Research article

A new multistage technique for approximate analytical solution of nonlinear differential equations

Saheed Ojo Akindeinde *

Department of Mathematics, Obafemi Awolowo University, 220005, Ile-Ife, Nigeria

ARTICLE INFO

ABSTRACT

The article introduces a new multistage technique for solving a polynomial system of nonlinear initial and boundary value problems of differential equations. The radius of convergence R of the series solution to the problem is derived a-priorly in terms of the parameters of the polynomial system. Then guided by the convergence-control parameter $h < R$, the domain of the problem is split into subintervals. By stepping out in a multistage manner, corresponding subproblems are defined which are then subsequently solved with conventional Parker-Sochacki method to get a piecewise continuous solution with very high accuracy. The method is applied to SIR epidemic model, stiff differential equation modelling combustion, Lorenz chaotic problem, and the Troesch's boundary value problem. The results obtained showed a remarkable accuracy when compared with Runge-Kutta Method of order 4. The article showcased the proposed method as a simple, yet accurate approximate analytical technique for nonlinear differential equations.

1. Introduction

Nonlinear ordinary differential equations appear naturally either in the modelling of physical phenomena in nature, or because of a transformation which forms part of solution technique for solving some partial differential equations. The often desired exact solutions to such differential equations are mostly not available. Researchers often resort to numerical and approximate analytical techniques for solving differential equations. However, for problems being considered in this paper, namely the chaotic initial value problems and the Troesch's problem, existing numerical solvers have failed to produce accurate solutions for certain *choices* of the parameters describing the problem, see e.g. [1] (Troesch's problem) and [2] (Chaotic problem).

On the other hand, approximate analytical techniques such as Homotopy Analysis Method (HAM), Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), including their various modifications have been successfully applied and solutions got for nonlinear differential equations in the literature. Among the major concerns reported for such solutions is their limited convergence horizon, beyond which approximate analytical solution diverges. Recently, the authors in [3] showed that approximate analytical solution, convergent on the entire domain, can not be got for SIR epidemic model using conventional HAM. Stiff differential equations are a class of differen-

tial equations whose solutions, as reported in the literature, can not be efficiently computed using the aforementioned techniques. Therefore, speeding up the convergence or extending the domain of convergence of such series solution has been examined by several authors and different post-processing techniques have been proposed. A quite unifying approach for extending the domain of convergence of approximate analytical solution is to implement the method in a multistage manner. This procedure was used to improve the Adomian Decomposition Method in [4], the Variational Iteration Method in [5], Differential Transformation Method [6], Homotopy Perturbation Method in [7], and the Homotopy Analysis Method in [3]. The cost arising from computational complexities of the underlying method remains a major concern in any multistage implementation. It is therefore desirable for numerical analysts to have access to a method that is both computationally simple and yet highly accurate.

Motivated by the above concerns on existing techniques, we devise, in this article, a new multistage technique which is based on Parker-Sochacki method. The Parker-Sochacki method is a simple technique for computing iteratively, the coefficients of the Maclaurin series solution to a polynomial system of strongly nonlinear ordinary and partial differential equations whose solutions are analytic. The ease of application of this method to both initial and boundary value problems has been showed by the authors in [8, 9]. Building on the conventional

* Corresponding author.
E-mail addresses: saheed.akindeinde@gmail.com, soakindeinde@oauife.edu.ng.

Parker-Sochacki Method, the radius of convergence R of the series solution to the problem is derived a-priorly in terms of the parameters of the polynomial system. Then guided by the convergence-control parameter $h < R$, the domain of the problem is split into subintervals. By stepping out in a multistage manner, corresponding subproblems are defined which are then solved with conventional Parker-Sochacki method to get a piecewise continuous solution with very high accuracy.

It was demonstrated in [10] that computational complexity of Parker-Sochacki method compares well with that of Runge-Kutta method of order four. Thus in this article, relying on both the favourable computational complexity as well as computational simplicity of the method, a multistage implementation of Parker-Sochacki shall be used to derive approximate analytical solution to SIR epidemiological model, a stiff differential equation modelling combustion, Lorenz chaotic problem, and the Troesch's boundary value problem with Troesch's parameter $\lambda = 30$, which is convergent on the entire domain of the problem.

2. Basics of the proposed method (Multistage Parker-Sochacki Method)

Here, the basics of the proposed method are presented. We describe the conventional Parker-Sochacki method and introduce the domain split technique of the new approach.

2.1. Polynomial projection and Parker-Sochacki method

The Parker-Sochacki Method (PSM) [8, 9, 11, 12, 13, 14] extends the conventional power series method to solving nonlinear initial value problems and initial value problems involving transcendental functions. The method relies on polynomial projection, which is described below. To understand the concept of polynomial projection or projectively polynomial functions, consider the initial value problem

$$y'(x) = K \sin y(x), \quad y(0) = y_0. \tag{1}$$

The above problem is not directly amenable to the conventional power series method. However, introducing the new variables $u = \sin y$, $v = \cos y$ reduces the problem to

$$y' = Ku, \quad y(0) = y_0, \tag{2}$$

$$u' = Kuv, \quad u(0) = 0, \tag{3}$$

$$v' = -Ku^2, \quad v(0) = 1, \tag{4}$$

which is now a polynomial system in the variables y, u and v. As we shall shortly show, power series solution of this system can be straightforwardly computed through simple recursions. Notice that the solution y to (1) is now embedded in the solution $\begin{pmatrix} y \\ u \\ v \end{pmatrix}$ of (2)-(3). A differential equation for which a variable y is a component of the solution is called *projection* for y, or that y is *projectively polynomial*. In [11, 15], the authors showed that many differential equations can be projected into polynomial systems, where power series solution can easily be computed.

The basic computational idea of Parker-Sochacki method is now as follows. Given a first order non-autonomous problem of the form

$$\mathbf{y}' = f(x, \mathbf{y}), \quad x_0 \leq x \leq x_f, \quad \mathbf{y}(x_0) = \mathbf{y}_0, \tag{5}$$

the first step of the method is to recast the problem as a projectively polynomial system through appropriate auxiliary variables, to the form

$$\mathbf{y}' = f(\mathbf{y}), \quad x_0 \leq x \leq x_f, \quad \mathbf{y}(x_0) = \mathbf{y}_0. \tag{6}$$

In the above, the function f is a vector of polynomials, that is, $f : \mathbb{R}^d \to \mathbb{R}^d$ where d is the dimension of the polynomial system. Without loss of generality, let us assume that $x_0 = 0$ and write $\mathbf{y}(x)$ as

$$\mathbf{y}(x) = \sum_{i=0}^{m} \mathbf{y}_i x^i \tag{7}$$

where the coefficients $\mathbf{y}_0 = \mathbf{y}(0)$, $\mathbf{y}_1 = \mathbf{y}'(0)$, $\mathbf{y}_2 = \frac{1}{2!}\mathbf{y}''(0), \dots$ are to be determined. The corresponding expression for the derivative of y is obtained as

$$\mathbf{y}'(x) = \sum_{i=0}^{m} (i+1)\mathbf{y}_{i+1} x^i. \tag{8}$$

Now, the function f being a polynomial function of y, can similarly be written as

$$f(\mathbf{y}(x)) = \sum_{i=0}^{m} f_i x^i. \tag{9}$$

Upon substituting (8) and (9) into (6) we obtain

$$\mathbf{y}_{i+1} = \frac{f_i}{i+1} \tag{10}$$

where the coefficients f_i are computed by the repeated use of Cauchy product [16] e.g. if $f(y) = y^2$, then $f_i = \sum_{j=0}^{i} y_j y_{i-j}$. The obtained coefficients y_{i+1} above are then used in (7) to obtain the series solution of the problem (5) as

$$\mathbf{y}(x) = y_0 + \sum_{i=1}^{m} \frac{f_{i-1}}{i} x^i.$$

Now in the case $x_0 \neq 0$, the series solution is computed normally as described above. Let this series solution be denoted by $\phi(x)$. The solution to the original problem is then

$$\mathbf{y}(x) = \phi(x - x_0) = y_0 + \sum_{i=1}^{m} \frac{f_{i-1}}{i}(x - x_0)^i. \tag{11}$$

As can be seen above, the PSM is very easy to implement. This makes the method applicable to a wide range of differential equations that can be written in the form (6) including strongly nonlinear coupled differential equations.

2.2. Convergence and error analysis of the method

Following the notations in [17], we define the norm $|| \cdot ||$ as

$$||(y_1, y_2, \dots, y_n)|| = \max_{1 \leq i \leq n} |y_i|$$

If $p : \mathbb{R}^d \to \mathbb{R}^d$ is a vector of polynomials and $Y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2} \dots y_d^{\alpha_d}$, we write

$$p(Y_1, Y_2, \dots, Y_d) = \sum_{|\alpha| \leq k} a_\alpha Y^\alpha$$

where $k = \deg(p_i)$ is the degree of p_i and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. Furthermore, we define

$$\Sigma p_i = \sum_{|\alpha| \leq k} |a_{\alpha,i}|,$$

$\deg(p) = \max\{\deg(p_1), \deg(p_2), \dots, \deg(p_d)\}$ and $\Sigma p = \max\{\Sigma p_1, \Sigma p_2, \dots, \Sigma p_d\}$.

With these preliminaries, we now state the theorem on convergence of series solution by PSM.

Theorem 2.1. [18] If y satisfies (5), $k = \deg(p) \geq 2$, $\alpha = \max\{1, ||y_0||\}$, $M = (k-1)\Sigma p\, \alpha^{k-1}$, $x_0 = 0$, and $|x| < \frac{1}{M}$ then

$$\left\| y(x) - \sum_{i=0}^{m} y_i x^i \right\| \leq \frac{\alpha |Mx|^{m+1}}{1 - |Mx|}.$$

Now, to examine the order of convergence and the general behaviour of the global error, the following preliminaries are in order. First, let us consider the discretization of the problem domain $[x_0, x_f]$ which is obtained by introducing the nodes $x_i = x_0 + i\,h$, $i = 0, 1, \ldots, N = \frac{x_f - x_0}{h}$, where h is a fixed step-size conveniently chosen as $1/(2M)$ [17, Section 4.1].

For clarity, let us also recall the general problem (6) under consideration, namely

$$\mathbf{y}' = f(\mathbf{y}), \quad x_0 \leq x \leq x_f, \quad \mathbf{y}(x_0) = \mathbf{y}_0. \tag{12}$$

Recall that if f satisfies the existence and uniqueness conditions of Picard-Lindelöf Theorem [19], and $f \in C^{n+1}([x_0, x_f])$ then the unique solution y to (6) can be expressed as

$$y(x) = \sum_{j=0}^{n} y_j (x - c)^j + \frac{y^{(n+1)}(\xi(x))}{(n+1)!}(x - c)^{n+1}$$

for all $c \in (a, b)$ with $\xi(x) \in (c, x)$. Further, as f is assumed to be real analytic on $[x_0, x_f]$, then we can write (see also Section 2.1)

$$y(x) = \sum_{j=0}^{\infty} y_j (x - c)^j = y_0 + y_1(x - c) + y_2(x - c)^2 + y_3(x - c)^3 + \ldots$$

for all $c \in [x_0, x_f]$ so that for all $x = x_i + h \in [x_0, x_f]$, with $c = x_i$ in the above, it follows that

$$y(x_i + h) = \sum_{j=0}^{n} y_j h^j + \frac{y^{(n+1)}(\xi(h))}{(n+1)!} h^{n+1} \tag{13}$$

where $\xi(h) \in (x_i, x_i + h)$.

We also recall that the PSM technique described in Section 2.1 is designed in such a way that the approximation of $y(x_i + h)$ exactly matches, to a given degree say m, the Taylor polynomial of the unknown solution y expanded about x_i. Thus, the PSM can be viewed as an explicit one-step method of the form [20]

$$w_{i+1} = w_i + h(y_1 + y_2 h + y_3 h^2 + \cdots + y_m h^{m-1})$$

where $w_i \approx y(x_i)$ and $y_1, y_2, \ldots y_m$ are the Taylor coefficients of the solution to (6) obtained via the recursion (10). Note that for one-step methods, the global error $\epsilon_m = |y(x) - \sum_{j=0}^{m} y_j x^j| \to 0$ if the local truncation errors $\tau_i \to 0$ on the subintervals decay to zero as $h \to 0$ [19].

Theorem 2.2. *The absolute error ϵ_m for the PSM series solution defined in (7) has exponential decay for step-size $h < 1$.*

Proof. If we assume that $y(x_i)$ and $y(x_i + h)$ are exact, then the local truncation error for PSM is given by

$$\tau_{i+1}(h) = \frac{y(x_i + h) - y(x_i) - h(y_1 + y_2 h + y_3 h^2 + \cdots + y_m h^{m-1})}{h}.$$

Note that for one-step implementation, only x_i and $y(x_i)$ are required to compute $y(x_i + h)$. Thus, we choose $y_0 = y(x_i)$ so that with $n = m + 1$ in (13), $h < 1$, the local truncation error reduces to

$$\tau_{i+1}(h) = y_{m+1} h^m + \frac{y^{(m+2)}(\xi(h))}{(m+2)!} h^{m+1} \leq y_{m+1} h^m.$$

Thus the method is consistent. Consequently, the global error behaves as

$$\epsilon_m \leq C h^m$$

for some positive constant C. $\quad\square$

Remark 2.3. We also conclude from Theorem 2.2 that the PSM is also convergent of order m.

2.3. Multistage implementation

Theorem 2.1 above allows us to compute the radius and interval of convergence of the series solution derived through the conventional PSM, namely $1/M$. Unfortunately, most times this number is usually small and the series solution is not convergent on the entire integration interval $[x_0, x_f]$. Therefore, to extend the interval of convergence of the series solution computed via the method described above, we follow the idea in [17] and consider the problem on smaller subintervals where high accuracy of the method and convergence of the method is assured.

Introducing the nodes $x_i = x_0 + i\,h$ where $h = 1/(2M)$ allows us to divide the integration domain $[x_0, x_f]$ into non-overlapping subintervals $I_k = [x_{k-1}, x_k]$, $k = 1, 2, 3, \ldots, N$ such that $\cup [x_{k-1}, x_k] = [x_0, x_f]$. The PSM is implemented in a multistage fashion as follows. In the first subinterval I_1, starting with the given initial condition $\mathbf{y}_0^{(1)} = \mathbf{y}_0$, the problem is solved using the Parker-Sochacki method to get an approximate solution given by (11), denoted here as $\mathbf{y}^{(1)}(x)$. For the next subinterval I_2, the initial condition $\mathbf{y}_0^{(2)}$ is obtained by evaluating (11) at h, which is then used in the computation of new approximation $\mathbf{y}^{(2)}(x)$ on I_2, i.e. $\mathbf{y}_0^{(2)} = \mathbf{y}^{(1)}(h)$. The function evaluation of the solution at the end of one interval becomes the initial condition for the next subinterval. This procedure is repeated until the last subinterval $[(N-1)h, \beta]$, giving a highly accurate piecewise continuous solution of the form

$$\mathbf{y}(x) = \begin{cases} \mathbf{y}^{(1)}(x) & \text{if} \quad x \in I_1 \\ \mathbf{y}^{(2)}(x) & \text{if} \quad x \in I_2, \\ \mathbf{y}^{(3)}(x) & \text{if} \quad x \in I_3, \\ \quad\vdots \\ \mathbf{y}^{(N-1)}(x) & \text{if} \quad x \in I_{N-1}, \\ \mathbf{y}^{(N)}(x) & \text{if} \quad x \in I_N. \end{cases}$$

It should be noted that one advantage of Parker-Sochacki method is that it allows on-the-fly change of the order of approximation m. Therefore, besides reducing the length of the subintervals to achieve better results, the overall accuracy of the method can be improved by increasing the order of approximation in each subinterval.

3. Application to SIR disease epidemic model

The spread and control of infectious diseases are usually modelled by epidemic models. Mathematical modelling of progression of infectious diseases dates back to the work of Bernuolli [21]. In 1927 Kermack and McKendrick [22] proposed a model for a non-fatal disease in a population which is assumed to have a constant size over the period of epidemic. The model is termed the SIR model. The model decomposed the population into three compartments, the susceptible S, the infected I and the recovered population R. The movement of the population from one compartment to the other is described by the initial value problem given in [23] as

$$\frac{dS}{dt} = -\beta SI, \tag{14}$$

$$\frac{dI}{dt} = \beta SI - \gamma I, \tag{15}$$

$$\frac{dR}{dt} = \gamma I, \tag{16}$$

with initial conditions

$$S(0) = N_S, \quad I(0) = N_I, \quad R(0) = N_R. \tag{17}$$

The parameters $\beta, \gamma > 0$ denote the rate of change of susceptible population to infectious population and infectious population to immune population, respectively. In the above, at a time t, the population is assumed to comprise $S(t)$ susceptible - those so far uninfected and therefore liable to infection; $I(t)$ infective - those who have the disease and are still at large; $R(t)$ who are isolated, or who have recovered and are therefore immune.

The above problem has been widely considered in the literature, regarding analytical techniques or numerical approaches for finding disease progression pattern of the population, see [24], [25], [26], [27]. It has often been reported that approximate analytical solutions to such nonlinear differential equations are only valid for small finite intervals. Beyond such intervals, the approximate solutions diverge and thus not useful. Recently, the authors in [3] showed that, without multistage implementation, the famous Homotopy Analysis Method (HAM) also cannot solve the SIR model over a long time horizon. Here, the problem will be solved using the proposed multistage Parker-Sochacki Method.

3.1. Convergence and solution by Multistage Parker-Sochacki Method

According as Theorem 2.1, the smallest interval for which the series solution to the epidemic model (14)-(16) is expected to converge is given by $(-h, h)$ where $h < \frac{1}{M}$ and $M = (k - 1)\Sigma p \alpha^{k-1}$. Observe that the bound provided in Theorem 2.1 is a rather tight bound and so the actual interval of convergence could be larger. However, the result of the theorem provides the safest theoretical bound without computing the interval of convergence of the series solution. From the system (14)-(16), it is clear that

$$\Sigma p = \max\{\beta, \beta + \gamma, \gamma\} = \beta + \gamma, \quad ||y_0|| = \max\{N_S, N_I, N_R\},$$

$$\alpha = \max\{1, ||y_0||\}, \quad k = 2.$$

With $\beta = 0.001, \gamma = 0.1, N_S = 499, N_I = 1, N_R = 1$, we obtained $\Sigma p = 0.101, \alpha = 499, k = 2$ so that $M = 50.399$ and consequently the smallest interval for which the series solutions of the SIR model are theoretically convergent is $(-h, h)$ where $h < 1/M = 0.0198$.

Now, with the step-size $h > 0$ chosen within the interval of convergence, as $h = 0.01$ we divide the problem domain, chosen here as $[0, 100]$ into subintervals $I_i = [ih, (i + 1)h], \quad i = 0, 1, \ldots$ On the first subinterval $I_1 = [0, 0.01]$, the problem is solved as follows. Following the discussion of Section 2.1 above, we write

$$S(t) = \sum_{i=0}^{m} S_i t^i, \quad I(t) = \sum_{i=0}^{m} I_i t^i, \quad R(t) = \sum_{i=0}^{m} R_i t^i. \tag{18}$$

Thus by the PSM, the coefficients in the expansion are thus obtained through the recursion

$$S_{i+1} = -\frac{\beta \sum_{j=0}^{i} S_j I_{i-j}}{i + 1}, \quad I_{i+1} = \frac{\beta \sum_{j=0}^{i} S_j I_{i-j} - \gamma I_i}{i + 1}, \quad R_{i+1} = \frac{\gamma I_i}{i + 1} \tag{19}$$

with initial conditions

$$S_0 = S(0) = N_S, \quad I_0 = I(0) = N_I, \quad R_0 = R(0) = N_R.$$

Solving these with parameter values $\beta = 0.001, \gamma = 0.1, N_S = 499, N_I = 1, N_R = 1$ yield the solution

$$S^{(1)}(t) = 499 - 0.499\, t - 0.09930100000\, t^2 - 0.01309924900\, t^3$$
$$- 0.001281084280\, t^4 - 0.00009784814872\, t^5 - 0.000005908988970\, t^6$$
$$- 0.0000002687103489\, t^7 - 0.000000006753653701\, t^8$$
$$+ 0.0000000002645523368\, t^9 + 5.226667371 \times 10^{-11}\, t^{10} + \ldots \tag{20}$$

$$I^{(1)}(t) = 1 + 0.399\, t + 0.07935100000\, t^2 + 0.01045421567\, t^3$$
$$+ 0.001019728889\, t^4 + 0.00007745357094\, t^5 + 0.000004618096122\, t^6$$
$$+ 0.0000002027375471\, t^7 + 0.000000004219434362\, t^8$$
$$- 0.0000000003114349408\, t^9 - 4.915232430 \times 10^{-11}\, t^{10} + \ldots \tag{21}$$

$$R^{(1)}(t) = 1 + 0.1\, t + 0.01995000000\, t^2 + 0.002645033333\, t^3$$
$$+ 0.0002613553918\, t^4 + 0.00002039457778\, t^5$$
$$+ 0.000001290892849\, t^6 + 0.00000006597280174\, t^7$$
$$+ 0.000000002534219339\, t^8 + 4.688260402 \times 10^{-11}\, t^9$$
$$- 3.114349408 \times 10^{-12}\, t^{10} + \ldots \tag{22}$$

These results (using PSM) are plotted and compared against those obtained with Runge-Kutta method of order 4 (RK4) in Figs. 1 (a)-(e). The blowup of these solutions over a relatively short time horizon can be seen from the figures. Hence, the need for multistage implementation. Therefore, on the second subinterval $[0.01, 0.02]$, the same PSM computational algorithm is run but with a new initial conditions $S^{(2)}(0) = S^{(1)}(0.01), I^{(2)}(0) = I^{(1)}(0.01) =, R^{(2)}(0) = R^{(1)}(0.01)$. In a multi-stage fashion described in Section 2.3, further subproblems are solved from one interval to the other to get a piece-wise convergent solution, which is valid in the entire domain. These approximate analytical solutions are shown in Table 1. It is worthwhile to mention that only cubic polynomial is used, in each subinterval, to obtain the solution presented in the table.

To enable comparison of present results with existing results in the literature e.g. [24, 25, 26], we also consider other choice of the model parameters, namely $N_S = 20, N_I = 15, N_R = 10, \beta = 0.01, \gamma = 0.02$. Again with PSM, this choice produced the five-term approximations

$$S(t) = 20 - 3.0\, t - 0.04500000000\, t^2 + 0.02805000000\, t^3$$
$$+ 0.0007953750000\, t^4 - 0.0003165502500\, t^5, \tag{23}$$

$$I(t) = 15 + 2.70\, t + 0.01800000000\, t^2 - 0.02817000000\, t^3$$
$$- 0.0006545250000\, t^4 + 0.0003191683500\, t^5, \tag{24}$$

$$R(t) = 10 + 0.30\, t + 0.02700000000\, t^2 + 0.0001200000000\, t^3$$
$$- 0.0001408500000\, t^4 - 0.000002618100000\, t^5 \tag{25}$$

which coincide with those in [24, 25, 26]. These results are displayed in Fig. 2(a). When compared with Fig. 2(b), the blow-up of the five-term approximate analytical solution beyond some certain values of t is obvious from the graphical illustrations in Fig. 2(a).

4. Application to combustion equation

Consider the simple experiment of lighting a match stick. As described in [28], the ball of flame produced initially increases in size until a particular critical size where the size remains constant, because of a balance of oxygen available for combustion in the ball's interior and that available on its surface. The above phenomenon is described by the stiff ordinary differential equation

$$\frac{dy}{dt} = y^2(1 - y), \quad y(0) = \delta, \quad t \in [0, 2/\delta]. \tag{26}$$

The differential equation (26) belongs to a class of stiff differential equations, and arises in the modelling of trimolecular equations [28, 29]. Here, the variable $y(t) \geq 0$ denotes concentration while the initial radius $\delta > 0$ of the ball of flame describes a small disturbance of the pre-ignition state $y = 0$. The state $y = 1$ is termed the explosion state. Several approaches which have been proposed to solve such a problem include the Asymptotic Expansion Technique [29], Nonstandard Finite Difference Method [28], Singular Perturbation Method [30], Wavelets [31] and Boundary Value Methods [32]. Here, the problem is elegantly solved using multistage PSM.

4.1. Convergence and solution by Multistage Parker-Sochacki Method

In view of the convergence Theorem 2.1, for the combustion equation, we have

$$k = 3, \quad \alpha = \max\{1, |\delta|\}, \quad \Sigma p = 2$$

so that for the choice $\delta < 1$, we obtain $\alpha = 1$. Thus, $M = 4$ and the smallest interval of convergence is $(-h, h)$ with $h < 0.25$. So choosing $h = 0.2$, the integration interval is divided into smaller intervals $[ih, i(i + 1)], i = 0, 1, \ldots$ over which subproblems are defined. On the first interval $I_1 = [0, 0.2]$, the subproblem is solved as follows. Observe that

(a)

(b)

(c)
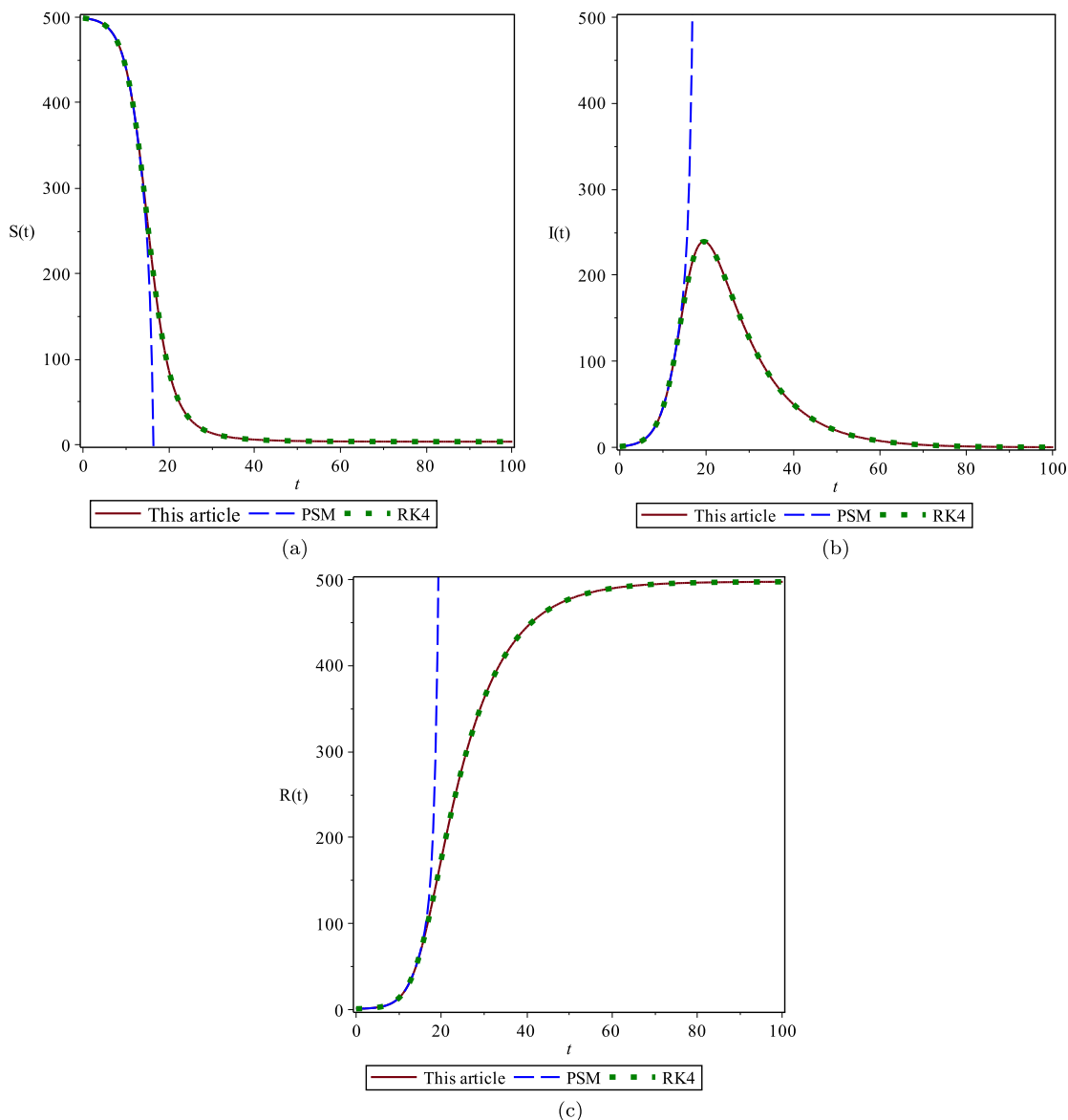
Fig. 1. Long-term behaviour of the solutions $S(t), I(t), R(t)$ of the epidemic model using the proposed Multistage-PSM (solid line), PSM (dash) and Runge-Kutta method (dots) for $\beta = 0.001, \gamma = 0.1, N_S = 499, N_I = 1, N_R = 1$.

Table 1. Convergent approximate analytical solution of the SIR model.

t	Convergent Approximate Analytical Solution $S(t)$, $I(t)$, $R(t)$
$0 \leq t \leq 0.01$	$S(t) = 499 - 0.499\,t - 0.099301\,t^2 - 0.013099249\,t^3$
	$I(t) = 1 + 0.399\,t + 0.079351\,t^2 + 0.01045421567\,t^3$
	$R(t) = 1 + 0.1\,t + 0.01995\,t^2 + 0.002645033333\,t^3$
$0.01 \leq t \leq 0.02$	$S(t) = 499.00499 - 0.499\,t - 0.099301\,(t - 0.01)^2 - 0.013099249\,(t - 0.01)^3$
	$I(t) = 0.99601 + 0.399\,t + 0.079351\,(t - 0.01)^2 + 0.01045421567\,(t - 0.01)^3$
	$R(t) = 0.999 + 0.1\,t + 0.01995\,(t - 0.01)^2 + 0.002645033333\,(t - 0.01)^3$
$\vdots$	$\vdots$
$99.99 \leq t \leq 100$	$S(t) = 548.89501 - 0.499\,t - 0.099301\,(t - 99.99)^2 - 0.013099249\,(t - 99.99)^3$
	$I(t) = -38.89601 + 0.399\,t + 0.079351\,(t - 99.99)^2 + 0.01045421567\,(t - 99.99)^3$
	$R(t) = -8.999 + 0.1\,t + 0.01995\,(t - 99.99)^2 + 0.002645033333\,(t - 99.99)^3$

the right-hand side of (26) is already a polynomial function so that on letting

$$y(t) = \sum_{i=0}^{m} y_i t^i,$$

the recursive relation for computing the series coefficients follows from (26) as

$$y_{i+1} = \frac{\sum_{j=0}^{i} y_j y_{i-j} - \sum_{j=0}^{i} \left(\sum_{k=0}^{j} y_k y_{j-k} \right) y_{i-j}}{i+1}, \quad y_0 = \delta.$$

Thus, we obtain

$$y_1 = -\delta^2(\delta - 1), \quad y_2 = \frac{1}{2}\delta^3(3\delta - 2)(\delta - 1),$$

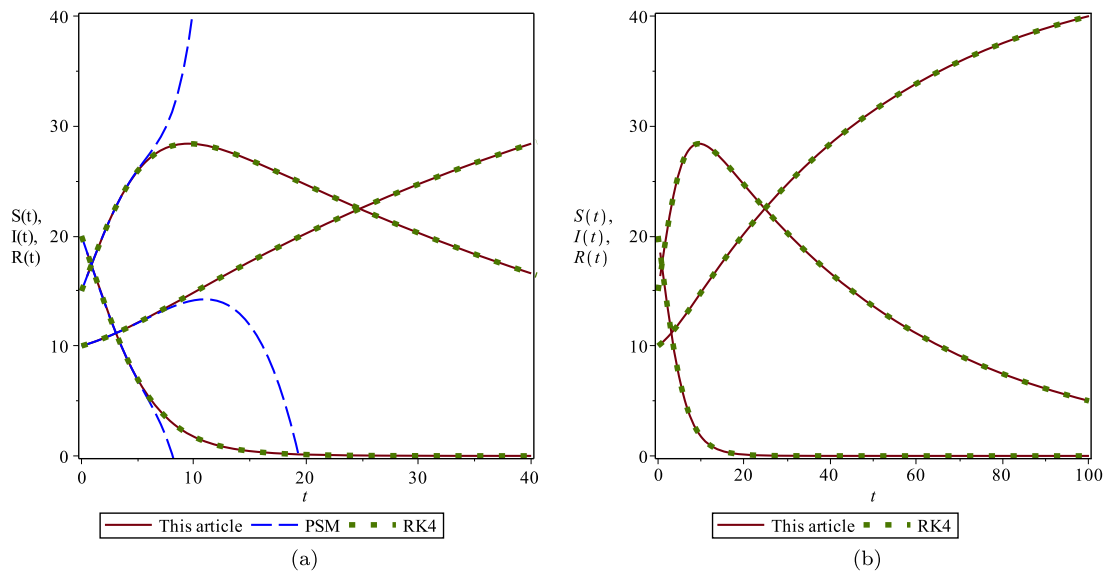$$y_3 = -\frac{1}{6}\delta^4\left(15\delta^2 - 20\delta + 6\right)(\delta - 1)$$

Fig. 2. Comparison of the long-term behaviour of the solutions $S(t), I(t), R(t)$ of the epidemic model using the proposed Multistage-PSM (solid line), PSM (dash) and Runge-Kutta method (dots) for $N_S = 20, N_I = 15, N_R = 10, \beta = 0.01, \gamma = 0.02$.

$$y_4 = \frac{1}{24}\delta^5\left(105\,\delta^3 - 210\,\delta^2 + 130\,\delta - 24\right)(\delta - 1), \ldots$$

which yields the approximate analytical solution

$$y(t) = \delta - \delta^2(\delta - 1)t + \frac{1}{2}\delta^3\left(3\,\delta - 2\right)(\delta - 1)t^2$$
$$- \frac{1}{6}\delta^4\left(15\,\delta^2 - 20\,\delta + 6\right)(\delta - 1)t^3 + \ldots$$

For $\delta = 0.01$, we get the analytical solution

$$y^{(1)}(t) = 0.01 + 9.9 \times 10^{-5}\,t + 9.7515 \times 10^{-7}\,t^2 + 9.572475 \times 10^{-9}\,t^3$$
$$+ 9.372369188 \times 10^{-11}\,t^4 + 9.157127880 \times 10^{-13}\,t^5 + \ldots$$

on the first subinterval $[0, 0.2]$. Graphical illustration of the above solution is displayed in Fig. 3. The series solution diverges for $t \geq 70$, making multistage implementation imperative to obtain a convergent solution in the entire domain of the problem. On the next subinterval $[0.2, 0.4]$, with initial condition $y^{(2)}(0) = y^{(1)}(0.2) = 0.01001983909$, the PSM algorithm is repeated to get the next approximate analytical solution.

5. Application to chaotic Lorenz problem

In 1963, Lorenz [33], proposed a model in form of nonlinear differential equations describing the flow of two-dimensional fluid cell between two parallel plates at different temperatures called the Lorenz system

$$\frac{dx}{dt} = a(y - x),$$
$$\frac{dy}{dt} = -xz + bx - y,$$
$$\frac{dz}{dt} = xy - cz$$

subject to initial conditions $x(0) = x_0, y(0) = y_0, z(0) = z_0$. Common techniques for solving this chaotic system are numerical integrators and multistage methods such as multistage spectral relaxation method [2]. Here, efficacy of our method is shown by solving the system with parameter values $a = 10, b = 28, c = 8/3$, and initial conditions $x_0 = 1, y_0 = 5, z_0 = 10$.

5.1. Convergence and solution by Multistage Parker-Sochacki Method

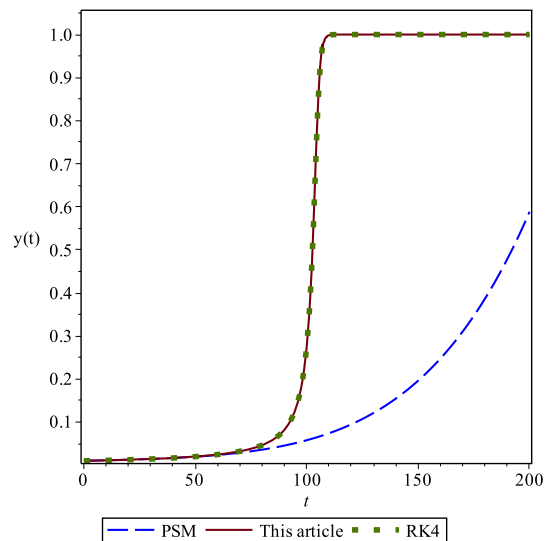Following Theorem 2.1, the convergence parameters are computed as

Fig. 3. Comparison of the long-term behaviour of the solution $y(t)$ of the combustion equation using the proposed Multistage Parker-Sochacki Method (solid line), the conventional Parker-Sochacki Method (dash) and Runge-Kutta method (dots).

$$k = 2, \quad \Sigma p = \max\{2|a|, |b| + 2, |c| + 1\}, \quad \alpha = \max\{1, |x_0|, |y_0|, |z_0|\}$$

so that the smallest interval of convergence for the series solution of the Lorenz problem is $\left(-\frac{1}{M}, \frac{1}{M}\right)$ where $M = (k - 1)\Sigma p\,\alpha^{k-1}$. With $a = 10, b = 28, c = 8/3$, and initial conditions $x_0 = 1, y_0 = 5, z_0 = 10$, we get $\Sigma p = \max\{20, 30, 11/3\} = 30$ and $\alpha = \max\{1, 1, 5, 10\} = 10$ so that $M = 300$ and the smallest interval of convergence obtained as $(-h, h)$ with $h < 0.003$. Thus, with $h = 0.002$, the integration interval is subdivided into subintervals, over which the problem is solved following the proposed multistage implementation. It is pertinent to remark that although chosen heuristically, the step-size h was also chosen to be 0.001 in [2], well within the now a-priorly computed interval of convergence.

Consequently, on the first subinterval, the subproblem is solved using PSM as follows. On noting that the Lorenz problem is already in the required polynomial form, the coefficients in the expansions

$$x(t) = \sum_{i=0}^{m} x_i t^i, \quad y(t) = \sum_{i=0}^{m} y_i t^i, \quad z(t) = \sum_{i=0}^{m} z_i t^i$$

are obtained directly through the recursions

$$x_{i+1} = \frac{10(y_i - x_i)}{i+1}, \quad x_0 = 1,$$

$$y_{i+1} = \frac{-\sum_{j=0}^{i} x_j z_{i-j} + 28x_i - y_i}{i+1}, \quad y_0 = 5,$$

$$z_{i+1} = \frac{\sum_{j=0}^{i} x_j y_{i-j} - \frac{8}{3} z_i}{i+1}, \quad z_0 = 10.$$

Approximate analytical solutions to the Lorenz problem are thus obtained as

$$x(t) = 1 + 40t - 135t^2 + \frac{14980}{9}t^3 - \frac{635075}{108}t^4 + \frac{7432397}{324}t^5 + \dots$$

$$y(t) = 5 + 13t + \frac{1093}{3}t^2 - \frac{37135}{54}t^3 + \frac{3621947}{648}t^4 - \frac{14633219}{1215}t^5 + \dots$$

$$z(t) = 10 - \frac{65}{3}t + \frac{2437}{18}t^2 - \frac{4096}{81}t^3 + \frac{10005631}{1944}t^4 - \frac{107999275}{5832}t^5 + \dots$$

These approximate solutions are plotted and displayed (dash) in Fig. 4(a), Fig. 4(c) and Fig. 4(e). The result of the multistage implementation (solid line) is also displayed against numerical result obtained via the Runge-Kutta Method of order 4 (dots) in Fig. 4(b), Fig. 4(d) and Fig. 4(f).

6. Application to the Troesch's problem

We consider the Troesch's boundary value problem arising in the study of confinement of a plasma column by a radiation pressure [34], and in the theory of gas porous electrodes [35, 36] given by

$$y''(x) = \lambda \sinh \lambda y(x), \quad y(0) = 0, \ y(1) = 1. \tag{27}$$

The Troesch's problem has been reported to have singularity which lies within the interval of integration $[0,1]$ whenever $y'(0) > 8e^{-\lambda}$, [1]. Consequent on this observation, both the numerical and approximate analytical solutions to the Troesch's problem are highly sensitive to the Troesch parameter λ. Several methods have been proposed to solving the Troesch's problem, see [37], [38], [39], [40], [41] and [42] for particular cases of small values of λ. Recently, a numerical method was proposed in [1], for the first time, to solve the Troesch's problem for large value $\lambda = 100$.

In this section, the strength of our method will be shown by solving the Troesch's problem with $\lambda = 30$. We consider the initial-valued equivalent formulation of the problem as

$$y''(x) = \lambda \sinh \lambda y(x), \quad y(0) = 0, \ y'(0) = \gamma \tag{28}$$

where for $\lambda > 20$, the unknown $\gamma = y'(0)$ will be computed using an approximation formula as reported in [43] as

$$y'(0) = 10^{-12} e^{29.71 - \lambda}, \quad \lambda > 20, \quad y'(0) < 10^{-4}.$$

In fact, for $\lambda = 30$, γ is computed as $7.482635676 \times 10^{-13}$.

6.1. Convergence and solution by Multistage Parker-Sochacki Method

The function $\lambda \sinh \lambda y$ on the right side of (28) is not a polynomial function. The equation has to be recast as a projectively polynomial system of differential equation. For that purpose, we introduce the variables

$$y' = u, \quad v = \sinh \lambda y, \quad w = \cosh \lambda y$$

so that from (28) one obtains

$$y' = u, \quad u' = \lambda v, \quad v' = \lambda u w, \quad w' = \lambda u v \tag{29}$$

with initial conditions

$$y(0) = 0, \quad u(0) = \gamma, \quad v(0) = 0, \quad w(0) = 1. \tag{30}$$

Now from the polynomial system (29)-(30), with $\lambda = 30$, $\gamma = 7.482635676 \times 10^{-13}$ the interval of convergence for series solution to (28) are computed. Here,

$$k = 2, \quad \alpha = \max\{1, \gamma\} = 1, \quad \Sigma p = \max(1, \lambda) = \lambda = 30.$$

Hence, $M = 30$, so that the step-size $h < 1/30 = 0.033$ must be chosen to have a convergent approximate analytical solution to the Troesch's problem. Here, we chose $h = 10^{-4}$ so that on the first subinterval, the problem is solved using the PSM as follows.

If we let

$$y = \sum_{i=0}^{m} y_i x^i, \quad u = \sum_{i=0}^{m} u_i x^i, \quad v = \sum_{i=0}^{m} v_i x^i, \quad w = \sum_{i=0}^{m} w_i x^i, \tag{31}$$

then it follows from the polynomial system (29)-(30) that

$$y_{i+1} = \frac{u_i}{i+1}, \quad u_{i+1} = \lambda \frac{v_i}{i+1}, \quad v_{i+1} = \lambda \frac{\sum_{j=0}^{i} u_{i-j} w_j}{i+1},$$

$$w_{i+1} = \lambda \frac{\sum_{j=0}^{i} u_{i-j} v_j}{i+1}, \quad i = 0, 1, 2, \dots, m. \tag{32}$$

Solving these recursions with the initial conditions (30), and substituting into (31), we obtain

$$y(x) = 7.482635676 \times 10^{-13} x + 0.0000000000112239535140000 x^3$$

$$+ 0.0000000000505077908130000 x^5 + \dots$$

The above approximate analytical solution as well as result of multistage implementation are shown in Fig. 5(a) and Fig. 5(b).

7. Conclusion

Based on Parker-Sochacki Method, a new multistage technique has been developed in this article to find an approximate analytical solution of initial and boundary valued problems of nonlinear differential equations. By computing a-priorly a convergence quantity in terms of the parameters of the problem, subproblems are defined and solved from one interval to the next in a multistage fashion. Within the interval of convergence, the method was highly accurate and efficient.

The proposed method has been applied to both the initial value problem and boundary value problems. The results obtained, shown graphically, showed that the proposed method gives appreciable accuracy when compared with results obtained via the numerical Runge-Kutta method of order 4 (RK4). The computational simplicity, and yet high accuracy of the proposed method makes it a viable alternative to solving highly nonlinear initial value problem, and even boundary value problems via the shooting technique.

Declarations

Author contribution statement

Competing interest statement

The authors declare no conflict of interest.

Additional information
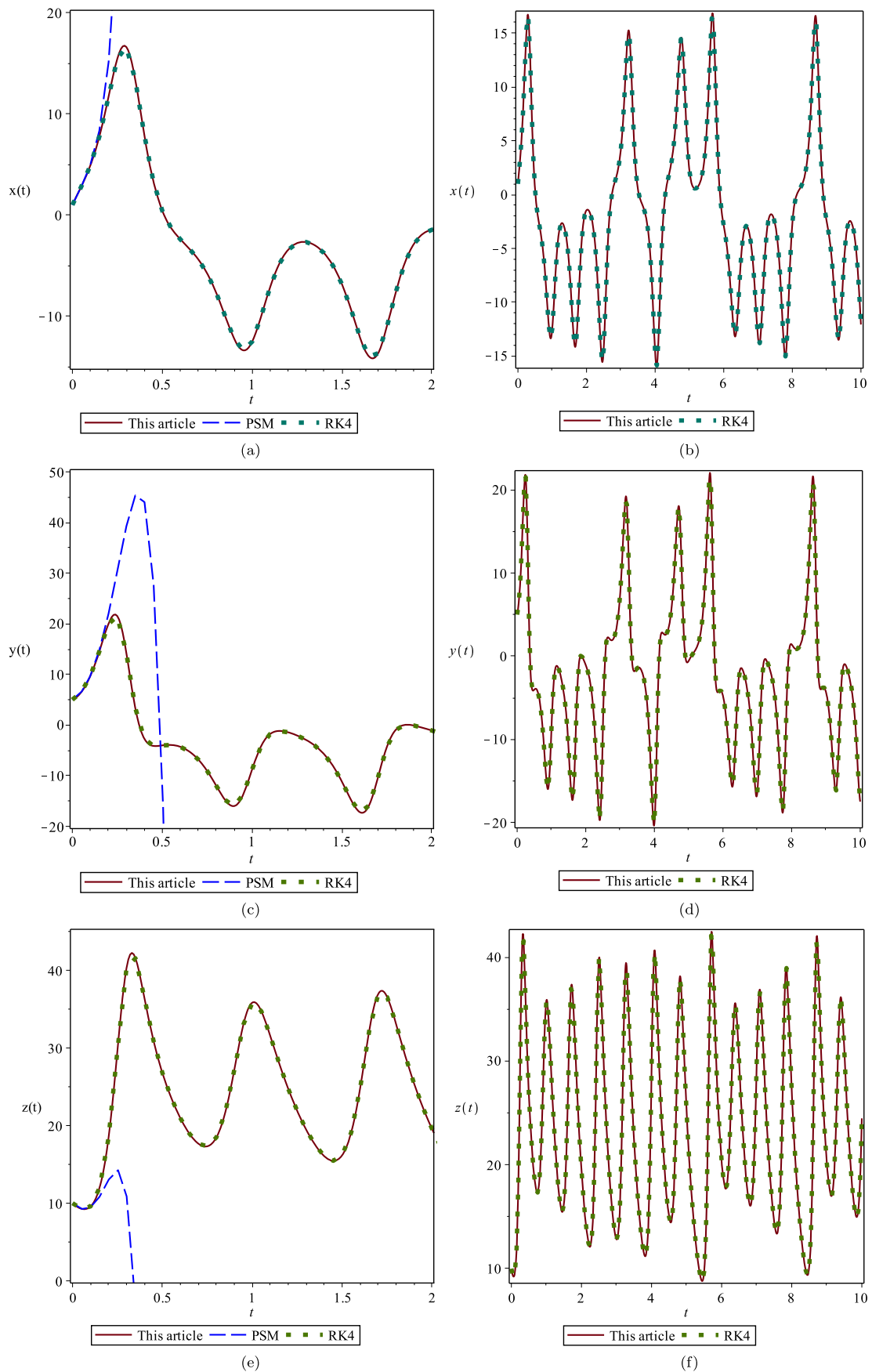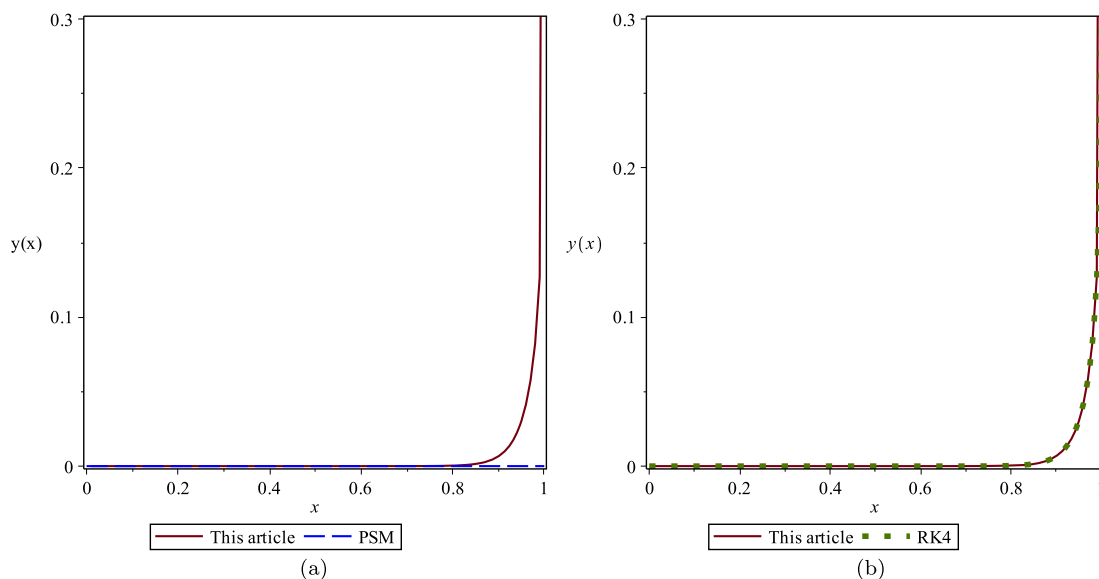
No additional information is available for this paper.

Fig. 4. Comparison of the long-term behaviour of the solution $x(t)$, $y(t)$ and $z(t)$ of the Lorenz's problem using the proposed Multistage Parker-Sochacki Method (solid line), the conventional Parker-Sochacki Method (dash) and Runge-Kutta method (dots).

Fig. 5. (a.) Comparison of the solution $y(x)$ of the Troesch's problem for $\lambda = 30$ using the proposed Multistage Parker-Sochacki Method (solid line) and the conventional Parker-Sochacki Method (dash). (b.) Comparison of the solution $y(x)$ of the Troesch's problem for $\lambda = 30$ using the proposed Multistage Parker-Sochacki Method (solid line) and Runge-Kutta method (dots).

Acknowledgements

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