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Research Article

Multiple Positive Solutions to Nonlinear Boundary Value Problems of a System for Fractional Differential Equations

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By using Krasnoselskii's fixed point theorem, we study the existence of at least one or two positive solutions to a system of fractional boundary value problems given by $-D_{0^+}^{\nu_1}y_1(t) = \lambda_1 a_1(t)f(y_1(t), y_2(t))$, $-D_{0^+}^{\nu_2}y_2(t) = \lambda_2 a_2(t)g(y_1(t), y_2(t))$, where $D_{0^+}^{\nu}$ is the standard Riemann-Liouville fractional derivative, $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in N$, subject to the boundary conditions $y_1^{(i)}(0) = 0 = y_2^{(i)}(0)$, for $0 \le i \le n-2$, and $[D_{0^+}^{\alpha}y_1(t)]_{t=1} = 0 = [D_{0^+}^{\alpha}y_2(t)]_{t=1}$, for $1 \le \alpha \le n-2$, or $y_1^{(i)}(0) = 0 = y_2^{(i)}(0)$, for $0 \le i \le n-2$, and $[D_{0^+}^{\alpha}y_1(t)]_{t=1} = \phi_1(y_1)$, $[D_{0^+}^{\alpha}y_2(t)]_{t=1} = \phi_2(y_2)$, for $1 \le \alpha \le n-2$, $\phi_1, \phi_2 \in C([0,1], R)$. Our results are new and complement previously known results. As an application, we also give an example to demonstrate our result.

1. Introduction

The purpose of this paper is to consider the existence of multiple positive solutions for the following system of nonlinear fractional differential equations:

$$
\begin{aligned}
-D_{0^+}^{\nu_1}y_1(t) &= \lambda_1 a_1(t) f(y_1(t), y_2(t)), \\
-D_{0^+}^{\nu_2}y_2(t) &= \lambda_2 a_2(t) g(y_1(t), y_2(t)),
\end{aligned} \tag{1}
$$

where $t \in (0,1)$, $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in N$, and $\lambda_1, \lambda_2 > 0$, subject to a couple of boundary conditions. In particular, we first consider (1) subject to

$$
\begin{aligned}
y_1^{(i)}(0) = 0 = y_2^{(i)}(0), &\quad 0 \le i \le n-2, \\
[D_{0^+}^{\alpha}y_1(t)]_{t=1} = 0 = [D_{0^+}^{\alpha}y_2(t)]_{t=1}, &\quad 1 \le \alpha \le n-2,
\end{aligned} \tag{2}
$$

where $f, g \in C([0,\infty) \times [0,\infty), [0,\infty))$, and $a_1, a_2 \in C([0,1], [0,\infty))$. We then consider the case in which the boundary conditions are changed to

$$
\begin{aligned}
y_1^{(i)}(0) = 0 = y_2^{(i)}(0), &\quad 0 \le i \le n-2, \\
[D_{0^+}^{\alpha}y_1(t)]_{t=1} &= \phi_1(y_1), \\
[D_{0^+}^{\alpha}y_2(t)]_{t=1} &= \phi_2(y_2), \\
1 \le \alpha &\le n-2,
\end{aligned} \tag{3}
$$

where $\phi_1, \phi_2 \in C([0,1], R)$.

Fractional differential equations arise in many fields, such as physics, mechanics, chemistry, economics, and engineering and biological sciences; see [1–11] for example. In recent years, the study of positive solutions for fractional differential equation boundary value problems has attracted considerable attention, and fruits from research into it emerge continuously. For a small sample of such work, we refer the reader to [12–20] and the references therein. The situation of at least one positive solution has been studied in many excellent monograph; see [12–19, 21] and other references therein. In [22], by means of Schauder fixed point theorem, Su investigated the existence of one positive solution to the following boundary value problem for a coupled system of nonlinear fractional differential equations:

$$
\begin{aligned}
D_{0^+}^{\alpha}y_1(t) = f\left(t, y_2(t), D_{0^+}^{\mu}y_2(t)\right), &\quad 0 < t < 1, \\
-D_{0^+}^{\beta}y_2(t) = g\left(t, y_1(t), D_{0^+}^{\nu}y_1(t)\right), &\quad 0 < t < 1,
\end{aligned}
$$

$$y_1(0) = y_1(1) = y_2(0) = y_2(1) = 0, \tag{4}$$

where $1 < \alpha,\ \beta < 2,\ \mu, \nu > 0,\ \alpha - \nu \geq 1,\ \beta - \mu \geq 1$.

In [21], Goodrich established the existence of one positive solution to problems (1)-(2) and (1), (3) by using Krasnoselskii's fixed point theorem. Different from the above works mentioned, in this paper we will present the existence of at least two positive solutions to problems (1)-(2) and (1), (3) by using the similar method presented in [21]. Moreover, under different conditions, we also present the existence of at least one positive solution to problems (1)-(2) and (1), (3) with $\lambda_1 = \lambda_2 = 1$.

2. Preliminaries

For the convenience of the reader, we present here some definitions, lemmas, and basic results that will be used in the proofs of our theorems.

Definition 1 (see [23]). Let $\nu > 0$ with $\nu \in R$. Suppose that $y : [a, +\infty) \to R$. Then the νth Riemann-Liouville fractional integral is defined to be

$$D_{a^+}^{-\nu} y(t) := \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1} ds, \tag{5}$$

whenever the right-hand side is defined. Similarly, with $\nu > 0$ and $\nu \in R$, we define the νth Riemann-Liouville fractional derivative to be

$$D_{a^+}^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}} ds, \tag{6}$$

where $n \in N$ is the unique positive integer satisfying $n - 1 \leq \nu < n$ and $t > a$.

Lemma 2 (see [24]). *Let $g \in C^n([0,1])$ be given. Then the unique solution to problem $-D_{0^+}^{\nu} y(t) = g(t)$ together with the boundary conditions $y^{(i)}(0) = 0 = [D_{0^+}^{\alpha} y(t)]_{t=1}$, where $1 \leq \alpha \leq n-2$ and $0 \leq i \leq n-2$, is*

$$y(t) = \int_0^1 G(t,s) g(s) ds, \tag{7}$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \leq s \leq t \leq 1 \\ \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & 0 \leq t \leq s \leq 1 \end{cases} \tag{8}$$

is the Green function for this problem.

Lemma 3 (see [24]). *Let $G(t,s)$ be as given in the statement of Lemma 2. Then one finds that*

(i) *$G(t,s)$ is a continuous function on the unit square $[0,1] \times [0,1]$;*

(ii) *$G(t,s) \geq 0$ for each $(t,s) \in [0,1] \times [0,1]$;*

(iii) *$\max_{t\in[0,1]} G(t,s) = G(1,s)$, for each $s \in [0,1]$.*

Lemma 4 (see [24]). *Let $G(t,s)$ be as given in the statement of Lemma 2. Then there exists a constant $\gamma \in (0,1)$ such that*

$$\min_{t\in[(1/2),1]} G(t,s) \geq \gamma \max_{t\in[0,1]} G(t,s) = \gamma G(1,s). \tag{9}$$

To prove our results, we need the following Krasnoselskii's fixed point theorem which can be seen in Guo and Lakshmikantham [25].

Lemma 5 (see [25]). *Let E be a Banach space, and let P be a cone. Assume that Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let $T : P \bigcap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator such that*

(i) *$\|Tu\| \leq \|u\|, \forall u \in P \bigcap \partial\Omega_1$, and $\|Tu\| \geq \|u\|, \forall u \in P \bigcap \partial\Omega_2$; or*

(ii) *$\|Tu\| \geq \|u\|, \forall u \in P \bigcap \partial\Omega_1$, and $\|Tu\| \leq \|u\|, \forall u \in P \bigcap \partial\Omega_2$.*

Then T has a fixed point in $P \bigcap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Main Results

In this section, we apply Lemma 5 to study problems (1)-(2) and (1), (3), and we obtain some new results on the existence of multiple positive solutions.

3.1. Problem (1)-(2) in the General Case. In our considerations, let E represent the Banach space of $C([0,1])$ when equipped with the usual supremum norm, $\|\cdot\|$. Then put $X := E \times E$, where X is equipped with the norm $\|(y_1, y_2)\| := \|y_1\| + \|y_2\|$ for $(y_1, y_2) \in X$. Observe that X is also a Banach space (see [26]). In addition, we define two operators T_1, $T_2 : X \to E$ by

$$\begin{aligned} T_1(y_1, y_2)(t) &:= \lambda_1 \int_0^1 G_1(t,s) a_1(s) f(y_1(s), y_2(s)) ds, \\ T_2(y_1, y_2)(t) &:= \lambda_2 \int_0^1 G_2(t,s) a_2(s) g(y_1(s), y_2(s)) ds, \end{aligned} \tag{10}$$

where $G_1(t,s)$ is the Green function of Lemma 2 with ν replaced by ν_1 and, likewise, $G_2(t,s)$ is the Green function of Lemma 2 with ν replaced by ν_2. Now, we define an operator $S : X \to X$ by

$$\begin{aligned} S(y_1, y_2)(t) &:= (T_1(y_1, y_2)(t), T_2(y_1, y_2)(t)) \\ &= \left(\lambda_1 \int_0^1 G_1(t,s) a_1(s) f(y_1(s), y_2(s)) ds,\right. \\ &\qquad \left.\lambda_2 \int_0^1 G_2(t,s) a_2(s) g(y_1(s), y_2(s)) ds\right). \end{aligned} \tag{11}$$

We claim that whenever $(y_1, y_2) \in X$ is a fixed point of the operator defined in (11), it follows that $y_1(t)$ and $y_2(t)$ solve problems (1)-(2). That is, a pair of functions $y_1, y_2 \in X$ is a solution of problems (1)-(2) if and only if y_1, y_2 is a fixed point of the operator S defined in (11) (see [26]).

In the following, we will look for fixed points of the operator S, because these fixed points coincide with solutions of problems (1)-(2). For use in the sequel, let γ_1 and γ_2 be the constants given by Lemma 4 associated, respectively, with the Green functions G_1 and G_2, and define $\widetilde{\gamma}$ by $\widetilde{\gamma} := \min\{\gamma_1, \gamma_2\}$, and notice that $\widetilde{\gamma} \in (0, 1)$.

For the sake of convenience, we set

$$\begin{aligned} f_0 &= \lim_{(y_1,y_2)\to(0^+,0^+)} \frac{f(y_1, y_2)}{y_1 + y_2}, \\ g_0 &= \lim_{(y_1,y_2)\to(0^+,0^+)} \frac{g(y_1, y_2)}{y_1 + y_2}, \\ f_\infty &= \lim_{(y_1,y_2)\to(\infty,\infty)} \frac{f(y_1, y_2)}{y_1 + y_2}, \\ g_\infty &= \lim_{(y_1,y_2)\to(\infty,\infty)} \frac{g(y_1, y_2)}{y_1 + y_2}. \end{aligned} \tag{12}$$

Now we list some assumptions:

(F_1) $f_0, g_0 \in (0, +\infty)$;

(F_2) $f_\infty, g_\infty \in (0, +\infty)$;

(F_3) there are numbers Φ_1, Φ_2, where

$$\begin{aligned} \Phi_1 &:= \max\left\{\frac{1}{2}\left[\int_{1/2}^1 \widetilde{\gamma} G_1(1,s)\, a_1(s)\, f_\infty ds\right]^{-1}, \right.\\ &\qquad \left. \frac{1}{2}\left[\int_{1/2}^1 \widetilde{\gamma} G_2(1,s)\, a_2(s)\, g_\infty ds\right]^{-1}\right\}, \\ \Phi_2 &:= \min\left\{\frac{1}{2}\left[\int_0^1 G_1(1,s)\, a_1(s)\, f_0 ds\right]^{-1}, \right.\\ &\qquad \left. \frac{1}{2}\left[\int_0^1 G_2(1,s)\, a_2(s)\, g_0 ds\right]^{-1}\right\}, \end{aligned} \tag{13}$$

such that $\Phi_1 < \lambda_1, \lambda_2 < \Phi_2$.

Next, we define the cone K by

$$\begin{aligned} K := &\left\{(y_1, y_2) \in X : y_1, y_2 \geq 0, \right.\\ &\left. \min_{t\in[(1/2),1]} [y_1(t) + y_2(t)] \geq \widetilde{\gamma} \|(y_1, y_2)\|\right\}. \end{aligned} \tag{14}$$

Lemma 6 (see [21]). *Let S be the operator defined by* (11). *Then* $S : K \to K$.

Lemma 7. *S is a completely continuous operator.*

Proof. The operator $T_1 : K \to E$ is continuous in view of nonnegativeness and continuity of $G_1(t, s)$, $f(y_1, y_2)$ and $a_1(t)$.

Let $\Omega \subseteq K$ be bounded; that is, there exists a positive constant $M > 0$ such that $\|(y_1, y_2)\| \leq M$, for all $(y_1, y_2) \in \Omega$. Let $L = \max_{0\leq t\leq 1,\, 0\leq\|(y_1,y_2)\|\leq M} |a_1(t) f(y_1(t), y_2(t))| + 1$; then, for $(y_1, y_2) \in \Omega$, we have

$$\begin{aligned} &|T_1(y_1, y_2)(t)| \\ &\quad \leq \lambda_1 \int_0^1 G_1(1,s)\, a_1(s)\, f(y_1(s), y_2(s))\, ds \\ &\quad \leq \lambda_1 \int_0^1 G_1(1,s)\, L ds \\ &\quad \leq L\Phi_2 \int_0^1 G_1(1,s)\, ds < +\infty. \end{aligned} \tag{15}$$

Hence, $T_1(\Omega)$ is bounded.

On the other hand, given $\varepsilon > 0$, setting $\delta = \min\{(1/2)(\Gamma(\nu_1)\varepsilon/L\Phi_2)^{1/(\nu_1-1)}, \varepsilon\Gamma(\nu_1)/(\nu_1 - 1)L\Phi_2\}$, then, for each $(y_1, y_2) \in \Omega$, $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, and $t_2 - t_1 < \delta$, one has $|T_1(y_1, y_2)(t_2) - T_1(y_1, y_2)(t_1)| < \varepsilon$. That is to say, $T(\Omega)$ is equicontinuity. In fact,

$$\begin{aligned} &|T_1(y_1, y_2)(t_2) - T_1(y_1, y_2)(t_1)| \\ &= \left|\lambda_1 \int_0^1 G_1(t_2,s)\, a_1(s)\, f(y_1(s), y_2(s))\, ds \right.\\ &\qquad \left. -\lambda_1 \int_0^1 G_1(t_1,s)\, a_1(s)\, f(y_1(s), y_2(s))\, ds\right| \\ &= \left|\lambda_1 \int_0^{t_1} [G_1(t_2,s) - G_1(t_1,s)]\, a_1(s)\, f(y_1(s), y_2(s))\, ds \right.\\ &\qquad + \lambda_1 \int_{t_1}^{t_2} [G_1(t_2,s) - G_1(t_1,s)]\, a_1(s) \\ &\qquad\qquad \times f(y_1(s), y_2(s))\, ds \\ &\qquad + \lambda_1 \int_{t_2}^1 [G_1(t_2,s) - G_1(t_1,s)]\, a_1(s) \\ &\qquad\qquad \left. \times f(y_1(s), y_2(s))\, ds\right| \\ &\leq \frac{\Phi_2 L}{\Gamma(\nu_1)} \left[\int_0^{t_1} (1-s)^{\nu_1-\alpha-1} \left(t_2^{\nu_1-1} - t_1^{\nu_1-1}\right) ds \right.\\ &\qquad + \int_{t_2}^1 (1-s)^{\nu_1-\alpha-1} \left(t_2^{\nu_1-1} - t_1^{\nu_1-1}\right) ds \\ &\qquad \left. + \int_{t_1}^{t_2} (1-s)^{\nu_1-\alpha-1} \left(t_2^{\nu_1-1} - t_1^{\nu_1-1}\right) ds\right] \\ &= \frac{\Phi_2 L}{\Gamma(\nu_1)} \frac{1}{\nu_1 - \alpha} \left(t_2^{\nu_1-1} - t_1^{\nu_1-1}\right) \\ &\leq \frac{\Phi_2 L}{\Gamma(\nu_1)} \left(t_2^{\nu_1-1} - t_1^{\nu_1-1}\right). \end{aligned} \tag{16}$$

In the following, we divide the proof into two cases.

Case 1. If $\delta \le t_1 < t_2 < 1$, then we have

$$\begin{aligned} &\left|T_1(y_1,y_2)(t_2) - T_1(y_1,y_2)(t_1)\right| \\ &\quad \le \frac{\Phi_2 L}{\Gamma(\nu_1)}\left(t_2^{\nu_1-1} - t_1^{\nu_1-1}\right) \\ &\quad = \frac{\Phi_2 L}{\Gamma(\nu_1)}(\nu_1-1)(t_2-t_1)t_\xi^{\nu_1-2} \\ &\quad < \frac{\Phi_2 L}{\Gamma(\nu_1)}(\nu_1-1)\delta \le \varepsilon, \end{aligned} \tag{17}$$

where $t_\xi \in (t_1, t_2)$.

Case 2. If $0 \le t_1 < \delta,\ t_2 < 2\delta$, then we have

$$\begin{aligned} &\left|T_1(y_1,y_2)(t_2) - T_1(y_1,y_2)(t_1)\right| \\ &\quad \le \frac{\Phi_2 L}{\Gamma(\nu_1)}\left(t_2^{\nu_1-1} - t_1^{\nu_1-1}\right) \\ &\quad \le \frac{\Phi_2 L}{\Gamma(\nu_1)}t_2^{\nu_1-1} < \frac{\Phi_2 L}{\Gamma(\nu_1)}(2\delta)^{\nu_1-1} \le \varepsilon. \end{aligned} \tag{18}$$

By the means of the Arzela-Ascoli theorem, we have that T_1 is completely continuous. Similarly, T_2 is completely continuous. Consequently, $S : K \to K$ is a completely continuous operator. This completes the proof. □

In [21], Goodrich established the following result.

Theorem 8 (see Theorem 3.3 in [21]). *Suppose that* (F_1)–(F_3) *are satisfied. Then problem* (1)-(2) *has at least one positive solution.*

From Theorem 8, the following problem is natural: whether we can obtain some conclusions or not, if $f_0 = f_\infty = g_0 = g_\infty = 0$ or $f_0 = f_\infty = g_0 = g_\infty = \infty$? In the rest of this paper, we give some answers to this problem.

For the sake of convenience, we make some assumptions:

(H_1) there exist constants $\rho_1, A_1 > 0$, such that

$$f(y_1,y_2), g(y_1,y_2) < A_1^{-1}\rho_1 \quad \text{for } 0 \le \|(y_1,y_2)\| \le \rho_1; \tag{19}$$

(H_2) there exist constants $\rho_2, A_2 > 0$, such that

$$f(y_1,y_2), g(y_1,y_2) \ge A_2^{-1}\rho_2 \quad \text{for } \widetilde{\gamma}\rho_2 \le \|(y_1,y_2)\| \le \rho_2; \tag{20}$$

(P_1) there are numbers Λ_1, Λ_2, where

$$\begin{aligned} \Lambda_1 &:= \max\left\{\frac{1}{2}\left[\int_{1/2}^1 \widetilde{\gamma}G_1(1,s)a_1(s)\,ds\right]^{-1}, \frac{1}{2}\left[\int_{1/2}^1 \widetilde{\gamma}G_2(1,s)a_2(s)\,ds\right]^{-1}\right\}, \\ \Lambda_2 &:= \min\left\{\frac{A_1}{2}\left[\int_0^1 G_1(1,s)a_1(s)\,ds\right]^{-1}, \frac{A_1}{2}\left[\int_0^1 G_2(1,s)a_2(s)\,ds\right]^{-1}\right\}, \end{aligned} \tag{21}$$

such that $\Lambda_1 < \lambda_1,\ \lambda_2 < \Lambda_2$;

(P_2) there are numbers $\Lambda_3,\ \Lambda_4$, where

$$\begin{aligned} \Lambda_3 &:= \max\left\{\frac{A_2}{2}\left[\int_{1/2}^1 G_1(1,s)a_1(s)\,ds\right]^{-1}, \frac{A_2}{2}\left[\int_{1/2}^1 G_2(1,s)a_2(s)\,ds\right]^{-1}\right\}, \\ \Lambda_4 &:= \min\left\{\frac{1}{2}\left[\int_0^1 G_1(1,s)a_1(s)\,ds\right]^{-1}, \frac{1}{2}\left[\int_0^1 G_2(1,s)a_2(s)\,ds\right]^{-1}\right\}, \end{aligned} \tag{22}$$

such that $\Lambda_3 < \lambda_1,\ \lambda_2 < \Lambda_4$.

Theorem 9. *Suppose that* $f_0 = f_\infty = g_0 = g_\infty = \infty$ *and* $(H_1), (P_1)$ *are satisfied. Then problem* (1)-(2) *has at least two positive solutions* (y_1^0, y_2^0), $(\overline{y}_1, \overline{y}_2)$, *such that* $0 < \|(y_1^0, y_2^0)\| < \rho_1 < \|(\overline{y}_1, \overline{y}_2)\|$.

Proof. From Lemma 7, S is a completely continuous operator. At first, in view of $f_0 = g_0 = \infty$, we have $f(y_1,y_2) \ge M(y_1 + y_2)$, for $0 < \|(y_1,y_2)\| < r_1^* < \rho_1$; $g(y_1,y_2) \ge M(y_1+y_2)$, for $0 < \|(y_1,y_2)\| < r_2^* < \rho_1$, where M satisfies $M \ge 1$. Set $\rho_0 := \min\{r_1^*, r_2^*\}$. So we define Ω_{ρ_0} by $\Omega_{\rho_0} := \{(y_1,y_2) \in X : \|(y_1,y_2)\| < \rho_0\}$. Then for each $(y_1,y_2) \in K \bigcap \partial\Omega_{\rho_0}$, we find that

$$\begin{aligned} T_1(y_1,y_2)(1) &= \lambda_1 \int_0^1 G_1(1,s)a_1(s)f(y_1(s),y_2(s))\,ds \\ &\ge \lambda_1 \int_{1/2}^1 G_1(1,s)a_1(s)f(y_1(s),y_2(s))\,ds \end{aligned}$$

$$\begin{aligned}&\geq \lambda_1 \int_{1/2}^{1} G_1(1,s)\, a_1(s)\, M\left(y_1(s)+y_2(s)\right) ds\\&\geq \lambda_1 \int_{1/2}^{1} \tilde{\gamma} G_1(1,s)\, a_1(s)\, M \left\|(y_1,y_2)\right\| ds\\&\geq \frac{M}{2}\left\|(y_1,y_2)\right\| \geq \frac{1}{2}\left\|(y_1,y_2)\right\|.\end{aligned} \tag{23}$$

So $\|T_1(y_1,y_2)\| \geq (1/2)\|(y_1,y_2)\|$ for $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_0}$.

Similarly, we find that $\|T_2(y_1,y_2)\| \geq (1/2)\|(y_1,y_2)\|$ for $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_0}$. Consequently,

$$\begin{aligned}\left\|S(y_1,y_2)\right\| &= \left\|\left(T_1(y_1,y_2), T_2(y_1,y_2)\right)\right\|\\&= \left\|T_1(y_1,y_2)\right\| + \left\|T_2(y_1,y_2)\right\|\\&\geq \left\|(y_1,y_2)\right\|,\end{aligned} \tag{24}$$

whenever $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_0}$. Thus, S is cone expansion on $K\bigcap \partial\Omega_{\rho_0}$.

Next, since $f_\infty = g_\infty = \infty$, we have $f(y_1,y_2) \geq M_1(y_1 + y_2)$ for $y_1 + y_2 \geq r_1^{**} > \rho_1$; $g(y_1,y_2) \geq M_1(y_1 + y_2)$ for $y_1 + y_2 \geq r_2^{**} > \rho_1$, where M_1 satisfies $M_1 \geq 1$. Set $\rho_{10} := \max\{r_1^{**}, r_2^{**}\}$. Let $\rho_0^* = \max\{2\rho_1, (\rho_{10}/\tilde{\gamma})\}$ and $\Omega_{\rho_0^*} := \{(y_1,y_2) \in X : \|(y_1,y_2)\| < \rho_0^*\}$. Then $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_0^*}$ implies

$$\begin{aligned}y_1(t) + y_2(t) &\geq \min_{t\in[(1/2),1]} \left[y_1(t) + y_2(t)\right]\\&\geq \tilde{\gamma}\left\|(y_1,y_2)\right\| = \tilde{\gamma}\rho_0^* \geq \rho_{10}.\end{aligned} \tag{25}$$

So we obtain

$$\begin{aligned}T_1(y_1,y_2)(1) &= \lambda_1 \int_0^1 G_1(1,s)\, a_1(s)\, f\left(y_1(s), y_2(s)\right) ds\\&\geq \lambda_1 \int_{1/2}^{1} G_1(1,s)\, a_1(s)\, f\left(y_1(s), y_2(s)\right) ds\\&\geq \lambda_1 \int_{1/2}^{1} G_1(1,s)\, a_1(s)\, M_1\left(y_1(s)+y_2(s)\right) ds\\&\geq \lambda_1 \int_{1/2}^{1} \tilde{\gamma} G_1(1,s)\, a_1(s)\, M_1 \left\|(y_1,y_2)\right\| ds\\&\geq \frac{M_1}{2}\left\|(y_1,y_2)\right\| \geq \frac{1}{2}\left\|(y_1,y_2)\right\|.\end{aligned} \tag{26}$$

So $\|T_1(y_1,y_2)\| \geq (1/2)\|(y_1,y_2)\|$ for $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_0^*}$.

Similarly, we find that $\|T_2(y_1,y_2)\| \geq (1/2)\|(y_1,y_2)\|$ for $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_0^*}$.

Consequently, $\|S_2(y_1,y_2)\| \geq \|(y_1,y_2)\|$, whenever $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_0^*}$. Thus, S is cone expansion on $K\bigcap \partial\Omega_{\rho_0^*}$.

Finally, let $\Omega_{\rho_1} := \{(y_1,y_2) \in X : \|(y_1,y_2)\| < \rho_1\}$. For $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_1}$, from (H_1), (P_1), we have

$$\begin{aligned}\left\|T_1(y_1,y_2)\right\| &= \lambda_1 \int_0^1 G_1(1,s)\, a_1(s)\, f\left(y_1(s), y_2(s)\right) ds\\&\leq \frac{A_1}{2}\left[\int_0^1 G_1(1,s)\, a_1(s)\, ds\right]^{-1}\\&\quad\times \int_0^1 G_1(1,s)\, a_1(s)\, ds A_1^{-1}\rho_1\\&= \frac{\rho_1}{2} = \frac{1}{2}\left\|(y_1,y_2)\right\|.\end{aligned} \tag{27}$$

Similarly, we find that $\|T_2(y_1,y_2)\| \leq (1/2)\|(y_1,y_2)\|$ for $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_1}$.

Consequently, $\|S(y_1,y_2)\| \leq \|(y_1,y_2)\|$, whenever $(y_1,y_2) \in K\bigcap \partial\Omega_{\rho_1}$. Thus, S is cone compression on $K\bigcap \partial\Omega_{\rho_1}$.

So, from Lemma 5, S has a fixed point $(y_1^0, y_2^0) \in K\bigcap(\overline{\Omega}_{\rho_1} \setminus \Omega_{\rho_0})$ and a fixed point $(\overline{y}_1, \overline{y}_2) \in K\bigcap(\overline{\Omega}_{\rho_0^*} \setminus \Omega_{\rho_1})$. Both are positive solutions of BVP (1)-(2) with

$$0 < \left\|\left(y_1^0, y_2^0\right)\right\| < \rho_1 < \left\|\left(\overline{y}_1, \overline{y}_2\right)\right\|. \tag{28}$$

The proof is complete. □

Theorem 10. *Suppose that* $f_0 = f_\infty = g_0 = g_\infty = 0$ *and* (H_2), (P_2) *are satisfied. Then problem* (1)-(2) *has at least two positive solutions* (y_1^0, y_2^0), $(\overline{y}_1, \overline{y}_2)$, *such that* $0 < \|(y_1^0, y_2^0)\| < \rho_2 < \|(\overline{y}_1, \overline{y}_2)\|$.

Proof. At first, in view of $f_0 = g_0 = 0$, we have $f(y_1,y_2) < \varepsilon(y_1 + y_2)$, $g(y_1,y_2) < \varepsilon(y_1 + y_2)$, for $0 < \|(y_1,y_2)\| \leq \rho < \rho_2$, where ε satisfies $\varepsilon \leq 1$. Let $\Omega_\rho := \{(y_1,y_2) \in X : \|(y_1,y_2)\| < \rho\}$.

Then for each $(y_1,y_2) \in K\bigcap \partial\Omega_\rho$, we find that

$$\begin{aligned}\left\|T_1(y_1,y_2)\right\| &= \lambda_1 \int_0^1 G_1(1,s)\, a_1(s)\, f\left(y_1(s), y_2(s)\right) ds\\&\leq \lambda_1 \int_0^1 G_1(1,s)\, a_1(s)\, \varepsilon\left(y_1(s)+y_2(s)\right) ds\\&\leq \lambda_1 \int_0^1 G_1(1,s)\, a_1(s)\, \varepsilon \left\|(y_1,y_2)\right\| ds\\&\leq \frac{\varepsilon}{2}\left\|(y_1,y_2)\right\| \leq \frac{1}{2}\left\|(y_1,y_2)\right\|.\end{aligned} \tag{29}$$

Like Theorem 9, we get $\|S(y_1,y_2)\| \leq \|(y_1,y_2)\|$ for $(y_1,y_2) \in K\bigcap \partial\Omega_\rho$.

Next, in view of $f_\infty = g_\infty = 0$, we have $f(y_1, y_2) < \varepsilon_1(y_1 + y_2)$, $g(y_1, y_2) < \varepsilon_1(y_1 + y_2)$, for $y_1 + y_2 \geq \rho' > \rho_2$, where ε_1 satisfies $\varepsilon_1 \leq 1$. We consider two cases.

Case 1. Suppose that f is unbounded; there exists $\rho^* > \rho'$ such that

$$f(y_1, y_2) \leq f(y_1^*, y_2^*) \quad \text{for } 0 \leq \|(y_1, y_2)\| \leq \rho^*, \qquad \|(y_1^*, y_2^*)\| = \rho^*. \tag{30}$$

Since $\rho^* > \rho'$, one has $f(y_1, y_2) \leq f(y_1^*, y_2^*) < \varepsilon_1(y_1^* + y_2^*)$ for $0 \leq \|(y_1, y_2)\| \leq \rho^*$. Then, for $(y_1, y_2) \in K$ and $\|(y_1, y_2)\| = \rho^*$, we obtain

$$\begin{aligned}\|T_1(y_1, y_2)\| &= \lambda_1 \int_0^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s)) ds \\ &\leq \lambda_1 \int_0^1 G_1(1, s) a_1(s) \varepsilon_1 (y_1(s) + y_2(s)) ds \\ &\leq \lambda_1 \int_0^1 G_1(1, s) a_1(s) \varepsilon_1 \|(y_1, y_2)\| ds \\ &\leq \frac{\varepsilon_1}{2} \|(y_1, y_2)\| \leq \frac{1}{2} \|(y_1, y_2)\|.\end{aligned} \tag{31}$$

Case 2. Suppose that f is bounded; there exists L_1 such that $f(y_1, y_2) \leq L_1$ for all $(y_1, y_2) \in K$. Taking $\rho^* \geq \max\{2\rho_2, L_1\}$, for $(y_1, y_2) \in K$ and $\|(y_1, y_2)\| = \rho^*$, we obtain

$$\begin{aligned}\|T_1(y_1, y_2)\| &= \lambda_1 \int_0^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s)) ds \\ &\leq \lambda_1 \int_0^1 G_1(1, s) a_1(s) L_1 ds \\ &\leq \frac{L_1}{2} \leq \frac{\rho^*}{2} = \frac{1}{2} \|(y_1, y_2)\|.\end{aligned} \tag{32}$$

Hence, in either case, we always may set $\Omega_{\rho^*} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho^*\}$ such that $\|T_1(y_1, y_2)\| \leq (1/2)\|(y_1, y_2)\|$ for $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho^*}$. Like Theorem 9, we get $\|S(y_1, y_2)\| \leq \|(y_1, y_2)\|$, for $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho^*}$.

Finally, set $\Omega_{\rho_2} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho_2\}$. Then $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho_2}$ implies

$$\begin{aligned}y_1(t) + y_2(t) &\geq \min_{t \in [(1/2), 1]} [y_1(t) + y_2(t)] \\ &\geq \tilde{\gamma} \|(y_1, y_2)\| = \tilde{\gamma}\rho_2.\end{aligned} \tag{33}$$

Hence we have

$$\begin{aligned}T_1(y_1, y_2)(1) &= \lambda_1 \int_0^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s)) ds \\ &\geq \lambda_1 \int_{1/2}^1 G_1(1, s) a_1(s) A_2^{-1} \rho_2 ds \\ &\geq \frac{A_2}{2} A_2^{-1} \rho_2 = \frac{\rho_2}{2} = \frac{1}{2} \|(y_1, y_2)\|.\end{aligned} \tag{34}$$

Consequently, $\|T_1(y_1, y_2)\| \geq (1/2)\|(y_1, y_2)\|$ for $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho_2}$. Like Theorem 9, we get $\|S(y_1, y_2)\| \geq \|(y_1, y_2)\|$ for $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho_2}$.

So, from Lemma 5, S has a fixed point $(y_1^0, y_2^0) \in K \bigcap (\overline{\Omega}_{\rho_2} \setminus \Omega_\rho)$ and a fixed point $(\overline{y}_1, \overline{y}_2) \in K \bigcap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_2})$. Both are positive solutions of BVP (1)-(2) with

$$0 < \|(y_1^0, y_2^0)\| < \rho_2 < \|(\overline{y}_1, \overline{y}_2)\|, \tag{35}$$

which complete the proof. □

3.2. Problem (1)–(3) *in Case* $\lambda_1 = \lambda_2 = 1$. In the following, for the sake of convenience, set

$$\begin{aligned}B_1 &:= \max \left\{2 \int_0^1 G_1(1, s) a_1(s) ds, 2 \int_0^1 G_2(1, s) a_2(s) ds\right\}, \\ B_2 &:= \min \left\{2 \int_{1/2}^1 G_1(1, s) a_1(s) ds, 2 \int_{1/2}^1 G_2(1, s) a_2(s) ds\right\}.\end{aligned} \tag{36}$$

Assume that there exist two positive constants $\rho_1 \neq \rho_2$ such that

(H_3) $f(y_1, y_2), g(y_1, y_2) \leq B_1^{-1}\rho_1$, for $0 \leq \|(y_1, y_2)\| \leq \rho_1$;

(H_4) $f(y_1, y_2), g(y_1, y_2) \geq B_2^{-1}\rho_2$, for $\tilde{\gamma}\rho_2 \leq \|(y_1, y_2)\| \leq \rho_2$.

Theorem 11. *Suppose that* (H_3) *and* (H_4) *are satisfied. Then problem* (1)-(2), *in the case where* $\lambda_1 = \lambda_2 = 1$, *has at least one positive solution* (y_1^0, y_2^0) *such that* $\|(y_1^0, y_2^0)\|$ *between* ρ_1 *and* ρ_2.

Proof. With loss of generality, we may assume that $\rho_1 < \rho_2$.

Let $\Omega_{\rho_1} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho_1\}$. For $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho_1}$, one has

$$\begin{aligned}\|T_1(y_1, y_2)\| &= \lambda_1 \int_0^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s)) ds \\ &\leq \frac{B_1}{2} B_1^{-1} \rho_1 = \frac{\rho_1}{2} = \frac{1}{2} \|(y_1, y_2)\|.\end{aligned} \tag{37}$$

Like Theorem 9, we get $\|S(y_1, y_2)\| \leq \|(y_1, y_2)\|$ for $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho_1}$.

Now, set $\Omega_{\rho_2} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho_2\}$. Then for $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho_2}$, one has

$$\begin{aligned}y_1(t) + y_2(t) &\geq \min_{t \in [(1/2), 1]} [y_1(t) + y_2(t)] \\ &\geq \tilde{\gamma} \|(y_1, y_2)\| = \tilde{\gamma}\rho_2.\end{aligned} \tag{38}$$

Thus, we get

$$\begin{aligned}T_1(y_1, y_2)(1) &= \lambda_1 \int_0^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s)) ds \\ &\geq \lambda_1 \int_{1/2}^1 G_1(1, s) a_1(s) B_2^{-1} \rho_2 ds \\ &\geq \frac{B_2}{2} B_2^{-1} \rho_2 = \frac{\rho_2}{2} = \frac{1}{2} \|(y_1, y_2)\|.\end{aligned} \tag{39}$$

Like Theorem 9, we get $\|S(y_1, y_2)\| \geq \|(y_1, y_2)\|$ for $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho_2}$. Hence, from Lemma 5, we complete the proof. □

Remark 12. In [21], problem (1)-(2) with $\lambda_1 = \lambda_2 = 1$ is not considered.

3.3. Problem (1), (3) *in the General Case.* Consider the following.

Lemma 13 (see [21]). *A pair of functions $(y_1, y_2) \in X$ is a solution of* (1), (3) *if and only if (y_1, y_2) is a fixed point of the operator $U : X \to X$ defined by*

$$\begin{aligned} U(y_1, y_2)(t) &:= (U_1(y_1, y_2)(t), U_2(y_1, y_2)(t)) \\ &= \Big(\beta_1(t)\phi_1(y_1) + \lambda_1 \int_0^1 G_1(t,s)a_1(s) f \\ &\qquad \times (y_1(s), y_2(s))\,ds, \\ &\qquad \beta_2(t)\phi_2(y_2) + \lambda_2 \int_0^1 G_2(t,s)a_2(s) g \\ &\qquad \times (y_1(s), y_2(s))\,ds\Big), \end{aligned} \tag{40}$$

where $\beta_1, \beta_2 : [0,1] \to [0,1]$ are defined by

$$\begin{aligned} \beta_1(t) &:= \frac{\Gamma(\nu_1 - \alpha)}{\Gamma(\nu_1)} t^{\nu_1 - 1}, \\ \beta_2(t) &:= \frac{\Gamma(\nu_2 - \alpha)}{\Gamma(\nu_2)} t^{\nu_2 - 1}. \end{aligned} \tag{41}$$

Lemma 14 (see [21]). *Each of $\beta_1(t)$ and $\beta_2(t)$ is strictly increasing in t and satisfies $\beta_1(0) = \beta_2(0) = 0$ and $\beta_1(1)$, $\beta_2(1) \in (0,1)$. Moreover, there exist constants M_{β_1} and M_{β_2} satisfying $M_{\beta_1}, M_{\beta_2} \in (0,1)$ such that $\min_{t\in[(1/2),1]}\beta_1(t) \geq M_{\beta_1}\|\beta_1\|$ and $\min_{t\in[(1/2),1]}\beta_2(t) \geq M_{\beta_2}\|\beta_2\|$.*

Let one define a new cone K_1 by

$$\begin{aligned} K_1 := \Big\{&(y_1, y_2) \in X : y_1, y_2 \geq 0, \\ &\min_{t\in[(1/2),1]} [y_1(t) + y_2(t)] \geq \gamma_0 \|(y_1, y_2)\|\Big\}, \end{aligned} \tag{42}$$

where $\gamma_0 := \min\{\tilde{\gamma}, M_{\beta_1}, M_{\beta_2}\}$. It is obvious that $\gamma_0 \in (0,1)$.

Lemma 15 (see [21]). *$U : K_1 \to K_1$ is a completely continuous operator.*

Now, one assumes

(D_1) *$\phi_1(y_1) \leq \|y_1\|/4$, $\phi_2(y_2) \leq \|y_2\|/4$ for each $(y_1, y_2) \in K_1$;*

(P_3) *There are numbers Λ_5, Λ_6, where*

$$\begin{aligned} \Lambda_5 &:= \max\left\{\frac{1}{2}\left[\int_{1/2}^1 \gamma_0 G_1(1,s)a_1(s)\,ds\right]^{-1}, \frac{1}{2}\left[\int_{1/2}^1 \gamma_0 G_2(1,s)a_2(s)\,ds\right]^{-1}\right\}, \\ \Lambda_6 &:= \min\left\{\frac{A_1}{4}\left[\int_0^1 G_1(1,s)a_1(s)\,ds\right]^{-1}, \frac{A_1}{4}\left[\int_0^1 G_2(1,s)a_2(s)\,ds\right]^{-1}\right\}, \end{aligned} \tag{43}$$

such that $\Lambda_5 < \lambda_1, \lambda_2 < \Lambda_6$.

(P_4) *There are numbers Λ_7, Λ_8, where*

$$\begin{aligned} \Lambda_7 &:= \max\left\{\frac{A_2}{2}\left[\int_{1/2}^1 G_1(1,s)a_1(s)\,ds\right]^{-1}, \frac{A_2}{2}\left[\int_{1/2}^1 G_2(1,s)a_2(s)\,ds\right]^{-1}\right\}, \\ \Lambda_8 &:= \min\left\{\frac{1}{4}\left[\int_0^1 G_1(1,s)a_1(s)\,ds\right]^{-1}, \frac{1}{4}\left[\int_0^1 G_2(1,s)a_2(s)\,ds\right]^{-1}\right\}, \end{aligned} \tag{44}$$

such that $\Lambda_7 < \lambda_1, \lambda_2 < \Lambda_8$.

Theorem 16. *Suppose that $f_0 = f_\infty = g_0 = g_\infty = \infty$ and (H_1), (D_1), (P_3) are satisfied. Then problem* (1), (3) *has at least two positive solutions (y_1^0, y_2^0), $(\overline{y}_1, \overline{y}_2)$, such that*

$$0 < \left\|(y_1^0, y_2^0)\right\| < \rho_1 < \left\|(\overline{y}_1, \overline{y}_2)\right\|. \tag{45}$$

Proof. At first, in view of $f_0 = g_0 = \infty$, we have $f(y_1, y_2) \geq M(y_1 + y_2)$, $g(y_1, y_2) \geq M(y_1 + y_2)$, for $0 < \|(y_1, y_2)\| \leq \rho_0 < \rho_1$, where M satisfies $M \geq 1$.

Let $\Omega_{\rho_0} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho_0\}$. Then for each $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_0}$, we find that

$$\begin{aligned} U_1(y_1, y_2)(1) &\geq \lambda_1 \int_{1/2}^1 G_1(1,s)a_1(s) f(y_1(s), y_2(s))\,ds \\ &\geq \lambda_1 \int_{1/2}^1 G_1(1,s)a_1(s) M(y_1(s) + y_2(s))\,ds \\ &\geq \lambda_1 \int_{1/2}^1 \gamma_0 G_1(1,s)a_1(s) M\|(y_1, y_2)\|\,ds \\ &\geq \frac{M}{2}\|(y_1, y_2)\| \geq \frac{1}{2}\|(y_1, y_2)\|. \end{aligned} \tag{46}$$

So $\|U_1(y_1, y_2)\| \geq (1/2)\|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_0}$.

Similarly, we find that $\|U_2(y_1, y_2)\| \geq (1/2)\|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_0}$. Consequently,

$$\begin{aligned}\|U(y_1, y_2)\| &= \|(U_1(y_1, y_2), U_2(y_1, y_2))\| \\ &= \|U_1(y_1, y_2)\| + \|U_2(y_1, y_2)\| \\ &\geq \|(y_1, y_2)\|,\end{aligned} \tag{47}$$

whenever $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_0}$. Thus, U is cone expansion on $K_1 \bigcap \partial\Omega_{\rho_0}$.

Next, since $f_\infty = g_\infty = \infty$, we get $f(y_1, y_2) \geq M_1(y_1 + y_2)$, $g(y_1, y_2) \geq M_1(y_1 + y_2)$, for $y_1 + y_2 \geq \rho_{10} > \rho_1$, where M_1 satisfies $M_1 \geq 1$. Let $\rho_0^* = \max\{2\rho_1, (\rho_{10}/\gamma_0)\}$ and $\Omega_{\rho_0^*} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho_0^*\}$; then, $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho_0^*}$ implies

$$\begin{aligned}y_1(t) + y_2(t) &\geq \min_{t\in[(1/2),1]} [y_1(t) + y_2(t)] \\ &\geq \gamma_0 \|(y_1, y_2)\| = \gamma_0\rho_0^* \geq \rho_{10}.\end{aligned} \tag{48}$$

So for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_0^*}$, we obtain

$$\begin{aligned}U_1(y_1, y_2)(1) &\geq \lambda_1 \int_{1/2}^1 G_1(1, s) a_1(s) f(y_1(s), y_2(s)) ds \\ &\geq \lambda_1 \int_{1/2}^1 G_1(1, s) a_1(s) M_1(y_1(s) + y_2(s)) ds \\ &\geq \lambda_1 \int_{1/2}^1 \gamma_0 G_1(1, s) a_1(s) M_1 \|(y_1, y_2)\| ds \\ &\geq \frac{M_1}{2} \|(y_1, y_2)\| \geq \frac{1}{2} \|(y_1, y_2)\|.\end{aligned} \tag{49}$$

That is, $\|U_1(y_1, y_2)\| \geq (1/2)\|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_0^*}$.

Similarly, we find that $\|U_2(y_1, y_2)\| \geq (1/2)\|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_0^*}$. Consequently, $\|U(y_1, y_2)\| \geq \|(y_1, y_2)\|$, whenever $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_0^*}$. Thus, U is cone expansion on $K_1 \bigcap \partial\Omega_{\rho_0^*}$.

Finally, let $\Omega_{\rho_1} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho_1\}$. For $(y_1, y_2) \in K \bigcap \partial\Omega_{\rho_1}$, from (H_1), (D_1), and (P_3), we have

$$\begin{aligned}\|U_1(y_1, y_2)\| &\leq \phi_1(y_1) + \lambda_1 \int_0^1 G_1(t, s) a_1(s) \\ &\quad \times f(y_1(s), y_2(s)) ds \\ &\leq \frac{\|y_1\|}{4} + \frac{A_1}{4} A_1^{-1} \rho_1 \\ &\leq \frac{\rho_1}{2} = \frac{1}{2} \|(y_1, y_2)\|.\end{aligned} \tag{50}$$

Similarly, we find that $\|U_2(y_1, y_2)\| \leq (1/2)\|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_1}$. Consequently, $\|U(y_1, y_2)\| \leq \|(y_1, y_2)\|$, whenever $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_1}$. Thus, U is cone compression on $K_1 \bigcap \partial\Omega_{\rho_1}$.

So, from Lemma 5, U has a fixed point $(y_1^0, y_2^0) \in K_1 \bigcap (\overline{\Omega}_{\rho_1} \setminus \Omega_{\rho_0})$ and a fixed point $(\overline{y}_1, \overline{y}_2) \in K_1 \bigcap (\overline{\Omega}_{\rho_0^*} \setminus \Omega_{\rho_1})$. Both are positive solutions of BVP (1), (3) with

$$0 < \left\|\left(y_1^0, y_2^0\right)\right\| < \rho_1 < \left\|\left(\overline{y}_1, \overline{y}_2\right)\right\|. \tag{51}$$

The proof is complete. □

Theorem 17. *Suppose that $f_0 = f_\infty = g_0 = g_\infty = 0$ and (H_2), (D_1), (P_4) are satisfied. Then problem (1), (3) has at least two positive solutions (y_1^0, y_2^0), $(\overline{y}_1, \overline{y}_2)$, such that*

$$0 < \left\|\left(y_1^0, y_2^0\right)\right\| < \rho_2 < \left\|\left(\overline{y}_1, \overline{y}_2\right)\right\|. \tag{52}$$

Proof. At first, in view of $f_0 = g_0 = 0$, we have $f(y_1, y_2) < \varepsilon(y_1 + y_2)$, $g(y_1, y_2) < \varepsilon(y_1 + y_2)$ for $\|(y_1, y_2)\| \leq \rho < \rho_2$, where ε satisfies $\varepsilon \leq 1$. Let $\Omega_\rho := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho\}$. Then for each $(y_1, y_2) \in K_1 \bigcap \partial\Omega_\rho$, we find that

$$\begin{aligned}&\|U_1(y_1, y_2)\| \\ &\quad \leq \phi_1(y_1) + \lambda_1 \int_0^1 G_1(t, s) a_1(s) f(y_1(s), y_2(s)) ds \\ &\quad \leq \frac{\|y_1\|}{4} + \frac{\varepsilon}{4} \|(y_1, y_2)\| \leq \frac{1}{2} \|(y_1, y_2)\|.\end{aligned} \tag{53}$$

Like Theorem 16, we get $\|U(y_1, y_2)\| \leq \|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_\rho$.

Next, in view of $f_\infty = g_\infty = 0$, we have $f(y_1, y_2) < \varepsilon_1(y_1 + y_2)$, $g(y_1, y_2) < \varepsilon_1(y_1 + y_2)$, for $y_1 + y_2 \geq \rho' > \rho_2$, where ε_1 satisfies $\varepsilon_1 \leq 1$. We consider two cases.

Case 1. Suppose that f is unbounded and there exists $\rho^* > \rho'$ such that

$$\begin{aligned}f(y_1, y_2) \leq f(y_1^*, y_2^*) \quad &\text{for } 0 \leq \|(y_1, y_2)\| \leq \rho^*, \\ &\|(y_1^*, y_2^*)\| = \rho^*.\end{aligned} \tag{54}$$

Since $\rho^* > \rho'$, one has $f(y_1, y_2) \leq f(y_1^*, y_2^*) < \varepsilon_1(y_1^* + y_2^*)$ for $0 \leq \|(y_1, y_2)\| \leq \rho^*$.

Then, for $(y_1, y_2) \in K_1$ and $\|(y_1, y_2)\| = \rho^*$, we obtain

$$\begin{aligned}&\|U_1(y_1, y_2)\| \\ &\quad \leq \phi_1(y_1) + \lambda_1 \int_0^1 G_1(t, s) a_1(s) f(y_1(s), y_2(s)) ds \\ &\quad \leq \frac{\|y_1\|}{4} + \frac{\varepsilon_1}{4} \|(y_1, y_2)\| \leq \frac{1}{2} \|(y_1, y_2)\|.\end{aligned} \tag{55}$$

Case 2. Suppose that f is bounded; there, exists L_1 such that $f(y_1, y_2) \leq L_1$ for all $(y_1, y_2) \in K_1$. Taking $\rho^* \geq \max\{2\rho_2, L_1\}$, for $(y_1, y_2) \in K_1$ and $\|(y_1, y_2)\| = \rho^*$, we obtain

$$\begin{aligned} &\|U_1(y_1, y_2)\| \\ &\quad \leq \phi_1(y_1) + \lambda_1 \int_0^1 G_1(t,s) a_1(s) f(y_1(s), y_2(s)) ds \\ &\quad \leq \phi_1(y_1) + \lambda_1 \int_0^1 G_1(t,s) a_1(s) L_1 ds \\ &\quad \leq \frac{\|y_1\|}{4} + \frac{L_1}{4} \leq \frac{\rho^*}{2} = \frac{1}{2} \|(y_1, y_2)\|. \end{aligned} \tag{56}$$

Hence, in either case, we always may set $\Omega_{\rho^*} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho^*\}$ such that $\|U_1(y_1, y_2)\| \leq (1/2)\|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho^*}$.

Like Theorem 16, we get $\|U(y_1, y_2)\| \leq \|(y_1, y_2)\|$, for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho^*}$.

Finally, set $\Omega_{\rho_2} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho_2\}$. Then $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_2}$ implies

$$\begin{aligned} y_1(t) + y_2(t) &\geq \min_{t\in[(1/2),1]} [y_1(t) + y_2(t)] \\ &\geq \gamma_0 \|(y_1, y_2)\| = \gamma_0 \rho_2. \end{aligned} \tag{57}$$

Hence we have

$$\begin{aligned} U_1(y_1, y_2)(1) &\geq \lambda_1 \int_{1/2}^1 G_1(1,s) a_1(s) A_2^{-1} \rho_2 ds \\ &\geq \frac{A_2}{2} A_2^{-1} \rho_2 = \frac{\rho_2}{2} = \frac{1}{2} \|(y_1, y_2)\|. \end{aligned} \tag{58}$$

So, $\|U_1(y_1, y_2)\| \geq (1/2)\|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_2}$.

Like Theorem 16, we get $\|U(y_1, y_2)\| \geq \|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_2}$. So, from Lemma 5, U has a fixed point $(y_1^0, y_2^0) \in K_1 \bigcap (\overline{\Omega}_{\rho_2} \setminus \Omega_\rho)$ and a fixed point $(\overline{y}_1, \overline{y}_2) \in K_1 \bigcap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_2})$. Both are positive solutions of BVP (1), (3) with $0 < \|(y_1^0, y_2^0)\| < \rho_2 < \|(\overline{y}_1, \overline{y}_2)\|$, which complete the proof. □

3.4. Problem (1), (3) *in Case* $\lambda_1 = \lambda_2 = 1$. In [21], the author obtained that problem (1), (3) with $\lambda_1 = \lambda_2 = 1$ having at least one positive solution. In the following, we also establish the existence of one positive solution to problem (1), (3) with $\lambda_1 = \lambda_2 = 1$ under different conditions.

For the sake of convenience, set

$$B_3 := \max\left\{4 \int_0^1 G_1(1,s) a_1(s) ds, 4 \int_0^1 G_2(1,s) a_2(s) ds\right\}. \tag{59}$$

Assume that there exist two positive constants $\rho_1 \neq \rho_2$ such that

(H_5) $f(y_1, y_2), g(y_1, y_2) \leq B_3^{-1}\rho_1$ for $0 \leq \|(y_1, y_2)\| \leq \rho_1$;

(H_6) $f(y_1, y_2), g(y_1, y_2) \geq B_2^{-1}\rho_2$ for $\gamma_0\rho_2 \leq \|(y_1, y_2)\| \leq \rho_2$.

Theorem 18. *Suppose that* (H_5), (H_6), *and* (D_1) *are satisfied. Then problem* (1), (3), *in the case where* $\lambda_1 = \lambda_2 = 1$, *has at least one positive solution* (y_1^0, y_2^0) *such that* $\|(y_1, y_2)\|$ *between* ρ_1 *and* ρ_2.

Proof. With loss of generality, we may assume that $\rho_1 < \rho_2$. Let $\Omega_{\rho_1} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho_1\}$. For $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_1}$, from (H_7), (D_1), one has

$$\begin{aligned} \|U_1(y_1, y_2)\| &\leq \phi_1(y_1) + \int_0^1 G_1(t,s) a_1(s) \\ &\qquad \times f(y_1(s), y_2(s)) ds \\ &\leq \frac{\rho_1}{4} + \frac{B_3}{4} B_3^{-1} \rho_1 = \frac{\rho_1}{2} = \frac{1}{2} \|(y_1, y_2)\|. \end{aligned} \tag{60}$$

Like Theorem 16, we get $\|U(y_1, y_2)\| \leq \|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_1}$.

Now, set $\Omega_{\rho_2} := \{(y_1, y_2) \in X : \|(y_1, y_2)\| < \rho_2\}$. For $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_2}$, one has

$$\begin{aligned} y_1(t) + y_2(t) &\geq \min_{t\in[(1/2),1]} [y_1(t) + y_2(t)] \\ &\geq \gamma_0 \|(y_1, y_2)\| = \gamma_0 \rho_2. \end{aligned} \tag{61}$$

Thus, from (H_8), we get

$$\begin{aligned} U_1(y_1, y_2)(1) &\geq \lambda_1 \int_{1/2}^1 G_1(1,s) a_1(s) B_2^{-1} \rho_2 ds \\ &\geq \frac{B_2}{2} B_2^{-1} \rho_2 = \frac{\rho_2}{2} = \frac{1}{2} \|(y_1, y_2)\|. \end{aligned} \tag{62}$$

Like Theorem 16, we get $\|U(y_1, y_2)\| \geq \|(y_1, y_2)\|$ for $(y_1, y_2) \in K_1 \bigcap \partial\Omega_{\rho_2}$. Hence, from Lemma 5, we complete the proof. □

4. An Example

To illustrate how our main results can be used in practice, we present one example.

Example 19. Consider the following BVP, for $t \in (0, 1)$:

$$\begin{aligned} -D_{0^+}^{5.2} y_1(t) &= 164500 e^{-2t} \left[(y_1(t) + y_2(t))^{1/2}\right. \\ &\qquad \left. + (y_1(t) + y_2(t))^2\right], \\ -D_{0^+}^{5.95} y_2(t) &= 164000 e^{-3t} \left[(y_1(t) + y_2(t))^{1/3}\right. \\ &\qquad \left. + (y_1(t) + y_2(t))^3\right], \end{aligned} \tag{63}$$

subject to the boundary conditions

$$y_1^{(i)}(0) = y_2^{(i)}(0) = 0, \quad 0 \leq i \leq 4, \tag{64}$$

$$D_{0+}^{1.5}[y_1(t)]_{t=1} = D_{0+}^{1.5}[y_2(t)]_{t=1} = 0. \tag{65}$$

Obviously, problem (63)–(65) fits the framework of problem (1)-(2) with

$$\nu_1 := 5.2, \quad \nu_2 := 5.95, \quad \alpha = 1.5,$$
$$\lambda_1 = 164500, \quad \lambda_2 = 164000, \quad n = 6. \tag{66}$$

In addition, we have set

$$f(y_1, y_2) := (y_1 + y_2)^{1/2} + (y_1 + y_2)^2,$$
$$g(y_1, y_2) := (y_1 + y_2)^{1/3} + (y_1 + y_2)^3, \tag{67}$$
$$a_1(t) := e^{-2t}, \quad a_2(t) := e^{-3t}.$$

We can see that $f, g : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ and are continuous. The functions $a_1(t)$ and $a_2(t)$ are obviously nonnegative.

Now, observe that $f_0 = f_\infty = g_0 = g_\infty = \infty$ holds. Again set $A_1 = 1/850$, because $f(y_1, y_2)$, $g(y_1, y_2)$ is monotone increasing function for $(y_1, y_2) \geq 0$, taking $\rho_1 = 4$; then, when $\|(y_1, y_2)\| \in [0, \rho_1]$, we get

$$\begin{aligned} f(y_1, y_2) &\leq \|(y_1, y_2)\|^{1/2} + \|(y_1, y_2)\|^2 \\ &\leq 2 + 16 = 18 < 4 \times 850 = A_1^{-1}\rho_1 \\ g(y_1, y_2) &\leq \|(y_1, y_2)\|^{1/3} + \|(y_1, y_2)\|^3 \\ &\leq 4^{1/3} + 64 < 4 \times 850 = A_1^{-1}\rho_1, \end{aligned} \tag{68}$$

which implies that (H_1) holds.

On the other hand, to calculate the admissible range of the eigenvalues λ_1, λ_2, as given by condition (P_1), observe by numerical approximation, we find that

$$\Lambda_1 \approx 163530, \quad \Lambda_2 \approx 164547.25. \tag{69}$$

Thus, for any λ_1, λ_2 satisfying $163530 < \lambda_1, \lambda_2 < 164547.25$, condition (P_1) will be satisfied.

Consequently, by Theorem 9, problem (63)–(65) has at least two positive solutions.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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