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FOUNDATIONS

# Approximation operators via TD-matroids on two sets

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**Abstract**

Rough set theory is an extension of set theory with two additional unary set-theoretic operators known as approximation in order to extract information and knowledge. It needs the basic, or say definable, knowledge to approximate the undefinable knowledge in a knowledge space using the pair of approximation operators. Many existed approximation operators are expressed with unary form. How to mine the knowledge which is asked by binary form with rough set has received less research attention, though there are strong needs to reveal the answer for this challenging problem. There exist many information with matroid constraints since matroid provides a platform for combinatorial algorithms especially greedy algorithm. Hence, it is necessary to consider a matroidal structure on two sets no matter the two sets are the same or not. In this paper, we investigate the construction of approximation operators expressed by binary form with matroid theory, and the constructions of matroidal structure aided by a pair of approximation operators expressed by binary form.

- First, we provide a kind of matroidal structure—TD-matroid defined on two sets as a generalization of Whitney classical matroid.
- Second, we introduce this new matroidal construction to rough set and construct a pair of approximation operators expressed with binary form.
- Third, using the existed pair of approximation operators expressed with binary form, we build up two concrete TD-matroids.
- Fourth, for TD-matroid and the approximation operators expressed by binary form on two sets, we seek out their properties with aspect of posets, respectively.
- Through the paper, we use some biological examples to explain and test the correct of obtained results. In summary, this paper provides a new approach to research rough set theory and matroid theory on two sets, and to study on their applications each other.

**Keywords** TD-matroid · Approximation operator · Semiconcept · Two sets

## 1 Introduction

Rough set theory, which was proposed by Pawlak (1982), can be viewed as a successful mathematical approach to deal with situations in which every object of a given universe can be identified only within the limits of imprecise data by an indiscernibility relation. It associates some information table such as data and knowledge. With this philosophy idea, rough set theory has been applied to many fields, for instance, computer science, expert system, classification theory, artificial intelligence, and so on. (Acharjya and Ahmed 2020; Błaszczyński et al. 2021; Kauser and Acharjya 2021; Lei et al. 2021; Pawlak 1982, 1991; Pawlak and Skowron 2007a, b, c; Penmatsa et al. 2020; Qu et al. 2020; Selvi and Chandrasekaran 2019; Silfia et al. 2021; Wang et al. 2021; Wei et al. 2021; Yoshifumi et al. 2021). The lower and upper approximation operators are the basic notions in rough set

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theory. Since the basic notions in Pawlak classical rough set model (Pawlak 1982, 1991) asked the requirement to be an equivalence relation which restricted the applied fields of rough set, many authors have hoped to change the equivalence relation in order to extend the approximation operators of Pawlak classical rough set model. And some extended approximation operators are provided by the combining of rough set with some other theories and some other mathematical structures, for example, fuzzy theory (Cattanco 1998; Mi et al. 2008; Yao 1998a, 2001; Yao et al. 2019), covering theory (Bonikowski et al. 1998; Yao 1998b; Yao and Yao 2012; Zhang et al. 2019), lattice (Li et al. 2016; Pawlak and Skowron 2007c; Wang et al. 2019; Yao 1998b), matroid theory (Li and Liu 2012; Marek and Skowron 2014), three-way decisions (Huang and Zhu 2016; Zhao and Hu 2020), and some others (Li et al. 2019; Mao et al. 2021; Xu and Wang 2018; Zhang et al. 2016).

As Yao (1996, 2015) said, an interesting research direction in rough set theory is to extend Pawlak classical rough set model to more general status in order to be applied to more situations. All of the above results with respect to rough set are defined on one set. Pawlak classical rough set model is established on *one universe*, i.e., one non-empty set, which limits its application. In other words, another interesting generalization of Pawlak classical rough set model is to generalize the one universe to more than ones. For example, Yao et al. (1995) provided two-universe rough set model. After that, it was born many results of rough set on two universes (Pedrycz and Bargiela 2002; Shao et al. 2018; Sun et al. 2017; Yao 1996, 2001, 2015).

In fact, the rough set model of Yao et al. (1995) on two universes and the other results mentioned above for rough set on two universes were actually dealt on "one universe". The two universes are different, but there are some relevances with some functions between the two universes. For example, let $(OB, AT, R)$ be an information table, in which $OB$ is a set of objects, $AT$ is a set of attributes, $R$ is a family of relations between $OB$ and $AT$ such as that in Pedrycz and Bargiela (2002) and Shao et al. (2018), and Yao (1996), respectively. Let $(\underline{APR}, \overline{APR})$ be the pair of lower and upper approximation operators defined for $(OB, AT, R)$ such as that of Shao et al. (2018) and the others above. $\underline{APR}$ and $\overline{APR}$ are defined on "one universe", respectively, since the domains of definition of $\underline{APR}$ and $\overline{APR}$ are on $2^{AT}$, respectively. At the same time, the codomains of $\underline{APR}$ and $\overline{APR}$ are on another universe such as on $2^{AT}$ or $2^{OB}$. Both $\underline{APR}$ and $\overline{APR}$ are expressed by unary form. In addition, some kind of lower and upper approximation operators are produced with aid of the pairwise form during the productive process such as that in Pei and Xu (2004), but the final expressions of lower and upper approximation operators are with the form of unary. Actually, the process of "pairwise" is realized on one set also.

As a matter of fact, some information extracted from an information table is expressed as a pair, that is, a binary form. For example, in a kind of information table—a formal context $(OB, AT, I)$, a formal concept extracted from $(OB, AT, I)$ is a pair $(X, Y)$ where $I \subseteq OB \times AT$, $X \subseteq OB$, $Y \subseteq AT$ (Ganter and Wille 1999). Actually, in our real life, many facts are needed to be expressed by pairwise on two different sets. For instance, in classification of insects, or cluster analysis of insects, if the set $OB$ of objects is consisted of the specimens of insects and the set $AT$ of attributes is consisted of some biological characteristics relative to the objects in $OB$, then to obtain the dendrogram, or the cladogram, of the specimens in $OB$, is a main duty for biologists to do their study on $OB$. The dendrogram and cladogram should be simple and intuitive so as easily for some biologists to analyze the biological properties existed in $OB$. When searching those diagrams, some biologists consider every point in a dendrogram or a cladogram by a pair $(X, Y)$ where $X \subseteq OB$ and $Y \subseteq AT$, though some points are obtained by the help of rough set approximations.

With assistance of rough set theory or formal concept analysis (simply FCA), some biological ideas have been explored and some of biological thoughts have been obtained (Mao 2018; Shang et al. 2010; Sinha and Namdev 2020; Wang and Mao 2020; Wang et al. 2020; Ytow et al. 2006).

Another instance is to determine whether a person has illness like the current COVID-19 pandemic for a doctor. To find the patients is important for the tracing investigation and further prevention of COVID-19 (Apolloni 2021; Zhu et al. 2021). Let $OB$ be the set of citizen of a city and $AT$ be the characteristics of COVID-19. It is easy to see that $OB$ and $AT$ are two different sets. Let $A \subseteq OB$. $A' \subseteq AT$ stands for the characteristics common to the elements in $A$. Then, $(a, a')$ implies that a person $a \in A$ has the family $a'$ of characteristics. Using $a'$, a doctor can determine the person $a$ to have COVID-19 or not. Here, $(a, a')$ is expressed by pairwise on two different sets $OB$ and $AT$. If $A$ is a family of citizens at the same residence zone, a person $a$ belongs to $A$ with "+" for nucleic acid testing and every $x \in A \setminus a$ has "−" result for nucleic acid testing. Then, the doctor can use the other characteristics for any $x \in A \setminus a$ to infer the situations of health for $x$ with aspect of COVID-19 in order to find the potential patients. The inference is to approximate the fact based on the known knowledge of a doctor with assistance of $(x, x')$ expressed by the way of pairwise on two different sets $OB$ and $AT$. Since every characteristic $y \in x'$ has different efficacy to affect COVID-19, the decidable process of the doctor is actually "greedy" process.

From the above analysis for rough set and the need of information and knowledge expressed by pairwise, we confirm that it is necessary to consider to build up a knowledge system and rough set approximation operators expressed by pairwise on two sets no matter the two are the same or not.

Matroid, coined by Whitney (1935), is a mathematical abstract structure for the combination of linear algebra and graph theory. It solves combinatorial optimization problems since its structure is better to design algorithms especially greedy algorithm (Oxley 2011; Welsh 1976). Many merits of matroid make it be applied in many fields such as FCA, three-way decisions and granular computing (Li 2019; Mao 2018; Wang and Mao 2020), coding and cryptology (Ambadi 2019; Liu et al. 2017), rough set (Hu and Yao 2019; Li et al. 2016, 2017, 2019; Marek and Skowron 2014; Restrepo and Aguilar 2019; Wang et al. 2019; Zhu and Wang 2011), submodular maximization (Corah and Michael 2019; Hou and Clark 2021), gamble (Kleinberg and Weinberg 2019), and graph theory (Huang and Zhu 2016). Whitney classical matroidal model (Whitney 1935) has been generalized in many ways. Among them, the combination between rough sets and matroids has been obtained many interesting results such as Zhu and Wang (2011) established a matroidal structure using the upper approximation number and studied generalized rough sets with matroidal methods. Li et al. (2016) investigated rough sets using matroidal approaches. Restrepo and Aguilar (2019) presented the matroidal structures obtained from different partitions, coverings of a specific set, and covering-based rough sets. With the assistance of rough sets and three-way decisions, Li et al. (2017) proposed three-way matroids. We observe that any of the above results about matroids is discussed on one universe and expressed as one dimension. One dimension space limits the development of matroid theory since the world of real life not only existed in one dimension. Some researchers have found this problem and tried to solve it. For example, Im et al. (2021) introduced a matroidal structure—matroid cup game on $n$ cup, i.e., on $\mathbb{R}^n$. In fact, this matroidal structure is expressed on the "same" ground universe $\mathbb{R}$ and displays with unary form. Additionally, we observe that many information and knowledge are with matroid constraints such as that existed in Corah and Michael (2019) and He and Shi (2019), respectively. Hence, to consider extract information from an information system, it is better to consider matroid theory under some situations.

Recently, we have discussed to extract information from an information table such as formal context by pairwise with rough set approximations (Mao 2019; Wang and Mao 2020). However, how to reveal the dendrogram or the cladogram of some biological specimens with an effective algorithm, up to now, we do not find a way to get success. With biology knowledge, we know that these algorithms are actually in an optimal and "greedy" process. Matroidal structure perhaps will give a hand since matroid structure provides a good platform for designing greedy algorithm (Im et al. 2021; Oxley 2011; Welsh 1976). But we cannot deal with the relationships between rough set approximations and matroidal structures expressed by pairwise on two sets especially two different sets. This situation reduces the speed of development to design an algorithm for some needs in real life such as biological research, and to give a decision for a doctor in searching the potential patients with some ill like COVID-19.

Based on the above expressions, we realize that the key challenge in the theories of rough set and matroid is to study their combination expressed by binary way on two sets especially two different sets. The solving of this problem will produce a way to extract information from an information table under a specially designed and more efficient algorithm such as greedy algorithm to approximate undefinable knowledge with the definable knowledge. According to the views of Watt and Berg (2002) and Yao (2015), we know "concepts are the building blocks of scientific theories. A scientific concept consists of a theoretical definition and an operational definition". Hence, in this paper, we will make the following contributions specially.

- The basic work in this paper is to consider matroid theory by binary form on two sets, that is, to generalize Whitney classical matroid model from one set to two sets. We will call this new matroidal structure as TD-matroid and express the feasible sets of a TD-matroid by binary way.
- A pair of operators will be defined with aid of a TD-matroid. And furthermore, it will find the pair of operators roughly to be a pair of lower and upper approximation operators, which is a main context in rough set theory. These operators are expressed by "pairwise" form.
- With the approximation operators provided by Mao (2019), this paper will establish two concrete TD-matroids.
- Some biological examples show the necessity and correct of all of the above results. All of these biological examples also indicate that the idea and discussion in this paper are based on practical needs.

We will see in Sect. 3 that up to isomorphism, the classical matroid is a special case of TD-matroid. This implies that TD-matroid can solve the problems that Whitney classical matroid does. In addition, TD-matroid surmounts the weakness of Whitney classical matroid since the classical matroid can only solve the problem or extract information expressed on one set. However, TD-matroid has the ability for users to mine the useful information hoped to express with binary form on two sets, no matter the two sets are the same or not. Furthermore, TD-matroid is different from some existed matroidal structures defined on two sets such as matroid cup game if $n = 2$ in Im et al. (2021). In one word, TD-matroid is a new matroidal structure in order to extract information defined on two sets by binary form. The approximation operators defined in this paper are different from those approximation operators defined on one set since the operators here are defined on two sets and expressed by binary form, and from the existed approximation operators defined on two sets such as that in Shao et al. (2018), and also from ones expressed by binary form such as that in Mao (2019) and that in Im et al. (2021) if $n = 2$, since the operators here based on the feasible sets of a TD-matroid, that is, the approximation operators given in this paper are constrained by a TD-matroid. These consequences show that this paper provides a new approach to study rough set and matroid theory.

The remainder of this paper is organized as follows. In Sect. 2, we present some notions and properties of matroid, FCA and rough set. In Sect. 3, we provide the definition of TD-matroid and find a pair of lower and upper approximation operators. After that, using a pair of lower and upper approximation operators based on the set of semiconcepts in a formal context, we construct two TD-matroids. Finally, a summary of this paper is offered and future work is discussed in the last section—Sect. 4.

## 2 Some notions and properties

Below we review basic notions used in this paper. For more details, matroid is referred to Oxley (2011) and Welsh (1976), FCA is referred to Ganter and Wille (1999), semiconcept is seen (Vormbrock and Wille 2005), and rough set is seen (Pawlak 1982, 1991), poset and lattice are seen (Grätzer 2011).

### 2.1 Some notations

Let $S$, $T$ and $U$ be three sets. Then, we will use the following notations in this paper for $X_1, X_2 \subseteq S$ and $Y_1, Y_2 \subseteq T$.

(1) $(X_1, Y_1) \subseteq (X_2, Y_2) :\Leftrightarrow X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$.
(2) $(X_1, Y_1) \sqsubseteq (X_2, Y_2) :\Leftrightarrow X_1 \subseteq X_2$ and $Y_1 \supseteq Y_2$.
(3) $(X_1, Y_1) \cup (X_2, Y_2) :\Leftrightarrow (X_1 \cup X_2, Y_1 \cup Y_2)$.
(4) $(X_1, Y_1) \cap (X_2, Y_2) :\Leftrightarrow (X_1 \cap X_2, Y_1 \cap Y_2)$.
(5) $(X_1, Y_1)\backslash(X_2, Y_2) :\Leftrightarrow (X_1\backslash X_2, Y_1\backslash Y_2)$.
(6) $|(X, Y)| := |X| + |Y|$.
(7) $2^U$ represents the power set of $U$.

***Remark 1*** (1) This paper writes $y$ for $\{y\}$ if $\{y\}$ is singleton.
(2) Since $(X_2 \cup a, Y_2 \cup b) = (X_2, Y_2) \cup (a, b)$ for any $(X_2, Y_2) \subseteq S \times T$ and $(a, b) \in S \times T$, we often write $(X_2 \cup a, Y_2 \cup b)$ when we consider $(X_2, Y_2) \cup (a, b)$.
(3) Pawlak classical rough set model is defined on one universe. Whitney classical matroid model is defined on one set. In fact, this paper says a universe as a non-empty set. Hence, to discuss identity, sometimes, we also say that Pawlak classical rough set model is defined on one set if there is no confusion from the text.
(4) Let $f : S \to U$ be a bijection. Then, we say that $S$ is isomorphic to $U$ from the idea of sets.
(5) All of discussions are finite in this paper in what follows.

### 2.2 Matroid

**Definition 1** (1) (Oxley 2011, p. 7; Welsh 1976, p. 7) A *matroid* $M$ is a finite set $S$ and a collection $\mathcal{I}$ of subsets of $S$ (called *independent* sets) such that (I1)–(I3) are satisfied.

$$\begin{aligned}&(\text{I1})\emptyset \in \mathcal{I}.\\ &(\text{I2})X \in \mathcal{I} \text{ and } Y \subseteq X \Rightarrow Y \in \mathcal{I}.\\ &(\text{I3})X, Y \in \mathcal{I} \text{ and } |X| < |Y| \Rightarrow X \cup y \in \mathcal{I}\\ &\quad\text{for some } y \in Y \setminus X.\end{aligned}$$

(2) (Oxley 2011, p. 11; Welsh 1976, p. 9) Two matroids $M_1$ and $M_2$ on $S_1$ and $S_2$, respectively, are *isomorphic* if there is a bijection $\varphi : S_1 \to S_2$ which preserves independence. We write $M_1 \cong M_2$ if $M_1$ and $M_2$ are isomorphic.

### 2.3 FCA

**Definition 2** (1) (Ganter and Wille 1999, pp. 17–18) A *formal context* is a set structure $\mathbb{K} := (O, P, I)$ for which $O$ and $P$ are nonempty sets while $I$ is a binary relation between $O$ and $P$, i.e., $I \subseteq O \times P$; the elements of $O$ and $P$ are called *objects* and *attributes*, respectively, and $gIm$ is $(g, m) \in I$. The *derivation operators* of $\mathbb{K}$ are defined as follows ($X \subseteq O, Y \subseteq P$): $X' = \{m \in P \mid gIm$ for all $g \in X\}$ and $Y' = \{g \in O \mid gIm$ for all $m \in Y\}$.
(2) (Vormbrock and Wille 2005) In a formal context $\mathbb{K} = (O, P, I)$, a pair $(X, Y)$ with $X \subseteq O$ and $Y \subseteq P$ is called a $\sqcap$-*semiconcept* if $Y = X'$. Dually, a pair $(C, D)$ with $C \subseteq O$ and $D \subseteq P$ is called a $\sqcup$-*semiconcept* if $C = D'$.

**Lemma 1** (Ganter and Wille 1999, p. 19) *The two derivation operators in a formal context $\mathbb{K} = (O, P, I)$ satisfy the following conditions for any $A_j, Z, Z_1, Z_2 \subseteq O$ (or $A_j, Z, Z_1, Z_2 \subseteq P$) where $j \in J$ and $J$ is an index set.*

(1) $Z_1 \subseteq Z_2 \Rightarrow Z_1' \supseteq Z_2'$.
(2) $(\cup_{j\in J} A_j)' = \cap_{j\in J} A_j'$.

***Remark 2*** (1) For a formal context $\mathbb{K} = (O, P, I)$, if $x \in O$ (or $x \in P$), then $\{x\}'$ is written as $x'$ for short.
(2) By the discussions in Ganter and Wille (1999, pp. 17 and 24), this paper considers the formal contexts with no full rows and no full columns, that we mean objects $g$ with $g' = P$ and attributes $m$ with $m' = O$, respectively.
(3) According to the above (2), it is easy to see

$$\emptyset' = P \text{ if } \emptyset \subseteq O;\ \emptyset' = O \text{ if } \emptyset \subseteq P.$$

(4) It is easy to know that the family of ⊓-semiconcepts is the dual family of ⊔-semiconcepts. Hence, we only consider the family of ⊓-semiconcepts, and simply call a *semiconcept* instead of a ⊓-semiconcept in what follows. All of semiconcepts in a formal context $\mathbb{K}$ is denoted as $\mathcal{B}(\mathbb{K})$.

## 2.4 Poset

**Definition 3** (1) (Welsh 1976, p. 45; Grätzer 2011) A *poset* is a set $S$ together with a binary relation $\leq$ such that the following properties for any $x, y, z \in S$

(p1) $x \leq x$.

(p2) $x \leq y$ and $y \leq x \Rightarrow x = y$.

(p3) $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

(2) (Welsh 1976, p. 3; Grätzer 2011) A map $\varphi : S_1 \to S_2$ between two posets $(S_1, \leq_1)$ and $(S_2, \leq_2)$ is called *order-preserving*, if $x \leq_1 y \Rightarrow \varphi(x) \leq_2 \varphi(y)$ for all $x, y \in S_1$.

## 2.5 Rough set

For a formal context $\mathbb{K} = (O, P, I)$, using the idea of Pawlak classical rough set approximation operators model, Mao (2019) provided a pair of operators $\underline{R}$ and $\overline{R}$ on $O \times P$, see Definition 4, and obtained Lemma 2. The definition of equivalence relation on a set is seen Pawlak (1991).

**Definition 4** Mao (2019) Let $\mathbb{K} = (O, P, I)$ be a formal context and $E \subseteq P \times P$ be defined as: $b_1 E b_2 \Leftrightarrow b_1' = b_2'$ for $b_1, b_2 \in P$. Let $[x]_E$ be the equivalence class containing $x$ for $x \in P$ since it is easy to see $E$ to be an equivalence relation on $P$. Let $(X, Y) \subseteq (O, P)$.

If $X \neq \emptyset$ and $Y \neq \emptyset$. Let $L(X, Y) = \{(x, [b]_E \cap Y) \mid x \in X, b \in P \text{ and } b \in x'\}$ and $U(X, Y) = \{(x, [b]_E \cup Y \mid x \in X, b \in P \text{ and } b \in x'\}$. Let $X = \{x_1, \ldots, x_n\}$. Define

$$\underline{R}(X, Y) = \left( \bigcup_{(x,[b]_E \cap Y) \in L(X,Y)} x, \; \bigcap_{j=1}^{n} \left( \bigcup_{(x_j,[b]_E \cap Y) \in L(X,Y)} ([b]_E \cap Y) \right) \right);$$

$$\overline{R}(X, Y) = \left( \bigcup_{(x,[b]_E \cup Y) \in U(X,Y)} x, \; \bigcap_{j=1}^{n} \left( \bigcup_{(x_j,[b]_E \cup Y) \in U(X,Y)} ([b]_E \cup Y) \right) \right);$$

If one of $X$ and $Y$ is $\emptyset$. Define

$\underline{R}(\emptyset, Y) = (\emptyset, Y)$ and $\overline{R}(\emptyset, Y) = (\emptyset, P)$;
$\underline{R}(X, \emptyset) = (X, \emptyset)$ and $\overline{R}(X, \emptyset) = (X, X')$.

**Lemma 2** Mao (2019) $(\underline{R}, \overline{R})$ *satisfies the following statements for any* $X \subseteq O$ *and* $Y \subseteq P$.

(1) $\underline{R}(X, Y) = (X, X' \cap Y)$ *and* $\overline{R}(X, Y) = (X, X' \cup Y)$.
(2) $\underline{R}(X, Y) \sqsubseteq (X, Y) \sqsubseteq \overline{R}(X, Y)$.
(3) *If* $X \neq \emptyset$ *and* $Y \neq \emptyset$, *then* $\underline{R}(X, Y) = \overline{R}(X, Y) = (X, Y) \Leftrightarrow (X, Y) \in \mathcal{B}(\mathbb{K})$.
(4) $\underline{R}(\emptyset, Y) = \overline{R}(\emptyset, Y) \Leftrightarrow (\emptyset, Y) \in \mathcal{B}(\mathbb{K})$. $\underline{R}(X, \emptyset) = \overline{R}(X, \emptyset) \Leftrightarrow (X, \emptyset) \in \mathcal{B}(\mathbb{K})$.
Yao and Yao (2012) *gave the following definition and properties.*

**Definition 5** Let $U$ be a non-empty set, and $\mathcal{C}$ be a family of subsets of $U$. If $\emptyset \notin \mathcal{C}$ and $\bigcup \mathcal{C} = U$, then $\mathcal{C}$ is called a *covering* of $U$.

**Lemma 3** *A pair of operators* $\underline{APR}$ *and* $\overline{APR}$ *on a non-empty set* $U$ *is a pair of rough set approximations, it should keep the following properties for all* $X \subseteq U$.

(1) $\underline{APR}(X) \subseteq X \subseteq \overline{APR}(X)$.
(2) $X is R-definable \Leftrightarrow \underline{APR}(X) = \overline{APR}(X)$.

*Considered the definition of R-definable by Pawlak (1982, 1991), Definition 5, Lemma 3 with the expression of approximations for knowledge spaces in Xu et al. (2008) and Stefanutti (2019), we can give the following definition and Proposition 1.*

**Definition 6** Let $U$ be a non-empty set, $\mathcal{S} \subseteq 2^U$ and $\mathcal{S} \neq \emptyset$. Then, $(U, \mathcal{S})$ is a *knowledge space* and $\mathcal{S}$ is called *basic knowledge*.

**Proposition 1** *Let* $(U, \mathcal{S})$ *be a knowledge space,* $\underline{APR}$ *and* $\overline{APR}$ *be a pair operators defined on* $2^U$. *Then,* $(\underline{APR}, \overline{APR})$ *is a pair of lower and upper approximation operators if and only if the following properties are satisfied by* $\underline{APR}$ *and* $\overline{APR}$ *with a partial order* $\leq$ *defined on* $2^U$ *for every* $X \subseteq U$.

(1) $\underline{APR} \leq X \leq \overline{APR}(X)$.
(2) $X \in \mathcal{S} \Leftrightarrow \underline{APR}(X) = X = \overline{APR}(X)$.

## 3 Generalized matroids and approximation operators

The first duty in this section is to generalize Whitney classical matroid model from one set to two sets, that is, to obtain a matroidal structure on two sets—TD-matroid. After that, it is to discuss how to generalize lower and upper approximation operators from one set to two sets with the assistance of a TD-matroid. Another duty is the following: in a given formal context $\mathbb{K} = (O, P, I)$, using $(\underline{R}, \overline{R})$ on $2^O \times 2^P$ defined in Definition 4, it will show how to construct TD-matroids. The two duties are the fundamental contents to realize the mutual applications between matroids and rough sets on two sets.

### 3.1 Fundamental notions and approximation operators for TD-matroids

Similar to the definition of Whitney classical matroid model on one set, we provide the definition of a generalized matroidal structure on two sets as follows.

**Definition 7** (1) Let $S$ and $T$ be two sets such that $S \times T \neq \emptyset$. A *two-dimensional matroid*, simply *TD-matroid*, $M$ is $S \times T$ and a collection $\mathcal{T} \subseteq 2^S \times 2^T$ (called *feasible* sets) such that (T1)–(T3) are satisfied.

(T1) $\mathcal{T} \neq \emptyset$.

(T2) $(X_1, Y_1) \in \mathcal{T}$and$(X_2, Y_2) \sqsubseteq (X_1, Y_1) \Rightarrow (X_2, Y_2) \in \mathcal{T}$.

(T3) Let $(X_1, Y_1), (X_2, Y_2) \in \mathcal{T}$.If$(X_1, Y_1) \neq (\emptyset, \emptyset)$and $|(X_2, Y_2)| < |(X_1, Y_1)|$, then$(X_2, Y_2) \cup (a, b) \in \mathcal{T}$ for some $(a, b) \in (X_1, Y_1) \setminus (X_2, Y_2)$and$(a, b) \neq (\emptyset, \emptyset)$.

(2) Two TD-matroids $M_1$ and $M_2$ on $S_1 \times T_1$ and $S_2 \times T_2$, respectively, are *isomorphic* if there are two bijections $\varphi_1 : S_1 \to S_2$ and $\varphi_2 : T_1 \to T_2$ such that $\varphi : S_1 \times T_1 \to S_2 \times T_2$ defined as $(x_1, y_1) \mapsto (\varphi_1(x_1), \varphi_2(y_1))$ preserves feasibility. We write $M_1 \cong M_2$ if $M_1$ and $M_2$ are isomorphic.

We use an example, in which the data are referred to Liu et al. (2011), to show the existence of TD-matroids.

***Example 1*** Table 1 is a table information expressed some specimens of genus *Uloma* Dejean of China in biology.

In Table 1, "yes" represents an insect has the correspondent characteristic;

"no" stands for an insect does not have the correspondent characteristic.

Let $S$ be some specimens of genus *Uloma* Dejean of China, i.e., $S = \{x_1 = U.compressa,\ x_2 = U.latimanus,\ x_3 = U.contracta,\ x_4 = U.quadratithoraca\}$.

Let $T$ be some biological characteristics, i.e.,

$$T = \{b_1 = \text{Posterior angles of pronotum obtuse},$$
$$b_2 = \text{Male anterior margin of pronotum not emarginate},$$
$$b_3 = \text{Anterior margin of clypeus straight}\}.$$

Let $\mathcal{T} = \{(\emptyset, \{b_1, b_2, b_3\}), (\emptyset, \{b_2, b_3\}), (x_1, \{b_1, b_2, b_3\}), (x_2, \{b_1, b_2, b_3\}), (\{x_1, x_2\}, \{b_1, b_2, b_3\}), (x_1, \{b_2, b_3\}), (x_2, \{b_2, b_3\}), (\{x_1, x_2\}, \{b_2, b_3\})\}$.

We may easily check $\mathcal{T}$ to satisfy the conditions (T1), (T2) and (T3). Hence, $(S \times T, \mathcal{T})$ is a TD-matroid.

Let 1 = yes and 0 = no. Then, Table 1 can be expressed as Table 2. Hence, $(S, T, I)$ is a formal context in which $I$ is shown as Table 2.

Let $OB$ be a non-empty set of some biological specimens such as insects, $AT$ be a non-empty set of some biological characteristics, and $I \subseteq OB \times AT$ be defined as follows for $(x, y) \in OB \times AT$:

$$(x, y) \in I \Leftrightarrow x \text{ owns the characteristic } y.$$

Let $(X, Y) \subseteq OB \times AT$. Let $X'$ stand for the family of characteristics who are commonly owned by any of specimens in $X$. Then, $(X, X') \in \mathcal{B}((OB, AT, I))$. In fact, sometimes, biologists just hope to search out $(X, X')$ for any $X \subseteq OB$ when they study the classification of $OB$.

Let $A_1 = \{x_3, x_4\}$ and $B_1 = \{b_2\}$. It is clear to know $A'_1 = x'_3 \cap x'_4 = \{b_1, b_2\} \cap \{b_2\} = \{b_2\} = B_1$ by Lemma 1(2) and Definition 2. Hence, $(A_1, B_1) \in \mathcal{B}((S, T, I))$ holds by Definition 2(2) and Remark 2.

Let $A_2 = \{x_3\}$ and $B_2 = B_1$. Then, we see $A_2 \subseteq A_1$ and $B_2 \supseteq B_1$. So, $(A_2, B_2) \sqsubseteq (A_1, B_1)$ holds. However, $A'_2 = x'_3 = \{b_1, b_2\} \neq B_2$ follows $(A_2, B_2) \notin \mathcal{B}((S, T, I))$ using Definition 2(2). Thus, $\mathcal{B}((S, T, I))$ does not satisfy (T2).

Using Table 2, we see $(x_2, x'_2 = b_3), (x_3, x'_3 = \{b_1, b_2\}) \in \mathcal{B}((S, T, I))$ such that $|(x_2, x'_2)| = 1 + 1 = 2 < 3 = |(x_3, x'_3)|$. However, $(x_2, x'_2) \cup (a, b) \notin \mathcal{B}((S, T, I))$ holds for any $(a, b) \in (x_3, x'_3) \setminus (x_2, x'_2)$ and $(a, b) \neq (\emptyset, \emptyset)$. That is to say, (T3) does not hold for $\mathcal{B}((S, T, I))$.

In other words, $(S \times T, \mathcal{B}((S, T, I)))$ is not a TD-matroid by Definition 7.

In addition, we may easily know $(S \times T, \mathcal{T}) \neq \mathcal{B}((S, T, I))$ since $(x_2, x'_2) \notin \mathcal{T}$ though $(x_2, x'_2) \in \mathcal{B}((S, T, I))$.

With Example 1, we can indicate the following statements.

(1) The structure of $\mathcal{T}$ is different from that of $\mathcal{B}((S, T, I))$, though the two structures are based on the same ground set $S \times T$. This statement is the same with the general idea since the background knowledge for everyone is perhaps different to solve the same problem based on the same context.

(2) We will find $X'_2 \subseteq X'_1$ if $X_1 \subseteq X_2 \subseteq S$ according to biological knowledge. This is just the same as that demonstrated by Lemma 1(1). Considering this result with the basic properties for semiconcept in Vormbrock and Wille 2005, we can indicate that (T2) is correct for the classification of biological specimens, though it has $(S \times T, \mathcal{T}) \neq \mathcal{B}((S, T, I))$.

***Remark 3*** Let $(S \times T, \mathcal{T})$ be a TD-matroid.

(1) We analyze the definition of a TD-matroid, i.e., Definition 7(1), as follows.

**Table 1** Taxa and data matrix used in the cladistic analysis of some species of genus *Uloma* Dejean of China

| | Posterior angles of pronotum obtuse | Male anterior margin of pronotum not emarginate | Anterior margin of clypeus straight |
|---|---|---|---|
| *U.compressa* | Yes | No | No |
| *U.latimanus* | No | No | Yes |
| *U.contracta* | Yes | Yes | No |
| *U.quadratithoraca* | No | Yes | No |

**Table 2** The mathematical expression of Table 1

| | $b_1$ | $b_2$ | $b_3$ |
|---|---|---|---|
| $x_1$ | 1 | 0 | 0 |
| $x_2$ | 0 | 0 | 1 |
| $x_3$ | 1 | 1 | 0 |
| $x_4$ | 0 | 1 | 0 |

(1.1) In Definition 1(1), if $S = \emptyset$, then $(S, \mathcal{I}_S) = (\emptyset, \{\emptyset\})$ is a matroid.

In biology, $S \times T = \emptyset$ means no specimens and no biological characteristics to be considered. This case has no valuable for biologists. Hence, Definition 7(1) asks $S \times T \neq \emptyset$. Correspondingly, this paper will suppose $S \neq \emptyset$ for any matroid $(S, \mathcal{I})$.

(1.2) Let $(X_1, Y_1), (X_2, Y_2) \in \mathcal{T}$ satisfy $|(X_2, Y_2)| < |(X_1, Y_1)|$ and $(X_1, Y_1) \neq (\emptyset, \emptyset)$.

If $Y_1 \backslash Y_2 = \emptyset$ and $Y_1 \cap Y_2 \neq \emptyset$, then $|X_2| < |X_1|$ holds since "$Y_1 \backslash Y_2 = \emptyset$ and $Y_1 \cap Y_2 \Rightarrow Y_1 \subseteq Y_2$" and $|X_2| + |Y_2| = |(X_2, Y_2)| < |(X_1, Y_1)| = |X_1| + |Y_1|$. No matter whether (T2) is satisfied, we can obtain $a \neq \emptyset$ satisfying $(a, \emptyset) \in (X_1 \backslash X_2, Y_1 \backslash Y_2)$, but we do not confirm $(X_2 \cup a, Y_2) \in \mathcal{T}$ since we do not know whether $X_2 \cup a \subseteq X_1$ even if $\mathcal{T}$ satisfies (T2). This shows the essentials of (T3) and the independence between (T2) and (T3) in Definition 7.

(2) Next, for the restrictive conditions (I1)-(I3) and (T1)-(T3) on matroid and TD-matroid, respectively, we analyze the relationships between them.

(2.1) (I1) and (T1) are the same.

(2.2) Let $(S \times T, \mathcal{I})$ be a matroid and $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \subseteq S \times T$.

If $Y \subseteq X \in \mathcal{I}$, then $Y \in \mathcal{I}$ since (I2). It is easy to obtain

$$(Y_1, Y_2) \subseteq (X_1, X_2) \Rightarrow Y_1 \subseteq X_1 \text{ and } Y_2 \subseteq X_2.$$

If $Y_2 \neq X_2$, then $Y_2 \supseteq X_2$ does not hold. This hints $(Y_1, Y_2) \sqsubseteq (X_1, X_2)$ not to be correct. And further, (I2) will not replace (T2).

If $Y \sqsubseteq X \in \mathcal{T}$, then $Y \in \mathcal{T}$ according to (T2). If $Y_2 \neq X_2$, then $Y_2 \subseteq X_2$ does not hold according to $Y_2 \supset X_2$. This means $(Y_1, Y_2) \subseteq (X_1, X_2)$ not to be correct. And further, (T2) will not replace (I2).

Hence, we confirm that (T2) is a new feature which is different from (I2).

(2.3) By (I1) and (I2), we know $(\emptyset, \emptyset) \in \mathcal{I}$. The TD-matroid in Example 1 satisfies $(\emptyset, \emptyset)$ not to be feasible. It is easy to see $(S \times T, \{(\emptyset, \emptyset)\})$ to be a matroid with $T \neq \emptyset$. By (T2), $(S \times T, \{(\emptyset, \emptyset)\})$ is not a TD-matroid. This means (I1)-(I2) together does not have ability to replace (T1)-(T2).

(2.4) If (I3) and (T3) can replace each other, then $|(\emptyset, \emptyset)| < |(X_1, X_2)|$ for $(X_1, X_2) \in \mathcal{T} \backslash (\emptyset, \emptyset)$. Using (I3), we obtain $(a, b) \in \mathcal{T}$ for $\forall (a, b) \in (X_1, X_2)$, a contradiction to Example 1 since $(x_1, b_j) \notin \mathcal{T}$, $(j = 1, 2, 3)$ in Example 1. Hence, (I3) and (T3) cannot replace each other in general.

(3) Every feasible set in a TD-matroid is expressed by "pairwise" which is different from the classical matroid in which any independent set is expressed by unary form.

In other words, even if $(U = S \times T, \mathcal{I} \subseteq 2^U)$ be a matroid, then $(S \times T, \mathcal{I})$ may not be a TD-matroid since (T2) cannot be expressed by (I1)-(I3).

Next, we consider the relationships in respect of properties between matroids and TD-matroids. For this purpose, we need the following series lemmas.

**Lemma 4** *Let $S$ and $T$ be two sets. Let $(X_j, Y_j) \subseteq S \times T, (j = 1, 2)$. If $(X_1, Y_1) \neq (\emptyset, \emptyset)$ and $|(X_2, Y_2)| < |(X_1, Y_1)|$, then $(X_1, Y_1) \backslash (X_2, Y_2) \neq (\emptyset, \emptyset)$.*

***Proof*** $(X, Y) \neq (\emptyset, \emptyset)$ means that at least one of $X$ and $Y$ is not to be $\emptyset$ for $(X, Y) \subseteq S \times T$. According to this meaning, we will use two cases to finish the proof.

Case 1. $(X_2, Y_2) = (\emptyset, \emptyset)$.

$(X_1, Y_1) \backslash (X_2, Y_2) \neq (\emptyset, \emptyset)$ is followed since $(X_1, Y_1) \backslash (X_2, Y_2) = (X_1, Y_1) \neq (\emptyset, \emptyset)$.

Case 2. $(X_2, Y_2) \neq (\emptyset, \emptyset)$.

Suppose $(X_1, Y_1) \backslash (X_2, Y_2) = (\emptyset, \emptyset)$. Then, there are $X_1 \backslash X_2 = \emptyset$ and $Y_1 \backslash Y_2 = \emptyset$ since $(X_1, Y_1) \backslash (X_2, Y_2) = (X_1 \backslash X_2, Y_1 \backslash Y_2)$. Thus, it is easy to find $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. And further, we receive $|X_1| \leq |X_2|$ and $|Y_1| \leq |Y_2|$. So, it follows $|(X_1, Y_1)| = |X_1| + |Y_1| \leq |X_2| + |Y_2| = |(X_2, Y_2)|$. This is a contradiction to $|(X_2, Y_2)| < |(X_1, Y_1)|$.

Lemma 4 implies the existence of $(a, b) \in (X_1, Y_1) \backslash (X_2, Y_2) \neq (\emptyset, \emptyset)$. It also gives a chance of the possibility for (T3) in Definition 7(1). □

**Lemma 5** *Let $E$, $S$ and $T$ be three sets. Then, the following statements are correct.*

(1) *Let $S \neq \emptyset$. If $(S \times T, \mathcal{T}_S)$ is a TD-matroid, then $(S, \mathcal{I}_S)$ is a matroid, where $\mathcal{I}_S = \{X \subseteq S \mid (X, Y) \in \mathcal{T}_S$ for some $Y \subseteq T\}$.*

**Table 3** Comparison the structures between matroid and TD-matroid

| | Background set | Dimension of background set | Decisive conditions | Replacement |
|---|---|---|---|---|
| $(E,\mathcal{I})$, a matroid | $E$ | One | (I1)–(I3) | $(E\times\emptyset,\mathcal{I}\times\emptyset)$, a TD-matroid with $E\neq\emptyset$ |
| $(S\times T,\mathcal{T})$, a TD-matroid | $S\times T$ | Two | (T1)–(T3) | $(S\times\emptyset,\mathcal{T})$,a matroid with $S\neq\emptyset$ |

(2) *Let $E\neq\emptyset$ and $(E,\mathcal{I}_E)$ be a matroid. Then, $(E\times\emptyset,\mathcal{T}_E)$ is a TD-matroid where $\mathcal{T}_E=\{(X,\emptyset)\mid X\subseteq E, X\in\mathcal{I}_E\}$.*

*Lemma 5 can be easily verified by checking $\mathcal{I}_S$ and $\mathcal{T}_E$ to satisfy (I1)–(I3) and (T1)–(T3), respectively. Its proof is omitted.*

***Remark 4*** (1) We analyze Lemma 5 as follows

Let $(S\times T,\mathcal{T})$ be a TD-matroid. If $S=\emptyset$, then $T\neq\emptyset$ holds since $S\times T\neq\emptyset$. By Lemma 5, we can obtain $\mathcal{I}_T=\{Y\subseteq T\mid (X,Y)\in\mathcal{T}$ for some $X\subseteq S\}$ to be a matroid since "$S\times T\neq\emptyset$ and $S=\emptyset''\Rightarrow$ "$T\neq\emptyset$". Combining with Lemma 5, no matter $S=\emptyset$ or $S\neq\emptyset$, we always construct a matroid based on $(S\times T,\mathcal{T})$. Hence, we only need to pay attention to the case of $S\neq\emptyset$.

According to the above and item (1) in Remark 3, we ask $S\neq\emptyset$ and $E\neq\emptyset$ in Lemma 5.

(2) Let $(S,\mathcal{I})$ be a matroid with $S\neq\emptyset$.

Let $E=S\times\emptyset$. Then, we may obtain $(E,\mathcal{I}\times\emptyset=\{(X,\emptyset)\mid X\in\mathcal{I}\})$ to be a matroid according to Definition 1.

Let $f:E\to E\times\emptyset$ as $x\mapsto(x,\emptyset)$. Then, it is easy to see $f$ to be an isomorphic map between $(S,\mathcal{I})$ and $(E,\mathcal{I}\times\emptyset)$ using item (2) in Definition 1.

Combining the above and Lemma 5, we can obtain Table 3.

**Lemma 6** *Let $(S_j\times T_j,\mathcal{T}_{S_j})$ be a TD-matroid, $(E_j,\mathcal{I}_{E_j})$ be a matroid, and $\mathcal{I}_{S_j}$, $\mathcal{T}_{E_j}$ be defined as Lemma 5 on the sets $S_j$ and $E_j\times\emptyset$, respectively, where $S_j\neq\emptyset$ and $E_j\neq\emptyset$, $(j=1,2)$. Then,*

(1) $(S_1\times T_1,\mathcal{T}_{S_1})\cong(S_2\times T_2,\mathcal{T}_{S_2})\Rightarrow(S_1,\mathcal{I}_{S_1})\cong(S_2,\mathcal{I}_{S_2})$.
(2) $(E_1,\mathcal{I}_{E_1})\cong(E_2,\mathcal{I}_{E_2})\Rightarrow(E_1\times\emptyset,\mathcal{T}_{E_1})\cong(E_2\times\emptyset,\mathcal{T}_{E_2})$.

*Lemma 6 can be easily verified by combining Lemma 5 and Definition 1(2) with Definition 7(2) and its proof is omitted.*

**Theorem 1** *Under the idea of isomorphism of matroids in Definition 1(2) and the idea of isomorphism of TD-matroids in Definition 7(2), the correspondence between a matroid $(E,\mathcal{I}_E)$ and the TD-matroid $(E\times\emptyset,\mathcal{T}_E)$ is a bijection between the set of matroids and the set of TD-matroids in the forms $(S\times\emptyset,\mathcal{T})$ where $S\neq\emptyset$.*

***Proof*** It is straightforward by Definition 1(2) and Definition 7(2), Lemmas 5 and 6. □

***Remark 5*** However, a TD-matroid on $S\times\emptyset$ is a special kind of TD-matroids where $S\neq\emptyset$. Considered Theorem 1, we can say that the definition of a TD-matroid is an extension of the definition of Whitney classical matroid from one set $S$ to two sets $S$ and $T$. In other words, TD-matroid is a matroidal structure. Hence, Definition 7 is meaningful.

For matroidal structures on one set, some researchers use rough set to study on them (Hu and Yao 2019; Li et al. 2016, 2019; Wang et al. 2019; Zhu and Wang 2011). Here, for a TD-matroid on two sets, we will apply rough set to research on it since TD-matroid is a matroidal structure by Theorem 1. The basic work of the application is to explore the relationships between TD-matroids and approximation operators. Hence, our work now is to construct a pair of approximation operators using a TD-matroid.

According to Definition 6, a TD-matroid $(S\times T,\mathcal{T})$ is a knowledge space with $\mathcal{T}$ as the set of basic knowledge. Hence, we will seek approximation operators based on the knowledge space $(S\times T,\mathcal{T})$, i.e., on a TD-matroid.

**Definition 8** Let $(S\times T,\mathcal{T})$ be a TD-matroid. Let $(A,B)\subseteq(S,T)$. Define

$$low(A,B)=\{(X,Y)\mid(X,Y)\in\mathcal{T},(X,Y)\sqsubseteq(A,B)\};$$
$$upr(A,B)=\{(X,Y)\mid(X,Y)\in\mathcal{T},(A,B)\sqsubseteq(X,Y)\};$$
$$\underline{MR}(A,B)=(\cup_{(X,Y)\in low(A,B)}X,\cap_{(X,Y)\in low(A,B)}Y);$$
$$\overline{MR}(A,B)=(\cap_{(X,Y)\in upr(A,B)}X,\cup_{(X,Y)\in upr(A,B)}Y)\text{ if } upr(A,B)\neq\emptyset.$$
$$\overline{MR}(A,B)=(S,\emptyset)\text{ if } upr(A,B)=\emptyset.$$

We analyze Definition 8 as follows.

***Remark 6*** Let $(S\times T,\mathcal{T})$ be a TD-matroid.

On one hand, $\mathcal{T}\neq\emptyset$ is correct by (T1). We confirm $(\emptyset,T)\in\mathcal{T}$ since $(\emptyset,T)\sqsubseteq(X,Y)\in\mathcal{T}$ and (T2). And further, it induces $low(A,B)\neq\emptyset$ since $(\emptyset,T)\sqsubseteq(A,B)$ and $(\emptyset,T)\in\mathcal{T}$. Hence, the definition of $\underline{MR}(A,B)$ is effective and well defined.

On the other hand, if $upr(A,B)\neq\emptyset$, then $\overline{MR}(A,B)$ is well defined. If $upr(A,B)=\emptyset$, then by the definition of $\overline{MR}$ for $upr(A,B)\neq\emptyset$, there is $\overline{MR}(A,B)=(S,\emptyset)$ since $\cap_{\emptyset\subseteq S}\emptyset=S$ and $\cup_{\emptyset\subseteq T}\emptyset=\emptyset$. This also shows the correct of $\overline{MR}(A,B)=(S,\emptyset)$ if $upr(A,B)=\emptyset$ in Definition 8. Definition 8 provides the different expressions for $upr(A, B) \neq \emptyset$ and $upr(A, B) = \emptyset$ so as to express $\overline{MR}(A, B)$ more clearly. In one word, $\overline{MR}(A, B)$ is effective.

Considering Definition 8, we will find the following lemma.

**Lemma 7** *Let $(S \times T, \mathcal{T})$ be a TD-matroid. Then, the following statements are correct for any $(A, B) \subseteq (S, T)$.*

(1) $\underline{MR}(A, B) \sqsubseteq (A, B) \sqsubseteq \overline{MR}(A, B)$.
(2) $(A, B) \in \mathcal{T} \Rightarrow \underline{MR}(A, B) = (A, B) = \overline{MR}(A, B)$.
(3) *If* $upr(A, B) \neq \emptyset$, *then* $\underline{MR}(A, B) = (A, B) = \overline{MR}(A, B) \Rightarrow (A, B) \in \mathcal{T}$.

*The proof of Lemma 7 can be found in Appendix.*

***Remark 7*** We analyze the condition $upr(A, B) \neq \emptyset$ in Lemma 7 as follows.

Let $upr(A, B) = \emptyset$. We see $\overline{MR}(A, B) = (S, \emptyset)$ by Definition 8. If $\underline{MR}(A, B) = (A, B) = \overline{MR}(A, B)$, then $(A, B) = (S, \emptyset)$. However, in some TD-matroid, $(S, \emptyset) \in \mathcal{T}$ does not hold. For example, the TD-matroid $(S \times T, \mathcal{T})$ in Example 1 satisfies $(S, \emptyset) = (\{x_1, x_2, x_3, x_4\}, \emptyset) \notin \mathcal{T}$.

In other words, the pre-condition $upr(A, B) \neq \emptyset$ is necessary for $\underline{MR}(A, B) = (A, B) = \overline{MR}(A, B) \Rightarrow (A, B) \in \mathcal{T}$ in Lemma 7 (3).

Combining Remarks 6 and 7 with Lemma 7, we can demonstrate the following theorem.

**Theorem 2** *Let $(S \times T, \mathcal{T})$ be a TD-matroid. If $\underline{MR}$ and $\overline{MR}$ are defined as Definition 8, then for $(A, B) \subseteq (S, T)$ and $upr(A, B) \neq \emptyset$, there are the following statements.*

(1) $\underline{MR}(A, B) \sqsubseteq (A, B) \sqsubseteq \overline{MR}(A, B)$.
(2) $\underline{MR}(A, B) = (A, B) = \overline{MR}(A, B) \Leftrightarrow (A, B) \in \mathcal{T}$.

***Proof*** It is routine verification from Lemma 7. □

***Remark 8*** We analyze $\underline{MR}$ and $\overline{MR}$ as follows.

(1) For any $(A, B) \in \mathcal{T}$, we confirm $upr(A, B) \neq \emptyset$ since $(A, B) \sqsubseteq (A, B) \in \mathcal{T} \Rightarrow (A, B) \in upr(A, B)$. Hence, using Theorem 2, $\underline{MR}$ and $\overline{MR}$ together provides an idea to characterize $\mathcal{T}$.

Under the case of $upr(A, B) = \emptyset$ for $(A, B) \subseteq S \times T$, we can use $\underline{MR}(A, B)$ to approximate $(A, B)$, namely, use $\underline{MR}(A, B)$ to infer the information existed in $(A, B)$. If $upr(A, B) \neq \emptyset$, then Theorem 2 shows $\underline{MR}$ and $\overline{MR}$ to be approximation operators by Proposition 1.

Additionally, $\underline{MR}$ and $\overline{MR}$ are described as binary form which is different from the known rough set approximation operators defined on one set and that on two non-empty sets. It is a new expression with an information table and knowledge database. Hence, it is better to say that a pair of operators by binary form is to be approximation operators if the pair satisfies most of properties in Proposition 1 which is satisfied by approximation operators on one set.

Based on the above analysis, $(\underline{MR}, \overline{MR})$ is roughly called *lower* and *upper* approximation operators defined by the basic knowledge $\mathcal{T}$ in knowledge space $(S \times T, \mathcal{T})$—a TD-matroid using Definition 6 and Proposition 1. In other words, $(\underline{MR}, \overline{MR})$ is a pair of approximation operators in a knowledge space $(S \times T, \mathcal{T})$, or said an information system, with TD-matroid constraints. Using $(\underline{MR}, \overline{MR})$, we can extract information from $(S \times T, \mathcal{T})$ by rough set theory.

(2) For a family $OB$ of collected biological specimens such as insects and a family $AT$ of some biological characteristics, sometimes biologists hope to consider to approximate some information based on the known information $\mathcal{T} \subseteq 2^{OB} \times 2^{AT}$ which is the family of feasible sets of a TD-matroid. Under this view, the lower $\underline{MR}$ and upper $\overline{MR}$ of approximation operators may play an important role.

Let $upr(A, B) = \emptyset$ for some $(A, B) \subseteq OB \times AT$. Definition 8 points $\overline{MR}(A, B) = (S, \emptyset)$. This means that there does not exist any known biological information to give a hand for biologists to guess the biological knowledge existed in $(A, B)$ with aspect of $\overline{MR}(A, B)$. Under this case, if biologists hope to obtain the results by their guesses using the known information $\mathcal{T}$, then $\underline{MR}$ may give their help. Thus, $(\underline{MR}, \overline{MR})$ will assist biologists to do their research.

The following example illustrates Lemma 7 and Theorem 2.

***Example 2*** Let $S$, $T$ and $\mathcal{T}$ be given in Example 1.

Let $A_1 = \{x_1, x_2\}$ and $B_1 = \{b_1, b_2\}$. Then,

$$\begin{aligned} low(A_1, B_1) &= \{(X, Y) \in \mathcal{T} \mid (X, Y) \sqsubseteq (A_1, B_1)\} \\ &= \{(\emptyset, \{b_1, b_2, b_3\}), (x_1, \{b_1, b_2, b_3\}), \\ &\quad (x_2, \{b_1, b_2, b_3\}), (\{x_1, x_2\}, \{b_1, b_2, b_3\})\}, \\ upr(A_1, B_1) &= \{(X, Y) \in \mathcal{T} \mid (A_1, B_1) \sqsubseteq (X, Y)\} = \emptyset. \end{aligned}$$

Thus,

$$\begin{aligned} &\underline{MR}(A_1, B_1) = (\emptyset \cup x_1 \cup x_2 \cup \{x_1, x_2\}, \\ &\{b_1, b_2, b_3\} \cap \{b_1, b_2, b_3\} \\ &\cap\{b_1, b_2, b_3\} \cap \{b_1, b_2, b_3\}) \\ &= (\{x_1, x_2\}, \{b_1, b_2, b_3\}) \end{aligned}$$

and $\overline{MR}(A_1, B_1)) = (\{x_1, x_2, x_3, x_4\}, \emptyset)$.

Let $A_2 = \{x_1\}$ and $B_2 = \{b_2, b_3\}$. Then,

$$\begin{aligned} low(A_2, B_2) &= \{(X, Y) \in \mathcal{T} \mid (X, Y) \sqsubseteq (A_2, B_2)\} \\ &= \{(\emptyset, \{b_1, b_2, b_3\}), (\emptyset, \{b_2, b_3\}), \\ &\quad (x_1, \{b_2, b_3\}), (x_1, \{b_1, b_2, b_3\})\} \\ upr(A_2, B_2) &= \{(X, Y) \in \mathcal{T} \mid (A_2, B_2) \sqsubseteq (X, Y)\} \end{aligned}$$

$= \{(x_1, \{b_2, b_3\}), (\{x_1, x_2\}, \{b_2, b_3\})\}$.

So, $\underline{MR}(A_2, B_2) = (\emptyset \cup \emptyset \cup x_1 \cup x_1, \{b_1, b_2, b_3\} \cap \{b_2, b_3\} \cap \{b_2, b_3\} \cap \{b_1, b_2, b_3\})$
$= (x_1, \{b_2, b_3\})$ and $\overline{MR}(A_2, B_2) = (x_1 \cap \{x_1, x_2\}, \{b_2, b_3\} \cup \{b_2, b_3\}) = (x_1, \{b_2, b_3\})$.

From the above, we summarize the results as follows:

(1) $(A_1, B_1) \notin \mathcal{T}, upr(A_1, B_1) = \emptyset, \overline{MR}(A_1, B_1) = (S, \emptyset)$ and $\underline{MR}(A_1, B_1) \neq \overline{MR}(A_1, B_1)$;
(2) $(A_2, B_2) \in \mathcal{T}$ and $\underline{MR}(A_2, B_2) = \overline{MR}(A_2, B_2) = (A_2, B_2)$;
(3) $\underline{MR}(A_i, B_i) \sqsubseteq (A_i, B_i) \sqsubseteq \overline{MR}(A_i, B_i)$, $(i = 1, 2)$.

The above results not only show the condition $upr(A, B) \neq \emptyset$ to be a necessary condition for $\underline{MR}(A, B) = (A, B) = \overline{MR}(A, B) \Rightarrow (A, B) \in \mathcal{T}$ where $(A, B) \subseteq S \times T$, but also examine the correct of Theorem 2. In addition, $upr(A_1, B_1) = \emptyset$ demonstrates the correct of the description in Remark 8(2) for $upr(A_1, B_1) = \emptyset$.

***Remark 9*** In Example 2, we find $(A_1, B_1) \notin \mathcal{T}, upr(A_1, B_1) = \emptyset$ and $\bigcup \mathcal{T} \neq (S, T)$. In Pawlak classical rough set approximation operators model, the family of all of classes for an equivalent relation $R$ on a non-empty set $U$ satisfies $\bigcup_{x \in U} [x]_R = U$. That is to say, $\{[x]_R \mid x \in U\}$ is a covering of $U$ in view of Definition 5. Some researchers considered rough set approximation operators relative to matroid $(U, \mathcal{I})$ under the supposition $\bigcup \mathcal{I} = U$ (e.g.,Li et al. 2016 ), i.e., $\mathcal{I}$ is a covering of $U$ by Definition 5. In fact, if the family of specimens collected by biologists is in notations $S$, then under some cases, the set $\mathcal{T}$ of known knowledge of biologists does not always satisfy $\bigcup \mathcal{T} = (S, T)$, where $T$ is the set of biological characteristics considered by biologists for every specimen in $S$. For example, in Example 1, there is $\bigcup \mathcal{T} = (\{x_1, x_2\}, \{b_1, b_2, b_3\}) \neq (\{x_1, x_2, x_3, x_4\}, \{b_1, b_2, b_3\}) = (S, T)$. Hence, we do not ask $\bigcup \mathcal{T} = (S, T)$ for TD-matroid. It is easy to know the relevant results of TD-matroid here to be correct if $\bigcup \mathcal{T} = (S, T)$ according to the definition of TD-matroid in Definition 7.

The following is to consider some further properties for $\underline{MR}$ and $\overline{MR}$.

**Lemma 8** *Let $(S \times T, \mathcal{T})$ be a TD-matroid. If $(A_1, B_1) \sqsubseteq (A_2, B_2) \subseteq S \times T$, then the following statements hold.*
*(1) $\underline{MR}(A_1, B_1) \sqsubseteq \underline{MR}(A_2, B_2)$.*
*(2) $\overline{MR}(A_1, B_1) \sqsubseteq \overline{MR}(A_2, B_2)$.*
*The proof of Lemma 8 can be found in Appendix.*
*The following example shows the correct of Lemma 8.*

***Example 3*** Let $S, T$ and $\mathcal{T}$ be given in Example 1, and $(A_1, B_1)$ be given in Example 2.

Let $A_3 = \{x_1\}$ and $B_3 = \{b_1, b_2, b_3\}$. Then, $(A_3, B_3) \sqsubseteq (\{x_1, x_2\}, \{b_1, b_2\}) = (A_1, B_1)$ holds.

Using Example 1, we see $(A_3, B_3) \in \mathcal{T}$. So, combining with Lemma 7, we receive $\underline{MR}(A_3, B_3) = \overline{MR}(A_3, B_3) = (A_3, B_3)$.

Using Example 2, we see

$$\underline{MR}(A_1, B_1) = (\{x_1, x_2\}, \{b_1, b_2, b_3\});$$
$$\overline{MR}(A_1, B_1) = (\{x_1, x_2, x_3, x_4\}, \emptyset).$$

Hence, we obtain $\underline{MR}(A_3, B_3) \sqsubseteq \underline{MR}(A_1, B_1)$ and $\overline{MR}(A_3, B_3) \sqsubseteq \overline{MR}(A_1, B_1)$.

By an extension of Lemma 8, we find the following corollary.

**Corollary 1** *Let $\underline{MR}$ and $\overline{MR}$ be defined as Definition 8. Then, $\underline{MR}$ and $\overline{MR}$ are two order-preserving mappings from $(2^S \times 2^T, \sqsubseteq)$ to $(2^S \times 2^T, \sqsubseteq)$, and further from $(\mathcal{T}, \sqsubseteq)$ to $(\mathcal{T}, \sqsubseteq)$.*

*Corollary 1 is straightforward to be verified by Definition 3 and the definition of $\sqsubseteq$. Its proof is omitted.*

***Remark 10*** In the research of biology, sometimes biologists express their results with the form of posets, for example, phylogenetic tree is a construction of poset. Lemma 8 and Corollary 1 imply that the two approximation operators $\underline{MR}$ and $\overline{MR}$ preserve the "construction" of a known information system $\mathcal{T}$ with respect of posets where $\mathcal{T}$ is the set of feasible sets of a TD-matroid. In fact, this assertion shows that in some biological research, $\underline{MR}$ and $\overline{MR}$ own the ability to keep the structure of the whole space which is considered by biologists and the structure of basic knowledge of biologists.

## Comparison and analysis

There are many models for rough set approximation operators. We just compare $(\underline{MR}, \overline{MR})$ to some other famous models.

(1) Table 4 is the comparison between $(\underline{MR}, \overline{MR})$, or say TD-matroid approximation operators model, and the pair $(\underline{R}, \overline{R})$, the approximation operators model defined in Pawlak classical rough set.

Let $\underline{MR}$ and $\overline{MR}$ be produced by a TD-matroid $(S \times T, \mathcal{T})$, respectively.

Let $U$ be a non-empty set, $R$ be an equivalence relation on $U$, $[x]_R$ be a category in $R$ containing $x \in U$.

With the aspect of express form, $(\underline{MR}, \overline{MR})$ and $(\underline{R}, \overline{R})$ cannot replace each other.

Suppose $U = S \times T$. $(\underline{R}, \overline{R})$ cannot be replaced by $(\underline{MR}, \overline{MR})$ since $2^{S \times T} \neq 2^S \times 2^T$ in general. For example, let $S$ and $T$ be given in Example 1. We may easily know $2^{S \times T} \neq 2^S \times 2^T$.

Generally, $\{[x]_R \mid x \in U\}$ does not satisfy (I3) since $0 < |[x]_R| < |[y]_R| \Rightarrow |[x]_R| \neq |[y]_R|$ for $x, y \in U$. And further, $[x]_R \cup a \notin R$ holds for any $a \in [y]_R$. For example, let $S$ and $T$ be given in Example 1. Let $U_0 = S \cup \{x_5 = U.formosana\}$. Let $x, y \in U_0$. Define $R_0$ as:

**Table 4** Comparison between $(\underline{MR}, \overline{MR})$ and Pawlak classical $(\underline{R}, \overline{R})$

| Operator | Domain | Range | Based space | Basic knowledge | Express form |
|---|---|---|---|---|---|
| $(\underline{MR}, \overline{MR})$ | $2^S \times 2^T$ | $2^S \times 2^T$ | $(S \times T, \mathcal{T})$ | $\mathcal{T}$ | Binary |
| $(\underline{R}, \overline{R})$ | $2^U$ | $2^U$ | $(U, R)$ | $\{[x]_R \mid x \in U\}$ | Unary |

$xR_0y \Leftrightarrow x' = y'$. It is easy to see the binary relation $R_0$ given here to be an equivalence one on $U_0$. Thus, $[x]_{R_0} = \{y \in U_0 \mid x' = y'\}$ holds for $x \in U_0$. Using Liu et al. (2011), we know the relationships between *U.formosana* and $T$ as: corresponding to "Posterior angles of pronotum obtuse" is "yes"; for the other two "Male anterior margin of pronotum not emarginate" and "Anterior margin of clypeus straight", they are "not," respectively. $(S \cup x_5 = U_0, T, I)$ can be expressed in Table 5.

**Table 5** The mathematical expression of $(S \cup x_5 = U_0, T, I)$

| | $b_1$ | $b_2$ | $b_3$ |
|---|---|---|---|
| $x_1$ | 1 | 0 | 0 |
| $x_2$ | 0 | 0 | 1 |
| $x_3$ | 1 | 1 | 0 |
| $x_4$ | 0 | 1 | 0 |
| $x_5$ | 1 | 0 | 0 |

Then, we obtain $[x_1]_{R_0} = \{x_1, x_5\}$ and $[x_2]_{R_0} = \{x_2\}$ satisfying $|[x_2]_{R_0}| = 1 < 2 = |[x_1]_{R_0}|$. However, $[x_2]_{R_0} \cup x_1 = \{x_1, x_2\} \notin R_0$, $[x_2]_{R_0} \cup x_5 = \{x_2, x_5\} \notin R_0$ since $[x_j]_{R_0} = \{x_j\}, (j = 2, 3, 4)$.

Hence, in general, $\{[x]_R \mid x \in U\}$ cannot produce a matroid on $U$ according to (I3). And further, $(U \times \emptyset, \{[x]_R \times \emptyset \mid x \in U\})$ is not a TD-matroid since Theorem 1 and the above closest result. Moreover, $(U, R)$ cannot produce $(\underline{MR}, \overline{MR})$ though $(\underline{R}, \overline{R})$ is produced by $(U, R)$. Therefore, even under the idea of Theorem 1, $(\underline{MR}, \overline{MR})$ cannot be replaced by $(\underline{R}, \overline{R})$.

Let $S_1 = \{s\} \neq \emptyset$ and $T_1 = \emptyset$. Then, we may easily obtain $(S_1 \times T_1, \mathcal{T}_1 = 2^{S_1} = \{(\emptyset, \emptyset), (s, \emptyset)\})$ to be a TD-matroid, and $(S_1, \mathcal{I}_{S_1} = \{\emptyset, s\})$ to be a matroid by Theorem 1. Under this case, $\underline{MR}(\emptyset, \emptyset) = (\emptyset, \emptyset) = \overline{MR}(\emptyset, \emptyset)$ and $\underline{MR}(s, \emptyset) = (s, \emptyset) = \overline{MR}(s, \emptyset)$ hold. That is, $\underline{MR}(A, B) = (A, B) = \overline{MR}(A, B)$ holds for any $(A, B) \subseteq S_1 \times T_1$. Define a relation $R_1$ on $S_1 : xR_1y \Leftrightarrow x = y$. Thus, $R_1$ is an equivalence relation on $S_1$. For any $X \subseteq S_1$, there are $\underline{R_1}(X) = X = \overline{R_1}(X)$ where $(\underline{R_1}, \overline{R_1})$ is defined by Pawlak (1982, 1991). Since $S_1 \times \emptyset$ is isomorphic to $S_1$ under the isomorphic of sets, we can state $(\underline{MR}, \overline{MR})$ expressed here for $S_1 \times T_1$ to be $(\underline{R_1}, \overline{R_1})$ under the idea of isomorphism of sets. This means that $(\underline{MR}, \overline{MR})$ and $(\underline{R}, \overline{R})$ can replace each other for some cases under isomorphism.

(2) For rough set model over two non-empty sets, we will compare $(\underline{MR}, \overline{MR})$ with the other three famous approximation operators. One is the model of Yao (2015), the second is that of Yao et al. (1995), and the third is the one of Pedrycz and Bargiela (2002).

Let $(\underline{MR}, \overline{MR})$ be produced by a TD-matroid $(S \times T, \mathcal{T})$.

Let $U, V$ be two non-empty sets.

Let $(\underline{apr}_A, \overline{apr}_A)$ be the approximation operators defined by Yao (2015). Let $IS = (U, V, \{v_a \mid a \in A\}, \{I_a \mid a \in A\})$ be an information system expressed by Yao (2015), $\mathrm{DEF}_A(IS)$ be the family of all $A$-definable sets in $IS$ and $\emptyset \neq A \subseteq V$ such that $\emptyset, U \in \mathrm{DEF}_A(IS)$. Let $\{\mathrm{DEF}_A(IS) \mid A \subseteq V\}$ be $\mathcal{DS}$ for short.

Let $(\underline{R}_F, \overline{R}_F)$ be the approximation operators defined by Yao et al. (1995). Let $R$ be a compatibility relation from $U$ to $V$, $F : U \to 2^V$, $F(u) = \{v \in V \mid (u, v) \in R\}$, $\underline{R}_F = \{x \in U \mid F(x) \subseteq Y\}$, $\overline{R}_F = \{x \in U \mid F(x) \cap Y \neq \emptyset\}$.

Let $(\underline{R}_r, \overline{R}_r)$ be the approximation operators defined by Pedrycz and Bargiela (2002). Let $F : 2^U \to 2^V$ as $X \mapsto \bigcup\{F(x) \mid x \in X\}$.

The comparisons between $(\underline{MR}, \overline{MR})$ and $(\underline{apr}_A, \overline{apr}_A)$, $(\underline{R}_F, \overline{R}_F)(\underline{R}_r, \overline{R}_r)$ are shown as Table 6.

Using Table 6, we can obtain the following results by comparing.

With the aspect of express form, $(\underline{MR}, \overline{MR})$ cannot be replaced by any of the other three pairs of rough set approximation operators. At the same time, any of the other three operators $(\underline{apr}_A, \overline{apr}_A)$, $(\underline{R}_F, \overline{R}_F)$ and $(\underline{R}_r, \overline{R}_r)$ cannot be replaced by $(\underline{MR}, \overline{MR})$.

In an information system on $U \times V$, it has $U \neq \emptyset$, $V \neq \emptyset$ and $U \cap V = \emptyset$. We confirm that between $(\underline{MR}, \overline{MR})$ and one of the three $(\underline{apr}_A, \overline{apr}_A)$, $(\underline{R}_F, \overline{R}_F)$ and $(\underline{R}_r, \overline{R}_r)$, it cannot replace each other since $2^U \neq 2^U \times 2^V$ and $2^V \neq 2^U \times 2^V$.

It will do more detail comparing and analysis for the relationships between $(\underline{MR}, \overline{MR})$ and one of the three approximation operators.

(2.1) Comparing: $(\underline{MR}, \overline{MR})$ and $(\underline{apr}_A, \overline{apr}_A)$.

Though under isomorphism of sets, $2^U$ is isomorphic to $2^U \times 2^V$ if $V = \emptyset$. With the aspect of knowledge space, $(\underline{apr}_A, \overline{apr}_A)$ ask $U \in \mathrm{DEF}_A(IS)$. Under isomorphism of sets, $U$ is isomorphic to $(U, \emptyset)$. $(U, \emptyset)$ cannot always belong to $\mathcal{T}$ according to Definition 1, Definition 7 and Theorem 1. For example, in Example 1, $(U, \emptyset) \notin \mathcal{T}$ holds if $U = S$. Hence, $(\underline{apr}_A, \overline{apr}_A)$ cannot replace $(\underline{MR}, \overline{MR})$.

$(\underline{apr}_A, \overline{apr}_A)$ is based on $A \subseteq V$. It cannot determine $\mathrm{DEF}_{A_1}(IS) = \mathrm{DEF}_{A_2}(IS)$ for $A_1, A_2 \subseteq V$ and $A_1 \neq A_2$. So, if $A_1, A_2 \subseteq V$ and $A_1 \neq A_2$, then it cannot determine $\underline{apr}_{A_1}(X) = \underline{apr}_{A_2}(X)$ and $\overline{apr}_{A_1}(X) = \overline{apr}_{A_2}(X)$ for $X \subseteq U$. However, for one $X \subseteq U$, $\underline{MR}(X, A)$

**Table 6** Comparison between $(\underline{MR}, \overline{MR})$ and $(\underline{apr}_A, \overline{apr}_A)$, $(\underline{R}_F, \overline{R}_F)$, $(\underline{R}_r, \overline{R}_r)$

| Operator | Domain | Range | Based space | Basic knowledge | Express form |
|---|---|---|---|---|---|
| $(\underline{MR}, \overline{MR})$ | $2^S \times 2^T$ | $2^S \times 2^T$ | $(S \times T, \mathcal{T})$ | $\mathcal{T}$ | Binary |
| $(\underline{apr}_A, \overline{apr}_A)$ | $2^U$ | $2^U$ | $(IS, \mathcal{DS})$ | $\mathcal{DS}$ | Unary |
| $(\underline{R}_F, \overline{R}_F)$ | $2^V$ | $2^U$ | $(U, V, R)$ | $\{F(x) \mid x \in U\}$ | Unary |
| $(\underline{R}_r, \overline{R}_r)$ | $2^V$ | $2^V$ | $(U, V, R)$ | $\{F(X) \mid X \subseteq U\}$ | Unary |

and $\overline{MR}(X, A)$ are deterministic according to Definition 8. Hence, $(\underline{MR}, \overline{MR})$ cannot replace $(\underline{apr}_A, \overline{apr}_A)$.

The above comparison and analysis together mean that generally, $(\underline{MR}, \overline{MR})$ and $(\underline{apr}_A, \overline{apr}_A)$ cannot replace each other.

Let $\emptyset \neq V \subseteq V$, and every $X \subseteq V$ be an $A$-definable for any $A \subseteq V$. Then, it is easy to see $\mathrm{DEF}_A(IS) = 2^U$, and further $\{\mathrm{DEF}_A(IS) \mid A \subseteq V\} = 2^U$, and $\underline{apr}_A(X) = X = \overline{apr}_A(X)$ for any $A \subseteq V$ and $X \subseteq U$. Using Definition 1, we obtain $(U, 2^U)$ to be a matroid. So, $(U \times \emptyset, 2^U \times 2^\emptyset = (2^U, \emptyset) = \mathcal{T}_U)$ is a TD-matroid by Definition 7 or by Theorem 1. Under this case, we receive $\underline{MR}(X, \emptyset) = (X, \emptyset)$ and $\overline{MR}(X, \emptyset) = (X, \emptyset)$. Thus, under the idea of isomorphism of sets, we can say $\underline{apr}_A(X) = \underline{MR}(X, \emptyset)$ and $\overline{apr}_A(X) = \overline{MR}(X, \emptyset)$ for any $X \subseteq U$ since $X$ is isomorphic to $(X, \emptyset)$. This status indicates that for some cases, $(\underline{MR}, \overline{MR})$ and $(\underline{apr}_A, \overline{apr}_A)$ can replace each other under isomorphism of sets.

(2.2) Comparing: $(\underline{MR}, \overline{MR})$ and $(\underline{R}_F, \overline{R}_F)$.

It does not confirm the basic knowledge $\{F(x) \mid x \in U\}$ to be the family of independent sets of a matroid on $V$. For example, let $(U, V, R)$ be a formal context. Define $F(u) = u'$ for any $u \in U$. So, it gets $\{F(x) \mid x \in U\} = \{x' \mid x \in U\}$. Using Definitions 1 and 2, we may easily know $\{F(x) \mid x \in U\}$ not always to be the family of a matroid on $V$, though sometimes it is. Hence, in general, $(\underline{R}_F, \overline{R}_F)$ cannot replace $(\underline{MR}, \overline{MR})$ even by the idea of isomorphism of matroids and by Theorem 1. Meanwhile, $(\underline{MR}, \overline{MR})$ cannot replace $(\underline{R}_F, \overline{R}_F)$ generally.

If $R$ is defined as $(x, y) \notin R$ for any $x \in U$ and every $y \in V$, then $F(u) = \{v \in V \mid (u, v) \in R\} = \emptyset$ holds for any $u \in U$. And further, if $Y \subseteq V$, we obtain $\underline{R}_F(Y) = \{x \in U \mid F(x) \subseteq Y\} = U$ and $\overline{R}_F(Y) = \{x \in U \mid F(x) \cap Y \neq \emptyset\} = \emptyset$. So, it gets $\{F(x) \mid x \in U\} = \{\emptyset\} \subseteq 2^V$.

It is easy to know $(V, \{\emptyset\})$ to be a matroid. By Theorem 1, this matroid corresponds to a TD-matroid $(V \times \emptyset, \mathcal{T}_V = \{(\emptyset, \emptyset)\})$.

Let $(A, B) \subseteq T \times \emptyset$. Then, by Definition 8, there are $low(A, B) = low(A, \emptyset) = \{(\emptyset, \emptyset)\}$ and $upr(A, B) = upr(A, \emptyset) = \emptyset$, and further, $\underline{MR}(A, B) = (\bigcup \emptyset, \bigcap \emptyset) = (\emptyset, \emptyset)$ and $\overline{MR}(A, B) = (\bigcap \emptyset, \bigcup \emptyset) = (V, \emptyset)$.

Since $V$ given above is the set of attributes and $U$ is the set of objects in the space $(U, V, R)$ provided by Yao et al. (1995). However, in the set $V \times \emptyset$, $V$ is the family of objects. In order to make it intuitive with aspect of corresponding $(\underline{R}_F, \overline{R}_F)$ and the approximation operators given in Definition 8, we use the dual operators $(\underline{MR}^*, \overline{MR}^*)$ of $(\underline{MR}, \overline{MR})$ on $\emptyset \times U$, we can obtain $\underline{MR}^*(\emptyset, B) = (\emptyset, U)$ and $\overline{MR}^*(\emptyset, B) = (\emptyset, \emptyset)$ for any $B \subseteq U$.

Therefore, under the isomorphism of set, it has $(\emptyset, U)$ to be isomorphic to $U$. We obtain $(\underline{R}_F, \overline{R}_F)$ to be $(\underline{MR}^*, \overline{MR}^*)$. In other words, under the duality and the isomorphism, $(\underline{R}_F, \overline{R}_F)$ and $(\underline{MR}, \overline{MR})$ are the same for some special cases. Hence, for some special cases, $(\underline{MR}, \overline{MR})$ and $(\underline{R}_F, \overline{R}_F)$ can replace each other under isomorphism of sets and the duality.

(2.3) Comparing: $(\underline{MR}, \overline{MR})$ and $(\underline{R}_r, \overline{R}_r)$.

From the descriptions in Pedrycz and Bargiela (2002) and Shao et al. (2018), we know $(\underline{R}_r, \overline{R}_r)$ to be the revised version of $(\underline{R}_F, \overline{R}_F)$ on the space $(U, V, R)$ where $R$ is an arbitrary compatibility relation. Hence, similarly to (2.2), we confirm that $(\underline{R}_r, \overline{R}_r)$ and $(\underline{MR}, \overline{MR})$ cannot replace each other in general, though $(\underline{R}_r, \overline{R}_r)$ and $(\underline{MR}, \overline{MR})$ are the same for some special cases under some ideas.

(2.4) Consequence.

Summarizing all of the above discussions for the relationships between $(\underline{MR}, \overline{MR})$ and any of $(\underline{apr}_A, \overline{apr}_A)$, $(\underline{R}_F, \overline{R}_F)$ and $(\underline{R}_r, \overline{R}_r)$, we find that each of $(\underline{MR}, \overline{MR})$, $(\underline{apr}_A, \overline{apr}_A)$, $(\underline{R}_F, \overline{R}_F)$ and $(\underline{R}_r, \overline{R}_r)$ has its own features. Between $(\underline{MR}, \overline{MR})$ and one of the other three, any of them cannot replace the other one, but they have some intersection up to some ideas and for some cases.

These comparisons and analysis indicate that $(\underline{MR}, \overline{MR})$ is different from the existed rough set approximation operators on two sets. $(\underline{MR}, \overline{MR})$ is a new pair of rough set approximation operators on two sets and has the value of its own development. It is a complement for the research of rough set approximation operators.

The research route in this subsection is as follows:

$$
\begin{aligned}
&(S \times T, \mathcal{T}),\ \text{a TD-matroid,}\\
&\Longrightarrow (\underline{MR}, \overline{MR}),\\
&\quad \text{a pair of approximation operators, roughly say}
\end{aligned}
$$

This route is exactly to apply matroid theory to the research of rough set theory.

It is probably more interesting to find the converse of the above route. That is, on two sets, how to establish TD-matroids using a pair of given approximation operators? Since any pair of approximation operators are defined by some given information. Hence, the above question can be expressed in detail as: how to establish TD-matroids using a pair of rough set approximation operators based on some given information on two sets.

The two approximation operators $\underline{MR}$ and $\overline{MR}$ are based on the known information system $\mathcal{T}$ which is a family of feasible sets of a TD-matroid. However, some information system is not expressed as the family $\mathcal{T}$ mentioned above. For example, as one of the main tool to extract information—FCA, the set $\mathcal{B}(\mathbb{K})$ in a formal context $\mathbb{K}$ is perhaps not to be the family of feasible sets of a TD-matroid, for instance, $(S \times T, \mathcal{B}((S, T, I)))$ in Example 1 is not a TD-matroid. Hence, it is necessary to answer the above question for a pair of approximation operators which are based on $\mathcal{B}(\mathbb{K})$. Namely, it is necessary to find a way to establish some TD-matroids with a pair of approximation operators based on $\mathcal{B}(\mathbb{K})$. This will be done in the coming subsection.

### 3.2 TD-matroids based on approximation operators $\underline{R}$ and $\overline{R}$

In this subsection, how to establish TD-matroids with $\underline{R}$ and $\overline{R}$ will be solved, where $\underline{R}$ and $\overline{R}$ are given in Definition 4. $(\underline{R}, \overline{R})$ is a pair of approximation operators on $O \times P$ based on $\mathcal{B}(\mathbb{K})$ in which $\mathbb{K} = (O, P, I)$ according to items (2)-(4) in Lemma 2 and Proposition 1 for $(X, Y) \subseteq O \times P$ and $X \neq \emptyset$ and $Y \neq \emptyset$. In other words, it will realize the converse route in the last subsection, that is,

$$\underline{R} \text{ and } \overline{R}, \text{ a pair of approximation operators on } O \times P, \quad \text{roughly say} \Longrightarrow \text{some TD-matroids}.$$

The realization of this route belongs to the applied fields of rough set theory into matroid theory.

**Theorem 3** *Let $\mathbb{K} = (O, P, I)$ be a formal context, and $\underline{R}$ and $\overline{R}$ be given in Definition 4. For $(A, B) \subseteq (O, P)$, let*

$$\underline{\mathcal{T}_{(A,B)}}(R) = \{(X, Y) \mid (X, Y) \subseteq (O, P), \underline{R}(X, Y) \sqsubseteq (A, B)\};$$

$$\overline{\mathcal{T}_{(A,B)}}(R) = \{(X, Y) \mid (X, Y) \subseteq (O, P), \overline{R}(X, Y) \sqsubseteq (A, B)\}.$$

*Then, the following statements are correct.*

(1) *$(O \times P, \underline{\mathcal{T}_{(A,B)}}(R))$ is a TD-matroid.*

(2) *$(O \times P, \overline{\mathcal{T}_{(A,B)}}(R))$ is a TD-matroid if $B \subseteq A'$.*

*Item (1) in Theorem 3 is easily verified by Definition 7(1) and its proof is omitted. The proof of Theorem 3(2) can be found in Appendix.*

*The following example shows that for some $(A, B) \subseteq O \times P$, $(O \times P, \overline{\mathcal{T}_{(A,B)}}(R))$ can be a TD-matroid even if $B \nsubseteq A'$.*

***Example 4*** Let $O = \{x_1, x_2, x_3, x_4\}$, $P = \{b_1, b_2, b_3\}$ and $I$ be shown in Table 2.

Let $A_1 = \{x_1, x_2\}$ and $B_1 = \{b_3\}$. Let $\overline{\mathcal{T}_{(A_1,B_1)}}(R) = \{(X, Y) \subseteq O \times P \mid \overline{R}(X, Y) \sqsubseteq (A_1, B_1)\}$. Then, we obtain the following results:

(1) $A_1' = \{x_1, x_2\}' = x_1' \cap x_2' = \emptyset \Rightarrow B_1 \nsubseteq A_1'$;
(Definition 2 and Lemma 1(2)).

(2) $(\emptyset, Y), (x_2, Y) \in \overline{\mathcal{T}_{(A_1,B_1)}}(R)$ for $Y \in 2^{\{b_1,b_2,b_3\}}$;
$(\emptyset' = P \supseteq Y, x_2' = b_3$, Lemma 2(1)).

(3) $(x_1, Y), (\{x_1, x_2\}, Y) \in \overline{\mathcal{T}_{(A_1,B_1)}}(R)$
for $Y \in \{b_3, \{b_1, b_3\}, \{b_2, b_3\}, \{b_1, b_2, b_3\}\}$;
$(x_1' = b_1, \{x_1, x_2\}' = \emptyset$, Lemma 2(1)).

Since $\overline{R}(X, Y) \sqsubseteq (A_1, B_1)$ and Lemma 2(1) together ask $(X, X' \cup Y) \sqsubseteq (A_1, B_1)$, namely, $X \subseteq A_1$ and $X' \cup Y \supseteq B_1$. $X \subseteq A_1$ implies $X \in 2^{A_1} = \{\emptyset, x_1, x_2, \{x_1, x_2\}\}$. Using Table 2, Definition 2 and Remark 2, we obtain $\emptyset' = P$, $x_1' = \{b_1\}$, $x_2' = \{b_3\}$ and $\{x_1, x_2\}' = x_1' \cap x_2' = \emptyset$. $X' \cup Y \supseteq B_1$ implies $b_3 \in X' \cup Y$. Hence, we infer that

$$\begin{aligned}
&\text{if } X = \emptyset, \text{ then } b_3 \in P \cup Y = P \text{ for any } Y \in 2^{\{b_1,b_2,b_3\}};\\
&\text{if } X = \{x_1\}, \text{ then } b_3 \in b_1 \cup Y \Rightarrow b_3 \in Y;\\
&\text{if } X = \{x_2\}, \text{ then } b_3 \in x_3 \cup Y \text{ for any} Y \in 2^{\{b_1,b_2,b_3\}};\\
&\text{if } X = \{x_1, x_2\}, \text{ then } b_3 \in \emptyset \cup Y \Rightarrow b_3 \in Y.
\end{aligned}$$

Moreover, we obtain $\overline{\mathcal{T}_{(A_1,B_1)}}(R) = \{(\emptyset, Y) \mid Y \in 2^{\{b_1,b_2,b_3\}}\} \cup \{(x_1, Y) \mid Y \in \{b_3, \{b_1, b_3\}, \{b_2, b_3\}, \{b_1, b_2, b_3\}\}\} \cup \{(x_2, Y) \mid Y \in 2^{\{b_1,b_2,b_3\}}\} \cup \{(\{x_1, x_2\}, Y) \mid Y \in \{b_3, \{b_1, b_3\}, \{b_2, b_3\}, \{b_1, b_2, b_3\}\}\}$.

Here, $B_1 \nsubseteq A_1'$ holds, but we may easily prove $(O \times P, \overline{\mathcal{T}_{(A_1,B_1)}}(R))$ to be a TD-matroid.

***Remark 11*** (1) In Theorem 3(2), it asks $B \subseteq A'$. Actually, the condition $B \subseteq A'$ is only used in the proof of Case 2 when we check (T3) for $\overline{\mathcal{T}_{(A,B)}}(R)$. Next, we analyze the corresponding proof as follows:

Assume $Y_1 \setminus Y_2 = \emptyset$. Then, $(a, b) \in (X_1, Y_1) \setminus (X_2, Y_2) = (X_1 \setminus X_2, \emptyset)$ holds. Under this assumption, if the following $(\star 1)$ and $(\star 2)$, two possible and reasonable suppositions, happen

$(\star 1)$ If $(X_2' \cap a') \cup Y_2 \subset X_2' \cup Y_2$
holds since $(X_2' \cap a') \cup Y_2 \subseteq X_2' \cup Y_2$.

$(\star 2)$ If $X_2' \cup Y_2 = B$ holds since the following statements

$$\overline{R}(X_2, Y_2) \sqsubseteq (A, B)$$
$$\Rightarrow \overline{R}(X_2, Y_2) = (X_2, X_2' \cup Y_2) \sqsubseteq (A, B)(\text{Lemma } 2(1))$$
$$\Rightarrow X_2' \cup Y_2 \supseteq B(\text{Definition of } \sqsubseteq)$$

Then, it is clear $(X_2' \cap a') \cup Y_2 \subset X_2' \cup Y_2 = B$. So, $\overline{R}(X_2 \cup a, Y_2) = (X_2 \cup a, (X_2 \cup a)' \cup Y_2) = (X_2 \cup a, (X_2' \cap a') \cup Y_2) \not\sqsubseteq (A, B)$ is correct by Lemma 2(1), Lemma 1(1) and the definition of $\sqsubseteq$. That is to say, $(X_2 \cup a, Y_2) \notin \overline{\mathcal{T}_{(A,B)}}(R)$ holds. This means $\overline{\mathcal{T}_{(A,B)}}(R)$ not to satisfy (T3). In other words, $(S \times T, \overline{\mathcal{T}_{(A,B)}}(R))$ is not a TD-matroid. In addition,

$$\begin{aligned}&(X_2, Y_2) \in \overline{\mathcal{T}_{(A,B)}}(R) \Rightarrow \overline{R}(X_2, Y_2) \sqsubseteq (A, B)\\ &(\text{definition of } \overline{\mathcal{T}_{(A,B)}}(R))\\ &\Rightarrow (X_2, X_2' \cup Y_2) \sqsubseteq (A, B)\\ &(\text{Lemma } 2(1))\\ &\Rightarrow X_2 \subseteq A \text{ and } X_2' \cup Y_2 \supseteq B(\text{definition of } \sqsubseteq)\\ &\Rightarrow X_2' \supseteq A' \text{ and } X_2' \cup Y_2 \supseteq B\\ &(\text{Lemma } 1(1))\\ &\Rightarrow B = X_2' \cup Y_2 \supseteq X_2' \supseteq A'((\star 2))\\ &\Rightarrow B \nsubseteq A'\end{aligned}$$

The above analysis shows that $B \nsubseteq A'$ holds, and $\overline{\mathcal{T}_{(A,B)}}(R)$ is not the family of feasible sets of any of TD-matroids on $O \times P$ under the assumption and suppositions.

(2) Combining the above analysis (1) and Example 4 with Theorem 3, we confirm $B \subseteq A'$ to be a necessary condition for deciding $(S \times T, \overline{\mathcal{T}_{(A,B)}}(R))$ to be a TD-matroid.

(3) The results of Theorem 3 also illustrate the importance to study on TD-matroids since $\underline{\mathcal{T}_{(A,B)}}(R)$ is the family of feasible sets of a TD-matroid, and $\overline{\mathcal{T}_{(A,B)}}(R)$ is also the family of feasible sets of a TD-matroid if $B \subseteq A'$.

**Corollary 2** *Let $\mathbb{K} = (O, P, I)$ be a formal context, and $\underline{R}$ and $\overline{R}$ be given in Definition 4. Let $(A, B) \subseteq O \times P$. Let $\underline{\mathcal{T}_{(A,B)}}(R)$ and $\overline{\mathcal{T}_{(A,B)}}(R)$ be given in Theorem 3. Then, $\underline{\mathcal{T}_{(A,B)}}(R) \subseteq \overline{\mathcal{T}_{(A,B)}}(R)$.*

***Proof*** Let $(X, Y) \subseteq O \times P$. If $(X, Y) \in \underline{\mathcal{T}_{(A,B)}}(R)$. Then, by Lemma 2(1), this means $(X, X' \cap Y) = \underline{R}(X, Y) \sqsubseteq (A, B)$. And further, $X \subseteq A$ and $X' \cap Y \supseteq B$ hold using Lemma 2(1) and the definition of $\sqsubseteq$. In addition, $X' \cap Y \subseteq X' \cup Y$ holds. So, it gets $B \subseteq X' \cup Y$. Moreover, we receive $(X, X' \cup Y) = \overline{R}(X, Y) \sqsubseteq (A, B)$ by Lemma 2(1). That is to say, $(X, Y) \in \overline{\mathcal{T}_{(A,B)}}(R)$ holds. □

For $\underline{\mathcal{T}_{(A,B)}}(R)$ and $\overline{\mathcal{T}_{(A,B)}}(R)$ given in Theorem 3, we continue to discuss their properties.

**Theorem 4** *Let $\mathbb{K} = (O, P, I)$ be a formal context. Let $(A, B), (X_i, Y_i), (A_i, B_i) \subseteq O \times P$ $(i = 1, 2)$. Let $\underline{R}$ and $\overline{R}$ be given in Definition 4. Let $\underline{\mathcal{T}_{(A,B)}}(R) = \{(X_j, Y_j), j \in \mathcal{J}\}$ and $\overline{\mathcal{T}_{(A,B)}}(R) = \{(X_p, Y_p), p \in \mathcal{P}\}$, where $\underline{\mathcal{T}_{(A,B)}}(R)$ and $\overline{\mathcal{T}_{(A,B)}}(R)$ are given in Theorem 3. Then, there are the following statements.*

1. *$(\underline{\mathcal{T}_{(A,B)}}(R), \sqsubseteq)$ and $(\overline{\mathcal{T}_{(A,B)}}(R), \sqsubseteq)$ are a poset, respectively.*
2. *$(X_1, Y_1) \sqsubseteq (X_2, Y_2) \Rightarrow \underline{R}(X_1, Y_1) \sqsubseteq \underline{R}(X_2, Y_2)$; $(X_1, Y_1) \sqsubseteq (X_2, Y_2) \Rightarrow \overline{R}(X_1, Y_1) \sqsubseteq \overline{R}(X_2, Y_2)$.*
3. *$(\emptyset, P)$ is the minimum element in $(\underline{\mathcal{T}_{(A,B)}}(R), \sqsubseteq)$; $(\cup_{j\in\mathcal{J}} X_j, \cap_{j\in\mathcal{J}} Y_j)$ is the maximum element in $(\underline{\mathcal{T}_{(A,B)}}(R), \sqsubseteq)$.*

*Suppose $B \subseteq A'$. Then, $(\emptyset, P)$ is the minimum element in $(\overline{\mathcal{T}_{(A,B)}}(R), \sqsubseteq)$; $(\cup_{p\in\mathcal{P}} X_p, \cap_{p\in\mathcal{P}} Y_p)$ is the maximum element in $(\overline{\mathcal{T}_{(A,B)}}(R), \sqsubseteq)$.*

(4) *$\underline{\mathcal{T}_{(A_1,B_1)}}(R) \cap \underline{\mathcal{T}_{(A_2,B_2)}}(R) = \underline{\mathcal{T}_{(A_1\cap A_2, B_1\cup B_2)}}(R)$; $\overline{\mathcal{T}_{(A_1,B_1)}}(R) \cap \overline{\mathcal{T}_{(A_2,B_2)}}(R) = \overline{\mathcal{T}_{(A_1\cap A_2, B_1\cup B_2)}}(R)$.*

*Item (1) in Theorem 4 is easily verified by checking (p1)–(p3) in Definition 3 and its proof is omitted. The proofs of the other items in Theorem 4 can be found in Appendix.*

***Remark 12*** (1) For a formal context $\mathbb{K} = (OB, AT, I)$, where $OB$ is a set of collected biological specimens such as insects, $AT$ is a set of biological characteristics, and $I \subseteq OB \times AT$ is a binary relation such that "$(x, y) \in I \Leftrightarrow x$ owns $y$", we can obtain $\mathcal{B}(\mathbb{K})$ by Definition 2 or by the way in Vormbrock and Wille 2005. But, sometimes, biologists hope to know how to explore the information not in $\mathcal{B}(\mathbb{K})$ with the assistance of $\mathcal{B}(\mathbb{K})$.

For $(A, B) \subseteq OB \times AT$, $\underline{\mathcal{T}_{(A,B)}}(R)$ is obtained using Theorem 3, and so as to $\overline{\mathcal{T}_{(A,B)}}(R)$. Applying the known information $\underline{\mathcal{T}_{(A,B)}}(R)$ and $\overline{\mathcal{T}_{(A,B)}}(R)$, one can explore the biological information contained in $(A, B)$ with respect to the order $\sqsubseteq$. And further, one can find out which elements in $\underline{\mathcal{T}_{(A,B)}}(R)$ and $\overline{\mathcal{T}_{(A,B)}}(R)$ to be the closest for explaining the nature of $(A, B)$, respectively. In fact, Theorem 4(3) shows that $(\cup_{j\in\mathcal{J}} X_j, \cap_{j\in\mathcal{J}} Y_j)$, and $(\cup_{p\in\mathcal{P}} X_p, \cap_{p\in\mathcal{P}} Y_p)$ if $B \subseteq A'$, will be up to the requirements, respectively.

(2) If the set $A_i$ of the specimens such as insects is collected at the same region $C_i$, $(i = 1, 2)$, and the two regions $C_1$ and $C_2$ are not adjacent. How to use the known $A_1$ and $A_2$ to infer the situation of insects living at the middle zone $C$ between $C_1$ and $C_2$? This question is often considered by biologists. Actually, using $A_i$, biologists can find the set $B_i$ of biological characteristics relative to $A_i$, $(i = 1, 2)$. This means that $\underline{\mathcal{T}_{(A_i,B_i)}}(R)$ and $\overline{\mathcal{T}_{(A_i,B_i)}}(R))$ will be obtained $(i = 1, 2)$. Sometimes, $A_1 \cap A_2$ is con-

sidered as the set of insects in the region $C$ since biologists guess the insects in $C$ to be the common insects in both $C_1$ and $C_2$. Under this guess, with the known information $\underline{\mathcal{T}_{(A_i,B_i)}}(R)$ and $\overline{\mathcal{T}_{(A_i,B_i)}}(R)$ $(i = 1, 2)$, Theorem 4(4) indicates $\underline{\mathcal{T}_{(A_1\cap A_2, B_1\cup B_2)}}(R)$ and $\overline{\mathcal{T}_{(A_1\cap A_2, B_1\cup B_2)}}(R)$ perhaps to assist to arrive the goal of biologists' guess.

Here, we provide two kinds of information systems $\underline{\mathcal{T}_{(A,B)}}(R)$ and $\overline{\mathcal{T}_{(A,B)}}(R)$. After using Theorem 4(4), if some biologists hope to know which one in

$\underline{\mathcal{T}_{(A_1\cap A_2, B_1\cup B_2)}}(R)$ and $\overline{\mathcal{T}_{(A_1\cap A_2, B_1\cup B_2)}}(R)$ to be better for biologists, it needs to analyze the results obtained here combining with some other biological ideas such as morphology for the two known information systems $\underline{\mathcal{T}_{(A,B)}}(R)$ and $\overline{\mathcal{T}_{(A,B)}}(R)$.

(3) It is well known that biological ideas are the best way to research on the study of insects. The way provided in this paper is just to help biologists to speed up the process of their research. It is an auxiliary method for biologists under some cases.

## 4 Conclusion

As an extension of Whitney classical matroid model on one set, this paper provides a matroidal structure on two sets—TD-matroid. Using the family of feasible sets of a TD-matroid, the lower and upper approximation operators are constructed. For the existed pair of lower and upper approximation operators based on the family of semiconcepts for a formal context, it constructs two concrete TD-matroids and deals their some properties with aspect of poset theory. Some examples used biological data examine the correct of all of results in this paper. These results indicate the important to research on TD-matroids and rough sets on two sets.

TD-matroid is effective only within limits on two-dimensional space, and also one-dimensional space since one-dimensional space is a subspace of two-dimensional space up to isomorphism of spaces. According to the structure of TD-matroid, TD-matroid can reveal some properties for the same existed phenomena in real world as that Whitney classical matroid did. However, our real world is in three-dimensional space. This implies that some contents in our real world should be expressed by ternary form. Under this analysis, we believe that it is not only that TD-matroid will not play a role to solve those problems needed by ternary form, but also that the approximation operators aided by TD-matroid in this paper will not approximate the knowledge expressed by ternary form. However, the idea in this paper will perhaps assist to define a "matroidal structure" on three-dimensional space, and further find approximation operators aided by the above "matroidal structure". In addition, TD-matroid is a discrete structure, and the correspondent approximation operator is also discrete. Both of them have no ability to face to a continuous process.

In the future, we hope to continue the research of TD-matroid with rough set. For instance,

(1) How to generalize the other axioms for matroids defined on one set such as the axioms of greedy algorithm to TD-matroids. With assistance of these axioms, how to find out rough set approximation operators, and further, use the found approximation operators to research on TD-matroids.
(2) How to use the results in (1) to solve some problems in real life such as biology.
(3) How to generalize the train of thought in this paper to $n$-dimensional space ($n \geq 3$). How to find their correspondent greedy algorithm for the new matroidal structure on $n$-dimensional space ($n \geq 3$).
(4) How to combine the results here with that of Im et al. (2021) to simulate a continuous process.

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**Data Availability** All of authors confirm that this statement is accurate.

## Declarations

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## Appendix

### 1. Proof of Lemma 7

***Proof*** Let $low(A, B) = \{(X_j, Y_j), j \in \mathcal{J}\}$ and $upr(A, B) = \{(X_p, Y_p), p \in \mathcal{P}\}$. It follows $|\mathcal{J}|, |\mathcal{P}| < \infty$ since $|S|, |T| < \infty$.

Next, we prove item (1).

$X_j \subseteq A$ and $B \subseteq Y_j$ hold since $(X_j, Y_j) \sqsubseteq (A, B)$ $(j \in \mathcal{J})$. So, we obtain $\cup_{j\in\mathcal{J}} X_j \subseteq A$ and $B \subseteq \cap_{j\in\mathcal{J}} Y_j$. And further, we get $\underline{MR}(A, B) = (\cup_{j\in\mathcal{J}} X_j, \cap_{j\in\mathcal{J}} Y_j) \sqsubseteq (A, B)$.

If $|\mathcal{P}| = 0$, then $upr(A, B) = \emptyset$. Under this case, we know $\overline{MR}(A, B) = (S, \emptyset)$ by Definition 8. So, $(A, B) \sqsubseteq \overline{MR}(A, B)$ holds since $A \subseteq S$ and $B \supseteq \emptyset$.

If $|\mathcal{P}| \neq \emptyset$, then $A \subseteq X_p$ and $B \supseteq Y_p$ hold since $(A, B) \sqsubseteq (X_p, Y_p)$ $(p \in \mathcal{P})$. So, we obtain $A \subseteq \cap_{p\in\mathcal{P}} X_p$ and $B \supseteq \cup_{p\in\mathcal{P}} Y_p$. And further, we get $(A, B) \sqsubseteq (\cap_{p\in\mathcal{P}} X_p, \cup_{p\in\mathcal{P}} Y_p) = \overline{MR}(A, B)$.

Next, we prove item (2).

Let $(A, B) \in \mathcal{T}$. Then, $A \subseteq A$ and $B \subseteq B$ together mean $(A, B) \in low(A, B)$ and $(A, B) \in upr(A, B)$.

On the one hand, there is one and only one $j_0 \in \mathcal{J}$ satisfying $X_{j_0} = A$ and $Y_{j_0} = B$ since $(A, B) \in low(A, B)$. So, we get $A = X_{j_0} \subseteq \cup_{j\in\mathcal{J}} X_j \subseteq A$ and $B = Y_{j_0} \supseteq \cap_{j\in\mathcal{J}} Y_j \supseteq B$ according to the definition of $\underline{MR}(A, B)$ and item (1). Thus, we obtain $\underline{MR}(A, B) = (\cup_{j\in\mathcal{J}} X_j, \cap_{j\in\mathcal{J}} Y_j) = (A, B)$.

On the other hand, there is one and only one $p_0 \in \mathcal{P}$ satisfying $X_{p_0} = A$ and $Y_{p_0} = B$ since $(A, B) \in upr(A, B)$. So, we get $A \subseteq \cap_{p\in\mathcal{P}} X_p \subseteq X_{p_0} = A$ and $B \supseteq \cup_{p\in\mathcal{P}} Y_p \supseteq Y_{p_0} = B$ according to the definition of $\overline{MR}(A, B)$ and item (1). Thus, we obtain $\overline{MR}(A, B) = (\cap_{p\in\mathcal{P}} X_p, \cup_{p\in\mathcal{P}} Y_p) = (A, B)$.

Next, we prove item (3).

Suppose $\underline{MR}(A, B) = (A, B) = \overline{MR}(A, B)$. Then, it infers the following formulas

$$X_j \subseteq \cup_{j\in\mathcal{J}} X_j = A = \cap_{p\in\mathcal{P}} X_p \subseteq X_p \ (j \in \mathcal{J}, p \in \mathcal{P}) \ ①.$$
$$Y_p \subseteq \cup_{p\in\mathcal{P}} Y_p = B = \cap_{j\in\mathcal{J}} Y_j \subseteq Y_j \ (j \in \mathcal{J}, p \in \mathcal{P}) \ ②.$$

It induces $(A, Y_p) \sqsubseteq (X_p, Y_p) \in \mathcal{T}$ according to $Y_p \subseteq Y_p$ and the formula ① $(p \in \mathcal{P})$. Combining formula ② with (T2) and $(X_p, Y_p) \in \mathcal{T}$ $(p \in \mathcal{P})$, we get $(A, Y_p) \in \mathcal{T}$ satisfying $(A, B) \sqsubseteq (A, Y_p)$ for any $p \in \mathcal{P}$. Thus, $(A, Y_p) \in upr(A, B)$ holds for any $p \in \mathcal{P}$. That is to say, there is $p_1 \in \mathcal{P}$ such that $X_{p_1} = A$ satisfying $(X_{p_1}, Y_p) \in \mathcal{T}$ $(p \in \mathcal{P})$. Additionally, $(X_p, B) \sqsubseteq (X_p, Y_p) \in \mathcal{T}$ holds since $B \supseteq Y_p$ holds by formula ② $(p \in \mathcal{P})$. Taking this result with (T2), it induces $(X_p, B) \in \mathcal{T}$ $(p \in \mathcal{P})$. Combining formula ① with $(A, B) \sqsubseteq (X_p, B)$ $(p \in \mathcal{P})$, we get $(X_p, B) \in upr(A, B)$ for any $p \in \mathcal{P}$. That is to say, there is $p_2 \in \mathcal{P}$ such that $Y_{p_2} = B$ and $(X_p, Y_{p_2}) \in \mathcal{T}$ $(p \in \mathcal{P})$. Especially, $(X_{p_1}, Y_{p_2}) \in \mathcal{T}$ holds since $p_1 \in \mathcal{P}$. In other words, $(A, B) \in \mathcal{T}$ is followed since $X_{p_1} = A$ and $Y_{p_2} = B$.

## 2. Proof of Lemma 8

***Proof*** Let $low(A_i, B_i) = \{(X_{ij}, Y_{ij}), j \in \mathcal{J}_i\}$ and $upr(A_i, B_i) = \{(X_{ip}, Y_{ip}), p \in \mathcal{P}_i\}$ $(i = 1, 2)$.

To prove item (1).

$(X_{1j}, Y_{1j}) \sqsubseteq (A_1, B_1) \sqsubseteq (A_2, B_2)$ follows $(X_{1j}, Y_{1j}) \in low(A_2, B_2)$ $(j \in \mathcal{J}_1)$. So, we get $\cup_{j\in\mathcal{J}_1} X_{1j} \subseteq \cup_{j\in\mathcal{J}_2} X_{2j}$ and $\cap_{j\in\mathcal{J}_1} Y_{1j} \supseteq \cap_{j\in\mathcal{J}_2} Y_{2j}$. Furthermore, we obtain $\underline{MR}(A_1, B_1) = (\cup_{j\in\mathcal{J}_1} X_{1j}, \cap_{j\in\mathcal{J}_1} Y_{1j}) \sqsubseteq (\cup_{j\in\mathcal{J}_2} X_{2j}, \cap_{j\in\mathcal{J}_2} Y_{2j}) = \underline{MR}(A_2, B_2)$.

To prove item (2).

$(A_1, B_1) \sqsubseteq (A_2, B_2) \sqsubseteq (X_{2p}, Y_{2p})$ follows $(X_{2p}, Y_{2p}) \in upr(A_1, B_1)$ $(p \in \mathcal{P}_2)$. So, $upr(A_2, B_2) \subseteq upr(A_1, B_1)$ holds. This demonstrates that the case of $upr(A_1, B_1) = \emptyset$ and $upr(A_2, B_2) \neq \emptyset$ does not exist. In other words, it exists and only exists the following cases:

Case 1. $upr(A_i, B_i) \neq \emptyset$ $(i = 1, 2)$;
Case 2. $upr(A_i, B_i) = \emptyset$ $(i = 1, 2)$;
Case 3. $upr(A_1, B_1) \neq \emptyset$ and $upr(A_2, B_2) = \emptyset$.

Under Case 1, we get $\cap_{p\in\mathcal{P}_1} X_{1p} \subseteq \cap_{p\in\mathcal{P}_2} X_{2p}$ and $\cup_{p\in\mathcal{P}_1} Y_{1p} \supseteq \cup_{p\in\mathcal{P}_2} Y_{2p}$ according to $upr(A_2, B_2) \subseteq upr(A_1, B_1)$. Furthermore, we obtain $\overline{MR}(A_1, B_1) = (\cap_{p\in\mathcal{P}_1} X_{1p}, \cup_{p\in\mathcal{P}_1} Y_{1p}) \sqsubseteq (\cap_{p\in\mathcal{P}_2} X_{2p}, \cup_{p\in\mathcal{P}_2} Y_{2p}) = \overline{MR}(A_2, B_2)$.

Under Case 2, we obtain $\overline{MR}(A_i, B_i) = (S, \emptyset)$ $(i = 1, 2)$ in view of Definition 8. That is, $\overline{MR}(A_1, B_1) \sqsubseteq \overline{MR}(A_2, B_2)$ holds.

Under Case 3, we obtain $\overline{MR}(A_2, B_2) = (S, \emptyset)$ and the existence of $\overline{MR}(A_1, B_1)$ with $(\overline{MR}(A_1, B_1) \neq (S, \emptyset)$ in light of Definition 8. It is easy to see $\overline{MR}(A_1, B_1) \sqsubseteq (S, \emptyset)$. Namely, it has $\overline{MR}(A_1, B_1) \sqsubseteq \overline{MR}(A_2, B_2)$.

## 3. Proof of Theorem 3(2)

***Proof*** Since $\mathbb{K}$ is a formal context, it follows $O \neq \emptyset$ and $P \neq \emptyset$ by Definition 2. Using Definition 7, we only need to check the conditions (T1), (T2) and (T3) to be satisfied by $\overline{\mathcal{T}_{(A,B)}}(R)$, respectively.

First to check (T1) for $\overline{\mathcal{T}_{(A,B)}}(R)$.

Using Lemma 2(1), we receive $\overline{R}(\emptyset, B) = (\emptyset, \emptyset' \cup B) = (\emptyset, P)$ since $\emptyset' = P$ for $\emptyset \subseteq O$. Hence, $(\emptyset, B) \in \overline{\mathcal{T}_{(A,B)}}(R)$ holds since $\emptyset \subseteq A$ and $P \supseteq B$ together follow $(\emptyset, P) \sqsubseteq (A, B)$. Thus, $\overline{\mathcal{T}_{(A,B)}}(R) \neq \emptyset$ holds.

Second to check (T2) for $\overline{\mathcal{T}_{(A,B)}}(R)$.

Let $(X_2, Y_2) \sqsubseteq (X_1, Y_1) \in \overline{\mathcal{T}_{(A,B)}}(R)$. Then, we may easily find $X_2 \subseteq X_1$ and $Y_2 \supseteq Y_1$ by the definition of $\sqsubseteq$. In addition, we also find the following two facts.

Fact 1 : $(X_1, Y_1) \in \overline{\mathcal{T}_{(A,B)}}(R) \Rightarrow \overline{R}(X_1, Y_1) \sqsubseteq (A, B)$
(definition of $\overline{\mathcal{T}_{(A,B)}}(R)$)
$\Rightarrow (X_1, X_1' \cup Y_1) \sqsubseteq (A, B)$ (Lemma 2(1))
$\Rightarrow X_1 \subseteq A$ and $X_1' \cup Y_1 \supseteq B$. (definition of $\sqsubseteq$)

Fact 2 : $(X_2, Y_2) \sqsubseteq (X_1, Y_1)$
$\Rightarrow X_2 \subseteq X_1$ and $Y_2 \supseteq Y_1$ (definition of $\sqsubseteq$)
$\Rightarrow X_2' \supseteq X_1'$
and $Y_2 \supseteq Y_1$ (Lemma 1(1))
$\Rightarrow X_2' \cup$

$Y_2 \supseteq X_1' \cup Y_1$

Combining the above two facts and $X_2 \subseteq X_1 \subseteq A$, we obtain $X_2' \cup Y_2 \supseteq X_1' \cup Y_1 \supseteq B$. Thus, $\overline{R}(X_2, Y_2) = (X_2, X_2' \cup Y_2) \sqsubseteq (A, B)$ holds. So, $(X_2, Y_2) \in \overline{\mathcal{T}_{(A,B)}}(R)$ is followed.

Third to check (T3) for $\overline{\mathcal{T}_{(A,B)}}(R)$.

Let $(X_1, Y_1), (X_2, Y_2) \in \overline{\mathcal{T}_{(A,B)}}(R)$ satisfy $|(X_2, Y_2)| < |(X_1, Y_1)|$ and $(X_1, Y_1) \neq (\emptyset, \emptyset)$. Then, we get the following fact:

$$\begin{aligned}
&(X_j, Y_j) \in \overline{\mathcal{T}_{(A,B)}}(R)\\
&\Rightarrow \overline{R}(X_j, Y_j) \sqsubseteq (A, B)\ (j = 1, 2)\\
&\text{(definition of } \overline{\mathcal{T}_{(A,B)}}(R))\\
&\Rightarrow (X_j, X_j' \cup Y_j) \sqsubseteq (A, B)\ (j = 1, 2)\\
&\text{(Lemma 2(1))}\\
&\Rightarrow X_j \subseteq A \text{ and } X_j' \cup Y_j \supseteq B\ (j = 1, 2)\\
&\text{(definition of } \sqsubseteq)
\end{aligned}$$

Via Lemma 4, we confirm $(X_1, Y_1)\backslash(X_2, Y_2) \neq (\emptyset, \emptyset)$. Then, we select $(a, b) \in (X_1, Y_1) \setminus (X_2, Y_2)$. We divide two cases to finish the proof.

Case 1. $X_1 \setminus X_2 = \emptyset$.

It gets $a = \emptyset$ and $b \in Y_1 \backslash Y_2 \neq \emptyset$. Then, we obtain $\overline{R}((X_2, Y_2) \cup (a, b)) = \overline{R}(X_2, Y_2 \cup b) = (X_2, (X_2' \cup Y_2) \cup b)$ holds according to Lemma 2(1). Using $X_2' \cup Y_2 \supseteq B$, we obtain $(X_2' \cup Y_2) \cup b \supseteq B$. Thus, combining $X_2 \subseteq A$ with the results mentioned above, we confirm $\overline{R}((X_2, Y_2) \cup (a, b)) \sqsubseteq (A, B)$. Hence, we receive $(X_2, Y_2) \cup (a, b) \in \overline{\mathcal{T}_{(A,B)}}(R)$.

Case 2. $X_1 \setminus X_2 \neq \emptyset$.

Select $a \in X_1 \backslash X_2$ and $b \in Y_1 \backslash Y_2$. If $Y_1 \backslash Y_2 = \emptyset$ (or $Y_1 \backslash Y_2 \neq \emptyset$), then $b = \emptyset$ (or $b \neq \emptyset$). No matter which situation happens, it follows

$$\begin{aligned}
\overline{R}((X_2, Y_2) \cup (a, b)) &= \overline{R}(X_2 \cup a, Y_2 \cup b)\\
&\text{(definition of } \cup)\\
&= (X_2 \cup a, (X_2 \cup a)' \cup (Y_2 \cup b))\\
&\text{(Lemma 2(1))}\\
&= (X_2 \cup a, (X_2' \cap a') \cup (Y_2 \cup b))\\
&\text{(Lemma 1(2))}
\end{aligned}$$

Combining $a \in X_1 \subseteq A$ and $X_2 \subseteq A$, we obtain $X_2 \cup a \subseteq X_2 \cup X_1 \subseteq A$. So, $(X_2 \cup a)' = X_2' \cap a' \supseteq A'$ in light of Lemma 1. Thus, we confirm $(X_2' \cap a') \cup (Y_2 \cup b) \supseteq A' \cup (Y_2 \cup b) \supseteq A' \supseteq B$ since $B \subseteq A'$. Furthermore, we get $(X_2, Y_2) \cup (a, b) \in \overline{\mathcal{T}_{(A,B)}}(R)$.

Summing up the above two cases, (T3) is correct for $\overline{\mathcal{T}_{(A,B)}}(R)$.

## 4. Proofs of items (2), (3) and (4) in Theorem 4

***Proof*** The proof of item (2) will be finished by two parts.

Part 1. $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$

$$\begin{aligned}
&\Rightarrow X_1 \subseteq X_2 \text{ and } Y_1 \supseteq Y_2 \quad \text{(definition of } \sqsubseteq)\\
&\Rightarrow X_1 \subseteq X_2,\ X_1' \supseteq X_2' \text{ and } Y_1 \supseteq Y_2 \quad \text{(Lemma 1(1))}\\
&\Rightarrow X_1 \subseteq X_2 \text{ and } X_1' \cap Y_1 \supseteq X_2' \cap Y_2\\
&\Rightarrow (X_1, X_1' \cap Y_1) \sqsubseteq (X_2, X_2' \cap Y_2)\\
&\text{(defition of } \sqsubseteq)\\
&\Rightarrow \underline{R}(X_1, Y_1) \sqsubseteq \underline{R}(X_2, Y_2)\\
&\text{(Lemma 2(1))}
\end{aligned}$$

Part 2. $(X_1, Y_1) \sqsubseteq (X_2, Y_2)$

$$\begin{aligned}
&\Rightarrow X_1 \subseteq X_2 \text{ and } Y_1 \supseteq Y_2 && \text{(definition of } \sqsubseteq)\\
&\Rightarrow X_1 \subseteq X_2,\ X_1' \supseteq X_2' \text{ and } Y_1 \supseteq Y_2 && \text{(Lemma 1(1))}\\
&\Rightarrow X_1 \subseteq X_2 \text{ and } X_1' \cup Y_1 \supseteq X_2' \cup Y_2\\
&\Rightarrow (X_1, X_1' \cup Y_1) \sqsubseteq (X_2, X_2' \cup Y_2) && \text{(definition of } \sqsubseteq)\\
&\Rightarrow \overline{R}(X_1, Y_1) \sqsubseteq \overline{R}(X_2, Y_2) && \text{(Lemma 2(1))}
\end{aligned}$$

To prove item (3).

Combining Remark 2, we know $\emptyset' = P$ for $\emptyset \subseteq O$. This implies $(\emptyset, P) \in \mathcal{B}(\mathbb{K})$ by Definition 2(2) and Remark 2. So, it infers $\underline{R}(\emptyset, P) = \overline{R}(\emptyset, P) = (\emptyset, P)$ using Lemma 2(4).

It is clear $(\emptyset, P) \sqsubseteq (A, B)$. Moreover, we receive $(\emptyset, P) \in \underline{\mathcal{T}_{(A,B)}}(R)$ and $(\emptyset, P) \in \overline{\mathcal{T}_{(A,B)}}(R)$. Additionally, both $(\emptyset, P) \sqsubseteq (X_j, Y_j)$ and $(\emptyset, P) \sqsubseteq (X_p, Y_p)$ evidently hold ($j \in \mathcal{J}$; $p \in \mathcal{P}$). Considered item (1), we can indicate $(\emptyset, P)$ to be the minimum element in the poset $(\underline{\mathcal{T}_{A,B)}}(R), \sqsubseteq)$, and also the minimum element in the poset $(\overline{\mathcal{T}_{(A,B)}}(R), \sqsubseteq)$.

Additionally, we obtain the following expression Part I.

Part I. $(X_j, Y_j) \in \underline{\mathcal{T}_{(A,B)}}(R)\ (j \in \mathcal{J})$

$$\begin{aligned}
&\Rightarrow \underline{R}(X_j, Y_j) \sqsubseteq (A, B)\ (j \in \mathcal{J})\\
&\text{(definition of } \underline{\mathcal{T}_{(A,B)}}(R))\\
&\Rightarrow (X_j, X_j' \cap Y_j) \sqsubseteq (A, B)\ (j \in \mathcal{J})\\
&\text{(Lemma 2(1))}\\
&\Rightarrow X_j \subseteq A \text{ and } X_j' \cap Y_j \supseteq B\ (j \in \mathcal{J})\\
&\text{(definition of } \sqsubseteq)\\
&\Rightarrow \cup_{j\in\mathcal{J}} X_j \subseteq A\\
&\text{and} (\cup_{j\in\mathcal{J}} X_j)' \cap (\cap_{j\in\mathcal{J}} Y_j) = \cap_{j\in\mathcal{J}} (X_j' \cap Y_j) \supseteq B\\
&\text{(Lemma 1(2))}\\
&\Rightarrow (\cup_{j\in\mathcal{J}} X_j, (\cup_{j\in\mathcal{J}} X_j)' \cap (\cap_{j\in\mathcal{J}} Y_j)) \sqsubseteq (A, B)\\
&\text{(definition of } \sqsubseteq)
\end{aligned}$$

$\Rightarrow \underline{R}(\cup_{j\in\mathcal{J}}X_j, \cap_{j\in\mathcal{J}}Y_j) \sqsubseteq (A, B)$
(Lemma 2(1))
$\Rightarrow (\cup_{j\in\mathcal{J}}X_j, \cap_{j\in\mathcal{J}}Y_j) \in \underline{\mathcal{T}_{(A,B)}}(R)$
(definition of $\underline{\mathcal{T}_{(A,B)}}(R)$)

Using the definition of $\sqsubseteq$, we may easily obtain "$X_j \subseteq \cup_{j\in\mathcal{J}}X_j$ and $Y_j \supseteq \cap_{j\in\mathcal{J}}Y_j$"$\Rightarrow$ "$(X_j, Y_j) \sqsubseteq (\cup_{j\in\mathcal{J}}X_j, \cap_{j\in\mathcal{J}}Y_j)$" for any $j \in \mathcal{J}$. Considered this result with the result in Part I, we find $(\cup_{j\in\mathcal{J}}X_j, \cap_{j\in\mathcal{J}}Y_j)$ to be the maximum element in $(\underline{\mathcal{T}_{(A,B)}}(R), \sqsubseteq)$.

We can obtain the following expression Part II.

Part II. $(X_p, Y_p) \in \overline{\mathcal{T}_{(A,B)}}(R)\ (p \in \mathcal{P})$
$\Rightarrow \overline{R}(X_p, Y_p) \sqsubseteq (A, B)\ (p \in \mathcal{P})$
(definition of $\overline{\mathcal{T}_{(A,B)}}(R)$)
$\Rightarrow (X_p, X_p' \cup Y_p) \sqsubseteq (A, B)\ (p \in \mathcal{P})$
(Lemma 2(1))
$\Rightarrow X_p \subseteq A\ (p \in \mathcal{P})$
(definition of $\sqsubseteq$)
$\Rightarrow \cup_{p\in\mathcal{P}}X_p \subseteq A$
$\Rightarrow (\cup_{p\in\mathcal{P}}X_p)' \cup (\cap_{p\in\mathcal{P}}Y_p)$
$\supseteq (\cup_{p\in\mathcal{P}}X_p)' \supseteq A' \supseteq B$
(Lemma 1(1), and $A' \supseteq B$)
$\Rightarrow (\cup_{p\in\mathcal{P}}X_p, (\cup_{p\in\mathcal{P}}X_p)' \cup (\cap_{p\in\mathcal{P}}$
$Y_p)) \sqsubseteq (A, B)$ (definition of $\sqsubseteq$)
$\Rightarrow \overline{R}(\cup_{p\in\mathcal{P}}X_p, \cap_{p\in\mathcal{P}}Y_p) \sqsubseteq (A, B)$
(Lemma 2(1))
$\Rightarrow (\cup_{p\in\mathcal{P}}X_p, \cap_{p\in\mathcal{P}}Y_p) \in \overline{\mathcal{T}_{(A,B)}}(R)$
(definition of $\overline{\mathcal{T}_{(A,B)}}(R)$)

Using the definition of $\sqsubseteq$, we may easily obtain "$X_p \subseteq \cup_{p\in\mathcal{P}}X_p$ and $Y_p \supseteq \cap_{p\in\mathcal{P}}Y_p$"$\Rightarrow$ "$(X_p, Y_p) \sqsubseteq (\cup_{p\in\mathcal{P}}X_p, \cap_{p\in\mathcal{P}}Y_p)$" for any $p \in \mathcal{P}$. Combining the result of Part II with the above result, we find $(\cup_{p\in\mathcal{P}}X_p, \cap_{p\in\mathcal{P}}Y_p)$ to be the maximum element in $(\overline{\mathcal{T}_{(A,B)}}(R), \sqsubseteq)$.

To prove item (4).

We will use two parts to finish the proof.

Part 1. $(X, Y) \in \underline{\mathcal{T}_{(A_1,B_1)}}(R) \cap \underline{\mathcal{T}_{(A_2,B_2)}}(R)$
$\Leftrightarrow (X, Y) \in \underline{\mathcal{T}_{(A_i,B_i)}}(R)\ (i = 1, 2)$
$\Leftrightarrow \underline{R}(X, Y) \sqsubseteq (A_i, B_i)\ (i = 1, 2)$
(definition of $\underline{\mathcal{T}_{(A_i,B_i)}}(R)$)
$\Leftrightarrow X \subseteq A_i$ and $X' \cap Y \supseteq B_i\ (i = 1, 2)$
(Lemma 2(1), definition of $\sqsubseteq$)
$\Leftrightarrow X \subseteq A_1 \cap A_2$ and $X' \cap Y \supseteq B_1 \cup B_2$
$\Leftrightarrow \underline{R}(X, Y) \sqsubseteq (A_1 \cap A_2, B_1 \cup B_2)$
(Lemma 2(1))
$\Leftrightarrow (X, Y) \in \underline{\mathcal{T}_{(A_1\cap A_2,B_1\cup B_2)}}(R)$
(definition of $\underline{\mathcal{T}_{(A_1\cap A_2,B_1\cup B_2)}}(R)$)

Part 2. $(X, Y) \in \overline{\mathcal{T}_{(A_1,B_1)}}(R) \cap \overline{\mathcal{T}_{(A_2,B_2)}}(R)$
$\Leftrightarrow \overline{R}(X, Y) \sqsubseteq (A_i, B_i)\ (i = 1, 2)$
(definition of $\overline{\mathcal{T}_{(A_i,B_i)}}(R)$)
$\Leftrightarrow X \subseteq A_i$ and $X' \cup Y \supseteq B_i\ (i = 1, 2)$
(Lemma 2(1), definition of $\sqsubseteq$)
$\Leftrightarrow X \subseteq A_1 \cap A_2$ and $X' \cup Y \supseteq B_1 \cup B_2$
$\Leftrightarrow \overline{R}(X, Y) \sqsubseteq (A_1 \cap A_2, B_1 \cup B_2)$
(Lemma 2(1), definition of $\sqsubseteq$)
$\Leftrightarrow (X, Y) \in \overline{\mathcal{T}_{(A_1\cap A_2,B_1\cup B_2)}}(R)$
(definition of $\overline{\mathcal{T}_{(A_1\cap A_2,B_1\cup B_2)}}(R)$)

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