Heliyon 6 (2020) e04963

Contents lists available at ScienceDirect

Heliyon

journal homepage: www.cell.com/heliyon

Research article

On various Riesz-dual sequences for Schauder frames

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ARTICLE INFO

Keywords:
Mathematics
Markushevich basis
Riesz basis
Schauder frame
Riesz-dual sequence

ABSTRACT

In this paper, we introduce various definitions of R-duals, to be called R-duals of type I, II, which leads to a generalization of the duality principle in Banach spaces. A basic problem of interest in connection with the study of R-duals in Banach spaces is that of characterizing those R-duals which can essentially be regarded as M-basis. We give some conditions under which an R-dual sequence to be an M-basis for X.

1. Introduction

Duality principle [1] and the Wexler-Raz [2] biorthogonality relations play a fundamental role in analyzing Gabor systems. In [3], Casazza, Kutyniok, and Lammers raised the question of whether these results, which can be regarded as duality principles, can be generalized to abstract frame theory. They presented a general approach to derive duality principles in abstract frame theory in 2004. Recently, the various generalizations of duality principles have been proposed. For example, duality principle for g-frames in Hilbert spaces [4, 5, 6], the duality principle for p-frames [7], and various R-duals [8, 9]. In [10], the authors studied R-duals for the purpose of extending this to general sequences in arbitrary Banach spaces. This was referred to as an X_d-R-dual. If we would have general duality principles in Banach spaces, we could hope to get an abundance of new duality principles for shift-invariant subspaces of L^p by using the Banach frame theory.

In the current paper, we introduce certain variations of the R-duals (see Definitions 2.1, 2.2) and show that R-duals of type I, II cover the duality principle in Banach spaces. Then we characterize exactly the properties of the first sequence in terms of its R-dual sequence. For an R-dual sequence, a natural and important problem is that of determining when it is near to M-basis. We give some conditions under which an R-dual sequence to be an M-basis for X.

In the rest of this introduction, we state the key definitions and results from the literature concerning the frames and Riesz bases in Banach spaces. In Sect. 2 we introduce a modified version of the R-duals leads to a generalization of the duality principle that keeps all the attractive properties of the R-duals. In Sect. 3 we prove some properties of R-duals and we give some conditions under which an R-dual sequence to be an M-basis for X.

1.1. Review of Banach frames

Banach frames were introduced by Gröchenig [11] as a tool to express series expansions. An analysis of Banach frames in general Banach spaces appeared in [12, 13, 14]. In the following, after briefly recalling the basic definitions and notations of frames with respect to a certain sequence space X_d, the notion of a X_d-Riesz basis and a X_d^*-Riesz basis is introduced.

Definition 1.1. ([14]) Let X be separable Banach space and X^* be its dual space. Let X_d be a Banach space of scalar-valued sequences indexed by countable set I. Let $\{f_i\}_{i\in I}$ be a collection of vectors in the dual space X^* and $S : X_d \to X$ be given. The pair $(\{f_i\}_{i\in I}, S)$ is called a Banach frame for X w.r.t. X_d if

(i) $\{f_i(x)\}_{i\in I} \in X_d$, for all $x \in X$
(ii) the norms $\|x\|_X$ and $\|\{f_i(x)\}_{i\in I}\|_{X_d}$ are equivalent, i.e., there exist constants $0 < A \le B < \infty$ such that

$$A\|x\|_X \le \|\{f_i(x)\}_{i\in I}\|_{X_d} \le B\|x\|_X, \qquad \forall x \in X. \tag{1.1}$$

(iii) S is a bounded linear operator such that $S(\{f_i(x)\}) = x$ for all $x \in X$.

The positive constants A and B, respectively, are called the lower and the upper frame bounds of the Banach frame $(\{f_i\}_{i\in I}, S)$. If at least (i)

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[1] The authors' work was partially supported by the Central Tehran Branch of Islamic Azad University.

https://doi.org/10.1016/j.heliyon.2020.e04963
Received 10 January 2020; Received in revised form 9 August 2020; Accepted 14 September 2020

and the right-hand inequality in (1.1) are satisfied, $\{f_i\}_{i\in I}$ is called a X_d-Bessel sequence for X with Bessel bound B. The operator $S : X_d \to X$ is called the reconstruction operator (or, the pre-frame operator). The inequality (1.1) is called the frame inequality. The Banach frame $(\{f_i\}_{i\in I}, S)$ is called tight if $A = B$ and normalized tight if $A = B = 1$.

Definition 1.2. ([14]) Let X be Banach space and X^* be its dual space. Let X_d be a Banach space of scalar-valued sequences indexed by countable set I. Let $\{x_i\}_{i\in I}$ be a collection of vectors in X and $T : X_d^* \to X^*$ be given. The pair $(\{x_i\}_{i\in I}, T)$ is called a retro Banach frame for X^* w.r.t. X_d^* if

(i) $\{f(x_i)\}_{i\in I} \in X_d^*$, for all $f \in X^*$

(ii) the norms $\|f\|_{X^*}$ and $\|\{f(x_i)\}_{i\in I}\|_{X_d^*}$ are equivalent, i.e., there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|_{X^*} \le \|\{f(x_i)\}_{i\in I}\|_{X_d^*} \le B\|f\|_{X^*}, \qquad \forall x \in X. \tag{1.2}$$

(iii) T is a bounded linear operator such that $T(\{f(x_i)\}) = f$ for all $f \in X^*$.

The positive constants A and B, respectively, are called the lower and the upper frame bounds of the retro Banach frame $(\{x_i\}_{i\in I}, T)$. If at least (i) and the right-hand inequality in (1.2) are satisfied, $\{x_i\}_{i\in I}$ is called a X_d^*-Bessel sequence for X^* with Bessel bound B. The operator $T : X_d^* \to X^*$ is called the reconstruction operator (or, the pre-frame operator). The inequality (1.2) is called the retro frame inequality. The retro Banach frame $(\{x_i\}_{i\in I}, T)$ is called tight if $A = B$ and is called normalized tight if $A = B = 1$.

Definition 1.3. Let X be Banach space and X^* be its dual space. Let X_d be a Banach space of scalar-valued sequences indexed by countable set I. Let $u_i \in X$, $h_i \in X^*$ for all $i \in I$. Then

(i) $\{u_i\}_{i\in I}$ is called a X_d-Riesz basis for X, if it is complete in X and there exist constants $0 < A \le B < \infty$ such that

$$A\|\alpha\|_{X_d} \le \Big\| \sum_{i\in I} \alpha_i u_i \Big\|_X \le B\|\alpha\|_{X_d}, \quad \forall \alpha \in X_d. \tag{1.3}$$

(ii) $\{h_i\}_{i\in I}$ is called a X_d-Riesz basis for X^* if there exist constants $0 < A \le B < \infty$ such that

$$A\|\alpha\|_{X_d} \le \Big\| \sum_{i\in I} \alpha_i h_i \Big\|_{X^*} \le B\|\alpha\|_{X_d}, \quad \forall \alpha \in X_d. \tag{1.4}$$

The numbers A, B in (1.3) and (1.4) are called lower and upper X_d-Riesz basis bounds. If $\{u_i\}_{i\in I}$ or $\{h_i\}_{i\in I}$ are a X_d-Riesz basis only for its closed linear span in X or X^*, we call it a X_d-Riesz basic sequence in X or X^* respectively.

The X_d-Riesz bases are important in practice and are therefore studied widely by many authors, e.g., see [15, 16, 17, 18].

Definition 1.4. A Banach space X_d of scalar-valued sequences indexed by I is a BK-space if the coordinate linear functionals are continuous on X_d. A CB-space is a BK-space for which the canonical unit vectors constitute a Schauder basis. A BK-space is called an RCB-space if it is a reflexive CB-space.

By a result in [19], the dual space of a BK-space containing all canonical unit vectors is also a BK-space.

Definition 1.5. Let $x_i, u_i \in X$, $f_i, h_i \in X^*$ for all $i \in I$ and let X_d be a Banach space of scalar-valued sequences indexed by I. Then

(i) $\{(x_i, f_i)\}_{i\in I}$ is called a Bessel system for $X \times X^*$ w.r.t. X_d if $\{x_i\}_{i\in I}$ is a X_d^*-Bessel sequence for X^* and $\{f_i\}_{i\in I}$ is a X_d-Bessel sequence for X respectively.

(ii) $\{(x_i, f_i)\}_{i\in I}$ is called a frame system for $X \times X^*$ w.r.t. X_d when it satisfies only the frame inequality (1.1) and the retro frame inequality (1.2).

(iii) $\{(u_i, h_i)\}_{i\in I}$ is called a Riesz basis system for $X \times X^*$ w.r.t. X_d if $\{u_i\}_{i\in I}$ is a X_d-Riesz basis for X and $\{h_i\}_{i\in I}$ is a X_d^*-Riesz basis for X^* respectively. If only $\{u_i\}_{i\in I}$ is a X_d-Riesz basic sequence in X and $\{h_i\}_{i\in I}$ is a X_d^*-Riesz basic sequence in X^*, we call $\{(u_i, h_i)\}_{i\in I}$ a Riesz basic system for $X \times X^*$ w.r.t. X_d.

(iv) $\{(x_i, f_i)\}_{i\in I}$ with $x_i \neq 0$, $f_i \neq 0$ is called a Schauder frame for X if for every $x \in X$, $x = \sum_{i\in I} f_i(x)x_i$.

Definition 1.6. Let $x_i \in X$, $f_i \in X^*$. Then

(i) $\{(x_i, f_i)\}_{i\in I}$ is called a biorthogonal system for $X \times X^*$, if $f_i(x_j) = \delta_{ij}$ for all $i, j \in I$.

(ii) A biorthogonal system $\{(x_i, f_i)\}_{i\in I}$ is called fundamental if $X = \overline{\text{span}}\{x_i\}_{i\in I}$.

(iii) A biorthogonal system $\{(x_i, f_i)\}_{i\in I}$ is called total if $X^* = \overline{\text{span}}^{w^*}\{f_i\}_{i\in I}$.

(iv) A fundamental and total biorthogonal system $\{(x_i, f_i)\}_{i\in I}$ is called a Markushevich basis or M-basis for X.

Example 1.7. Let $X = X_d = c_0$ be the space of null sequences and $\{e_i\}_{i\in\mathbb{N}}$ be the standard basis of the unit vectors for c_0. Let $\{\lambda_i\}_{i\in\mathbb{N}}$ be a sequence of scalars such that $0 < A = \inf_{i\in\mathbb{N}} |\lambda_i| \le \sup_{i\in\mathbb{N}} |\lambda_i| = B < \infty$. For each $i \in \mathbb{N}$ define $u_i \in c_0$ by $u_i = \lambda_i e_i$. Then it is easily checked that $\{u_i\}_{i\in\mathbb{N}}$ is complete in X and

$$\Big\| \sum_{i\in\mathbb{N}} \alpha_i u_i \Big\|_{c_0} = \sup_{k\in\mathbb{N}} |\lambda_k \alpha_k| < \infty.$$

This yields

$$A\|\alpha\|_{X_d} \le \Big\| \sum_{i\in\mathbb{N}} \alpha_i u_i \Big\|_X \le B\|\alpha\|_{X_d}.$$

Thus $\{u_i\}_{i\in\mathbb{N}}$ is a c_0-Riesz basis for c_0 with bounds A and B.

Example 1.8. Let $X = X_d = c_0$ be the space of null sequences and let $\{\lambda_i\}_{i\in\mathbb{N}}$ be a sequence of scalars such that $0 < A = \inf_{i\in\mathbb{N}} |\lambda_i| \le \sup_{i\in\mathbb{N}} |\lambda_i| = B < \infty$. For each $i \in \mathbb{N}$ define $h_i \in X^* = \ell^1$ by $h_i(x) = \lambda_i x_i$, $(x \in X)$. Then it is easily checked that $h_i = \lambda_i e_i$, where $\{e_i\}_{i\in\mathbb{N}}$ is the standard basis of the unit vectors for ℓ^1. With a similar argument of Example 1.8, we can show that $\{h_i\}_{i\in\mathbb{N}}$ is a ℓ^1-Riesz basis for ℓ^1 with bounds A and B.

Example 1.9. Let $X = \ell^1$ and let $\{\tilde{e}_i\}_{i\in\mathbb{N}}$ be the sequence of the coefficient functionals associated to the canonical basis $\{e_i\}_{i\in\mathbb{N}}$ in X. Suppose that $\{\lambda_i\}_{i\in\mathbb{N}}$ is a sequence of scalars such that $\sum_{i\in\mathbb{N}} \sqrt{|\lambda_i|} < \infty$. For $n \in \mathbb{N}, y \in X$ define the following vectors $x_n \in X$ and $f_n \in X^*$ by

$$x_n = \begin{cases} \sqrt{|\lambda_i|}e_1 & n = 2i-1 \\ e_{i+1} & n = 2i. \end{cases} \quad \text{and} \quad f_n(y) = \begin{cases} \frac{\sqrt{|\lambda_i|}}{K}\tilde{e}_1(y) & n = 2i-1 \\ \tilde{e}_{i+1}(y) & n = 2i. \end{cases},$$

where $K = \sum_{i\in\mathbb{N}} |\lambda_i|$. Then we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} f_n(y)x_n &= \sum_{i\in\mathbb{N}} f_{2i-1}(y)x_{2i-1} + \sum_{i\in\mathbb{N}} f_{2i}(y)x_{2i} \\ &= \Big(\frac{1}{K}\sum_{i\in\mathbb{N}} |\lambda_i|\Big)\tilde{e}_1(y)e_1 + \sum_{i\in\mathbb{N}} \tilde{e}_{i+1}(y)e_{i+1} \\ &= \sum_{i\in\mathbb{N}} \tilde{e}_i(y)e_i = y.\end{aligned}$$

Therefore, $\{(x_n, f_n)\}_{n\in\mathbb{N}}$ is a Schauder frame for X.

2. Various Riesz-dual sequences and the duality principles

The notion of R-dual sequences was introduced and studied in [10] for the purpose of extending this to the general sequences in arbitrary Banach spaces.

Let $\{(u_i, v_i)\}_{i\in I}$ be a pair of X_d-Riesz bases for X, and let $\{f_i\}_{i\in I} \subset X^*$ be a X_d-Bessel sequence for X. Then a X_d^*-R-dual sequence of $\{f_i\}_{i\in I}$ with respect to $\{(u_i, v_i)\}_{i\in I}$ for X is a collection of vectors $\{\omega_i^f\}_{i\in I}$ in X which is defined by

$$\omega_i^f = \sum_{j\in I} f_j(u_i)v_j, \qquad \forall\, i \in I. \tag{2.1}$$

Similarly, given a pair of X_d^*-Riesz bases $\{(z_i, h_i)\}_{i\in I}$ for X^* and a X_d^*-Bessel sequence $\{x_i\}_{i\in I}$ for X^*. Then a X_d-R-dual sequence of $\{x_i\}_{i\in I}$ with respect to $\{(z_i, h_i)\}_{i\in I}$ is a collection of vectors $\{\psi_i^x\}_{i\in I}$ in X^* which is defined by

$$\psi_i^x = \sum_{j\in I} z_i(x_j)h_j, \qquad \forall\, i \in I. \tag{2.2}$$

In the following, we introduce two types of R-dual sequences that are available in the literature.

Definition 2.1. Let $\{(u_i, v_i)\}_{i\in I}$ be a pair of X_d-Riesz bases for X so that the biorthogonal sequences $\{\tilde{u}_i\}_{i\in I}, \{\tilde{v}_i\}_{i\in I} \subset X^*$ constitute X_d^*-Riesz bases for X^*. Suppose that $\{(x_i, f_i)\}_{i\in I}$ is a Bessel system for $X \times X^*$ w.r.t. X_d. Then a R-dual sequence of type I of $\{(x_i, f_i)\}_{i\in I}$ w.r.t. $\{(u_i, v_i)\}_{i\in I}$ is a collection of vectors $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$, where $\{\omega_i^f\}_{i\in I}$ is the X_d^*-R-dual sequence of $\{f_i\}_{i\in I}$ w.r.t. $\{(u_i, v_i)\}_{i\in I}$ and $\{\psi_i^x\}_{i\in I}$ is the X_d-R-dual sequence of $\{x_i\}_{i\in I}$ w.r.t. $\{(\tilde{u}_i, \tilde{v}_i)\}_{i\in I}$.

Definition 2.2. Let $\{(z_i, h_i)\}_{i\in I}$ be a pair of X_d^*-Riesz bases for X^* so that the biorthogonal sequences $\{\hat{z}_i\}_{i\in I}, \{\hat{h}_i\}_{i\in I} \subset X \subseteq X^{**}$ constitute X_d-Riesz bases for X. Suppose that $\{(x_i, f_i)\}_{i\in I}$ is a Bessel system for $X \times X^*$ w.r.t. X_d. Then a R-dual sequence of type II of $\{(x_i, f_i)\}_{i\in I}$ w.r.t. $\{(z_i, h_i)\}_{i\in I}$ is a collection of vectors $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$, where $\{\omega_i^f\}_{i\in I}$ is the X_d^*-R-dual sequence of $\{f_i\}_{i\in I}$ w.r.t. $\{(\hat{z}_i, \hat{h}_i)\}_{i\in I}$ and $\{\psi_i^x\}_{i\in I}$ is the X_d-R-dual sequence of $\{x_i\}_{i\in I}$ w.r.t. $\{(z_i, h_i)\}_{i\in I}$.

Example 2.3. Let $X = X_d = c_0$ and let $\{e_i\}_{i\in\mathbb{N}}$ be the standard basis of the canonical unit vectors in c_0. For each $i \in \mathbb{N}$ define the following vectors $u_i, v_i, x_i \in c_0$ and $f_i \in X^* = \ell^1$ by

$$u_i = \frac{i}{i+1}e_i, \quad v_i = \frac{i}{2i+1}e_i, \quad x_i = \frac{1}{2^i}e_1 + e_i, \quad f_i = \frac{1}{3i+1}(\tilde{e}_1 + i\tilde{e}_i),$$

where $\{\tilde{e}_i\}_{i\in\mathbb{N}} \subset (c_0)^* = \ell^1$ is the dual basic sequence of $\{e_i\}_{i\in\mathbb{N}}$. Then it is easily checked that $\{(u_i, v_i)\}_{i\in\mathbb{N}}$ is a pair of X_d-Riesz bases for X with bounds $A_u = \frac{1}{2}$, $B_u = 1$ and $A_v = \frac{1}{3}$, $B_v = \frac{1}{2}$ and $\{(x_i, f_i)\}_{i\in I}$ is a frame system for $X \times X^*$ w.r.t. X_d with frame bounds $A_x = \frac{1}{2}$, $B_x = 2$ and $A_f = \frac{1}{6}$, $B_f = \frac{7}{12}$, respectively. Moreover, for every $i \in \mathbb{N}$ we have

$$\begin{aligned}
\omega_1^f &= \sum_{j\in\mathbb{N}} f_j(u_1)v_j = \sum_{j\in\mathbb{N}} \frac{1}{2} f_j(e_1)v_j = \sum_{j\in\mathbb{N}} \frac{j}{2(2j+1)} f_j(e_1)e_j \\
&= \sum_{j\in\mathbb{N}} \frac{j}{2(2j+1)(3j+1)}\big(\tilde{e}_1(e_1) + j\tilde{e}_j(e_1)\big)e_j \\
&= \sum_{j\in\mathbb{N}} \frac{j}{2(2j+1)(3j+1)}\big(1 + j\delta_{1j}\big)e_j \\
&= \frac{e_1}{12} + \sum_{j=2}^{\infty} \frac{je_j}{2(2j+1)(3j+1)},
\end{aligned}$$

and for $i \ge 2$ we obtain

$$\begin{aligned}
\omega_i^f &= \sum_{j\in\mathbb{N}} f_j(u_i)v_j = \sum_{j\in\mathbb{N}} \frac{1}{3j+1}\big((u_i)_1 + j(u_i)_j\big)v_j = \sum_{j\in\mathbb{N}} \frac{j(u_i)_j}{3j+1}v_j \\
&= \sum_{j\in\mathbb{N}} \frac{ji\delta_{ij}}{(i+1)(3j+1)}v_j = \frac{i^3}{(i+1)(2i+1)(3i+1)}e_i.
\end{aligned}$$

We also have

$$\psi_1^x = \sum_{j\in\mathbb{N}} \tilde{u}_1(x_j)\tilde{v}_j = \sum_{j\in\mathbb{N}} 2\tilde{e}_1(x_j)\tilde{v}_j = \sum_{j\in\mathbb{N}} \frac{2j+1}{j2^{j-1}}\tilde{e}_j,$$

and

$$\begin{aligned}
\psi_i^x &= \sum_{j\in\mathbb{N}} \tilde{u}_i(x_j)\tilde{v}_j = \sum_{j\in\mathbb{N}} \frac{i+1}{i}\tilde{e}_i(x_j)\tilde{v}_j = \sum_{j\in\mathbb{N}} \frac{(i+1)(2j+1)}{ij}\tilde{e}_i(x_j)\tilde{e}_j \\
&= \sum_{j\in\mathbb{N}} \frac{(i+1)(2j+1)}{ij}\delta_{ij}\tilde{e}_j = \frac{(i+1)(2i+1)}{i^2}\tilde{e}_i, \qquad i \ge 2.
\end{aligned}$$

Therefore, $\{(\omega_i^f, \psi_i^x)\}_{i\in\mathbb{N}}$ is a R-dual of type I of $\{(x_i, f_i)\}_{i\in\mathbb{N}}$ w.r.t. $\{(u_i, v_i)\}_{i\in\mathbb{N}}$.

Example 2.4. Let $X = X_d = \ell^1$ and let $\{e_i\}_{i\in\mathbb{N}}$ be the standard basis of the canonical unit vectors in ℓ^1. For each $i \in \mathbb{N}$ define the following vectors $x_i \in \ell^1$ and $z_i, h_i, f_i \in X^* = \ell^\infty$ by

$$z_i = \frac{i}{i+2}\tilde{e}_i, \quad h_i = \frac{2i}{i+2}\tilde{e}_i, \quad x_i = e_1 + e_i, \quad f_i = \frac{2}{3^i}\tilde{e}_1 + \tilde{e}_i,$$

where $\{\tilde{e}_i\}_{i\in\mathbb{N}} \subset (\ell^1)^* = \ell^\infty$ is the dual basic sequence of $\{e_i\}_{i\in\mathbb{N}}$. Then it is easily checked that $\{(z_i, h_i)\}_{i\in\mathbb{N}}$ is a pair of X_d^*-Riesz bases for X^* with bounds $A_z = \frac{1}{3}$, $B_z = 1$ and $A_h = \frac{2}{3}$, $B_h = 2$ and $\{(x_i, f_i)\}_{i\in I}$ is a frame system for $X \times X^*$ w.r.t. X_d with frame bounds $A_x = \frac{1}{2}$, $B_x = 2$ and $A_f = \frac{1}{2}$, $B_f = 2$, respectively. Moreover, for every $i \in \mathbb{N}$ we have

$$\begin{aligned}
\omega_1^f &= \sum_{j\in\mathbb{N}} f_j(\hat{z}_1)\hat{h}_j = \sum_{j\in\mathbb{N}} 3f_j(e_1)\hat{z}_j = \sum_{j\in\mathbb{N}} \frac{3(j+2)}{2j} f_j(e_1)e_j \\
&= \sum_{j\in\mathbb{N}} \frac{3(j+2)}{2j}\Big(\frac{2}{3^j}\tilde{e}_1(e_1) + \tilde{e}_j(e_1)\Big)e_j \\
&= \frac{15}{2}e_1 + \sum_{j=2}^{\infty} \frac{j+2}{j3^{j-1}}e_j,
\end{aligned}$$

and for $i \ge 2$ we obtain

$$\begin{aligned}
\omega_i^f &= \sum_{j\in\mathbb{N}} f_j(\hat{z}_i)\hat{h}_j = \sum_{j\in\mathbb{N}} \frac{i+2}{i} f_j(e_i)\hat{h}_j \\
&= \sum_{j\in\mathbb{N}} \frac{(i+2)(j+2)}{2ij}\Big(\frac{2}{3^j}\tilde{e}_1(e_i) + \tilde{e}_j(e_i)\Big)e_j \\
&= \sum_{j\in\mathbb{N}} \frac{(i+2)(j+2)}{2ij}\delta_{ij}e_j = \frac{(i+2)^2}{2i^2}e_i.
\end{aligned}$$

We also have

$$\begin{aligned}
\psi_1^x &= \sum_{j\in\mathbb{N}} z_1(x_j)h_j = \sum_{j\in\mathbb{N}} \frac{1}{3}\tilde{e}_1(x_j)h_j \\
&= \sum_{j\in\mathbb{N}} \frac{2j}{3(j+2)}\big(\tilde{e}_1(e_1) + \tilde{e}_1(e_j)\big)\tilde{e}_j \\
&= \frac{4}{9}\tilde{e}_1 + \sum_{j=2}^{\infty} \frac{2j}{3(j+2)}\tilde{e}_j,
\end{aligned}$$

and

$$\begin{aligned}
\psi_i^x &= \sum_{j\in\mathbb{N}} z_i(x_j)h_j = \sum_{j\in\mathbb{N}} \frac{i}{i+2}\tilde{e}_i(x_j)h_j = \sum_{j\in\mathbb{N}} \frac{2ij}{(i+2)(j+2)}\tilde{e}_i(x_j)\tilde{e}_j \\
&= \sum_{j\in\mathbb{N}} \frac{2ij}{(i+2)(j+2)}\delta_{ij}\tilde{e}_j = \frac{2i^2}{(i+2)^2}\tilde{e}_i, \qquad i \ge 2.
\end{aligned}$$

Therefore, $\{(\omega_i^f, \psi_i^x)\}_{i\in\mathbb{N}}$ is a R-dual of type II of $\{(x_i, f_i)\}_{i\in\mathbb{N}}$ w.r.t. $\{(z_i, h_i)\}_{i\in\mathbb{N}}$.

To provide an algorithm for the purpose to reverse these processes, we present the following result that is a slight variation of [10, Theorems 4.3, 4.4].

Proposition 2.5. *Let X_d be a RCB-space. Then the following hold:*

(*i*) $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *is the R-dual of type I of* $\{(x_i, f_i)\}_{i\in I}$ *w.r.t.* $\{(u_i, v_i)\}_{i\in I}$ *if and only if* $\{(x_i, f_i)\}_{i\in I}$ *is the R-dual of type I of* $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *w.r.t.* $\{(v_i, u_i)\}_{i\in I}$.

(*ii*) $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *is the R-dual of type II of* $\{(x_i, f_i)\}_{i\in I}$ *w.r.t.* $\{(z_i, h_i)\}_{i\in I}$ *if and only if* $\{(x_i, f_i)\}_{i\in I}$ *is the R-dual of type II of* $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *w.r.t.* $\{(h_i, z_i)\}_{i\in I}$.

Proof. (*i*) By [10, Theorems 4.3, 4.4], $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ is the R-dual of type I of $\{(x_i, f_i)\}_{i\in I}$ w.r.t. $\{(u_i, v_i)\}_{i\in I}$ if and only if

$$x_i = \sum_{j\in I} \psi_j^x(v_i)u_j, \qquad \text{and} \qquad f_i = \sum_{j\in I} \tilde{v}_i(\omega_j^f)\tilde{u}_j,$$

for all $i \in I$. Hence $\{(x_i, f_i)\}_{i\in I}$ is the R-dual of type I of $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ w.r.t. $\{(v_i, u_i)\}_{i\in I}$.

(*ii*) Again, $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ is the R-dual of type II of $\{(x_i, f_i)\}_{i\in I}$ w.r.t. $\{(z_i, h_i)\}_{i\in I}$ if and only if for any $i \in I$

$$f_i = \sum_{j\in I} h_i(\omega_j^f)z_j, \qquad \text{and} \qquad x_i = \sum_{j\in I} \psi_j^x(\hat{h}_i)\hat{z}_j.$$

Therefore $\{(x_i, f_i)\}_{i\in I}$ is the R-dual of type II of $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ w.r.t. $\{(h_i, z_i)\}_{i\in I}$. □

In order to provide the frame properties and the duality principle for the R-dual of type I, we present the following result that is a slight variation of [10, Theorems 4.5, 4.6].

Proposition 2.6. *Let X_d be a RCB-space and let $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ be the R-dual of type I of $\{(x_i, f_i)\}_{i\in I}$ w.r.t. $\{(u_i, v_i)\}_{i\in I}$. Then the following statements hold.*

(*i*) *for any $\beta \in X_d^*$ with $g = \sum_{j\in I} \beta_j \tilde{u}_j$*

$$B^{-1}\|\{g(x_i)\}_{i\in I}\|_{X_d^*} \le \Big\| \sum_{j\in I} \beta_j \psi_j^x \Big\|_{X^*} \le A^{-1}\|\{g(x_i)\}_{i\in I}\|_{X_d^*},$$

(*ii*) *for any $\alpha \in X_d$ with $y = \sum_{j\in I} \alpha_j u_j$*

$$A\|\{f_i(y)\}_{i\in I}\|_{X_d} \le \Big\| \sum_{j\in I} \alpha_j \omega_j^f \Big\|_X \le B\|\{f_i(y)\}_{i\in I}\|_{X_d},$$

where A, B are the X_d-Riesz basis bounds for $\{v_i\}_{i\in I}$.

(*iii*) *for any $\beta \in X_d^*$ with $g = \sum_{j\in I} \beta_j \tilde{v}_j$*

$$D^{-1}\|\{g(\omega_i^f)\}_{i\in I}\|_{X_d^*} \le \Big\| \sum_{j\in I} \beta_j f_j \Big\|_{X^*} \le C^{-1}\|\{g(\omega_i^f)\}_{i\in I}\|_{X_d^*},$$

(*iv*) *for any $\alpha \in X_d$ with $y = \sum_{j\in I} \alpha_j v_j$*

$$C\|\{\psi_i^x(y)\}_{i\in I}\|_{X_d} \le \Big\| \sum_{j\in I} \alpha_j x_j \Big\|_X \le D\|\{\psi_i^x(y)\}_{i\in I}\|_{X_d},$$

where C, D are the X_d-Riesz basis bounds for $\{u_i\}_{i\in I}$.

Proof. (*i*) By the definition of ψ_j^x, we have $\psi_j^x = T_{\tilde{v}}(\{\tilde{u}_j(x_i)\}_{i\in I})$, where $T_{\tilde{v}}$ is the synthesis operator of $\{\tilde{v}_i\}_{i\in I}$ and by [20, Proposition 3.4], $T_{\tilde{v}}$ is an isomorphism of X_d^* onto X^*. Hence

$$\sum_{j\in I} \beta_j \psi_j^x = \sum_{j\in I} \beta_j T_{\tilde{v}}(\{\tilde{u}_j(x_i)\}_{i\in I}) = T_{\tilde{v}}(\{\sum_{j\in I} \beta_j \tilde{u}_j(x_i)\}_{i\in I})$$
$$= T_{\tilde{v}}(\{g(x_i)\}_{i\in I}) = \sum_{j\in I} g(x_j)\tilde{v}_j.$$

Now, the conclusion follows from [20, Proposition 4.9].

(*ii*) Similarly, the definition of ω_j^f implies that $\sum_{j\in I} \alpha_j \omega_j^f = \sum_{j\in I} f_j(y)v_j$. From this the result follows by the equation (1.4).

(*iii*), (*iv*) These are a consequence of Proposition 2.5 and [20, Proposition 4.9]. □

From the definitions, we immediately see that R-dual of type II has a similar characterization. The following are immediate consequences. We leave the proofs to interested readers.

Proposition 2.7. *Let X_d be a RCB-space and let $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ be the R-dual of type II of $\{(x_i, f_i)\}_{i\in I}$ w.r.t. $\{(z_i, h_i)\}_{i\in I}$. Then the following statements hold.*

(*i*) *for any $\beta \in X_d^*$ with $g = \sum_{j\in I} \beta_j z_j$*

$$A\|\{g(x_i)\}_{i\in I}\|_{X_d^*} \le \Big\| \sum_{j\in I} \beta_j \psi_j^x \Big\|_{X^*} \le B\|\{g(x_i)\}_{i\in I}\|_{X_d^*},$$

(*ii*) *for any $\alpha \in X_d$ with $y = \sum_{j\in I} \alpha_j \hat{z}_j$*

$$B^{-1}\|\{f_i(y)\}_{i\in I}\|_{X_d} \le \Big\| \sum_{j\in I} \alpha_j \omega_j^f \Big\|_X \le A^{-1}\|\{f_i(y)\}_{i\in I}\|_{X_d},$$

where A, B are the X_d^*-Riesz basis bounds for $\{h_i\}_{i\in I}$.

(*iii*) *for any $\beta \in X_d^*$ with $g = \sum_{j\in I} \beta_j h_j$*

$$C\|\{g(\omega_i^f)\}_{i\in I}\|_{X_d^*} \le \Big\| \sum_{j\in I} \beta_j f_j \Big\|_{X^*} \le D\|\{g(\omega_i^f)\}_{i\in I}\|_{X_d^*},$$

(*iv*) *for any $\alpha \in X_d$ with $y = \sum_{j\in I} \alpha_j \hat{h}_j$*

$$D^{-1}\|\{\psi_i^x(y)\}_{i\in I}\|_{X_d} \le \Big\| \sum_{j\in I} \alpha_j x_j \Big\|_X \le C^{-1}\|\{\psi_i^x(y)\}_{i\in I}\|_{X_d},$$

where C, D are the X_d^*-Riesz basis bounds for $\{z_i\}_{i\in I}$.

The next results show a kind of equilibrium between a sequence and its R-dual sequence. These can be viewed as a general version of the duality principle.

Corollary 2.8. *Let X_d be a RCB-space and let $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ be the R-dual of type I of $\{(x_i, f_i)\}_{i\in I}$ w.r.t. $\{(u_i, v_i)\}_{i\in I}$. Then the following statements hold.*

(*i*) $\{(x_i, f_i)\}_{i\in I}$ *is a frame system for $X \times X^*$ w.r.t. X_d if and only if* $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *is a Riesz basic system for $X \times X^*$ w.r.t. X_d.*

(*ii*) $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *is a frame system for $X \times X^*$ w.r.t. X_d if and only if* $\{(x_i, f_i)\}_{i\in I}$ *is a Riesz basic system for $X \times X^*$ w.r.t. X_d.*

Proof. The proof follows immediately from Proposition 2.6. □

A similar result holds for the R-dual of type II.

Corollary 2.9. *Let X_d be a RCB-space and let $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ be the R-dual of type II of $\{(x_i, f_i)\}_{i\in I}$ w.r.t. $\{(z_i, h_i)\}_{i\in I}$. Then the following statements hold.*

(*i*) $\{(x_i, f_i)\}_{i\in I}$ *is a frame system for $X \times X^*$ w.r.t. X_d if and only if* $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *is a Riesz basic system for $X \times X^*$ w.r.t. X_d.*

(*ii*) $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *is a frame system for $X \times X^*$ w.r.t. X_d if and only if* $\{(x_i, f_i)\}_{i\in I}$ *is a Riesz basic system for $X \times X^*$ w.r.t. X_d.*

Proof. The proof follows immediately from Proposition 2.7. □

3. Duality properties for Riesz-dual sequences

In this section, we study some properties for Riesz-dual sequences associated to Schauder frames. The first result is a slight variation of [10, Theorems 4.17]. Throughout this section X_d is an RCB-space

Proposition 3.1. *Let $\{(\omega_i^f,\psi_i^x)\}_{i\in I}$ be the R-dual of type I or II of $\{(x_i,f_i)\}_{i\in I}$. Then the following statements hold:*

(*i*) *$\{(x_i,f_i)\}_{i\in I}$ is a Schauder frame for X, if and only if $\{(\omega_i^f,\psi_i^x)\}_{i\in I}$ is a biorthogonal system for X.*
(*ii*) *$\{(\omega_i^f,\psi_i^x)\}_{i\in I}$ is a Schauder frame for X, if and only if $\{(x_i,f_i)\}_{i\in I}$ is a biorthogonal system for X.*

Proposition 3.2. *Let $\{(\omega_i^f,\psi_i^x)\}_{i\in I}$ and $\{(\omega_i^{Q^*f},\psi_i^{Q^{-1}x})\}_{i\in I}$ be the R-duals of type I or II of $\{(x_i,f_i)\}_{i\in I}$ and $\{(Q^{-1}(x_i),Q^*(f_i))\}_{i\in I}$, respectively. Suppose that $Q:X\to X$ is an invertible operator on X. Then the following statements hold:*

(*i*) *$\{(\omega_i^f,\psi_i^x)\}_{i\in I}$ is a Schauder frame for X, if and only if $\{(\omega_i^{Q^*f},\psi_i^{Q^{-1}x})\}_{i\in I}$ is a Schauder frame for X.*
(*ii*) *$\{(\omega_i^f,\psi_i^x)\}_{i\in I}$ is a biorthogonal system for X, if and only if $\{(\omega_i^{Q^*f},\psi_i^{Q^{-1}x})\}_{i\in I}$ is a biorthogonal system for X.*

Proof. This claim follows immediately from the fact that for each $i,j\in I$ we have

$$Q^*(f_i)\big(Q^{-1}(x_j)\big)=f_i\big(QQ^{-1}(x_j)\big)=f_i(x_j).$$

From this the result follows at once by Proposition 3.1. □

Definition 3.3. ([19]) A biorthogonal system $\{(x_i,f_i)\}_{i\in I}$ for X is called regular if the sequence $\{x_i\}_{i\in I}$ is a Schauder basis of the space X, otherwise $\{(x_i,f_i)\}_{i\in I}$ is said to be irregular.

To check the regularity of a biorthogonal system, we derive the following useful characterization.

Proposition 3.4. *Let X be Banach space and X^* be its dual space. Let $x_i\in X$, $f_i\in X^*$ with $x_i\neq 0$, $f_i\neq 0$ for all $i\in I$. Let $\{(x_i,f_i)\}_{i\in I}$ be a biorthogonal system for X. Then the following conditions are equivalent.*

(1) *$\{(x_i,f_i)\}_{i\in I}$ is regular.*
(2) *$\{(x_i,f_i)\}_{i\in I}$ is a Schauder frame for X.*

Proof. The implication $(1)\Rightarrow(2)$ is obvious. To prove $(2)\Rightarrow(1)$ suppose that $\{(x_i,f_i)\}_{i\in I}$ is a Schauder frame for X. If $\sum_{i\in I}c_ix_i=0$ with $c_i\in\mathbb{C}$, then by biorthogonality of $\{(x_i,f_i)\}$ we have $c_i=0$ for all $i\in I$ and so $\{x_i\}$ is a Schauder basis for X. Thus $\{(x_i,f_i)\}_{i\in I}$ is regular. □

Proposition 3.5. *Let $\{(\omega_i^f,\psi_i^x)\}_{i\in\mathbb{Z}}$ be the R-dual of type I or II of the regular biorthogonal system $\{(x_i,f_i)\}_{i\in\mathbb{Z}}$. Then $\{(X_i,F_i)\}_{i\in\mathbb{Z}}$ defined by*

$$X_i=\begin{cases}(x_k,0) & i=2k-1\\ (0,\omega_k^f) & i=2k,\end{cases}\quad\text{and}\quad F_i(s,t)=\begin{cases}f_k(s) & i=2k-1\\ \psi_k^x(t) & i=2k,\end{cases}\qquad \forall s,t\in X,$$

is a regular biorthogonal system for $X\times X$.

Proof. Since $\{(x_i,f_i)\}_{i\in\mathbb{Z}}$ is a regular biorthogonal system for X. By Propositions 3.1 and 3.4, $\{(\omega_i^f,\psi_i^x)\}_{i\in\mathbb{Z}}$ is a regular biorthogonal system for X. Thus for each $s,t\in X$ we have

$$\begin{aligned}\sum_{i\in\mathbb{Z}}F_i(s,t)X_i&=\sum_{k\in\mathbb{Z}}F_{2k-1}(s,t)X_{2k-1}+\sum_{k\in\mathbb{Z}}F_{2k}(s,t)X_{2k}\\&=\sum_{k\in\mathbb{Z}}f_k(s)(x_k,0)+\sum_{k\in\mathbb{Z}}\psi_k^x(t)(0,\omega_k^f)\\&=\Big(\sum_{k\in\mathbb{Z}}f_k(s)x_k,\sum_{k\in\mathbb{Z}}\psi_k^x(t)\omega_k^f\Big)=(s,t),\end{aligned}$$

which implies that $\{(X_n,F_n)\}_{n\in\mathbb{Z}}$ is a Schauder frame for $X\times X$. Obviously the condition $F_i(X_j)=\delta_{ij}$ for all $i,j\in\mathbb{Z}$ is satisfied. Therefore, $\{(X_i,F_i)\}_{i\in\mathbb{Z}}$ is a regular biorthogonal system for $X\times X$. □

Recall that the annihilators $M^\perp$ and $^\perp N$ from the subsets $M\subset X$, $N\subset X^*$ are defined as follows:

$$M^\perp=\big\{f\in X^*:\ f(x)=0\ \text{for all}\ x\in M\big\}$$
$$^\perp N=\big\{x\in X:\ f(x)=0\ \text{for all}\ f\in N\big\}.$$

Theorem 3.6. *Let $\{(x_i,f_i)\}_{i\in I}$ be a Schauder frame for X and let $\{(\omega_i^f,\psi_i^x)\}_{i\in I}$ be the R-dual of type I or II of $\{(x_i,f_i)\}_{i\in I}$. Then for any nonempty finite subset $J\subset I$*

(*i*) $X=\operatorname{span}\{\omega_j^f\}_{j\in J}\oplus{}^\perp\{\psi_j^x\}_{j\in J}$.
(*ii*) $^\perp\{\psi_j^x\}_{j\in J^c}=\operatorname{span}\{\omega_j^f\}_{j\in J}\oplus{}^\perp\{\psi_i^x\}_{i\in I}$.

Proof. Using Proposition 3.1 $\psi_i^x(\omega_j^f)=\delta_{ij}$, for all $i,j\in I$. Thus, if $y\in X$, then

$$y-\sum_{j\in J}\psi_j^x(y)\omega_j^f\in{}^\perp\{\psi_k^x\}_{k\in J}.$$

This immediately implies $X=\operatorname{span}\{\omega_j^f\}_{j\in J}+{}^\perp\{\psi_j^x\}_{j\in J}$. Also, if

$$y\in{}^\perp\{\psi_j^x\}_{j\in J}\cap\operatorname{span}\{\omega_j^f\}_{j\in J},$$

then $y=\sum_{j\in J}\psi_j^x(y)\omega_j^f=0$, hence (i) follows. To prove (ii) suppose that $y\in{}^\perp\{\psi_j^x\}_{j\in J^c}$. Then $y-\sum_{j\in J}\psi_j^x(y)\omega_j^f\in{}^\perp\{\psi_i^x\}_{i\in I}$. This yields

$$^\perp\{\psi_j^x\}_{j\in J^c}\subseteq\operatorname{span}\{\omega_j^f\}_{j\in J}+{}^\perp\{\psi_i^x\}_{i\in I}\subseteq{}^\perp\{\psi_j^x\}_{j\in J^c},$$

which implies that $^\perp\{\psi_j^x\}_{j\in J^c}=\operatorname{span}\{\omega_j^f\}_{j\in J}+{}^\perp\{\psi_i^x\}_{i\in I}$. Since we have

$$^\perp\{\psi_i^x\}_{i\in I}\cap\operatorname{span}\{\omega_j^f\}_{j\in J}=\{0\},$$

hence (ii) follows. □

Theorem 3.7. *Let $\{(x_i,f_i)\}_{i\in I}$ be a Schauder frame for X and let $\{(\omega_i^f,\psi_i^x)\}_{i\in I}$ be the R-dual of type I or II of $\{(x_i,f_i)\}_{i\in I}$. Then the following are equivalent:*

(*i*) *$\{\omega_i^f\}_{i\in I}$ is complete in X.*
(*ii*) *There exists a nonempty finite subset $J\subset I$ such that*

$$\{\omega_j^f\}_{j\in J^c}^\perp=\operatorname{span}\{\psi_j^x\}_{j\in J}.$$

(*iii*) *There exists a nonempty finite subset $J\subset I$ such that*

$$X=\operatorname{span}\{\omega_j^f\}_{j\in J}\oplus\overline{\operatorname{span}}\{\omega_j^f\}_{j\in J^c}.$$

Moreover, if (i) holds, then (ii) and (iii) hold for every nonempty finite subset $J\subset I$.

Proof. $(i)\Rightarrow(ii)$ Let $J\subset I$ be an arbitrary nonempty finite subset. By Proposition 3.1, for all $i,j\in I$, we have $\psi_i^x(\omega_j^f)=\delta_{ij}$, which implies

$$\operatorname{span}\{\psi_j^x\}_{j\in J}\subseteq\{\omega_j^f\}_{j\in J^c}^\perp.$$

For the opposite subset, we first show that $\{\omega_j^f\}_{j\in J}^\perp\cap\{\omega_j^f\}_{j\in J^c}^\perp=\{0\}$. To this end, let $f\in\{\omega_j^f\}_{j\in J}^\perp\cap\{\omega_j^f\}_{j\in J^c}^\perp$. Then we have $f(\omega_i^f)=0$, for all $i\in I$. Since $X=\overline{\operatorname{span}}\{\omega_i^f\}_{i\in I}$, it follows that $f=0$. Now, using Theorem 3.6 (i), we have $X^*=\operatorname{span}\{\psi_j^x\}_{j\in J}\oplus\{\omega_j^f\}_{j\in J}^\perp$, which implies that $\{\omega_j^f\}_{j\in J^c}^\perp\subseteq\operatorname{span}\{\psi_j^x\}_{j\in J}$, so (ii) follows.

$(ii)\Rightarrow(iii)$ If (ii) is satisfied, then $^\perp\big(\{\omega_j^f\}_{j\in J^c}^\perp\big)={}^\perp\big(\operatorname{span}\{\psi_j^x\}_{j\in J}\big)$. This immediately implies $\overline{\operatorname{span}}\{\omega_j^f\}_{j\in J^c}={}^\perp\{\psi_j^x\}_{j\in J}$. Now (iii) follows immediately from Theorem 3.6(i).

$(iii)\Rightarrow(i)$ is obvious.

For the moreover part, $(i)\Rightarrow(ii)$ holds for every nonempty finite subset J and (ii) for the same J implies (iii). Thus last statement holds. □

Theorem 3.8. *Let* $\{(x_i, f_i)\}_{i\in I}$ *be a Schauder frame for* X *and let* $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *be the R-dual of type I or II of* $\{(x_i, f_i)\}_{i\in I}$*. Suppose that* $\bigcap_{j\in J}\sigma_j = \emptyset$ *and* $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *is a M-basis for* X*. Then*

$$\bigcap_{j\in J}\overline{\operatorname{span}}\{\omega_i^f : i\in\sigma_j\} = \{0\}.$$

Proof. Let $y \in \bigcap_{j\in J}\overline{\operatorname{span}}\{\omega_i^f : i\in\sigma_j\}$. Choose an arbitrary $i_0 \in I$, then there exists $k \in J$ such that $i_0 \notin \sigma_k$ and $y \in \overline{\operatorname{span}}\{\omega_i^f : i \in \sigma_k\}$. Since $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ is a biorthogonal system for $X \times X^*$ by Proposition 3.1, we get $\psi_{i_0}^x(y) = 0$. This happens for every $i_0 \in I$. As $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ is a M-basis for X, then we have $y = 0$. □

Recall that a sequence $\{f_j\}_{j\in I}$ in X^* is said to be ω-independent w.r.t. X_d^*, if whenever the series $\sum_{j\in I} d_j f_j$ converges and equal to zero for some scalar coefficients $d \in X_d^*$ implies $d = 0$. The following result presents some conditions on a R-dual sequence to be a M-basis for X.

Theorem 3.9. *Let* $\{(x_i, f_i)\}_{i\in I}$ *be a Schauder frame for* X *and let* $\{f_i\}_{i\in I}$ *be* ω*-independent w.r.t.* X_d^**. Further, let* $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *be the R-dual of type I or II of* $\{(x_i, f_i)\}_{i\in I}$*. Suppose that* $\{\sigma_j\}_{j\in J}$ *is a family of subsets of* I *so that* $\tau_j = I \setminus \sigma_j$ *is finite for all* $j \in J$ *and* $\bigcap_{j\in J}\overline{\operatorname{span}}\{\omega_i^f : i\in\sigma_j\} = \{0\}$*. Then* $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ *is an M-basis for* X*.*

Proof. Using Proposition 3.1 and Theorems 4.7 in [10] $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ is a fundamental biorthogonal system for $X \times X^*$. Choose an arbitrary nonzero element $x \in X$, then there exists $j \in J$ such that $x \notin \overline{\operatorname{span}}\{\omega_i^f : i \in \sigma_j\}$. Now, using the part (iii) obtained in Theorem 3.7 we have $X = \operatorname{span}\{\omega_i^f : i\in\tau_j\} \oplus \overline{\operatorname{span}}\{\omega_i^f : i \in\sigma_j\}$. Thus we can write $x = y + z$, where $0 \neq y \in \operatorname{span}\{\omega_i^f : i\in\tau_j\}$ and $z \in \overline{\operatorname{span}}\{\omega_i^f : i\in\sigma_j\}$. So, we can find $k \in \tau_j$ such that $\psi_k^x(x) = \psi_k^x(y) \neq 0$. Hence $\{(\omega_i^f, \psi_i^x)\}_{i\in I}$ is an M-basis for X. □

Declarations

Author contribution statement

A. R. Neisi, M. S. Asgari: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Acknowledgements

The authors' work was partially supported by the Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University.

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