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Entanglement distribution in multi-particle systems in terms of unified entropy

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We investigate the entanglement distribution in multi-particle systems in terms of unified (q, s)-entropy. We find that for any tripartite mixed state, the unified (q, s)-entropy entanglement of assistance follows a polygamy relation. This polygamy relation also holds in multi-particle systems. Furthermore, a generalized monogamy relation is provided for unified (q, s)-entropy entanglement in the multi-qubit system.

Quantum entanglement is an important resource in quantum information theory. Different from classical correlations, this restricted shareability of entanglement in multi-particle systems is known as monogamy property. The more entanglement shared between two parties implies the less entanglement shared with the rest of the system. Monogamy property plays a crucial role in quantum cryptography: which restricts the quantity of information captured by an eavesdropper about the secret key to be extracted[1–3]. Monogamy property has also been discussed in the device-independent quantum information processing[4], condensed matter physics[5] and black-hole physics[6,7].

The study of monogamy property has a long history. The first monogamy relation was found by Coffman *et al.*, who considered a three-qubit system ABC[8], and showed that the amount of entanglement (which is quantified by the squared concurrence) between A and B, plus the amount of entanglement between A and C, cannot be greater than the amount of entanglement between A and the pair BC. Further, Osborne *et al.* proved the squared concurrence follows a general monogamy inequality for the N-qubit system[1]. Monogamy inequalities for different entanglement measures have been noted, such as concurrence[9–12], entanglement of formation[13,14], negativity[15–19], Rényi entropy entanglement[20,21], and Tsallis entropy entanglement[22–24]. For the other physical resources, such as discord and steering, the monogamy property of them has also been discussed[25–28].

As dual to monogamy property, polygamy property in multi-particle systems has arised many interests by researchers[15,19,22,29,30]. Polygamy property was first provided by using the concurrence of assistance to quantify the distributed bipartite entanglement in multi-qubit systems[29,30]. Polygamy property has also considered in many entanglement measures, such as Rényi entropy[20], Tsallis entropy[22,31] and convex-roof extended negativity[19].

Unified (q, s)-entropy is an important entropic measure, which can be used in many areas of quantum information theory. In this paper, we investigate the entanglement distribution in multi-particle systems in terms of unified (q, s)-entropy. We find that for any tripartite mixed state, the unified (q, s)-entropy entanglement of assistance follows a polygamy relation. This polygamy relation also holds in multi-particle systems. Furthermore, a generalized monogamy relation is provided for unified (q, s)-entropy entanglement in the multi-qubit system.

Results

This paper is organized as follows. In the first subsection, we recall the definition of unified (q, s)-entropy and discuss the properties of unified (q, s)-entropy entanglement. In the second subsection, we give our main results. We summarize our results in the third subsection.

Unified (q, s)-entropy entanglement and unified (q, s)-entropy entanglement of assistance. Given a quantum state ρ in the Hilbert space $\mathcal{H}$. The unified (q, s)-entropy is defined as[32]

$$S_{q,s}(\rho) = \frac{1}{(1-q)s}[Tr(\rho^q)^s - 1] \tag{1}$$

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for any $q, s \geq 0$ such that $q \neq 1$ and $s \neq 0$.

When s tends to 1, the unified (q, s)-entropy converges to Tsallis entropy $T_q(\rho)$[33]

$$\lim_{s \to 1} S_{q,s}(\rho) = \frac{1}{1-q}[Tr(\rho^q) - 1]. \tag{2}$$

When s tends to 0, the unified (q, s)-entropy converges to Rényi entropy $R_q(\rho)$[34]

$$\lim_{s \to 0} S_{q,s}(\rho) = \frac{1}{1-q} \ln Tr(\rho^q). \tag{3}$$

When q tends to 1, the unified (q, s)-entropy converges to von Neumann entropy $S(\rho)$[35]

$$\lim_{q \to 1} S_{q,s}(\rho) = -Tr\rho \ln \rho. \tag{4}$$

Because the limits exist in the case of $q \to 1$ and $s \to 0$, we will use $q = 1$ and $s = 1$ to represent the limits in this paper. Now, let's consider the entanglement in terms of the unified (q, s)-entropy. For any pure state $|\psi\rangle_{AB}$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ (it's does not matter for the sizes of subsystem A and B), the unified (q, s)-entropy entanglement is defined as[36]

$$E_{q,s}(|\psi\rangle_{AB}) = S_{q,s}(\rho_A) \tag{5}$$

for any $q, s \geq 0$.

For a mixed state ρ_{AB}, the unified (q, s)-entropy entanglement can be defined via the convex-roof extension

$$E_{q,s}(\rho_{AB}) = \min \sum_i p_i E_{q,s}\left(|\psi^i\rangle_{AB}\right), \tag{6}$$

where the minimum is taken over all possible ensembles $\left\{p_i, |\psi^i\rangle_{AB}\right\}$ of ρ_{AB} with $\sum_i p_i = 1$ and $p_i \geq 0$. It is straightforward to verify that $E_{q,s}(\rho_{AB})=0$ if and only if ρ_{AB} is a separable state for $q, s \geq 0$.

When s tends to 1, the unified (q, s)-entropy entanglement becomes Tsallis entanglement[31]. When s tends to 0, the unified (q, s)-entropy entanglement becomes Rényi entanglement[20]. Especially, The unified (q, s)-entropy entanglement becomes the entanglement of formation when q tends to 1. The entanglement of formation is defined as[37,38]

$$E_f(\rho_{AB}) = \min \sum_i p_i E_f\left(|\psi^i\rangle_{AB}\right), \tag{7}$$

where $E_f(|\psi^i_{AB}\rangle) = -Tr\rho_A^i \ln \rho_A^i = -Tr\rho_B^i \ln \rho_B^i$ is the von Neumann entropy, the minimum is taken over all possible ensembles $\left\{p_i, |\psi^i\rangle_{AB}\right\}$ of ρ_{AB} with $\sum_i p_i = 1$ and $p_i \geq 0$.

As a dual quantity to the unified (q, s)-entropy entanglement, the unified (q, s)-entropy entanglement of assistance ((q, s)-EOA) can be defined as

$$E^a_{q,s}(\rho_{AB}) = \max \sum_i p_i E_{q,s}\left(|\psi^i\rangle_{AB}\right), \tag{8}$$

where the maximum is taken over all possible ensembles $\left\{p_i, |\psi^i\rangle_{AB}\right\}$ of ρ_{AB} with $\sum_i p_i = 1$ and $p_i \geq 0$. To understand (q, s)-EOA better, consider a tripartite pure state $|\psi\rangle_{ABC}$ shared among three parties referred to as Alice, Bob, and Charlie[39]. The entanglement supplier, Charlie, performs a measurement on his share of the tripartite state, which yields a known bipartite entangled state for Alice and Bob. Tracing over Charlie's system yields the bipartite mixed state $\rho_{AB} = Tr_C(|\psi\rangle_{ABC}\langle\psi|)$ shared by Alice and Bob. Charlie's aim is to maximize entanglement for Alice and Bob, and the maximum average entanglement he can create is the (q, s)-EOA.

Concurrence and concurrence of assistance. For any pure state $|\psi\rangle_{AB}$ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the concurrence is defined as[40]

$$C(\rho_{AB}) = \sqrt{2(1 - Tr\rho_A^2)}, \tag{9}$$

where $\rho_A = Tr_B(\rho_{AB})$.

For a mixed state ρ_{AB}, the concurrence can be defined via the convex-roof extension

$$C(\rho_{AB}) = \min \sum_i p_i C\left(|\psi^i\rangle_{AB}\right), \tag{10}$$

where the minimum is taken over all possible ensembles $\left\{p_i, |\psi^i\rangle_{AB}\right\}$ of ρ_{AB} with $\sum_i p_i = 1$ and $p_i \geq 0$.

As a dual quantity to concurrence, the concurrence of assistance (COA) can be defined as

$$C^a(\rho_{AB}) = \max \sum_i p_i C\left(|\psi^i\rangle_{AB}\right), \tag{11}$$

where the maximum is taken over all possible ensembles $\left\{p_i, |\psi^i\rangle_{AB}\right\}$ of ρ_{AB} with $\sum_i p_i = 1$ and $p_i \geq 0$.

Analytical formula for two-qubit states. For a two-qubit mixed state ρ_{AB}, concurrence and COA are known to have analytic formula[30, 40]

$$\mathcal{C}(\rho_{AB}) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{12}$$

$$\mathcal{C}^a(\rho_{AB}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \tag{13}$$

where λ_i being the eigenvalues, in decreasing order, of matrix $\sqrt{\rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^*(\sigma_y \otimes \sigma_y)}$.

In ref. 40, Wootters derived an analytical formula of entanglement of formation for a two-qubit mixed state ρ_{AB}

$$E_f(\rho_{AB}) = h\left(\frac{1 + \sqrt{1 - \mathcal{C}^2(\rho_{AB})}}{2}\right), \tag{14}$$

where $h(x) = -x\ln x - (1-x)\ln(1-x)$ is the binary entropy.

In ref. 36, Kim found $E_{q,s}(\rho_{AB})$ has an analytical formula for a two-qubit mixed state, which can be expressed as a function of concurrence $\mathcal{C}_{AB}$ for $q \geq 1$, $0 \leq s \leq 1$ and $qs \leq 3$

$$E_{q,s}(\rho_{AB}) = f_{q,s}\left[\mathcal{C}(\rho_{AB})\right], \tag{15}$$

where the function $f_{q,s}(x)$ has the form

$$f_{q,s}(x) = \frac{\left[(1 + \sqrt{1 - x^2})^q + (1 - \sqrt{1 - x^2})^q\right]^s - 2^{qs}}{(1 - q)s2^{qs}}, \tag{16}$$

where $0 \leq x \leq 1$.

Main Results. In this section, we will provide our main results. First, we have following result:

Theorem 1. For any tripartite mixed state ρ_{ABC} in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, we have

$$E_{q,s}^a(\rho_{A|BC}) \leq E_{q,s}^a(\rho_{B|AC}) + E_{q,s}^a(\rho_{C|AB}), \tag{17}$$

where $q \geq 1$ and $qs \geq 1$.

Proof: Let $\rho_{ABC} = \max\sum_i p_i|\psi^i\rangle_{A|BC}\langle\psi^i|$ be an optimal decomposition of $E_{q,s}^a(\rho_{A|BC})$. That is

$$E_{q,s}^a(\rho_{A|BC}) = \max\sum_i p_i E_{q,s}\left(|\psi^i\rangle_{A|BC}\right). \tag{18}$$

For any bipartite pure state $|\psi^i\rangle_{A|BC}$, the unified (q, s)-entropy entanglement $E_{q,s}\left(|\psi^i\rangle_{A|BC}\right) = S_{q,s}(\rho_{BC}^i)$. In ref. 41, Rastegin proved that for any $q \geq 1$ and $qs \geq 1$, the unified (q, s)-entropy is subadditive, that is

$$S_{q,s}(\rho_{BC}^i) \leq S_{q,s}(\rho_B^i) + S_{q,s}(\rho_C^i). \tag{19}$$

Combining Eq. (18) with Eq. (19), we have

$$\begin{aligned} E_{q,s}^a(\rho_{A|BC}) &= \sum_i p_i S_{q,s}(\rho_{BC}^i) \\ &\leq \sum_i p_i S_{q,s}(\rho_B^i) + \sum_i p_i S_{q,s}(\rho_C^i) \\ &\leq E_{q,s}^a(\rho_{B|AC}) + E_{q,s}^a(\rho_{C|AB}). \end{aligned} \tag{20}$$

Thus, the proof is completed.

Theorem 1. Shows a simple but interesting polygamy relation of (q, s)-EOA in a tripartite quantum system. The upper bound of (q, s)-EOA of $A|BC$ can't be greater than the sum of (q, s)-EOA of $B|AC$ and (q, s)-EOA of $C|AB$. In particular, for a tripartite pure state $|\psi\rangle_{A|BC}$, the unified (q, s)-entropy entanglement $E_{q,s}\left(|\psi\rangle_{A|BC}\right) \leq E_{q,s}\left(|\psi\rangle_{B|AC}\right) + E_{q,s}\left(|\psi\rangle_{C|AB}\right)$.

We also have the following corollary:

Corollary 1. For any mixed state $\rho_{A_1|A_2\cdots A_n}$ in the Hilbert space $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$, we have

$$E_{q,s}^a\left(\rho_{A_1|A_2\cdots A_n}\right) \leq \sum_{i=2}^n E_{q,s}^a\left(\rho_{A_i|A_1\cdots A_{i-1}A_{i+1}\cdots A_n}\right), \tag{21}$$

where $q \geq 1$ and $qs \geq 1$.

Corollary 1. Shows a constrained relationship of (q, s)-EOA in the multi-particle system, and gives an upper bound of (q, s)-EOA of $A_1|A_2\cdots A_n$. In particular, for any pure state $|\psi\rangle_{A_1|A_2\cdots A_n}$, the unified (q, s)-entropy entanglement $E_{q,s}\left(|\psi\rangle_{A_1|A_2\cdots A_n}\right) \leq \sum_{i=2}^n E_{q,s}\left(|\psi\rangle_{A_i|A_1\cdots A_{i-1}A_{i+1}\cdots A_n}\right)$.

Example 1: Let's consider the general GHZ state $|GHZ\rangle = \alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n}$, where $|\alpha|^2 + |\beta|^2 = 1$ and $n \geq 3$. It's easy to show that $\sum_{i=2}^{n} E_{q,s}^{a}\left(\rho_{A_i|A_1\cdots A_{i-1}A_{i+1}\cdots A_n}\right) - E_{q,s}^{a}\left(|GHZ\rangle_{A_1|A_2\cdots A_n}\right) = \frac{n-2}{(1-q)s}[(|\alpha|^{2q} + |\beta|^{2q})^{s} - 1] \geq 0$.

Example 2: Consider a four-qubit cluster state $|C_4\rangle = \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$, which is a type of highly entangled state of four-qubit[42, 43]. The reduced states of $|C_4\rangle$ are $\rho_A = \rho_B = \rho_C = \rho_D = \frac{I}{2}$, thus $\sum_{i=2}^{n} E_{q,s}^{a}\left(\rho_{A_i|A_1\cdots A_{i-1}A_{i+1}\cdots A_n}\right) - E_{q,s}^{a}(|C_4\rangle) = \frac{2}{(1-q)s}\left[\frac{1}{(q-1)s} - 1\right]$, which is nonnegative for $q \geq 1$ and $qs \geq 1$.

We note that for any n-qubit mixed state $\rho_{AC_1\cdots C_n}$, the polygamy relation holds:

$$E_{q,s}^{a}\left(\rho_{A|C_1\cdots C_n}\right) \leq \sum_{i=1}^{n} E_{q,s}^{a}\left(\rho_{AC_i}\right) \tag{22}$$

for $1 \leq q \leq 2$ and $-q^2 + 4q - 3 \leq s \leq 1$[44]. Combining Eq. (17) with Eq. (22), we have

Corollary 2. For any multi-qubit mixed state $\rho_{ABC_1\cdots C_n}$, the following inequality holds

$$\begin{aligned} E_{q,s}^{a}\left(\rho_{AB|C_1\cdots C_n}\right) &\leq E_{q,s}^{a}\left(\rho_{A|BC_1\cdots C_n}\right) + E_{q,s}^{a}\left(\rho_{B|AC_1\cdots C_n}\right) \\ &\leq 2E_{q,s}^{a}(\rho_{AB}) + \sum_{i=1}^{n} E_{q,s}^{a}\left(\rho_{AC_i}\right) + \sum_{i=1}^{n} E_{q,s}^{a}\left(\rho_{BC_i}\right), \end{aligned} \tag{23}$$

where $1 \leq q \leq 2$, $s = 1$.In this case, (q, s)-EOA becomes Tsallis entropy entanglement of assistance which has discussed in ref. 22.

Before our second main result, we have following lemma:

Lemma 1. For $q = 2$ and $\frac{1}{2} \leq s \leq 1$, the function $f_{q,s}(x)$ in Eq. (16) satisfies

$$f_{q,s}(\sqrt{x^2 + y^2}) = f_{q,s}(x) + f_{q,s}(y). \tag{24}$$

Proof: For $q \geq 2$, $0 \leq s \leq 1$, and $qs \leq 3$, we have $f_{q,s}(\sqrt{x^2 + y^2}) \geq f_{q,s}(x) + f_{q,s}(y)$[36]. On the other hand, for $1 \leq q \leq 2$ and $0 \leq s \leq 1$, we have $f_{q,s}(\sqrt{x^2 + y^2}) \leq f_{q,s}(x) + f_{q,s}(y)$[44]. The equality holds if and only if $q = 2$ and $\frac{1}{2} \leq s \leq 1$. This completes the proofs.

Next, the following result will provide a lower bound of unified (q, s)-entropy entanglement of $|\psi\rangle_{AB|C_1\cdots C_n}$, with respect to the bipartition between AB and $C_1\cdots C_n$:

Theorem 2. For any multi-qubit pure state $|\psi\rangle_{ABC_1\cdots C_n}$ in the Hilbert space, we have

$$\begin{aligned} &E_{q,s}\left(|\psi\rangle_{AB|C_1\cdots C_n}\right) \\ &\geq \max\left\{\sum_{i=1}^{n}\left[E_{q,s}\left(\rho_{AC_i}\right) - E_{q,s}^{a}\left(\rho_{BC_i}\right)\right], \sum_{i=1}^{n}\left[E_{q,s}(\rho_{BC_i}) - E_{q,s}^{a}(\rho_{AC_i})\right]\right\} \end{aligned} \tag{25}$$

where $q = 2$ and $\frac{1}{2} \leq s \leq 1$.

Proof: Given a multi-qubit pure state $|\psi\rangle_{ABC_1\cdots C_n}$, the unified (q, s)-entropy is subadditive for any $q \geq 1$ and $qs \geq 1$. Thus, the following equality holds

$$\begin{aligned} S_{q,s}\left(\rho_{C_1\cdots C_n}\right) &= S_{q,s}(\rho_{AB}) \\ &\leq S_{q,s}(\rho_A) + S_{q,s}(\rho_B) \\ &= S_{q,s}(\rho_A) + S_{q,s}\left(\rho_{AC_1\cdots C_n}\right) \end{aligned} \tag{26}$$

which implies $S_{q,s}\left(\rho_{C_1\cdots C_n}\right) - S_{q,s}(\rho_A) \leq S_{q,s}\left(\rho_{AC_1\cdots C_n}\right)$, and similarly, $S_{q,s}(\rho_A) - S_{q,s}\left(\rho_{C_1\cdots C_n}\right) \leq S_{q,s}\left(\rho_{AC_1\cdots C_n}\right)$. Combine with the two equalities above, one obtain

$$\left|S_{q,s}(\rho_A) - S_{q,s}\left(\rho_{C_1\cdots C_n}\right)\right| \leq S_{q,s}\left(\rho_{AC_1\cdots C_n}\right). \tag{27}$$

From the definition of unified (q, s)-entropy entanglement of $|\psi\rangle_{AB|C_1\cdots C_n}$, with respect to the bipartition between AB and $C_1\cdots C_n$, we have

$$\begin{aligned} E_{q,s}\left(|\psi\rangle_{AB|C_1\cdots C_n}\right) &= S_{q,s}(\rho_{AB}) \\ &\geq S_{q,s}(\rho_A) - S_{q,s}(\rho_B) \\ &= E_{q,s}\left(\rho_{A|BC_1\cdots C_n}\right) - E_{q,s}\left(\rho_{B|AC_1\cdots C_n}\right). \end{aligned} \tag{28}$$

Note that for any pure state $|\psi\rangle_{ABC}$ in a $2 \otimes 2 \otimes d$ system, the following equality holds[45, 46]

$$\mathcal{C}^2(|\psi\rangle_{ABC}) = [\mathcal{C}^a(\rho_{AB})]^2 + \mathcal{C}^2(\rho_{AC}), \tag{29}$$

where ρ_{AB} and ρ_{AC} are the reduced matrices of state $|\psi\rangle_{ABC}$ respectively.

For $q=2$ and $\frac{1}{2}\le s\le 1$, we have

$$\begin{aligned} E_{q,s}\left(|\psi\rangle_{AB|C_1\cdots C_n}\right) &= f_{q,s}\left(\mathcal{C}\left(|\psi\rangle_{AB|C_1\cdots C_n}\right)\right)\\ &= f_{q,s}\left(\sqrt{[\mathcal{C}^a(\rho_{AB})]^2+\mathcal{C}^2(\rho_{AC})}\right)\\ &= f_{q,s}(\mathcal{C}^a(\rho_{AB}))+f_{q,s}(\mathcal{C}(\rho_{AC})), \end{aligned} \tag{30}$$

where we have used Eq. (29) in the second equality, the third equality holds is due to lemma 1. Therefore,

$$\begin{aligned} E_{q,s}\left(|\psi\rangle_{AB|C_1\cdots C_n}\right) &= f_{q,s}\left(\mathcal{C}\left(|\psi\rangle_{AB|C_1\cdots C_n}\right)\right)\\ &\le f_{q,s}\left(\sqrt{[\mathcal{C}^a(\rho_{AB})]^2+\sum_{i=1}^{n}\mathcal{C}^2(\rho_{AC_i})}\right)\\ &\le f_{q,s}(\mathcal{C}^a(\rho_{AB}))+f_{q,s}\left(\sqrt{\sum_{i=1}^{n}\mathcal{C}^2(\rho_{AC_i})}\right). \end{aligned} \tag{31}$$

Compare Eq. (30) with Eq. (31), it's easy to see that

$$\begin{aligned} E_{q,s}\left(|\psi\rangle_{A|BC_1\cdots C_n}\right)-E_{q,s}\left(|\psi\rangle_{B|AC_1\cdots C_n}\right) \ge\ & f_{q,s}\left(\mathcal{C}\left(\rho_{AC_1\cdots C_n}\right)\right)\\ &-f_{q,s}\left\{\sqrt{\sum_{i=1}^{n}\left[\mathcal{C}^a(\rho_{BC_i})\right]^2}\right\}. \end{aligned} \tag{32}$$

We also note that

$$\begin{aligned} f_{q,s}\left(\mathcal{C}\left(\rho_{AC_1\cdots C_n}\right)\right) &\ge f_{q,s}\left[\sqrt{\sum_{i=1}^{n}\mathcal{C}^2(\rho_{AC_i})}\right]\\ &= \sum_{i=1}^{n}f_{q,s}\left(\mathcal{C}\left(\rho_{AC_i}\right)\right)\\ &= \sum_{i=1}^{n}E_{q,s}\left(\rho_{AC_i}\right), \end{aligned} \tag{33}$$

where the first equality holds is due to the monogamy of concurrence[1] and $f_{q,s}(x)$ is an increasing function for $q\ge 2$, $0\le s\le 1$, and $qs\le 3$[36].

On the other hand, we have

$$\begin{aligned} f_{q,s}\left\{\sqrt{\sum_{i=1}^{n}\left[\mathcal{C}^a\left(\rho_{BC_i}\right)\right]^2}\right\} &= \sum_{i=1}^{n}f_{q,s}\left[\mathcal{C}^a\left(\rho_{BC_i}\right)\right]\\ &\le \sum_{i=1}^{n}E_{q,s}^a\left(\rho_{BC_i}\right) \end{aligned} \tag{34}$$

Combine Eqs (32) and (33) with Eq. (34), we have

$$E_{q,s}\left(|\psi\rangle_{A|BC_1\cdots C_n}\right)-E_{q,s}\left(|\psi\rangle_{B|AC_1\cdots C_n}\right)\ge\sum_{i=1}^{n}\left[E_{q,s}\left(\rho_{AC_i}\right)-E_{q,s}^a\left(\rho_{BC_i}\right)\right]. \tag{35}$$

Putting Eq. (35) into Eq. (32), we obtain our result. Similarly, we have

$$E_{q,s}\left(|\psi\rangle_{AB|C_1\cdots C_n}\right)\ge\sum_{i=1}^{n}\left[E_{q,s}\left(\rho_{BC_i}\right)-E_{q,s}^a\left(\rho_{AC_i}\right)\right] \tag{36}$$

Thus, the proof is completed.

Theorem 2 shows a monogamy relation for a multi-qubit pure state $|\psi\rangle_{ABC_1\cdots C_n}$. The lower bound of the unified (q, s)-entropy entanglement for $AB|C_1\cdots C_n$ can't be less than the sum of the two-qubit entanglement between bipartitions of the system. In particular, if $|\psi\rangle_{ABC_1\cdots C_n}=|\psi\rangle_{AC_1\cdots C_n}\otimes|\psi\rangle_B$, the entanglement of $AB|C_1\cdots C_n$ is equal to the entanglement of $A|C_1\cdots C_n$. In this case, $E_{q,s}\left(\rho_{BC_i}\right)=0$ for $i=1, 2, \ldots, n$. Theorem 2 becomes $E_{q,s}\left(|\psi\rangle_{A|C_1\cdots C_n}\right)\ge\sum_{i=1}^{n}E_{q,s}^a\left(\rho_{AC_i}\right)$, which is a CKW-type monogamy relation[1,8].

Example 3: Consider a pure state $|\phi\rangle_{ABC_1C_2}=\frac{1}{\sqrt{2}}(|0000\rangle+|1001\rangle)$ in the four-qubit system. for the range $q=2$ and $\frac{1}{2}\le s\le 1$, we have $E_{q,s}\left(\rho_{AC_1}\right)=E_{q,s}^a\left(\rho_{AC_1}\right)=0$, and $E_{q,s}\left(\rho_{AC_2}\right)=E_{q,s}^a\left(\rho_{AC_2}\right)=\frac{1}{s}\left(1-\frac{1}{2^s}\right)$.

$E_{q,s}\left(\rho_{BC_i}\right) = E^a_{q,s}\left(\rho_{BC_i}\right) = 0$ where $i = 1, 2$ and $E_{q,s}\left(|\phi\rangle_{ABC_1C_2}\right) = \frac{1}{s}\left(1 - \frac{1}{2^s}\right)$. Therefore, we can see $|\phi\rangle_{ABC_1C_2}$ saturates the inequality Eq. (25).

Example 4: Finally, let's consider a general W state $|W\rangle_{A_1A_2\cdots A_n} = a_1|00\cdots01\rangle + a_2|00\cdots10\rangle + \cdots + a_n|10\cdots00\rangle$ in the n-qubit system, where $\sum_i^n |a_i|^2 = 1$. The reduced state of subsystem A_1A_2 is

$$\rho_{A_1A_2} = \begin{pmatrix} 1 - |a_{n-1}|^2 - |a_n|^2 & 0 & 0 & 0 \\ 0 & |a_{n-1}|^2 & a_{n-1}a_n^* & 0 \\ 0 & a_{n-1}^*a_n & |a_n|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (37)$$

which implies $E_{q,s}\left(|W\rangle_{A_1A_2|A_3\ldots A_n}\right) \geq 0$. It's also easy to show that the reduced state $\rho_{A_iA_j}$ is separable, where $i, j = \{1, 2, \ldots, n\}$. Thus $E_{q,s}\left(\rho_{A_1A_i}\right) = E^a_{q,s}\left(\rho_{A_2A_i}\right) = E_{q,s}\left(\rho_{A_2A_i}\right) = E^a_{q,s}\left(\rho_{A_1A_i}\right) = 0$. We find that the right side of the inequality Eq. (25) is $\max\left\{\sum_{i=2}^n\left[E_{q,s}\left(\rho_{A_1A_i}\right) - E^a_{q,s}\left(\rho_{A_2A_i}\right)\right], \sum_{i=2}^n\left[E_{q,s}\left(\rho_{A_2A_i}\right) - E^a_{q,s}\left(\rho_{A_1A_i}\right)\right]\right\} = 0$. Which mean the inequality Eq. (25) holds for the general W state.

Conclusion

Unified (q, s)-entropy is an important generalized entropy in quantum information theory. Many entropies such as Tsallis entropy, Rényi entropy, and von Neumann entropy can be seen as a special case for unified (q, s)-entropy. In this paper, we have investigated the entanglement distribution in multi-particle systems in terms of unified (q, s)-entropy. We find that for any tripartite mixed state, the (q, s)-EOA follows a polygamy relation for $q \geq 1$ and $qs \geq 1$. This polygamy relation provides an upper bound for the bipartition $A|BC$, which also holds in multi-particle systems. Furthermore, for $q = 2$ and $\frac{1}{2} \leq s \leq 1$, a generalized monogamy relation is provided for unified (q, s)-entropy entanglement. This monogamy relation provides a lower bound for the bipartition $AB|C_1\cdots C_n$ in the multi-qubit system. In particular, if $|\psi\rangle_{ABC_1\cdots C_n} = |\psi\rangle_{AC_1\cdots C_n} \otimes |\psi\rangle_B$, the generalized monogamy relation becomes a CKW-type monogamy relation.

Both monogamy property and polygamy property are fundamental properties of multipartite entangled states. We have studied the properties above in detail, and provided a two-parameters entropy function to study the entanglement distribution. We believe our result provides a useful methodology to understand the entanglement distribution of multi-particle entanglement.

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Author Contributions

Y. Luo performed the calculations and wrote the main manuscript. F.-G. Zhang checked the calculations. Y. Li improved the manuscript. All authors contributed to the discussion and reviewed the manuscript.

Additional Information

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