1 Text, we provide examples of ZD strategies in simple imperfect-monitoring games.

Methods

Proof of Lemma 1

Assume to the contrary that $\upsilon \equiv \gamma \mathbf{1}_M \in \text{span } T$ with $\gamma \neq 0$. Taking the inner product of $\boldsymbol{v}$ with the stationary distribution $\boldsymbol{\rho}$, one has $\rho^\top \upsilon = 0$ since $\upsilon \in \text{span } T$ is represented as a linear combination of the strategy vectors and since the inner product of a strategy vector and the stationary distribution is zero. On the other hand, $\gamma \rho^\top \mathbf{1}_M = \gamma$ holds because of the normalization of the stationary distribution. Therefore we obtain $\gamma = 0$, leading to contradiction.

Proof of Proposition 2

We first show the following lemma.

Lemma 2. *Let N' be a subset of players. Assume that $\tilde{\boldsymbol{T}}_n(\sigma_n)$ does not have zero elements for any $n \in N'$ and any $\sigma_n \in \{1, \ldots, M_n\}$. For $n \in N'$, let $\boldsymbol{v}_n$ be an arbitrary non-zero vector in* span T_n. *Then $\{\mathbf{v}_n\}_{n\in N'}$ are linearly independent.*

Proof. We assume to the contrary that $\{\mathbf{v}_n\}_{n\in N'}$ are linearly dependent. Then there is a set of coefficients $\{a_n\}_{n\in N'}$ with which $\sum_{n\in N'} a_n \mathbf{v}_n = \mathbf{0}_M$ holds. Without loss of generality we assume $a_n \neq 0$ for $n \in N'$.

Since $\upsilon_n \in \text{span } T_n$, it is expressed as $\upsilon_n = T_n \boldsymbol{c}_n$ with a non-zero vector $\boldsymbol{c}_n = (c_{n,1}, \ldots, c_{n,M_n})^\top$. Let $\tilde{\sigma}_n \equiv \arg\min_{\sigma_n \in \{1,\ldots,M_n\}} \{a_n c_{n,\sigma_n}\}$, where ties may be broken arbitrarily, and $\tilde{c}_n \equiv c_{n,\tilde{\sigma}_n}$. With Eq (8), one obtains

$$\mathbf{v}_n = \mathcal{T}_n(\mathbf{c}_n - \tilde{c}_n \mathbf{1}_{M_n}), \tag{18}$$

and thus

$$a_n v_n(\boldsymbol{\sigma}') = \sum_{\sigma_n=1}^{M_n} a_n (c_{n,\sigma_n} - \tilde{c}_n) \tilde{T}_n(\sigma_n | \boldsymbol{\sigma}'). \tag{19}$$

We show that the inequality

$$a_n (c_{n,\sigma_n} - \tilde{c}_n) \tilde{T}_n(\sigma_n | \boldsymbol{\sigma}') \geq 0 \tag{20}$$

holds for any n, any $\sigma_n \in \{1, \ldots, M_n\}$, and any $\boldsymbol{\sigma}' \in \Sigma$ satisfying $\boldsymbol{\sigma}'_n = \tilde{\sigma}_n$. We first note that for any strategy vector $\tilde{\boldsymbol{T}}_n(\sigma_n)$ with action $\sigma_n \in \{1, \cdots, M_n\}$, one has, from Eq (6),

$$\tilde{T}_n(\sigma_n | \boldsymbol{\sigma}') \begin{cases} \leq 0, & \boldsymbol{\sigma}'_n = \sigma_n, \\ \geq 0, & \boldsymbol{\sigma}'_n \neq \sigma_n. \end{cases} \tag{21}$$

Fix any $\boldsymbol{\sigma}' \in \Sigma$ satisfying $\boldsymbol{\sigma}'_n = \tilde{\sigma}_n$ for a moment. Then, for $\sigma_n = \tilde{\sigma}_n$ one has $c_{n,\sigma_n} = \tilde{c}_n$ by definition, making the left-hand side of Eq (20) equal to zero. For $\sigma_n \neq \tilde{\sigma}_n$, on the other hand, one has $a_n(c_{n,\sigma_n} - \tilde{c}_n) \geq 0$ by definition. Also, since $\boldsymbol{\sigma}'_n = \tilde{\sigma}_n \neq \sigma_n$, from Eq (21) one has $\tilde{T}_n(\sigma_n | \sigma') \geq 0$. These imply that the inequality (20) holds for $\sigma_n \neq \tilde{\sigma}_n$. Putting the above arguments together, we have shown that the inequality (20) holds for any n, any $\sigma_n \in \{1, \ldots, M_n\}$, and any $\boldsymbol{\sigma}' \in \Sigma$ satisfying $\boldsymbol{\sigma}'_n = \tilde{\sigma}_n$.

Fix any $\boldsymbol{\sigma}' \in \Sigma$ satisfying $\boldsymbol{\sigma}'_n = \tilde{\sigma}_n$ for all $n \in N'$. The above argument has shown that the inequality (20) holds for any n and any $\sigma_n \in \{1, \ldots, M_n\}$. On the other hand, at the beginning of the proof we have assumed that

$$\sum_{n\in N'} a_n v_n(\boldsymbol{\sigma}') = \sum_{n\in N'} \sum_{\sigma_n=1}^{M_n} a_n (c_{n,\sigma_n} - \tilde{c}_n) \tilde{T}_n(\sigma_n | \boldsymbol{\sigma}') = 0 \tag{22}$$

holds, implying that the summand $a_n(c_{n,\sigma_n} - \tilde{c}_n)\tilde{T}_n(\sigma_n|\sigma')$ is equal to zero for any $n \in N'$ and any $\sigma_n \in \{1, \ldots, M_n\}$. By assumption, $a_n \neq 0$ and $\tilde{T}_n(\sigma_n|\sigma') \neq 0$, so that one has $c_{n,\sigma_n} = \tilde{c}_n$, and consequently, $\upsilon_n = \tilde{c}_n T_n \mathbf{1}_{M_n} = \mathbf{0}_M$, leading to contradiction.

The proof of Proposition 2 is straightforward by taking $\mathbf{v}_n$ as belonging to S in Lemma 2.

Proof of Proposition 3

We first show the following lemma.

Lemma 3. *Consider an N-player game which is symmetric with respect to a permutation π on* $\{1, \ldots, N\}$. *Assume that the column vectors of S are linearly independent. For any pair of players*

n and $\bar{n}$ satisfying $n \neq \pi(\bar{n})$, if the strategy vectors of these players contain no zero element, then it is impossible for these players to adopt ZD strategies with which player n enforces linear relation $\boldsymbol{e}^\top \alpha = 0$ with $\boldsymbol{\alpha} \neq \mathbf{0}_{N+1}$, and where player $\bar{n}$ enforces $\boldsymbol{e}^\top \alpha_\pi = 0$, where $\alpha_\pi \equiv (\alpha_0, \alpha_{\pi(1)}, \ldots, \alpha_{\pi(N)})^\top$.

Proof. We assume to the contrary that there exists $\boldsymbol{\alpha} \neq \mathbf{0}_{N+1}$ satisfying the properties stated in Lemma 3. By assumption, $S\alpha \in V_n = \text{span}\ T_n \cap \text{span}\ S$ and $S\alpha_\pi \in V\bar{n} = \text{span}\ T\bar{n} \cap \text{span}\ S$. There then exist $\mathbf{c}_n$ and $\bar{c}\bar{n}$ satisfying $T_n \boldsymbol{c}_n = S\alpha$ and $T\bar{n}\bar{\boldsymbol{c}}\bar{n} = S\alpha_\pi$. One has

$$\begin{aligned}(\mathcal{S}\boldsymbol{\alpha}_\pi)(\boldsymbol{\sigma}'_\pi) &= \alpha_0 + \sum_{n=1}^{N} \alpha_{\pi(n)} s_n(\boldsymbol{\sigma}'_\pi) \\ &= \alpha_0 + \sum_{n=1}^{N} \alpha_{\pi(n)} s_{\pi(n)}(\boldsymbol{\sigma}') = (\mathcal{S}\alpha)(\boldsymbol{\sigma}'),\end{aligned} \tag{23}$$

where the second equality is due to the assumed symmetry of the game with respect to π. Letting $\tilde{T}_{\bar{n},\pi}(\sigma_{\bar{n}}|\sigma') \equiv \tilde{T}_{\bar{n}}(\sigma_{\bar{n}}|\sigma'_\pi)$, $\tilde{\boldsymbol{T}}_{\bar{n},\pi}(\sigma_{\bar{n}}) \equiv (\tilde{T}_{\bar{n},\pi}(\sigma_{\bar{n}}|\sigma'))$, and $T_{\bar{n},\pi} \equiv (\tilde{\boldsymbol{T}}_{\bar{n},\pi}(1), \ldots, \tilde{\boldsymbol{T}}_{\bar{n},\pi}(M\bar{n}))$, one has

$$\begin{aligned}(\mathcal{T}_{\bar{n},\pi}\bar{\mathbf{c}}_{\bar{n}})(\boldsymbol{\sigma}') &= (\mathcal{T}_{\bar{n}}\bar{\mathbf{c}}_{\bar{n}})(\boldsymbol{\sigma}'_\pi) = (\mathcal{S}\alpha_\pi)(\boldsymbol{\sigma}'_\pi) \\ &= (\mathcal{S}\alpha)(\boldsymbol{\sigma}') = (\mathcal{T}_n\mathbf{c}_n)(\boldsymbol{\sigma}'),\end{aligned} \tag{24}$$

implying that $T_{\bar{n},\pi}\bar{\boldsymbol{c}}\bar{n} = T_n\boldsymbol{c}_n$ holds. Let $\upsilon = T_n\boldsymbol{c}_n = T_{\bar{n},\pi}\bar{\boldsymbol{c}}_{\bar{n}}$.

Let $\sigma_{n,\max} = \arg\max_{\sigma_n} c_{n,\sigma_n}$ and $\bar{\sigma}_{\bar{n},\min} = \arg\min_{\sigma_{\bar{n}}} \bar{c}_{\bar{n},\sigma_{\bar{n}}}$, where ties may be broken arbitrarily, and $c_{n,\max} = c_{n,\sigma_{n,\max}}$ and $\bar{c}_{\bar{n},\min} = \bar{c}_{\bar{n},\bar{\sigma}_{n,\min}}$. One then has

$$\boldsymbol{v} = \mathcal{T}_n(\mathbf{c}_n - c_{n,\max}\mathbf{1}_{M_n}) = \mathcal{T}_{\bar{n},\pi}(\bar{\mathbf{c}}_{\bar{n}} - \bar{c}_{\bar{n},\min}\mathbf{1}_{M\bar{n}}). \tag{25}$$

Recalling that we have assumed $n \neq \pi(\bar{n})$, let $\boldsymbol{\sigma}' \in \Sigma$ be an arbitrary state satisfying $\boldsymbol{\sigma}'_n = \sigma_{n,\max}$ and $\sigma'_{\pi(\bar{n})} = \bar{\sigma}_{\bar{n},\min}$. Then, in view of Eq (21), one has

$$\begin{aligned}v(\boldsymbol{\sigma}') &= \sum_{\sigma_n=1}^{M_n} (c_{n,\sigma_n} - c_{n,\max})\tilde{T}_n(\sigma_n|\boldsymbol{\sigma}') \leq 0, \\ &= \sum_{\sigma_{\bar{n}}=1}^{M_{\bar{n}}} (\bar{c}_{\bar{n},\sigma_{\bar{n}}} - \bar{c}_{\bar{n},\min})\tilde{T}_{\bar{n}}(\sigma_{\bar{n}}|\boldsymbol{\sigma}'_\pi) \geq 0,\end{aligned} \tag{26}$$

implying that $v(\boldsymbol{\sigma}') = 0$ holds. Since $(c_{n,\sigma_n} - c_{n,\max})\tilde{T}_n(\sigma_n|\sigma') \leq 0$ for all $\sigma_n \in \{1, \ldots, M_n\}$, they are all equal to zero. Since $\tilde{T}_n(\sigma_n|\sigma')$ is assumed non-zero, one has $c_{n,\sigma_n} = c_{n,\max}$ for all $\sigma_n \in \{1, \ldots, M_n\}$ and consequently $\boldsymbol{c}_n \propto \mathbf{1}_{M_n}$. One similarly has $\bar{\boldsymbol{c}}_{\bar{n}} \propto \mathbf{1}_{M_{\bar{n}}}$. Therefore, from Eq (8) one has $T_n\boldsymbol{c}_n = T_{\bar{n}}\bar{\boldsymbol{c}}_{\bar{n}} = \mathbf{0}_M$. Due to the assumption of linear independence of the columns of S, it in turn implies that $\boldsymbol{\alpha} = \mathbf{0}_{N+1}$ holds, leading to contradiction.

It should be noted that Lemma 3 holds even if one takes $\bar{n} = n$, in which case the Lemma implies that, if the game is symmetric with respect to π, player n with $\pi(n) \neq n$ cannot enforce linear relations $\boldsymbol{e}^\top\alpha = \boldsymbol{e}^\top\alpha_\pi = 0$ simultaneously. It should also be noted that Lemma 3 furthermore implies that it is impossible for that player to enforce a linear relation $\boldsymbol{e}^\top\alpha = 0$ satisfying $\boldsymbol{\alpha}_\pi = \boldsymbol{\alpha} \neq \mathbf{0}_{N+1}$. In other words, in a symmetric game no player to whom the game is symmetric can enforce a linear relation with the same symmetry as the game itself.

Proposition 3 is a direct consequence of Lemma 3 in weakly symmetric multi-player games.

Proof of Proposition 4

Without loss of generality, we assume that player k takes an N-dimensional ZD strategy determining the average payoffs e_n for $n = 1, \cdots, N$. Due to the above discussion, only $e_1 = \cdots = e_N = C$ is allowed. Letting $\alpha^{(n)} \equiv (-C, 0, \cdots, \underset{\hat{n}}{1}, \cdots, 0)$ for $n \in \{1, \ldots, N\}$, one can take $\{S\alpha^{(n)}\}_{n\in\{1,\ldots,N\}}$ as a basis of the N-dimensional ZD strategy. Let $\mathbf{c}^{(n)}$ be defined as

$$\mathcal{T}_k\boldsymbol{c}^{(n)} = \mathcal{S}\boldsymbol{\alpha}^{(n)} = \boldsymbol{s}_n - C\mathbf{1}_M, \quad n \in \{1, \ldots, N\}. \tag{27}$$

By the assumption of weak symmetry, for any player $n \neq k$, there exists a permutation π satisfying $\pi(n) = k$ such that the game is symmetric with respect to π. Noting that $\alpha_\pi^{(n)} = \alpha^{(k)}$, from Eq (23) one has

$$(\mathcal{T}_k\mathbf{c}^{(k)})(\boldsymbol{\sigma}'_\pi) = (\mathcal{T}_k\mathbf{c}^{(n)})(\boldsymbol{\sigma}'). \tag{28}$$

For $n \in \{1, \ldots, N\}$, define $\sigma_{\max}^{(n)} \equiv \arg\max_{\sigma_k} c_{\sigma_k}^{(n)}$ and $\sigma_{\min}^{(n)} \equiv \arg\min_{\sigma_k} c_{\sigma_k}^{(n)}$, where ties may be broken arbitrarily provided that $\sigma_{\max}^{(n)} \neq \sigma_{\min}^{(n)}$ holds, and $c_{\max}^{(n)} = c_{\sigma_{\max}}^{(n)}$ and $c_{\min}^{(n)} = c_{\sigma_{\min}}^{(n)}$. From Eq (7), one has

$$\begin{aligned} \mathcal{T}_k\mathbf{c}^{(n)} &= \mathcal{T}_k(\mathbf{c}^{(n)} - c_{\max}^{(n)}\mathbf{1}_{M_k}) \\ &= \mathcal{T}_k(\mathbf{c}^{(n)} - c_{\min}^{(n)}\mathbf{1}_{M_k}). \end{aligned} \tag{29}$$

Then, from Eqs (28) and (21), we obtain for an arbitrary $\boldsymbol{\sigma}^*$ satisfying $\boldsymbol{\sigma}_k^* = \sigma_{\max}^{(n)}$ and $\boldsymbol{\sigma}_n^* = \sigma_{\min}^{(k)}$

$$s_n(\boldsymbol{\sigma}^*) - C = (\mathcal{T}_k(\mathbf{c}^{(n)} - c_{\max}^{(n)}\mathbf{1}_{M_k}))(\boldsymbol{\sigma}^*) \leq 0 \tag{30}$$

$$= (\mathcal{T}_k(\mathbf{c}^{(k)} - c_{\min}^{(k)}\mathbf{1}_{M_k}))(\boldsymbol{\sigma}_\pi^*) \geq 0 \tag{31}$$

implying $s_n(\boldsymbol{\sigma}^*) = C$. On the other hand, we also obtain for an arbitrary $\boldsymbol{\sigma}^{**}$ satisfying $\boldsymbol{\sigma}_k^{**} = \sigma_{\min}^{(n)}$ and $\boldsymbol{\sigma}_n^{**} = \sigma_{\max}^{(k)}$

$$s_n(\boldsymbol{\sigma}^{**}) - C = (\mathcal{T}_k(\mathbf{c}^{(n)} - c_{\min}^{(n)}\mathbf{1}_{M_k}))(\boldsymbol{\sigma}^{**}) \geq 0 \tag{32}$$

$$= (\mathcal{T}_k(\mathbf{c}^{(k)} - c_{\max}^{(k)}\mathbf{1}_{M_k}))(\boldsymbol{\sigma}_\pi^{**}) \leq 0 \tag{33}$$

implying $s_n(\boldsymbol{\sigma}^{**}) = C$. Then, because we have assumed that all elements of the payoff vector $\mathbf{s}_n$ are different from each other, we have arrived at a contradiction.

Supporting information

S1 Text. Details of discussion.
(PDF)

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