



Complex dynamics and control of a novel physical model using nonlocal fractional differential operator with singular kernel

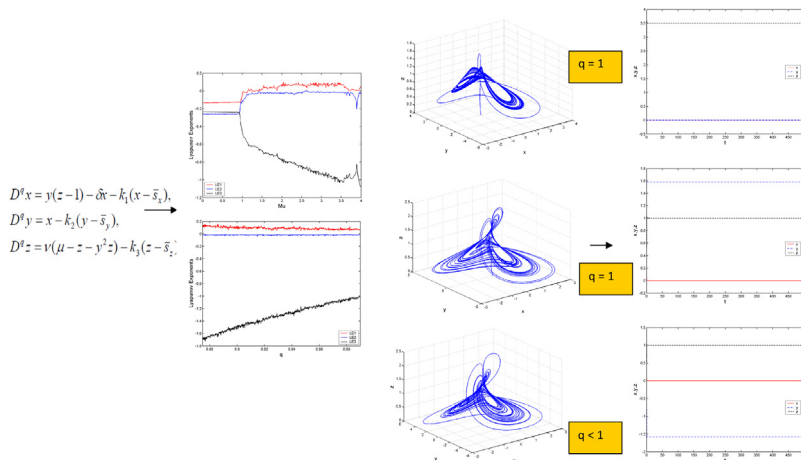
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GRAPHICAL ABSTRACT



ARTICLE INFO

Article history:

Received 9 March 2020

Revised 30 April 2020

Accepted 2 May 2020

Available online 2 June 2020

Keywords:

Nonlocal fractional differential operator

Stability

Hopf bifurcation

Chaos

Chaos control

ABSTRACT

Fractional calculus (FC) is widely used in many interdisciplinary branches of science due to its effectiveness in describing and investigating complicated phenomena. In this work, nonlinear dynamics for a new physical model using nonlocal fractional differential operator with singular kernel is introduced. New Routh-Hurwitz stability conditions are derived for the fractional case as the order lies in $[0,2)$. The new and basic Routh-Hurwitz conditions are applied to the commensurate case. The local stability of the incommensurate orders is also discussed. A sufficient condition is used to prove that the solution of the proposed system exists and is unique in a specific region. Conditions for the approximating periodic solution in this model via Hopf bifurcation theory are discussed. Chaotic dynamics are found in the commensurate system for a wide range of fractional orders. The Lyapunov exponents and Lyapunov spectrum of the model are provided. Suppressing chaos in this system is also achieved via two different methods.

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Peer review under responsibility of Cairo University.

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Introduction

Fractional calculus (FC) has recently been considered to be one of the powerful tools to describe a complex dynamical

<https://doi.org/10.1016/j.jare.2020.05.003>

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phenomenon [1–3] and is widely applied in different fields including physics, economics, combustion science, biology and engineering [4–15]. Indeed, FC provides a realistic description of a physical phenomenon and also helps to achieve greater degrees of freedom in physical models because the analysis in FC provides a generalization of the classical differentiation and integration to the arbitrary order (noninteger state). Thus, FC has been attracted a great deal of attention owing to its intrinsic advantages in modeling of natural phenomena involved with memory and hereditary properties. Moreover, FC has been utilized to define many physical models in which fractional differential and integral operators have been successfully used to describe their nature. However, to explain these physical phenomena in fractional language, authors have used several definitions. Among them are the well-known fractional derivative definitions used by Riemann-Liouville [16], Caputo [17] and Caputo-Fabrizio [18].

In fact, nonlocal differential and integral operators are better candidates for handling the chaotic behaviors of fractional derivatives, which are also classified based on their kernels. The Riemann-Liouville and Caputo derivatives possess nonlocal operators with singular kernels; however, the fractional derivative defined by Caputo and Fabrizio has a nonlocal operator with a non-singular kernel.

Recently, the applications of fractional derivatives in physical models have been widely examined owing to their usefulness in many fields of physics such as viscoelasticity, transient heat diffusion, steady-state heat conduction, electrochemical double layer capacitors, dielectric polarization, DNA chain, electromagnetic waves, hybrid nanofluid, quantum mechanics, and quantum evolution of complex systems. Moreover, the exhibition of chaos in a fractional physical model and its suppression in such a model are two of the main problems that have been encountered. Chaotic attractors have also been reported in some physical models with fractional order such as the Liu system [19], the Van der Pol-Duffing circuit [20], a Volta's model [21], and novel hyperchaotic circuits [22,23]. Furthermore, the suppression of chaos in differential models involving fractional derivatives has received increasing attention [24–28].

In [29], Constantinescu et al. proposed a model for quasi-periodic plasma perturbations that consists of an integer-order system of ordinary differential equations with two nonlinear terms. This low dimensional integer-order model for quasi-periodic plasma perturbations explores the dynamical behaviors of the amplitude of magnetic field displacement and the plasma pressure gradient in tokamaks. In addition, Constantinescu et al. [30] studied existence of Hopf bifurcation in this model of quasi-periodic plasma perturbations and analyzed the fast-slow dynamics of this model. Moreover in [31], qualitative dynamical study in this integer-order model of quasi-periodic plasma perturbations like existence of Bogdanov-Takens bifurcation, pitchfork bifurcation, homoclinic bifurcation and chaotic states, was reported by Elsadany et al.

In this work, we explore dynamics of the quasi-periodic plasma perturbations model with fractional derivatives. We use the Caputo type fractional differential operator, which is widely used in real applications. Indeed, imposing nonlocal fractional differential operators to the quasi-periodic plasma perturbations model allows us to obtain more accuracy and adequacy of describing the natural phenomena, and to obtain greater degrees of freedom in this model. Consequently, the proposed fractional form of the quasi-periodic plasma perturbations model is better candidate for describing the expected complex dynamics since it is defined by integration. However, the existence of unpredictable or complex dynamics is not desirable in many practical situations. Therefore, erasing the unpredictable dynamics that may arise from the fractional-order quasi-periodic plasma perturbations model

becomes a focal point of our interest. To the best of our knowledge, the results in this work are the first to report the complex dynamics and chaos suppression in the fractional-order quasi-periodic plasma perturbations model.

Here, new Routh-Hurwitz stability conditions in three dimensional fractional-order systems as the orders lie in the interval $[0,2)$, are proved and applied to the proposed model. A condition for the existence and uniqueness of the solution of the quasi-periodic plasma perturbations model is obtained. Conditions for the approximating periodic solutions in this system are also discussed. Chaos in the proposed model is also found for fractional orders above and less than 1. Furthermore, chaotic behaviors in this model are suppressed to its steady states as the orders lie in the interval $(0,2)$. Thus, our study helps to understand the complex dynamics arising from the quasi-periodic plasma perturbations model involving fractional derivatives based on Caputo nonlocal fractional operator which provides more appropriate and realistic description of the resulting complex dynamics and also our study helps to eliminate unpredictable dynamic behaviors of the proposed model.

Basic concepts of FC

The Caputo nonlocal fractional differential operator with singular kernel [17] is given as

$$D^q \phi(\tau) = \int_0^\tau (\tau - l)^{k-q-1} \phi^{(k)}(l) dl / \Gamma(k - q), \quad (1)$$

where $\phi^{(k)}$ refers to the k th-order derivative of $\phi(l)$. Moreover, the stability of nonlinear systems involving fractional derivatives is summarized by the following results:

Assume that

$$D^q X(t) = H(X(t)), \quad (2)$$

where $0 < q < 2, X \in R^3$, and the vector function H is nonlinear. If $X^{(e)}$ is an equilibrium point of (2) with the following eigenvalue equation:

$$\varphi(\lambda) = \lambda^3 + s_1 \lambda^2 + s_2 \lambda + s_3 = 0, \quad (3)$$

then the Matignon's inequalities [32] are used to discuss local stability of $X^{(e)}$ as follows

$$|\arg(\lambda_i)| > q\pi/2, \quad i = 1, 2, 3. \quad (4)$$

The corresponding region describing the local stability of $X^{(e)}$ is depicted in Fig. 1. Also, the following fractional Routh-Hurwitz (FRH) criterion [33] is obtained for $q \in [0, 2)$:

- (i) $X^{(e)}$ is locally asymptotically stable (LAS) for $q \in [0, 2)$, if the discriminant of $\varphi(\lambda)$ is positive in addition to $s_1 > 0$, $s_3 > 0$ and $s_1 s_2 - s_3 > 0$.
- (ii) $X^{(e)}$ is LAS for $q < \frac{2}{3}$ if the discriminant of $\varphi(\lambda)$ is negative in addition to $s_1 \geq 0$, $s_2 \geq 0$ and $s_3 > 0$.
- (iii) $X^{(e)}$ is LAS for $q \in (0, 1)$ if the discriminant of $\varphi(\lambda)$ is negative in addition to $s_1 > 0$, $s_2 > 0$ and $s_1 s_2 - s_3 = 0$.

Here, we also provide the following theorem.

Theorem 1 ((Matouk's)). For the eigenvalue equation (3);

- (a) If $q \in [1, 2)$, $s_3 = s_1 s_2$ and the discriminant of $\varphi(\lambda) < 0$, then the Matignon's inequalities (4) are not satisfied;

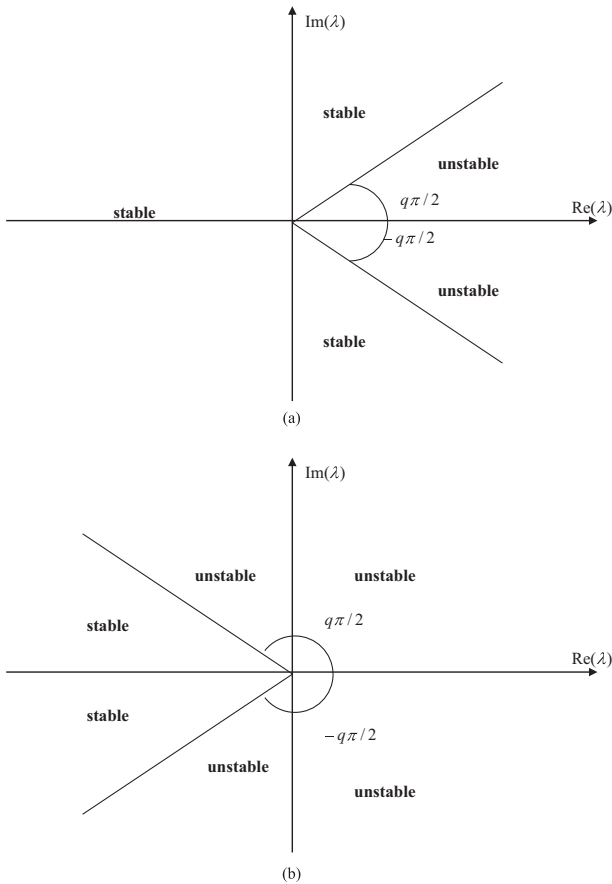


Fig. 1. Stability region of linear fractional-order system as: (a) $q \in [0, 1)$, (b) $q \in [1, 2)$.

(b) If $q \in [0, 2)$ then $s_3 > 0$ is a necessary condition for $X^{(e)}$ to be LAS.

Proof. To prove part (a), we recall that if discriminant of $\varphi(\lambda) < 0$, then Eq. (3) has the following eigenvalues

$$\lambda_1 = \lambda_0, \quad \lambda_{2,3} = p \pm \omega I, \quad I = \sqrt{-1}, \quad \lambda_0, p, \omega \in \mathbb{R}, \quad \omega \neq 0. \quad (5)$$

So, Eq. (3) has the following coefficients

$$s_1 = -2p - \lambda_0, \quad s_2 = |\lambda_2|^2 + 2p\lambda_0, \quad s_3 = -\lambda_0|\lambda_2|^2. \quad (6)$$

Consequently, Eq. (3) has two pure imaginary roots if and only if $s_3 = s_1s_2$, since the last condition implies that

$$(2p + \lambda_0)(|\lambda_2|^2 + 2p\lambda_0) = \lambda_0|\lambda_2|^2, \quad (7)$$

that is reduced to

$$p(\omega^2 + (\lambda_0 + p)^2) = 0. \quad (8)$$

It is now clear that $p = 0$ as $s_3 = s_1s_2$ which means that the eigenvalues $\lambda_{2,3}$ lie in the unstable region (See Fig. 1b) of the linearized fractional order system as $q \in [1, 2)$. \square

To prove part (b), we firstly assume that the discriminant of $\varphi(\lambda) < 0, q \in [0, 2)$ then $s_3 = -\lambda_0|\lambda_2|^2 < 0$ implies that $\lambda_0 > 0$ which also means that λ_0 lies in the unstable region of Fig. 1b. Secondly, we assume that the discriminant of $\varphi(\lambda) > 0$, then $s_3 = -\lambda_1\lambda_2\lambda_3 < 0, \lambda_i \in \mathbb{R}, i = 1, 2, 3$ implies that there exists at least one $\lambda_i > 0$ which also implies that λ_i lies in the unstable region of Fig. 1b. Also, the case $s_3 = 0$ is obviously belong to the unstable region. \square

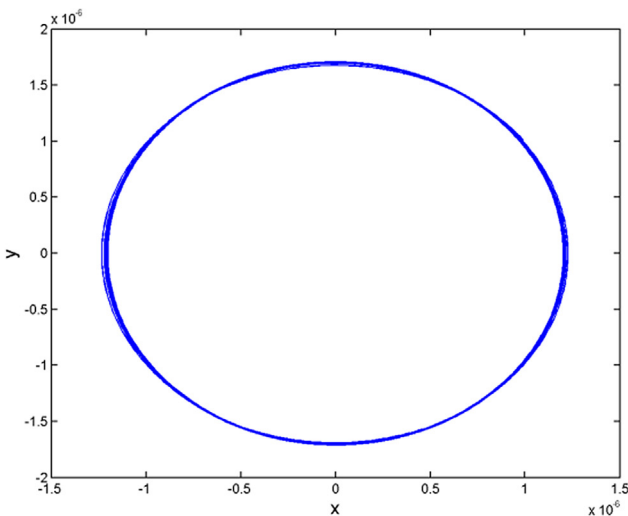
Theorem 2. (See [34]). Let system (2) be described as

$$D^q X(t) = AX(t) + B(X(t)), \quad (9)$$

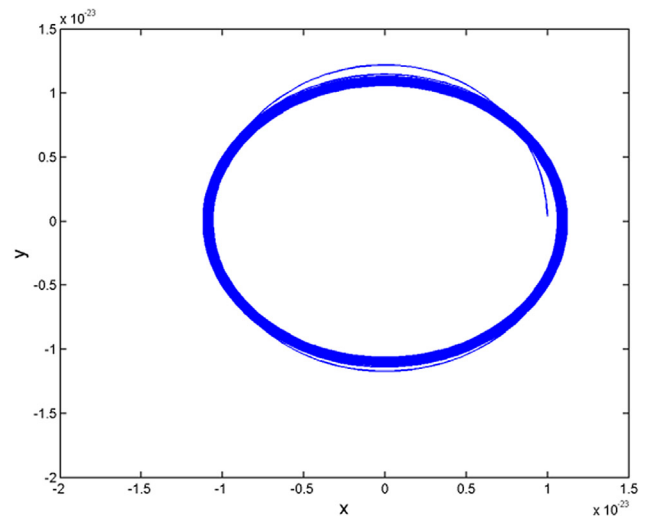
where $0 < q < 2, A \in \mathbb{R}^{3 \times 3}$, and B is a nonlinear function such that

$$\lim_{\|X(t)\| \rightarrow 0} \frac{\|B(X(t))\|}{\|X(t)\|} = 0, \quad (10)$$

then $X^{(e)} = 0$ is LAS if $|\arg(\lambda_i(A))| > q\pi/2, i = 1, 2, 3$, where $\|\cdot\|$ is the l_2 -norm.



(a)



(b)

Fig. 2. Asymptotically periodic signal near $S_1 = (0, 0, \mu)$: (a) 2D plot using $\mu = 0.5, \nu = 0.1, \delta = 0.0001$ and $q = 0.9999549838$; (b) 2D plot using $\mu = 0.001, \nu = 0.2, \delta = 0.0003140024694$ and $q = 0.9999$.

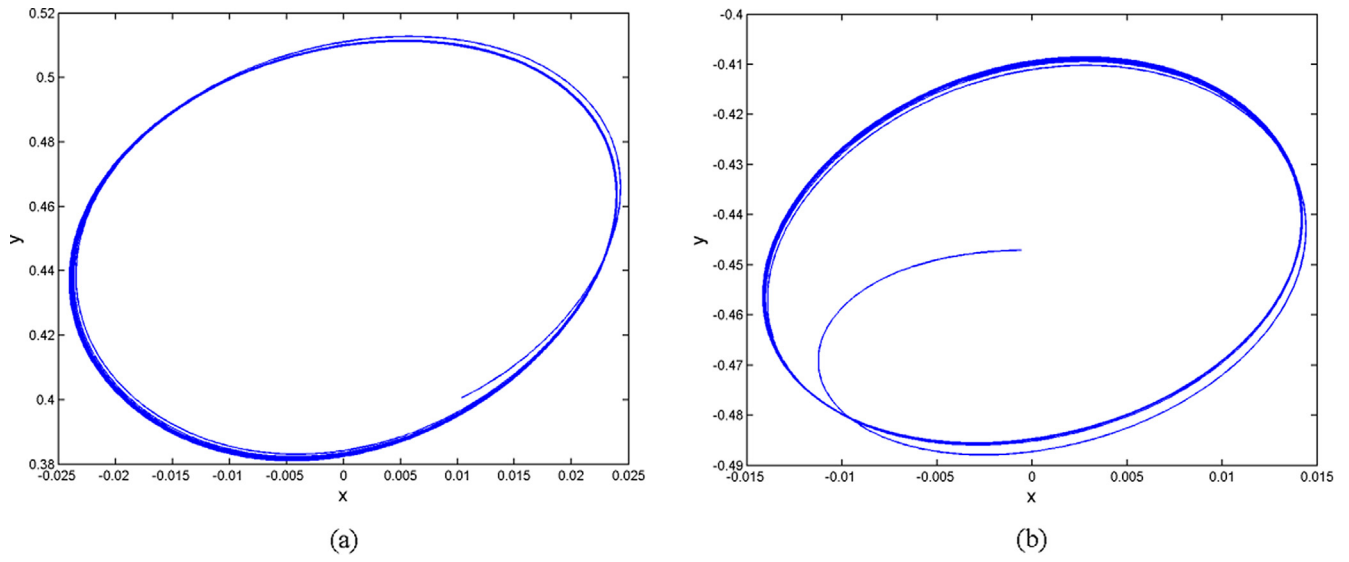


Fig. 3. 2D plot of an asymptotically periodic signal near: (a) $S_2(0, \sqrt{\mu-1}, 1)$ using $\mu = 1.2$, $\nu = 0.2$, $\delta = 0.2$ and $q = 0.8703517205$; (b) 2D plot $S_3(0, -\sqrt{\mu-1}, 1)$ using $\mu = 1.2$, $\nu = 0.2$, $\delta = 0.2$ and $q = 0.8703517205$.

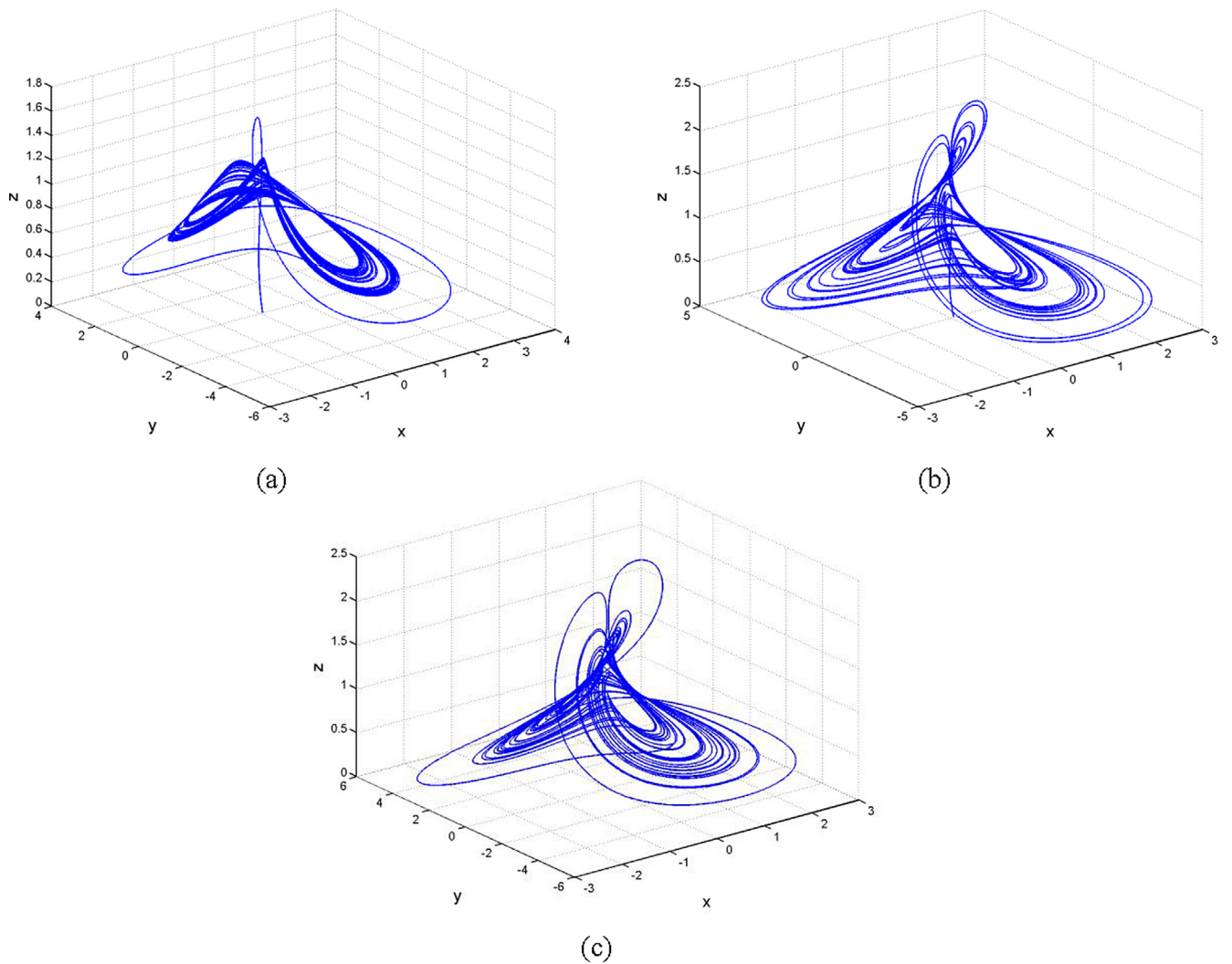
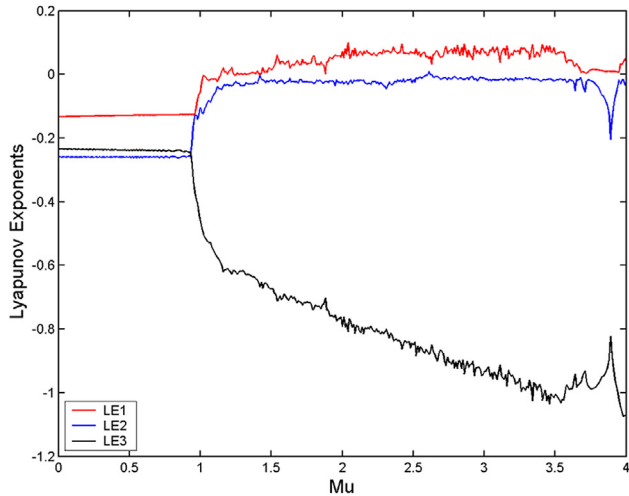
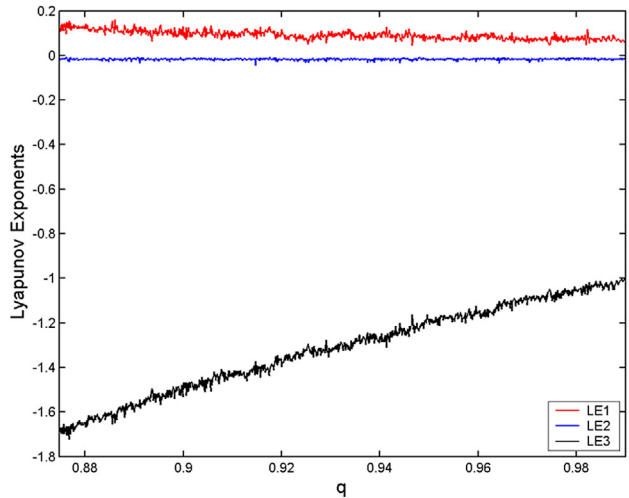


Fig. 4. Chaotic attractors appearing in the fractional model (12) using the parameter values $\mu = 3.5$, $\nu = 0.1$, $\delta = 0.5$ and the following fractional order: (a) $q = 1.1$, (b) $q = 1.0$ and (c) $q = 0.99$.



(a)



(b)

Fig. 5. Lyapunov spectrum of the fractional model (12) using $\nu = 0.1$ and $\delta = 0.5$: (a) The Lyapunov exponents are functions of μ with $q = 0.99$. (b) The Lyapunov exponents are functions of q with $\mu = 3.5$.

The model

Here, we introduce the integer-order form of the model as follows:

$$\begin{aligned} \frac{dx}{dt} &= y(z - 1) - \delta x, \\ \frac{dy}{dt} &= x, \\ \frac{dz}{dt} &= \nu(\mu - z - y^2 z), \end{aligned} \tag{11}$$

where δ, μ and ν are all positive real numbers, with δ denoting the relaxation of the perturbation; μ the input normalized power; and ν the characteristic relation of the heat diffusion coefficients [29]. In fact imposing the Caputo fractional differential operator to system (11), provides a generalization of the existing classical differentiation to the arbitrary order (noninteger state). The fractional-order form of model (11) is given as

$$\begin{aligned} D^q x(t) &= y(t)(z(t) - 1) - \delta x(t), \\ D^q y(t) &= x(t), \\ D^q z(t) &= \nu(\mu - z(t) - y^2(t)z(t)), \end{aligned} \tag{12}$$

where $q \in (0, 2)$. So, higher degrees of freedom in the quasi-periodic plasma perturbations model (12) are obtained than the integer-order counter-parts. Moreover, the resulting long-term memory effect and hereditary properties of this operator are very useful to describe complicated natural dynamical phenomena. Thus, it is shown that the quasi-periodic plasma perturbations model (12) generalizes the original integer-order models in [29,30] and helps to obtain more adequacy and realistic description of the resulting dynamical phenomena. The Model (12) has three equilibria, i.e., $S_1(0, 0, \mu)$, $S_2(0, \sqrt{\mu - 1}, 1)$, and $S_3(0, -\sqrt{\mu - 1}, 1)$, for $\mu > 1$. Moreover, it has the unique equilibrium $S_1(0, 0, \mu)$, where $\mu \in (0, 1]$.

Existence and uniqueness

According to the familiar existence and uniqueness procedure given in [22,35], the following conditions are straightforwardly obtained.

Lemma 1. A solution of the model for quasi-periodic plasma perturbations (12) exists and is unique in the region $\Omega \times (0, \tau]$ with the initial conditions $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ and $t \in (0, \tau]$ if

$$0 < \eta = \frac{\tau^q}{\Gamma(1 + q)} \max\{1 + \delta, 1 + \gamma + 2\nu\gamma^2, \nu + \gamma + \nu\gamma^2\} < 1. \tag{13}$$

Stability of the quasi-periodic plasma perturbations model (12)

The Jacobian of the fractional model for quasi-periodic plasma perturbations (12), computed at $\bar{S} = (\bar{s}_x, \bar{s}_y, \bar{s}_z)$, is described by

$$J(\bar{S}) = \begin{pmatrix} -\delta & \bar{s}_z - 1 & \bar{s}_y \\ 1 & 0 & 0 \\ 0 & -2\nu\bar{s}_y\bar{s}_z & -\nu(1 + \bar{s}_y^2) \end{pmatrix}. \tag{14}$$

Theorem 3. The equilibrium $S_1(0, 0, \mu)$ of the fractional the model for quasi-periodic plasma perturbations (12) is (i) a saddle point if $\mu > 1$ or (ii) LAS if $\delta^2 \geq 4(1 - \mu), \mu < 1$, or $\delta^2 < 4(1 - \mu), \mu < 1$, $q < \frac{2}{\pi} \arctan \frac{\sqrt{4(1 - \mu) - \delta^2}}{\delta}$.

Proof. The Jacobian (14) evaluated at $S_1(0, 0, \mu)$ is given by

$$J(S_1(0, 0, \mu)) = \begin{pmatrix} -\delta & \mu - 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\nu \end{pmatrix}.$$

The Jacobian matrix $J(S_1(0, 0, \mu))$ has the eigenvalues $\lambda_1 = -\nu < 0$, $\lambda_{2,3} = \frac{-\delta \pm \sqrt{\delta^2 + 4(\mu - 1)}}{2}$. Therefore, if $\mu > 1$, then $\lambda_2 > 0$, which implies that $S_1(0, 0, \mu)$ is a saddle point. Furthermore, if $\delta^2 \geq 4(1 - \mu), \mu < 1$, then $\lambda_i < 0$ for all $i = 1, 2, 3$, which implies that $S_1(0, 0, \mu)$ is LAS. Moreover, if $\delta^2 < 4(1 - \mu), \mu < 1$, then $J(S_1)$ possesses two complex conjugate eigenvalues and the condition $q < \frac{2}{\pi} \arctan \frac{\sqrt{4(1 - \mu) - \delta^2}}{\delta}$ implies that $|\arg(\lambda_i)| > q\pi/2, i = 2, 3$, which means that $S_1(0, 0, \mu)$ is LAS.

On the other hand, the Jacobian (14) evaluated at $S_{2,3}$ yields the same characteristic equation, i.e.,

$$\lambda^3 + (\mu\nu + \delta)\lambda^2 + (\mu\nu\delta)\lambda + 2\nu(\mu - 1) = 0. \tag{15}$$

Therefore, according to the FRH criterion, we obtain the following theorems which are easily to be proved by straightforward utilization of the classic FRH conditions (i)-(iii):

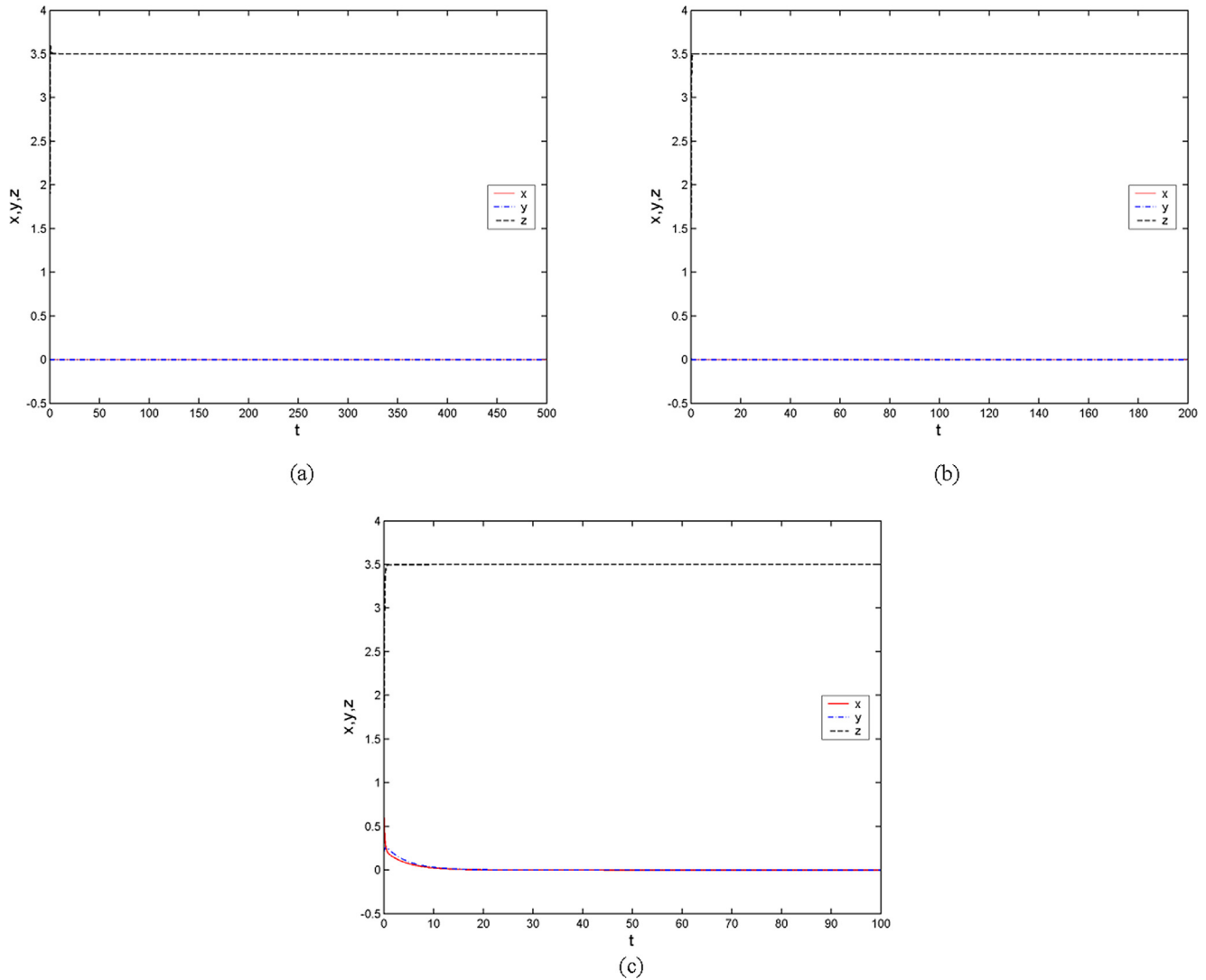


Fig. 6. Trajectories of the controlled fractional model (16) approach to the equilibrium point $S_1(0, 0, \mu)$ using $\mu = 3.5$, $v = 0.1$, $\delta = 0.5$ and the following fractional order: (a) $q = 1.1$, $k_1 = 300$, $k_2 = 200$, $k_3 = 10$; (b) $q = 1.0$, $k_1 = 300$, $k_2 = 200$, $k_3 = 10$; and (c) $q = 0.99$, $k_1 = 3$, $k_2 = 1$, $k_3 = 10$.

Theorem 4. If the discriminant of the polynomial given in Eq. (15) is positive, then $S_{2,3}$ are LAS for $0 < q < 2$ and $\delta > \sqrt{2}$. However, if this discriminant is negative, then $S_{2,3}$ are LAS for $q < \frac{2}{3}$ and also LAS for $0 < q < 1$ when $\delta < \sqrt{2 - \sqrt{8\delta v}}$, $\mu = \mu_1$, or $\delta < \sqrt{2 - \sqrt{8\delta v}}$, $\mu = \mu_2$, where $\mu_{1,2} = \frac{-\delta^2 + 2 \pm \sqrt{(2 - \delta^2)^2 - 8\delta v}}{2\delta v}$.

However, the results of applying the new FRH conditions given in Theorem 1 are summarized by the following lemma.

Lemma 2. If $q \in [1, 2)$ and the discriminant of $\varphi(\lambda) < 0$; then $S_1(0, 0, \mu)$ is not LAS for $\mu = v^2 + \delta v + 1$, and $S_{2,3}$ is not LAS for $\mu = \mu_1$ (or $\mu = \mu_2$). Moreover when $q \in [0, 2)$; $S_1(0, 0, \mu)$ is LAS only if $\mu < 1$, however $S_{2,3}$ are LAS only if $\mu > 1$.

For the incommensurate case of the model (12), we have the following theorem that is proved in [36].

Theorem 5. Consider the fractional model for quasi-periodic plasma perturbations (12) with incommensurate orders $q_i \in \mathbb{N}$, $0 < q_i < 1$, $i = 1, 2, 3$, where q_i is the fractional order on the i th equation of system (12). Also, define the ratio $q_i = \frac{g_i}{d_i}$, $d_i, g_i \in \mathbb{Z}^+$,

whose denominators have LCM = m , and $(d_i, g_i) = 1$. Hence, the equilibrium S_k , $k = 1, 2, 3$ of the fractional model for quasi-periodic plasma perturbations (12) with incommensurate orders are LAS iff

$$|\arg(\lambda_i)| > \frac{\pi}{2m},$$

where $\lambda_{i,s}$ and $b_{ij} = \frac{\partial H_i}{\partial v_j} \Big|_{S_k}$ must satisfy the following condition

$$\begin{vmatrix} \lambda^{mq_1} - b_{11} & -b_{12} & -b_{13} \\ -b_{21} & \lambda^{mq_2} - b_{22} & -b_{23} \\ -b_{31} & -b_{32} & \lambda^{mq_3} - b_{33} \end{vmatrix} = 0.$$

Conditions for the approximating periodic solution via Hopf bifurcation theory

In autonomous fractional-order system (AFOS), periodic solution cannot be analytically existed [37]. Only there are some asymptotically periodic signals satisfying the conditions of classical Hopf bifurcation theory, i.e. an approximation to the periodic solution around the steady state is expected as the AFQS, with

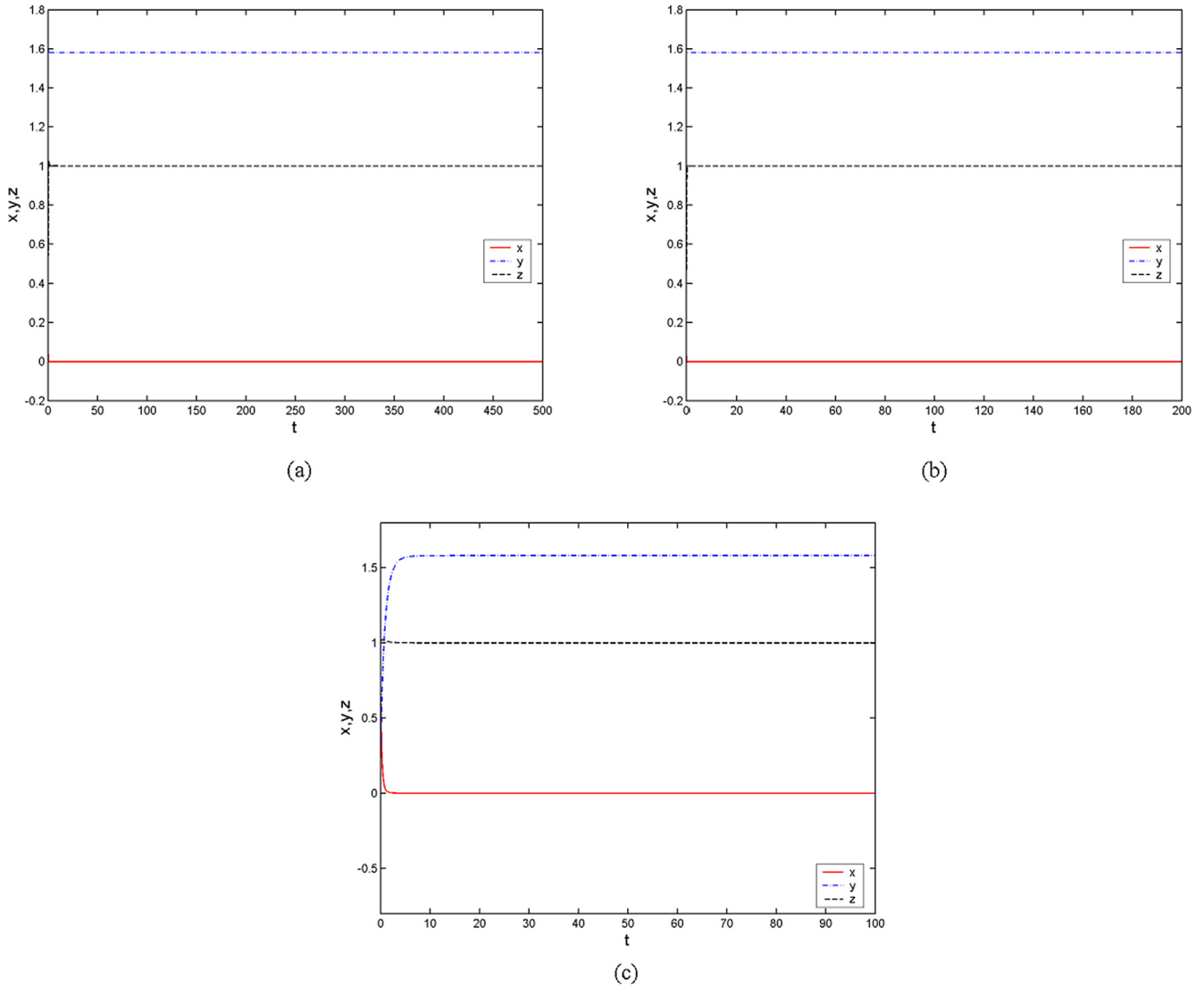


Fig. 7. Trajectories of the controlled fractional model (16) approach to the equilibrium point $S_2(0, \sqrt{\mu-1}, 1)$ using $\mu = 3.5$, $\nu = 0.1$, $\delta = 0.5$ and the following fractional order: (a) $q = 1.1$, $k_1 = 300$, $k_2 = 200$, $k_3 = 10$; (b) $q = 1.0$, $k_1 = 300$, $k_2 = 200$, $k_3 = 10$; and (c) $q = 0.99$, $k_1 = 3$, $k_2 = 1$, $k_3 = 10$.

order less than one, has negative real eigenvalues and a pair of complex conjugate eigenvalues $\lambda_{1,2} = u(t_{crh}) + Iw(t_{crh})$, $I = \sqrt{-1}$, $w \neq 0$, where t_{crh} is a critical value of the dynamical parameter, in addition to the existence of a function $\Xi(t)$ such that $\Xi(t_{crh}) = 0$ and $\frac{d\Xi(t)}{dt} \Big|_{t=t_{crh}} \neq 0$.

Asymptotically periodic signals near $S_1 = (0, 0, \mu)$

Obviously, the fractional parameter q affects the stability of the quasi-periodic plasma perturbations model (12). So, we can use it as a dynamical parameter. Now, let $\Xi(q) = \frac{q\pi}{2} - \arctan(\frac{\sqrt{-\delta^2 - 4(\mu-1)}}{\delta})$. Thus, $S_1 = (0, 0, \mu)$ changes its stability in the neighborhood of $q_{crh} = \frac{2}{\pi} \arctan(\frac{\sqrt{4(1-\mu)-\delta^2}}{\delta})$. Furthermore, the quantity $\frac{d\Xi(q)}{dq} \Big|_{q=q_{crh}}$ is not vanished. For $\mu = 0.5$, $\nu = 0.1$, $\delta = 0.0001$, the fractional parameter has the critical value $q_{crh} = 0.9999549838$. So, asymptotically periodic signal is expected near $S_1 = (0, 0, \mu)$ for these parameter values. In Fig. 2a, we summarize these results.

Moreover, the parameter δ can be selected as bifurcation parameter by setting $\Xi(\delta) = -\arctan(\frac{\sqrt{4(1-\mu)-\delta^2}}{\delta}) + \frac{q\pi}{2}$. In this case, the critical bifurcation value $\delta_{crh} = \pm 2 \frac{\sqrt{(1-\mu)(1+\tan^2(\frac{q\pi}{2}))}}{1+\tan^2(\frac{q\pi}{2})}$, $0 < q < 1$, $0 < \mu < 1$ and $\frac{d\Xi(\delta)}{d\delta} \Big|_{\delta=\delta_{crh}}$ is not vanished since it equals $\cot(\frac{q\pi}{2})/\delta_{crh}$, $\delta_{crh} \neq 0$. With the parameter selection $\mu = 0.001$, $\nu = 0.2$, $q = 0.9999$, the critical value δ_{crh} becomes 0.0003140024694. So, asymptotically periodic signal is expected near $S_1 = (0, 0, \mu)$. The indicated approximation to periodic signal is illustrated in Fig. 2b.

Asymptotically periodic signals near $S_{2,3}(0, \pm\sqrt{\mu-1}, 1)$

If the discriminant of the polynomial (15) is negative, then $S_{2,3}(0, \pm\sqrt{\mu-1}, 1)$ has a negative real root and a pair of complex conjugate roots. Then let $\Xi(q) = \frac{q\pi}{2} - \arctan(\frac{N}{L})$, where $N \neq 0$, $L^2 + N^2 = \varpi^3 - \mu\nu\delta\varpi^2 - 2(\delta + \mu\nu)(\nu - \mu\nu)\varpi - 4(\mu\nu - \nu)^2 = 0$. The equilibrium points $S_{2,3}(0, \pm\sqrt{\mu-1}, 1)$ change their stability near the critical fractional parameter $q_{crh} = 2\cos^{-1}(\frac{\sqrt{L^2}}{\varpi})/\pi$.

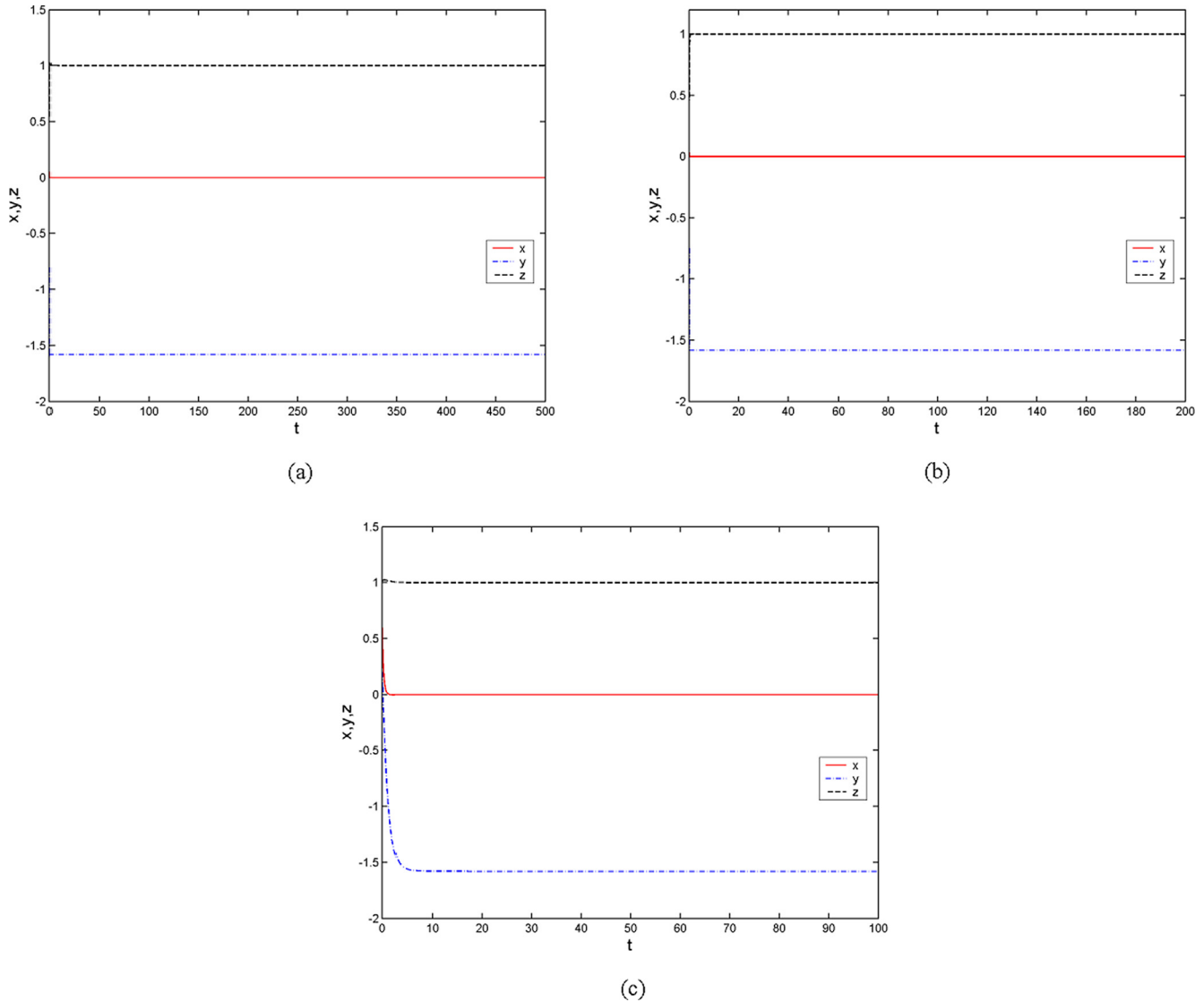


Fig. 8. Trajectories of the controlled fractional model (16) approach to the equilibrium point $S_3(0, -\sqrt{\mu-1}, 1)$ using $\mu = 3.5$, $\nu = 0.1$, $\delta = 0.5$ and the following fractional order: (a) $q = 1.1$, $k_1 = 300$, $k_2 = 200$, $k_3 = 10$; (b) $q = 1.0$, $k_1 = 300$, $k_2 = 200$, $k_3 = 10$; and (c) $q = 0.99$, $k_1 = 3$, $k_2 = 1$, $k_3 = 10$.

Obviously, $\left. \frac{d\Xi(q)}{dq} \right|_{q=q_{crh}}$ is not vanished. For $\mu = 1.2$, $\nu = 0.2$, $\delta = 0.2$, we get $q_{crh} = 0.8703517205$. Hence, periodic solutions are expected near $S_{2,3}(0, \pm\sqrt{\mu-1}, 1)$. The indicated approximation to periodic signals around S_2 and S_3 , are depicted in Fig. 3a and b, respectively.

Remark 1. According to Proposition 3 of [30], Hopf bifurcation occurs in the quasi-periodic plasma perturbations model (12) near $S_{2,3}(0, \pm\sqrt{\mu-1}, 1)$ for $q = 1$, $\delta \in (0, \sqrt{2})$, $\nu < \frac{(2-\delta^2)^2}{8\delta}$ and $\mu = \mu_1$ (or $\mu = \mu_2$).

Chaos in the fractional quasi-periodic plasma perturbations model

The fractional model for quasi-periodic plasma perturbations (12) is numerically integrated using $\mu = 3.5$, $\nu = 0.1$ and $\delta = 0.5$. Using the previous parameter values, the initial conditions (0.1, 0.1, 0.1), the fractional parameters $q = 1.1$, $q = 1.0$, $q = 0.99$ and $q = 0.13$, the system has a positive maximal Lyapunov expo-

nent $\Lambda_{max} = 0.0172, 0.0518, 0.0739$ and 0.1320 respectively, according to the algorithm given in [38]. The chaotic dynamics of system (12) are illustrated in Fig. 4. It can be seen that the lowest order in the commensurate fractional-order system for which chaos exists is approximately $3 \times 0.13 = 0.39$. Furthermore, we perform computations of the Lyapunov spectrum as the parameter (or the fractional order) are varied, as illustrated in Fig. 5, which also depicts the existence of a positive maximal Lyapunov exponent (MLE) that refers to the occurrence of a sensitive dependence on the initial conditions in the model.

Thus, it is shown that chaotic dynamics are found in the fractional quasi-periodic plasma perturbations model (12) for a wide scale of fractional orders $q \in (0, 2)$ which confirm that the proposed model exhibits more rich complex dynamics comparing to the models reported in previous literatures such as Refs. [30] and [31].

Achieving chaos control

Here, we will apply the stability results given by the FRH criterion and Theorems 2 to stabilize system (12) to its equilibrium points.

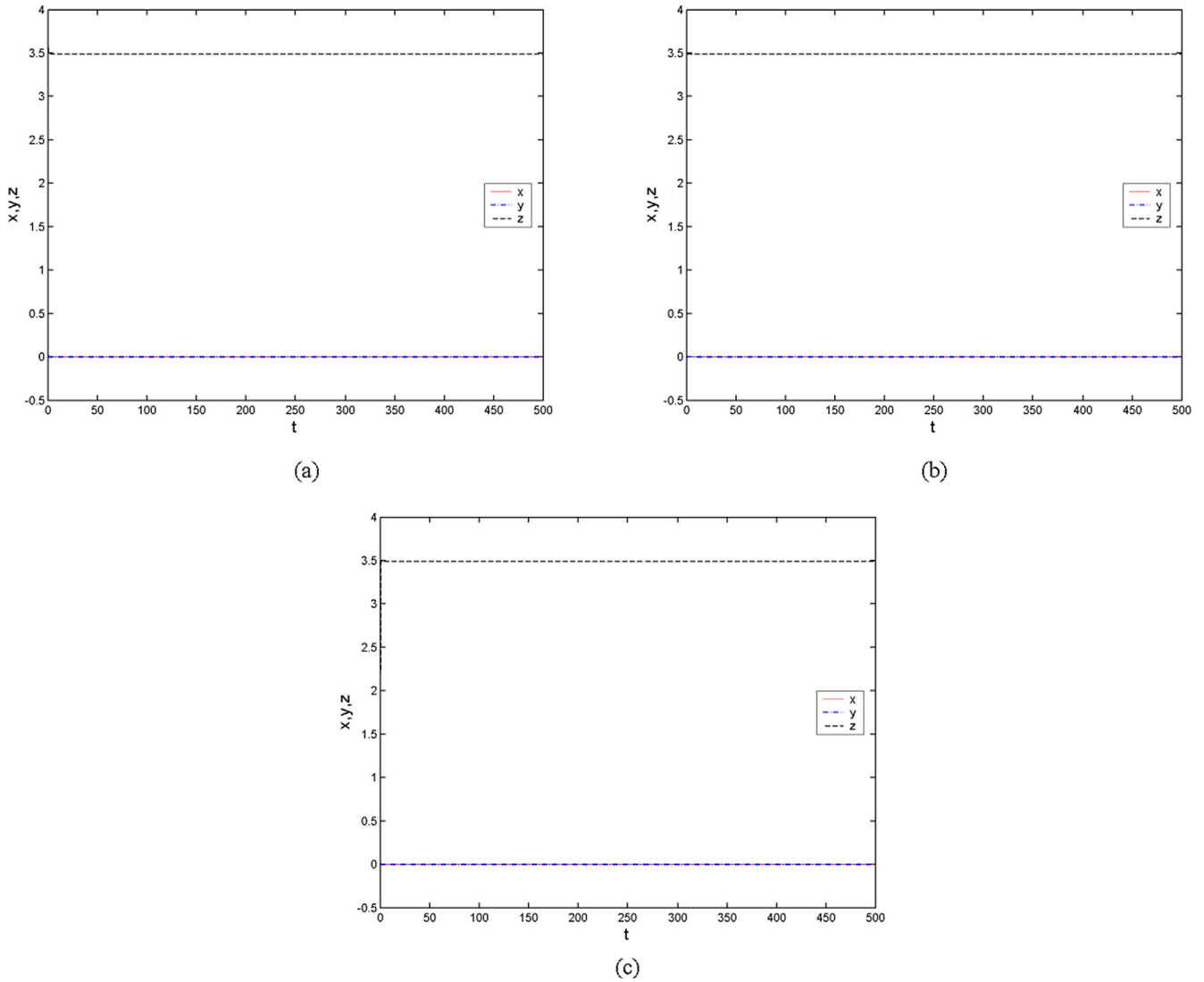


Fig. 9. Trajectories of the controlled fractional model (19) approach to the equilibrium point $S_1(0, 0, \mu)$ using $\mu = 3.5$, $v = 0.1$, $\delta = 0.5$, $\kappa_1 = 100$, $\kappa_2 = \kappa_3 = 30$ and the following fractional order: (a) $q = 1.1$, (b) $q = 1.0$ and (c) $q = 0.99$.

Stabilizing system (12) using the FRH criterion

We first consider the following controlled form of quasi-periodic plasma perturbations model (12):

$$\begin{aligned} D^q x &= y(z - 1) - \delta x - k_1(x - \bar{s}_x), \\ D^q y &= x - k_2(y - \bar{s}_y), \\ D^q z &= v(\mu - z - y^2 z) - k_3(z - \bar{s}_z), \end{aligned} \tag{16}$$

where $k_i \in R^+$, $i = 1, 2, 3$. In the case of the point $(\bar{s}_x, \bar{s}_y, \bar{s}_z) = S_1$, the characteristic polynomial of system (16) has the following coefficients:

$$\begin{aligned} s_1 &= \delta + v + \sum_{i=1}^3 k_i, \\ s_2 &= v(k_1 + k_2) + \delta(k_2 + k_3) + k_1 k_2 + k_2 k_3 + k_1 k_3 + 1 - \mu + \delta v, \\ s_3 &= \delta k_2(v + k_3) + v(k_1 k_2 + 1 - \mu) + k_3(1 - \mu) + k_1 k_2 k_3. \end{aligned} \tag{17}$$

However, the other equilibrium points S_2 and S_3 have the same coefficients of the eigenvalue equation of system (16). They are given as follows:

$$\begin{aligned} s_1 &= \delta + \mu v + \sum_{i=1}^3 k_i, \quad s_2 \\ &= \mu v(k_1 + k_2) + \delta(k_2 + k_3) + k_1 k_2 + k_2 k_3 + k_1 k_3 + \delta \mu v, \quad s_3 \\ &= \delta k_2(\mu v + k_3) + v(\mu k_1 k_2 - 2 + 2\mu) + k_1 k_2 k_3. \end{aligned} \tag{18}$$

For the parameter set $\mu = 3.5$, $v = 0.1$, $\delta = 0.5$ and $k_1 = 3$, $k_2 = 1$, $k_3 = 10$ or $k_1 = 300$, $k_2 = 200$, $k_3 = 10$, it is clear that the first FRH condition is satisfied for Eqs. (17) and (18). Therefore, system (16) is controlled to its equilibria. The simulation results that verified the stabilization of system (16) to the points S_1, S_2 , and S_3 are respectively illustrated in Figs. 6, 7, and 8 for $q = 1.1, 1$, and 0.99 .

Stabilizing system (12) using the results of Theorem 2

A controlled form of the fractional model for quasi-periodic plasma perturbations (12) is represented as

$$D^q X(t) = (A - K)X(t) + B(X(t)) - U, \tag{19}$$

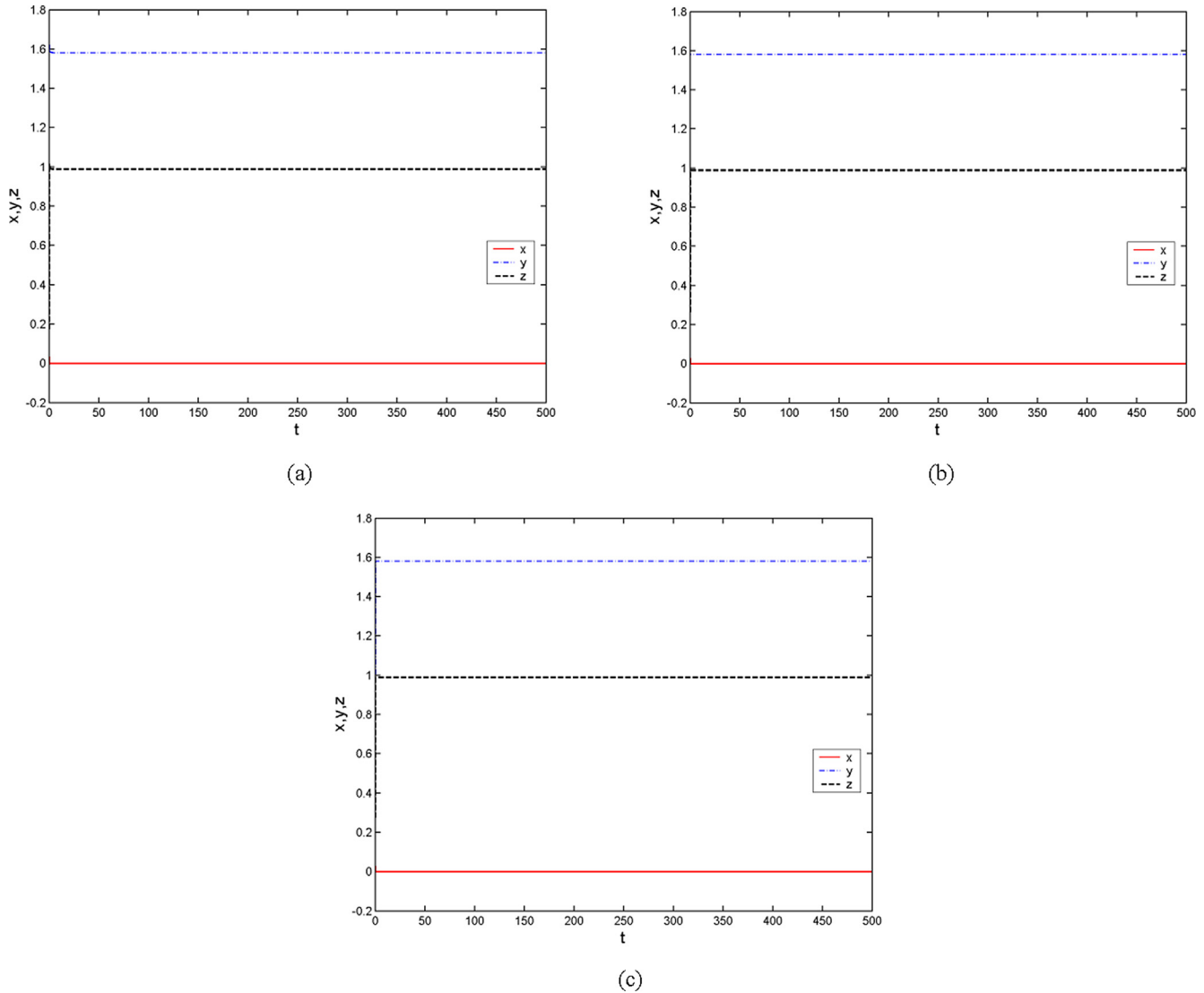


Fig. 10. Trajectories of the controlled fractional model (19) approach to the equilibrium point $S_2(0, \sqrt{\mu-1}, 1)$ using $\mu = 3.5$, $v = 0.1$, $\delta = 0.5$, $\kappa_1 = 100$, $\kappa_2 = \kappa_3 = 30$ and the following fractional order: (a) $q = 1.1$, (b) $q = 1.0$ and (c) $q = 0.99$.

given that

$$A = \begin{pmatrix} -\delta & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -v \end{pmatrix},$$

$$K = \text{diag}(\kappa_1 \bar{s}_x - \kappa_1 \bar{s}_x, \kappa_2 \bar{s}_y - \kappa_2 \bar{s}_y, \kappa_3 \bar{s}_z - \kappa_3 \bar{s}_z), \quad \kappa_i \in \mathbb{R}^+, \quad i = 1, 2, 3,$$

$$U = \begin{pmatrix} 0 \\ 0 \\ \mu v \end{pmatrix}.$$

For the point $S_1(0, 0, \mu)$, the condition $|\arg(\lambda_i(A - K))| > q\pi/2$, $i = 1, 2, 3$, is always satisfied and the nonlinear function has the form $B(X(t)) = \begin{pmatrix} yz \\ 0 \\ -vy^2z \end{pmatrix}$. Hence, condition

(10) is given by

$$\begin{aligned} \lim_{\|X(t)\| \rightarrow 0} \frac{\|B(X(t))\|}{\|X(t)\|} &= \lim_{\|X(t)\| \rightarrow 0} \frac{\sqrt{y^2z^2 + v^2y^4z^2}}{\sqrt{x^2 + y^2 + z^2}} \\ &\leq \lim_{\|X(t)\| \rightarrow 0} \frac{\sqrt{y^2(z^2 + v^2y^2z^2)}}{\sqrt{y^2}} \\ &= \lim_{\|X(t)\| \rightarrow 0} \sqrt{z^2 + v^2y^2z^2} = 0. \end{aligned}$$

Consequently, all the hypotheses of Theorem 2 are achieved, which implies that system (19) is controlled to $S_1(0, 0, \mu)$.

To stabilize system (19) to the other equilibrium points $S_2(0, \sqrt{\mu-1}, 1)$ and $S_3(0, -\sqrt{\mu-1}, 1)$, we utilize the transformation $X' = X - \bar{S}$, which transforms \bar{S} to the origin. Hence, it is clear that all the conditions of Theorem 2 are also satisfied. Consequently, $(x - \bar{s}_x, y - \bar{s}_y, z - \bar{s}_z)$ is stabilized to the origin according to the postulates of Theorem 2.

Now, the controlled system (19) with orders $q = 1.1$, 1, and 0.99 is numerically integrated using the selection $\mu = 3.5$, $v = 0.1$, $\delta = 0.5$, $\kappa_1 = 100$, $\kappa_2 = 30$ and $\kappa_3 = 30$. The numerical results show that system (19) is controlled to S_1 , S_2 , and S_3 , which are respectively depicted in Figs. 9, 10, and 11.

Conclusion

A novel model for quasi-periodic plasma perturbations using nonlocal fractional differential operator with singular kernel has been proposed. A sufficient condition has been used to show that the solution of the proposed system exists and is unique in a specific region. Local stability of the system's equilibria has been inves-

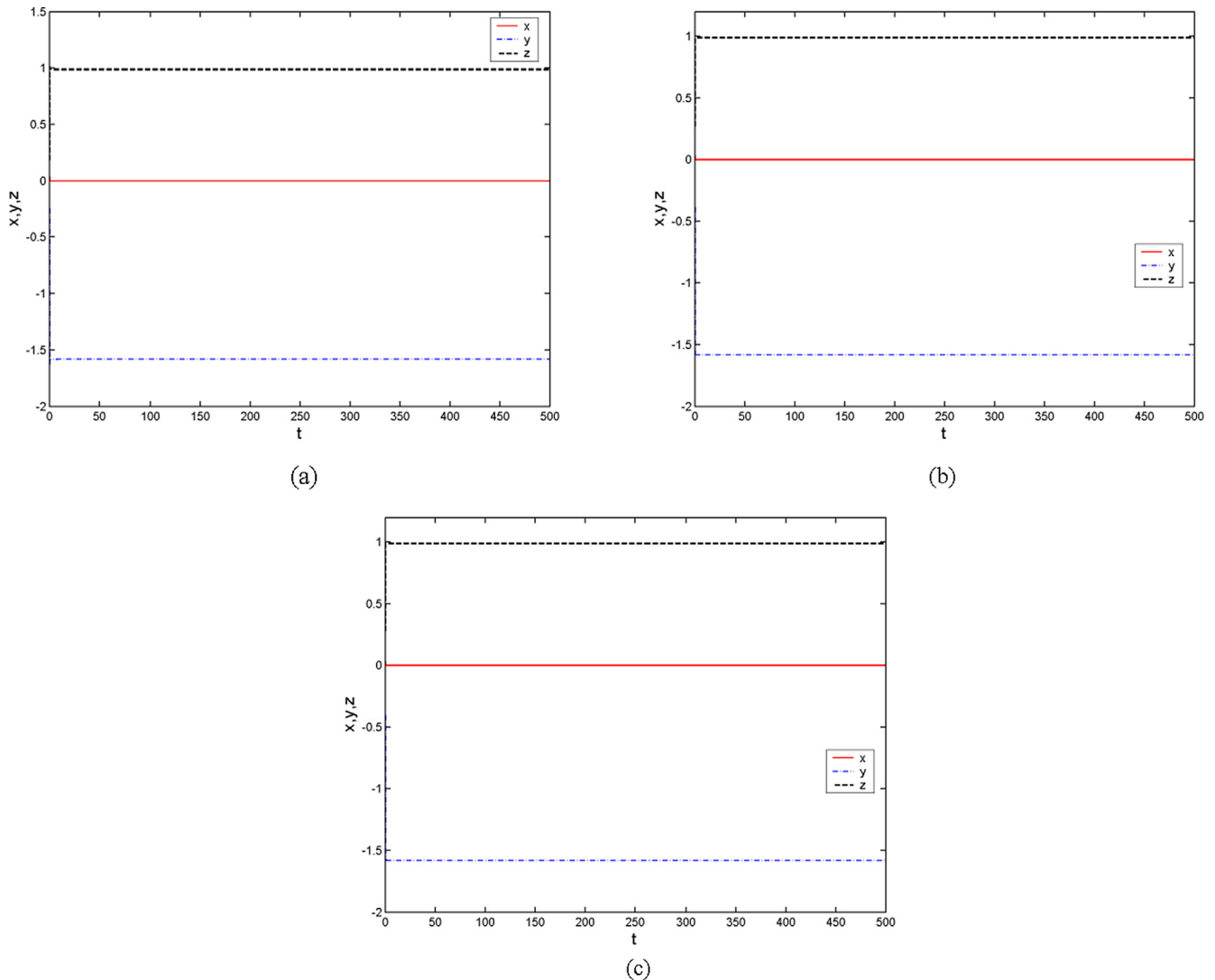


Fig. 11. Trajectories of the controlled fractional model (19) approach to the equilibrium point $S_3(0, -\sqrt{\mu-1}, 1)$ using $\mu = 3.5$, $\nu = 0.1$, $\delta = 0.5$, $\kappa_1 = 100$, $\kappa_2 = \kappa_3 = 30$ and the following fractional order: (a) $q = 1.1$, (b) $q = 1.0$ and (c) $q = 0.99$.

tigated with both commensurate and incommensurate orders. Conditions for the approximating periodic solution in this model via Hopf bifurcation theory have been obtained. Chaotic dynamics have been found in the commensurate system for a wide range of fractional orders. The Lyapunov exponents and Lyapunov spectrum of the model's parameters and fractional order have also been calculated. Suppressing chaos in this system has been achieved via two different approaches.

The obtained results provide us with fundamental and useful information to further better understand the complex dynamics arising from the quasi-periodic plasma perturbations model and also help to erase its unpredictable dynamical behaviors. In addition, our study provides more appropriate and realistic description of the proposed model. Therefore, our results might be very useful for the physicists who work with tokamaks models.

Compliance with ethics requirements

This work does not contain any studies with human or animal subjects.

Declaration of Competing Interest

The authors have declared no conflict of interest.

Acknowledgement

The authors would like to thank the Deanship of Scientific Research at Majmaah University, Saudi Arabia, for supporting this work under project no. 1440-42.

References

- [1] Matouk AE, Elsadany AA, Ahmed E, Agiza HN. Dynamical behavior of fractional-order Hastings-Powell food chain model and its discretization. *Commun Nonlin Sci Numer Simulat* 2015;27:153–67.
- [2] Matouk AE. Chaos synchronization of a fractional-order modified Van der Pol-Duffing system via new linear control, backstepping control and Takagi-Sugeno fuzzy approaches. *Complexity* 2016;21:116–24.
- [3] Matouk AE, Elsadany AA. Dynamical analysis, stabilization and discretization of a chaotic fractional-order GLV model. *Nonlinear Dyn* 2016;85:1597–612.
- [4] Laskin N. Fractional market dynamics. *Phys A* 2000;287:482–92.
- [5] Radwan AG, Shamim A, Salama KN. Theory of fractional order elements based impedance matching networks. *IEEE Microw Wirel Compon Lett* 2011. <https://doi.org/10.1109/LMWC.2010.2103051>.
- [6] Abdel Latif MS. Some exact solutions of KdV equation with variable coefficients. *Commun Nonlin Sci Numer Simulat* 2011;16(4):1783–6.
- [7] Radwan AG, Shamim A, Salama KN. Impedance matching through a single passive fractional element. *Antennas Propag Soc Int Symp* 2012.
- [8] Radwan AG. Resonance and quality factor of the RL alpha C alpha Fractional Circuit. *IEEE J Emerg Sel Top Circ Syst* 2013. <https://doi.org/10.1109/JETCAS.2013.2272838>.

- [9] Al-Khedhairi A, Matouk AE, Askar SS. Computations of synchronisation conditions in some fractional-order chaotic and hyperchaotic systems. *Pramana - J Phys* 2019;92(72):11 pages.
- [10] Ameen I, Novati P. The solution of fractional order epidemic model by implicit Adams methods. *Appl Math Model* 2017;43:78–84.
- [11] Radwan AG, Emira AA, AbdelAty AM, et al. Modeling and analysis of fractional order DC-DC converter. *ISA Trans* 2018. <https://doi.org/10.1016/j.isatra.2017.06.024>.
- [12] Abdel Latif MS, El-Shazly E, Baleanu D, Elsaid A, Nour HM. Some new soliton-like and doubly periodic-like solutions of Fisher equation with time-dependent coefficients. *Mod Phys Lett B* 2018;32(33):1850413.
- [13] Shamseldeen S. Approximate solution of space and time fractional higher order phase field equation. *Phys A* 2018;494:308–16.
- [14] Semary MS, Fouda ME, Hassan HN, Radwan AG. Realization of fractional-order capacitor based on passive symmetric network. *J Adv Res* 2019. <https://doi.org/10.1016/j.jare.2019.02.004>.
- [15] Ali HM, Ameen I. Mittag-Leffler approximation for the solution of nonlinear systems of fractional partial differential equations. *SYLWAN* 2019;163(9):17–32.
- [16] Riemann B. Versuch einer allgemeinen auffassung der integration und differentiation. In *Gesammelte mathematische Werke 1876* (H. Weber éd. avec le concours de R. Dedekind, 1847), Leipzig, p. 353–66..
- [17] Caputo M. Linear models of dissipation whose Q is almost frequency independent- II. *Geophys J R Astron Soc* 1967;13:529.
- [18] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. *Prog Fract Differ Appl* 2015;1:73.
- [19] Hegazi AS, Ahmed E, Matouk AE. On chaos control and synchronization of the commensurate fractional order Liu system. *Commun Nonlin Sci Numer Simulat* 2013;18:1193.
- [20] Matouk AE. Chaos, feedback control and synchronization of a fractional-order modified autonomous Van der Pol-Duffing circuit. *Commun Nonlin Sci Numer Simulat* 2011;16:975.
- [21] Petras I. Chaos in the fractional-order Volta's system: Modeling and simulation. *Nonlin Dyn* 2009;57:157–70.
- [22] El-Sayed AMA, Elsonbaty A, Elsadany AA, Matouk AE. Dynamical analysis and circuit simulation of a new fractional-order hyperchaotic system and its discretization Article ID 1650222. *Int J Bifurcat Chaos* 2016;26.
- [23] El-Sayed AMA, Nour HM, Elsaid A, Matouk AE, Elsonbaty A. Dynamical behaviors, circuit realization, chaos control and synchronization of a new fractional order hyperchaotic system. *Appl Math Model* 2016;40:3516.
- [24] Matouk AE. Stability conditions, hyperchaos and control in a novel fractional order hyperchaotic system. *Phys Lett A* 2009;373:2166.
- [25] Radwan AG, Moaddy K, Salama KN, Momani S, Hashim I. Control and switching synchronization of fractional order chaotic systems using active control technique. *J Adv Res* 2014;5:125.
- [26] Mahmoud GM, Ahmed ME, Abed-Elhameed TM. Active control technique of fractional-order chaotic complex systems. *Eur Phys J Plus* 2016;131:200. <https://doi.org/10.1140/epjp/i2016-16200-x>.
- [27] Mahmoud GM, Arafa AA, Abed-Elhameed TM, Mahmoud EE. Chaos control of integer and fractional orders of chaotic Burke-Shaw system using time delayed feedback control. *Chaos Solit Fract* 2017;104:680–92.
- [28] Matouk AE. Dynamical behaviors, linear feedback control and synchronization of the fractional order Liu system. *J Nonlinear Syst Appl* 2010;1(3):135–40.
- [29] Constantinescu D, Dumbrajs Q, Igochine V, Lackner K, Meyer-Spasche R, Zohm H. ASDEX Upgrade Team, A low-dimensional model system for quasi-periodic plasma perturbations. *Phys Plasmas* 2011;18(6):062307.
- [30] Constantinescu D, Dumbrajs Q, Igochine V, Lackner K, Zohm H. ASDEX Upgrade Team, Bifurcations and fast-slow dynamics in a low-dimensional model for quasi-periodic plasma perturbations. *Roman Rep Phys* 2015;67(3):1049–60.
- [31] Elsadany AA, Elsonbaty A, Agiza HN. Qualitative dynamical analysis of chaotic plasma perturbations model. *Commun Nonlin Sci Numer Simulat* 2018;59:409–23.
- [32] Matignon D. Stability results for fractional differential equations with applications to control processing. *Computational Engineering in System Application*, vol. 2. Lille: France; 1996. p. 963.
- [33] Ahmed E, Elgazzar AS. On fractional order differential equations model for nonlocal epidemics. *Phys A* 2007;379:607.
- [34] Zhang R, Tian G, Yang S, Cao H. Stability analysis of a class of fractional order nonlinear systems with order lying in $(0, 2)$. *ISA Trans* 2015;56:102.
- [35] Al-khedhairi A, Matouk AE, Khan I. Chaotic dynamics and chaos control for the fractional-order geomagnetic field model. *Chaos, Solit Fract* 2019;128:1–12.
- [36] Deng W, Li C, Lu J. Stability analysis of linear fractional differential system with multiple time delays. *Nonlinear Dyn* 2007;48:409.
- [37] Tavazoei MS, Haeri M. A proof for non existence of periodic solutions in time invariant fractional order systems. *Automatica* 2009;45(8):1886.
- [38] Wolf A, Swift JB, Swinney HL, Vastano JA. Determining Lyapunov exponents from a time series. *Phys D* 1985;16:285.