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FOUNDATIONS

Limit properties in the metric semi-linear space of picture fuzzy numbers

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Abstract

The picture fuzzy set (PFS) just appeared in 2014 and was introduced by Cuong, which is a generalization of intuitionistic fuzzy sets (Atanassov in Fuzzy Sets Syst 20(1):87–96, 1986) and fuzzy sets (Zadeh Inf Control 8(3):338–353, 1965). The picture fuzzy number (PFN) is an ordered value triple, including a membership degree, a neutral-membership degree, a non-membership degree, of a PFS. The PFN is a useful tool to study the problems that have uncertain information in real life. In this paper, the main aim is to develop basic foundations that can become tools for future research related to PFN and picture fuzzy calculus. We first establish a semi-linear space for PFNs by providing two new definitions of two basic operations, addition and scalar multiplication, such that the set of PFNs together with these two operations can form a semi-linear space. Moreover, we also provide some important properties and concepts such as metrics, order relations between two PFNs, geometric difference, multiplication of two PFNs. Next, we introduce picture fuzzy functions with a real domain that is also known as picture fuzzy functions with time-varying values, called geometric picture fuzzy function (GPFFs). In this framework, we give definitions about the limit of GPFFs and sequences of PFN. The important limit properties are also presented in detail. Finally, we prove that the metric semi-linear space of PFNs is complete, which is an important property in the classical mathematical analysis.

Keywords Picture fuzzy numbers · Metric semi-linear space · Picture fuzzy calculus · Geometric picture fuzzy functions

1 Introduction

In 1965, Zadeh (1965) first introduced the concept of fuzzy sets (FS) to deal with the uncertainties that appear in many real-world phenomena. It has become the focus of much research in both theoretical and applied fields. In 1986, Atanassov (1986) presented about intuitionistic fuzzy sets (IFS), he came up with the idea of defining a fuzzy set by giving a membership function and a non-member function such that the total between degrees of membership and non-membership is no more than 1. It is a generalization of Zadeh's fuzzy sets that can be a great idea when describing a problem with a variable language (fuzzy) and is pretty useful in situations when a description of a problem by a linguistic variable is given in terms of a membership function only seems too rough. Due to the flexibility of intuitionistic fuzzy sets in handling uncertainty, they are tools for human consistent reasoning under imperfectly defined facts and vague. In 1984, Takeuti and Titani (1984) by using the same terminology of "intuitionistic fuzzy sets" but with differences in meaning built the concept of intuitionistic fuzzy logic

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and intuitionistic fuzzy sets. In 1989, Atanassov and Gargov (1989) presented the concept of interval-valued intuitionistic fuzzy sets. In recent years, the intuitionistic fuzzy theory has been applied to many fields, such as decision-making (Atanassov et al. 2005; Chen 2011), medical diagnosis (De et al. 2001; Shinoj and Sunil 2012). Based on the intuitionistic fuzzy sets (IFS), Xu and Yager (2006) introduced the concept of intuitionistic fuzzy number (IFN), which is an ordered non-negative value pair consisting of a membership degree and the non-membership degree of an IFS, and their basic operations. From IFNs and their basic operations, Lei and Xu (2017) developed the fundamental theories of calculus, which are called intuitive fuzzy-calculus. Phu et al (2018, 2019, 2021) continued to develop a semi-linear space for IFNs and introduced its applications.

In 2014, Cuong (2014) introduced the concept of picture fuzzy sets (PFS), which is a generalization of the traditional fuzzy sets (FS) and the intuitionistic fuzzy sets (IFS). The PFS defines a fuzzy set by giving a membership function, a non-member function, and a neutral-membership function with a membership degree, a non-membership degree, and a neutral-membership degree, respectively. Although many uncertain problems in the real world have been effectively handled by the tools of Zadeh's fuzzy theory and Atanassov's intuitionistic fuzzy sets, many of them still need the tools of the picture fuzzy theory. Cuong (2014) gave a practical example, which is voting. The idea of the three membership degrees of a PFS can be seen in the case when a voter has to make his or her decision involving more answers like yes, abstain, no. In recent years, there have been many research directions on PFS such as: logical operations and algebraic (Cuong 2014; Cuong and Kreinovich 2013; Dutta and Ganju 2017), fuzzy clustering (Son 2016; Thong and Son 2015), decision-making (Khan et al. 2019; Si et al. 2019; Wei 2017), nonlinear programming Phu et al. (2021). As in a similar way, based on the PFS, we also get the definition of picture fuzzy number (PFN) as an ordered non-negative triple consisting of a membership degree, a non-membership degree, and a neutral-membership degree of a PFS. A PFN is a basic element of a PFS and is a useful tool to help us study more deeply the characteristics and properties of PFS. The PFNs are used to study decision-making theories in the picture fuzzy environment. For example, Wei (2017) studied multiple attribute decision-making (MADM) problems based on picture fuzzy information in the form of PFNs. Khan et al. (2019) presented a logarithmic approach to MADM problems with PFNs.

As we know along with the development of logic and algebraic theory for fuzzy-theory, then calculus-theory also had many powerful strides in recent years. For Zadeh's fuzzy theory, the calculus-theory in this fuzzy environment is also known by the name, fuzzy mathematics. In which, fuzzy numbers [can see Diamond and Kloeden (2000)] are basic and the main tool to develop for this field. For intuitionistic fuzzy theory, Lei and Xu (2017) used IFN as a tool for developing calculus theory in an intuitionistic fuzzy environment. For picture fuzzy-theory, development for the calculus-theory in picture fuzzy environments is still very new. Because basic operations (most importantly, addition and scalar multiplication) in Zadeh's fuzzy environment and Atanassov's intuitionistic environment have been defined and perfected in recent years [(can see in Dubois and Prade (1982), Phu et al. (2019), Xu and Yager (2006)], it has helped calculus theory that has a basis for development. Meanwhile, there are not perfect definitions for the basic operations in the picture fuzzy environment. Although Wei (2017) provided the basic operations of picture fuzzy numbers based on Xu's operation Xu and Yager (2006) for intuitionistic fuzzy number, some of them, namely scalar multiplication and addition, have some limitations. In this paper, we will show the limitation of Wei's two operations, addition, and scalar multiplication, and will provide two new operations with more advantages. The highlight is that the set of picture fuzzy numbers together with these two new operations can become a semi-linear space. This will be the framework for the development of future studies on picture fuzzy calculus theory. At the same time, we define the metric space for PFNs and present the concepts of limits and their properties because the limit is an important basis of picture fuzzy calculus. Finally, to confirm that the limit operators are well-defined, we have verified the completeness in metric space of PFNs.

The paper is organized as follows: In Sect. 2, we recall some knowledge to prepare for the next section. In Sect. 3, we divide the content into two subsections: For the first subsection, we point out some limitations in Wei's two operations and proceed to construct two new operations, addition and scalar multiplication, such that the set of PFNs together these operations becomes a semi-linear space for PFNs. On the other hand, we also present some related concepts and their properties such as order relations between two PFNs, metrics, geometric difference, multiplication of two PFNs. For the second subsection, we first define a function whose value changes over time, called the geometric picture fuzzy function. Next, we introduce the definition of limit for this function and the sequence of PFNs. Their properties are also shown. Finally, we prove that the metric space of PFNs is complete.

2 Preliminaries

For a start, we recall the concept of picture fuzzy sets, which was introduced by Cuong (2014).

Definition 2.1 (Cuong 2014) Let Ω be a universe set, then a set called a picture fuzzy set (PFS), which is defined as follows:

$$X = \{(\omega, \mu_X(\omega), \eta_X(\omega), \nu_X(\omega)) \mid \omega \in \Omega\}$$

where μ_X:$\Omega \to [0, 1]$ is called a membership function, ν_X:$\Omega \to [0, 1]$ is called a non-membership function, η_X: $\Omega \to [0, 1]$ is called a neutral-membership function with a membership degree $\mu_X(\omega)$, a non-membership degree $\nu_X(\omega)$ and a neutral-membership degree of element $\omega \in \Omega$, respectively, such that

$$0 \leq \mu_X(\omega) + \eta_X(\omega) + \nu_X(\omega) \leq 1.$$

Definition 2.2 a picture fuzzy number (PFN) x, which is defined as follows:

$$x = (\mu_x, \eta_x, \nu_x)$$

where μ_x, η_x, and ν_x are nonnegative real numbers such that $0 \leq \mu_x,\ \eta_x,\ \nu_x \leq 1$ and

$$0 \leq \mu_x + \eta_x + \nu_x \leq 1.$$

PFNs have been researched in recent years. For example, Wei (2017) researched multiple attribute decision-making (MADM) problems based on picture fuzzy information in the form of PFNs. He developed picture fuzzy aggregation operators for PFNs from geometric and arithmetic operations, then he used them to solve the picture fuzzy MADM problems. Khan et al. (2019) presented a logarithmic approach to MADM problem with picture fuzzy information in the form of PFNs. They developed a series of picture fuzzy logarithmic aggregation operators for PFNs and provided a novel algorithm technique to solve the MADM problems with picture fuzzy information. Si et al. (2019) provided a method for comparing and ranking PFNs. Furthermore, Wei (2017) also defined some basic operators of PFNs as follows:

Definition 2.3 (Wei 2017) Let $x = (\mu_x, \eta_x, \nu_x)$ and $y = (\mu_y, \eta_y, \nu_y)$ be two PFNs, then

1. $\overline{x} = (\nu_x, \eta_x, \mu_x)$ is called the reverse element of x;
2. $x \oplus y = \left(\mu_x + \mu_y - \mu_x\mu_y,\ \eta_x\eta_y,\ \nu_x\nu_y\right)$;
3. $x \otimes y = \left(\mu_x\mu_y,\ \eta_x + \eta_y - \eta_x\eta_y,\ \nu_x + \nu_y - \nu_x\nu_y\right)$;
4. $\lambda x = \left(1 - (1 - \mu_x)^\lambda,\ \eta_x^\lambda,\ \nu_x^\lambda\right)$ with $\lambda > 0$;
5. $x^\lambda = \left(\mu_x^\lambda,\ 1 - (1 - \eta_x)^\lambda,\ 1 - (1 - \nu_x)^\lambda\right)$ with $\lambda > 0$;
6. $x \cap y = \left(\min\{\mu_x, \mu_y\},\ \max\{\eta_x, \eta_y\},\ \max\{\nu_x, \nu_y\}\right)$;
7. $x \cup y = \left(\max\{\mu_x, \mu_y\},\ \min\{\eta_x, \eta_y\},\ \min\{\nu_x, \nu_y\}\right)$.

Next, we recall the common definition of a linear space (or vector space) over a scalar field (which may be real or complex).

Definition 2.4 Let F be a scalar field, then a linear space (or vector space) over the field F is a set A together with two operations, which addition and scalar multiplication are defined:

$$\begin{aligned}
\text{(addition)} + : A \times A &\longrightarrow A \\
x, y \in A &\longmapsto x + y \in A \\
\text{(scalar multiplication)} \cdot : F \times A &\longrightarrow A \\
\lambda \in F,\ x \in A &\longmapsto \lambda x \in A
\end{aligned}$$

that satisfy the eight axioms listed below. Let x, y, and z be belong to A, and λ and β scalars in F.

1. $x + (y + z) = (x + y) + z$;
2. $x + y = y + x$;
3. $x + 0 = x$ with $0 \in A$, called a neutral element of A;
4. $x + (\overline{x}) = 0$ with $\overline{x} \in A$, called a reverse element of x;
5. $\lambda(\beta x) = (\lambda\beta)x$;
6. $1x = x$, which 1 is identity element of F;
7. $\lambda(x + y) = \lambda x + \lambda y$;
8. $(\lambda +_F \beta)x = \lambda x +_F \beta x$, where $+_F$ is addition of the field F.

Remark 2.5 In fact, there are many sets with their two operations (addition and scalar multiplication) that do not satisfy the axiom in item (4.) of Definition 2.4. For example, for the set of interval numbers, let $X = [\underline{X}, \overline{X}]$ be a interval number with $\underline{X} < \overline{X}$ and $-X = [-\overline{X}, -\underline{X}]$ be reverse element of X, then $X + (-X) \neq 0 = [0, 0]$ [(can see in Moore et al. (2009)]. The same for the set of Zadeh's fuzzy numbers, let $\omega = (a, b, c)$ be a triangular fuzzy number with $a < b < c$ and $-\omega = (-c, -b, -a)$ be also reverse element of ω, then $\omega + (-\omega) \neq 0 = (0, 0, 0)$ [can see in Dijkman et al. (1983)]. If the axiom in item (4.) of Definition 2.4 does not satisfy, which is $x + (\overline{x}) \neq 0$, then the set A will be called a semi-linear space. We provide the following definition for a semi-linear space [can see in Galanis (2009); Phu et al. (2019); Worth (1970)] with scalar field F, which is the real number field $\mathbb{R}$.

Definition 2.6 (Galanis 2009; Phu et al. 2019; Worth 1970) A semi-linear space is a set B together two operations, addition and scalar multiplication with nonnegative reals, are defined:

- The first operation: addition, denoted by $+$, such that to every pair $x, y \in B$, there correspond a element $x + y \in B$,
- The second operation: scalar multiplication of $x \in B$ with an element $\lambda \in \mathbb{R}^+$, denoted by $\lambda x \in B$,

such that satisfy the following properties for every $x, y \in B$ and $\lambda, \beta \in \mathbb{R}^+$:

(1) $x + (y + z) = (x + y) + z$;
(2) $x + y = y + x$;
(3) $x + 0 = x$ with $0 \in B$, called a neutral element of B;
(4) $\lambda(x + y) = \lambda x + \lambda y$;
(5) $(\lambda +_{\mathbb{R}} \beta)x = \lambda x +_{\mathbb{R}} \beta x$, where $+_{\mathbb{R}}$ is addition of the field $\mathbb{R}$
(6) $(\lambda\beta)x = \lambda(\beta x)$;
(7) $1x = x$ and $0^* x = 0$ with $1 \in \mathbb{R}^+$ and $0^* \in \mathbb{R}^+$ are identity element and neutral element of scalar multiplication, respectively.

Remark 2.7 From Definition 2.6 and Remark 2.5, we can see that the set of interval numbers and the set of fuzzy numbers in Zadeh's sense together their two operations (addition and scalar multiplication) are semi-linear spaces. We recently defined the new addition and scalar multiplication for intuitionistic fuzzy numbers (IFNs), which are the basic elements of Atanassov's fuzzy sets Atanassov (1986) so that the set of IFNs becomes a semi-linear space [(can see in Phu and Hung (2018); Phu et al. (2019)]. In the next section, we will extend these two operations for picture fuzzy numbers (PFNs), which are also the basic elements of picture fuzzy sets, such that the set of PFNs also becomes a semi-linear space.

Definition 2.8 (Rudin 1976) A metric space is a set C together a metric or a distance function, which is defined:

$$\text{a mapping } \delta : C \times C \longrightarrow \mathbb{R}$$
$$x, y \in C \longmapsto \delta(x, y) \in \mathbb{R}$$

such that satisfy the following properties for every $x, y \in C$:

(1) $\delta(x, y) > 0$ if $x \neq y$ and $\delta(x, y) = 0$ if $x = y$;
(2) $\delta(x, y) = \delta(y, x)$;
(3) $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$ with $z \in C$

Definition 2.9 (Rudin 1976) Let $x \in D \subset \mathbb{R}$, Suppose the real-valued function $f(x)$ is defined when x is near the number $x_0 \in D$. Then, we define $\lim_{x \to x_0} f(x) = L$, or $f(x) \to L$ as $x \to x_0$, and say that the limit of $f(x)$, as x approaches x_0, equals L. Simultaneously, $\lim_{x \to x_0} f(x) = L$ if and only if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 \leq |x - x_0| \leq \delta$ then $|f(x) - L| \leq \varepsilon$.

Definition 2.10 (Rudin 1976) Let $\{x_n\} \subset \mathbb{R}$ be a sequence of n real numbers with $n \in \mathbb{N}$. Then, we define that the sequence $\{x_n\}$ has the limit L and is denoted by $\lim_{n \to +\infty} x_n = L$ or $x_n \to L$ as $n \to +\infty$. Simultaneously, if for every $\varepsilon > 0$ there is a positive integer N such that if $n > N$ then $|x_n - L| \leq \varepsilon$.

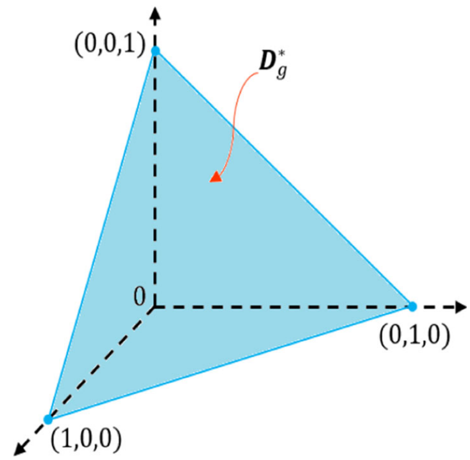

Fig. 1 The image illustrating for D_g^*, which is the set of PFNs and where the PFSs get values

3 Main result

3.1 The metric semi-linear space for PFNs

In this subsection, we will introduce some new concepts and definitions such as a set of PFNs, a neutral element and a reverse element of this set, new addition and scalar multiplication for PFNs. Then, we will prove this set together the new two operations to become a semi-linear space by verifying that these two new operations satisfy the seven axioms in Definition 2.6.

For convenience, we put $x = (x_1, x_2, x_3)$ instead of $x = (\mu_x, \eta_x, \nu_x)$ in Definition 2.2.

Definition 3.1 Let $x = (x_1, x_2, x_3)$ be an any PFN, then the following set

$$D_g^* = \{x = (x_1, x_2, x_3) |\, 0 \leq x_1 + x_2 + x_3 \leq 1\}, \qquad (3.1)$$

is called the set of PFNs. Where $x_1 = \mu_x \in [0, 1]$ correspond to a membership degree of x, $x_2 = \eta_x \in [0, 1]$ correspond to neutral-membership degree of x, and $x_3 = \nu_x \in [0, 1]$ correspond to non-membership degree of x.

In this study, we describe a PFN as an ordered non-negative triples (x_1, x_2, x_3) in $D_g^* \subset [0, 1]^3$. In addition, any PFN $x = (x_1, x_2, x_3)$, we have a box $B_x \subset D_g^*$ and is defined as the following form:

$$B_x = \{(v, w, u) | 0 \leq v \leq x_1, 0 \leq w \leq x_2, 0 \leq u \leq x_3\}.$$

Illustrations for D_g^* and an element $x = (x_1, x_2, x_3)$ in D_g^* are shown in Figs. 1 and 2 .

Definition 3.2 The reverse element of $x = (x_1, x_2, x_3)$ in D_g^* is the element $\overline{x} = (x_3, x_2, x_1)$ in D_g^*. Simultaneously, the neutral element in D_g^* is $\theta = (0, 0, 0)$.

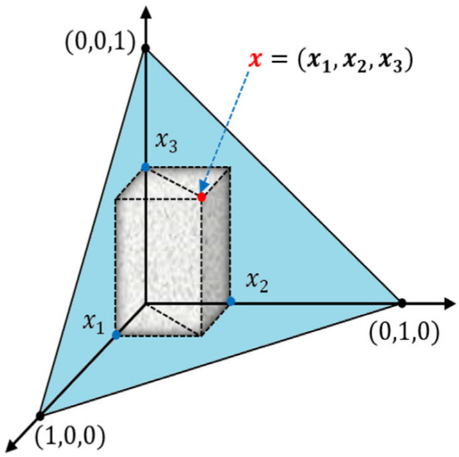

Fig. 2 The box B_x and a geometric interpretation of PFN $x = (x_1, x_2, x_3)$ in D_g^*

For two base operations of PFNs, addition and scalar multiplication, Wei provided these two operations in item (2.) and (4.) of Definition 2.3 [(can see in Wei (2017)]. However, they have many limitations to help the set of PFNs that become a semi-linear space. To see these limitations, we will test these two operations with the seven axioms in Definition 2.6.

Proposition 3.3 *Let two operations, addition and scalar multiplication with nonnegative reals, be defined:*

- *The addition: every two PFNs $x, y \in D_g^*$ with $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, there correspond a element $x \oplus y \in D_g^*$ and $x \oplus y = (x_1 + y_1 - x_1y_1,\ x_2y_2,\ x_3y_3)$, where addition is denoted by $\oplus$,*
- *The scalar multiplication: for a PFN $x \in D_g^*$ with $\lambda > 0$, there correspond a element $\lambda x \in D_g^*$ and $\lambda x = \left(1 - (1 - x_1)^{\lambda},\ x_2^{\lambda},\ x_3^{\lambda}\right)$.*

Then, they satisfy the following properties for every $x, y, z \in D_g^*$ and $\lambda, \beta > 0$:

(1.) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(2.) $x \oplus y = y \oplus x$;
(3.) $\lambda(x \oplus y) = \lambda x \oplus \lambda y$;
(4.) $(\lambda\beta)x = \lambda(\beta x)$;
(5.) $1x = x$.

Proof Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, and $z = (z_1, z_2, z_3)$ be the PFNs and $\lambda, \beta > 0$. We obtain

$$\begin{aligned}(1.)\ x \oplus (y \oplus z) &= x \oplus (y_1 + z_1 - y_1z_1,\ y_2z_2,\ y_3z_3)\\ &= (x_1 + (y_1 + z_1 - y_1z_1)\\ &\quad - x_1(y_1 + z_1 - y_1z_1),\ x_2y_2z_2,\ x_3y_3z_3)\\ &= (x_1 + y_1 + z_1 - y_1z_1\\ &\quad - x_1y_1 - x_1z_1 + x_1y_1z_1,\ x_2y_2z_2,\ x_3y_3z_3)\\ &= ((x_1 + y_1 - x_1y_1) + z_1\\ &\quad - z_1(x_1 + y_1 - x_1y_1),\ x_2y_2z_2,\ x_3y_3z_3)\\ &= (x \oplus y) \oplus z;\end{aligned}$$

$$\begin{aligned}(2.)\ x \oplus y &= (x_1 + y_1 - x_1y_1,\ x_2y_2, x_3y_3)\\ &= (y_1 + x_1 - y_1x_1,\ y_2x_2, y_3x_3) = y \oplus x;\end{aligned}$$

$$\begin{aligned}(3.)\ \lambda(x \oplus y) &= \lambda\,(x_1 + y_1 - x_1y_1,\ x_2y_2,\ x_3y_3)\\ &= \left(1 - (1 - x_1 - y_1 + x_1y_1)^{\lambda},\ (x_2y_2)^{\lambda},\ (x_3y_3)^{\lambda}\right)\\ &= \left(1 - ((1 - x_1)(1 - y_2))^{\lambda},\ (x_2y_2)^{\lambda},\ (x_3y_3)^{\lambda}\right)\\ &= \left(1 - (1 - x_1)^{\lambda}(1 - y_2)^{\lambda},\ x_2^{\lambda}y_2^{\lambda},\ x_3^{\lambda}y_3^{\lambda}\right)\\ &= \left(1 + [1 - (1 - x_1)^{\lambda} - (1 - y_2)^{\lambda} - 1 + (1 - x_1)^{\lambda}\right.\\ &\quad \left.+ (1 - y_2)^{\lambda}] - (1 - x_1)^{\lambda}(1 - y_2)^{\lambda},\ x_2^{\lambda}y_2^{\lambda},\ x_3^{\lambda}y_3^{\lambda}\right)\\ &= \left(1 - (1 - x_1)^{\lambda},\ x_2^{\lambda},\ x_3^{\lambda}\right)\\ &\quad \oplus \left(1 - (1 - y_1)^{\lambda},\ y_2^{\lambda},\ y_3^{\lambda}\right)\\ &= \lambda x \oplus \lambda y;\end{aligned}$$

$$\begin{aligned}(4.)\ (\lambda\beta)x &= \left(1 - (1 - x_1)^{\lambda\beta},\ x_2^{\lambda\beta},\ x_3^{\lambda\beta}\right)\\ &= \left(1 - (1 - x_1)^{\lambda\beta},\ x_2^{\lambda\beta},\ x_3^{\lambda\beta}\right)\\ &= \left(1 - (1 - 1 + (1 - x_1))^{\beta\lambda},\ \left(x_2^{\beta}\right)^{\lambda},\ \left(x_3^{\beta}\right)^{\lambda}\right)\\ &= \lambda(\beta x);\end{aligned}$$

$$(5.)\ 1x = \left(1 - (1 - x_1)^1,\ x_2^1,\ x_3^1\right) = (x_1,\ x_2,\ x_3) = x. \quad \square$$

Remark 3.4 In Proposition 3.3, we show the five axioms that Wei's two operations satisfy in the seven axioms of Definition 2.6. We now analyze the limitations of these two operations. Firstly, $(D_g^*, \oplus)$ is not a commutative cancellation semi-group with its neutral element because the D_g^* together the addition $\oplus$ in Proposition 3.3 has no a neutral element in D_g^*. This means that there is no neutral element 0^* in D_g^* such that $x \oplus 0^* = x$ with $x \in D_g^*$. Indeed, let us assume that there exists a 0^* element in D_g^* such that $x \oplus 0^* = x$ with $0^* = (\mu_{0*}, \eta_{0*}, \nu_{0*})$ and $x = (x_1, x_2, x_3)$. We get

$$\begin{cases} x_1 + \mu_{0*} - x_1\mu_{0*} &= x_1\\ x_2\eta_{0*} &= x_2\\ x_3\nu_{0*} &= x_3\end{cases} \Rightarrow \begin{cases}\mu_{0*} &= 0\\ \eta_{0*} &= 1\\ \nu_{0*} &= 1\end{cases}$$

However, 0^* does not belong to D_g^* since $\mu_{0*} + \eta_{0*} + \nu_{0*} = 2 > 1$, this contradicts the original assumption. Therefore, the neutral element does not exist in $(D_g^*, \oplus)$. Secondly, because $0^* = (0, 1, 1) \notin D_g^*$, $0x \notin D_g^*$, where 0 is neutral element of field $\mathbb{R}$. Indeed, we see that $0x = \left(1-(1-x_1)^0,\ x_2^0,\ x_3^0\right) = (0,1,1) = 0^*$. Finally, the scalar multiplication in Proposition 3.3 does not satisfy the axiom "distributivity of scalar multiplication concerning field addition", which means that the axiom (v) of Definition 2.6 does not satisfy. Indeed, let $x = (x_1, x_2, x_3)$ be a PFN and $\lambda, \beta > 0$. Then,

$$\begin{aligned}(\lambda +_{\mathbb{R}} \beta)x &= \left(1-(1-x_1)^{(\lambda +_{\mathbb{R}} \beta)},\ x_2^{(\lambda +_{\mathbb{R}} \beta)},\ x_3^{(\lambda +_{\mathbb{R}} \beta)}\right).\\ &= \left(1-(1-x_1)^{(\lambda+\beta)},\ x_2^{(\lambda+\beta)},\ x_3^{(\lambda+\beta)}\right).\\ \lambda x +_{\mathbb{R}} \beta x &= \left(1-(1-x_1)^{\lambda},\ x_2^{\lambda},\ x_3^{\lambda}\right)\\ &\quad +_{\mathbb{R}} \left(1-(1-x_1)^{\beta},\ x_2^{\beta},\ x_3^{\beta}\right)\\ &= \left(2-(1-x_1)^{\lambda}-(1-x_1)^{\beta},\ x_2^{\lambda}+x_2^{\beta},\right.\\ &\quad \left. x_3^{\lambda}+x_3^{\beta}\right).\end{aligned}$$

Therefore, $(\lambda +_{\mathbb{R}} \beta)x \neq \lambda x +_{\mathbb{R}} \beta x$ in general. For example, let $x = (0.25, 0.25, 0.25)$ and $\lambda = \beta = 2$. We get

$$\begin{aligned}(2+2)x &= \left(1-(1-0.25)^4,\ 0.25^4,\ 0.25^4\right)\\ &= (0.6836,\ 0.0039,\ 0.0039).\\ 2x+2x &= \left(1-(1-0.25)^2,\ 0.25^2,\ 0.25^2\right)\\ &\quad + \left(1-(1-0.25)^2,\ 0.25^2,\ 0.25^2\right)\\ &= (0.4375,\ 0.0625,\ 0.0625)\\ &\quad + (0.4375,\ 0.0625,\ 0.0625)\\ &= (0.875,\ 0.125,\ 0.125)\end{aligned}$$

and $(2+2)x \neq 2x+2x$. Thus, from the above limitations, we come to the conclusion that the set D_g^* together two base operations, addition and scalar multiplication, in Proposition 3.3 is not semi-linear space. Therefore, we will provide new two operations that make D_g^* to become a semi-linear space in what follows.

Definition 3.5 Let m elements $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(m)}$ in D_g^*. Then, the geometric addition, denoted by $\oplus_g$, of these m elements is a PFN $y = x^{(1)} \oplus_g x^{(2)} \oplus_g \ldots \oplus_g x^{(m)}$, if it exists, such that

$$y = \left(\frac{x_1^{(1)}+x_1^{(2)}+\cdots+x_1^{(m)}}{2^{m-p-1}},\ \frac{x_2^{(1)}+x_2^{(2)}+\cdots+x_2^{(m)}}{2^{m-p-1}},\ \frac{x_3^{(1)}+x_3^{(2)}+\cdots+x_3^{(m)}}{2^{m-p-1}}\right),$$

where p is the number of elements $x^{(k)} = (0,0,0) = \theta,\ k = 1, 2, \ldots, m$ and satisfy

(a) $\left(x^{(1)} \oplus_g x^{(2)} \oplus_g \ldots \oplus_g x^{(n)}\right) \oplus_g x^{(n+1)} \oplus_g \ldots \oplus_g x^{(m)}$

$$= \left(\frac{x_1^{(1)}+x_1^{(2)}+\cdots+x_1^{(m)}}{2^{m-p-1}},\ \frac{x_2^{(1)}+x_2^{(2)}+\cdots+x_2^{(m)}}{2^{m-p-1}},\ \frac{x_3^{(1)}+x_3^{(2)}+\cdots+x_3^{(m)}}{2^{m-p-1}}\right),$$

(b) $x^{(1)} \oplus_g x^{(2)} \oplus_g \ldots \oplus_g x^{(n-1)} \oplus_g \left(x^{(n)} \oplus_g \ldots \oplus_g x^{(m)}\right)$

$$= \left(\frac{x_1^{(1)}+x_1^{(2)}+\cdots+x_1^{(m)}}{2^{m-p-1}},\ \frac{x_2^{(1)}+x_2^{(2)}+\cdots+x_2^{(m)}}{2^{m-p-1}},\ \frac{x_3^{(1)}+x_3^{(2)}+\cdots+x_3^{(m)}}{2^{m-p-1}}\right),$$

with $n \leq m$.

Corollary 3.6 *If the geometric addition in Definition 3.5 of m elements $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(m)}$ in D_g^* exists, then*

(1) *We have a binary addition operation of two elements $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in D_g^* as follows:*

$$x \oplus_g y = \left(\frac{x_1+y_1}{2^{1-p}},\ \frac{x_2+y_2}{2^{1-p}},\ \frac{x_3+y_3}{2^{1-p}}\right), \tag{3.2}$$

where p is the number of neutral elements $\theta = (0,0,0)$ in this addition, $p = 2$ when $x = y = \theta$, $p = 1$ when $x = \theta,\ y \neq \theta$ or $x \neq \theta,\ y = \theta$, and $p = 0$ when $x \neq \theta$ and $y \neq \theta$.

(2) *Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, and $z = (z_1, z_2, z_3)$ be PFNs, then we have*

$$(x \oplus_g y) \oplus_g z = x \oplus_g (y \oplus_g z).$$

Proof Let us suppose that the geometric addition in Definition 3.5 of m elements $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(m)}$ in D_g^* exists. (1) with $m = 2$, we get

$$\begin{aligned}x^{(1)} \oplus_g x^{(2)} &= \left(\frac{x_1^{(1)}+x_1^{(2)}}{2^{2-p-1}},\ \frac{x_2^{(1)}+x_2^{(2)}}{2^{2-p-1}},\ \frac{x_3^{(1)}+y_3^{(2)}}{2^{2-p-1}}\right)\\ &= \left(\frac{x_1^{(1)}+x_1^{(2)}}{2^{1-p}},\ \frac{x_2^{(1)}+x_2^{(2)}}{2^{1-p}},\ \frac{x_3^{(1)}+y_3^{(2)}}{2^{1-p}}\right),\end{aligned}$$

we substitute $x^{(1)} = x = (x_1, x_2, x_3)$ and $x^{(2)} = y = (y_1, y_2, y_3)$. Thus, we obtain

$$x \oplus_g y = \left(\frac{x_1+y_1}{2^{1-p}},\ \frac{x_2+y_2}{2^{1-p}},\ \frac{x_3+y_3}{2^{1-p}}\right).$$

(2) From conditions (a) and (b) with $m = 3$ and $n = 2$, we obtain

$$\begin{cases} \left(x^{(1)} \oplus_g x^{(2)}\right) \oplus_g x^{(3)} \\ \quad = \left(\dfrac{x_1^{(1)} + x_1^{(2)} + x_1^{(3)}}{2^{3-p-1}}, \dfrac{x_2^{(1)} + x_2^{(2)} + x_2^{(3)}}{2^{3-p-1}}, \dfrac{x_3^{(1)} + x_3^{(2)} + x_3^{(3)}}{2^{3-p-1}}\right) \\ x^{(1)} \oplus_g \left(x^{(2)} \oplus_g x^{(3)}\right) \\ \quad = \left(\dfrac{x_1^{(1)} + x_1^{(2)} + x_1^{(3)}}{2^{3-p-1}}, \dfrac{x_2^{(1)} + x_2^{(2)} + x_2^{(3)}}{2^{3-p-1}}, \dfrac{x_3^{(1)} + x_3^{(2)} + x_3^{(3)}}{2^{3-p-1}}\right) \end{cases}$$
$$\Rightarrow \left(x^{(1)} \oplus_g x^{(2)}\right) \oplus_g x^{(3)} = x^{(1)} \oplus_g \left(x^{(2)} \oplus_g x^{(3)}\right)$$

we substitute $x^{(1)} = x = (x_1, x_2, x_3)$, $x^{(2)} = y = (y_1, y_2, y_3)$, $x^{(3)} = z = (z_1, z_2, z_3)$ and the proof is completed. □

Theorem 3.7 *Let m elements $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(m)}$ in D_g^*. Then, there is an element y in D_g^* such that $y = x^{(1)} \oplus_g x^{(2)} \oplus_g \ldots \oplus_g x^{(m)}$ with*

$$y = \left(\frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m)}}{2^{m-p-1}}, \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m)}}{2^{m-p-1}}, \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m)}}{2^{m-p-1}}\right),$$

where p is the number of elements $x^{(k)} = (0, 0, 0) = \theta,\ k = 1, 2, ..., m$.

Proof We will prove this theorem by mathematical induction method. Indeed, with $m = 1$, this theorem is true because we always have $x = (x_1, x_2, x_3) \in D_g^*$ and $0 \le x_1+x_2+x_3 \le 1$. Next, let us assume that this theorem is also true for $m-1$ elements $x^{(1)}, x^{(2)}, x^{(3)}, \ldots, x^{(m-1)}$ in D_g^*. Finally, we need to prove the theorem is true for m elements. From inductive assumption, we obtain an element z in D_g^* such that $z = x^{(1)} \oplus_g x^{(2)} \oplus_g \ldots \oplus_g x^{(m-1)}$ with

$$z = \left(\frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m-1)}}{2^{(m-1)-p-1}}, \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m-1)}}{2^{(m-1)-p-1}}, \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m-1)}}{2^{(m-1)-p-1}}\right),$$

where p is the number of elements $x^{(k)} = (0, 0, 0) = \theta,\ k = 1, 2, ..., m-1$. Because $z \in D_g^*$, we have

$$0 \le \frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m-1)}}{2^{(m-1)-p-1}} + \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m-1)}}{2^{(m-1)-p-1}} + \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m-1)}}{2^{(m-1)-p-1}} \le 1. \tag{3.3}$$

To prove an element y in D_g^* such that $y = x^{(1)} \oplus_g x^{(2)} \oplus_g \ldots \oplus_g x^{(m-1)} \oplus_g x^{(m)}$ with

$$y = \left(\frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m)}}{2^{m-p-1}}, \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m)}}{2^{m-p-1}}, \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m)}}{2^{m-p-1}}\right),$$

where p is the number of elements $x^{(k)} = (0, 0, 0) = \theta,\ k = 1, 2, ..., m-1, m$. We need to show that

$$0 \le \frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m)}}{2^{m-p-1}} + \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m)}}{2^{m-p-1}} + \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m)}}{2^{m-p-1}} \le 1.$$

Indeed, with case $p = m$: We have

$$\begin{aligned} &x^{(1)} = x^{(2)} = x^{(3)} = \cdots = x^{(m-1)} = x^{(m)} = \theta = (0, 0, 0) \\ &\Rightarrow y = x^{(1)} \oplus_g x^{(2)} \oplus_g x^{(3)} \oplus_g \cdots \oplus_g x^{(m-1)} \oplus_g x^{(m)} \\ &\quad = \theta = (0, 0, 0) \\ &\Rightarrow y = \theta \in D_g^*. \end{aligned}$$

With case $p = m-1$: We have

$$\begin{aligned} &x^{(1)} = x^{(2)} = \cdots = x^{(j-1)} = x^{(j+1)} = \cdots = x^{(m)} \\ &\quad = \theta = (0, 0, 0) \text{ and } x^{(j)} \ne \theta,\ j \in \overline{1, m} \\ &\Rightarrow y = \left(\frac{0 + \cdots + x_1^{(j)} + \cdots + 0}{2^{m-(m-1)-1}}, \frac{0 + \cdots + x_2^{(j)} + \cdots + 0}{2^{m-(m-1)-1}}, \frac{0 + \cdots + x_3^{(j)} + \cdots + 0}{2^{m-(m-1)-1}}\right) \\ &\quad = \left(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}\right) \\ &\Rightarrow y = x^{(1)} \oplus_g x^{(2)} \oplus_g x^{(3)} \\ &\quad \oplus_g \cdots \oplus_g x^{(m-1)} \oplus_g x^{(m)} = x^{(j)} \\ &\Rightarrow y = x^{(j)} \in D_g^* \text{ with } j \in \overline{1, m}. \end{aligned}$$

With case $0 \le p \le m-2$: From Eq 3.4, we have (Fig. 3)

$$0 \le \frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m-1)}}{2^{(m-1)-p-1}} + \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m-1)}}{2^{(m-1)-p-1}} + \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m-1)}}{2^{(m-1)-p-1}} \le 1.$$

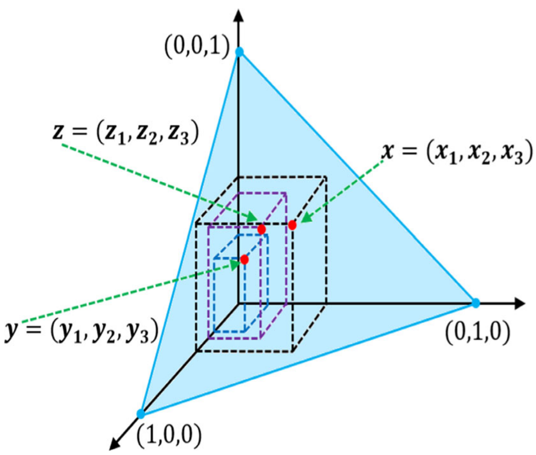

Fig. 3 Geometric interpretation of binary addition operation $x \oplus_G y = z \in D_g^*$ of the GPFNs in Definition 3.5, with both x and y are different θ

$$
\begin{aligned}
&\Rightarrow 0 \le \frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m-1)} + (x_1^{(m)} - x_1^{(m)})}{2^{(m-1)-p-1+(1-1)}} \\
&\quad + \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m-1)} + (x_2^{(m)} - x_2^{(m)})}{2^{(m-1)-p-1+(1-1)}} \\
&\quad + \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m-1)} + (x_3^{(m)} - x_3^{(m)})}{2^{(m-1)-p-1+(1-1)}} \le 1. \\
&\Rightarrow 0 \le \frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m-1)} + (x_1^{(m)} - x_1^{(m)})}{2^{m-p-1}.2^{-1}} \\
&\quad + \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m-1)} + (x_2^{(m)} - x_2^{(m)})}{2^{m-p-1}.2^{-1}} \\
&\quad + \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m-1)} + (x_3^{(m)} - x_3^{(m)})}{2^{m-p-1}.2^{-1}} \le 1. \\
&\Rightarrow 0 \le \frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m-1)} + (x_1^{(m)} - x_1^{(m)})}{2^{m-p-1}} \\
&\quad + \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m-1)} + (x_2^{(m)} - x_2^{(m)})}{2^{m-p-1}} \\
&\quad + \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m-1)} + (x_3^{(m)} - x_3^{(m)})}{2^{m-p-1}} \le \frac{1}{2}. \\
&\Rightarrow 0 \le \frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m-1)} + x_1^{(m)}}{2^{m-p-1}} \\
&\quad + \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m-1)} + x_2^{(m)}}{2^{m-p-1}} \\
&\quad + \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m-1)} + x_3^{(m)}}{2^{m-p-1}} \\
&\le \frac{x_1^{(m)} + x_2^{(m)} + x_3^{(m)}}{2^{m-p-1}} + \frac{1}{2}. \\
&\Rightarrow 0 \le \frac{x_1^{(1)} + x_1^{(2)} + \cdots + x_1^{(m-1)} + x_1^{(m)}}{2^{m-p-1}} \\
&\quad + \frac{x_2^{(1)} + x_2^{(2)} + \cdots + x_2^{(m-1)} + x_2^{(m)}}{2^{m-p-1}} \\
&\quad + \frac{x_3^{(1)} + x_3^{(2)} + \cdots + x_3^{(m-1)} + x_3^{(m)}}{2^{m-p-1}} \\
&\le \frac{1}{2^{m-p-1}} + \frac{1}{2} \le 1.
\end{aligned}
$$

So the proof is completed. □

Definition 3.8 Let $x = (x_1, x_2, x_3)$ be a PFN and $\lambda \in \mathbb{R}^+$, then the scalar multiplication, denoted by $\odot_g$, is a PFN $y = \lambda \odot_g x$ such that

$$y = (\lambda x_1, \lambda x_2, \lambda x_3)$$

and $0 \le \lambda x_1 + \lambda x_2 + \lambda x_3 \le 1$.

In our first idea, when we came up with this scalar multiplication, we only consider $\lambda \in [0, 1]$, because we want to make sure $\lambda \odot_g x \in D_g^*$. However, we realize that there are many cases $\lambda \in \mathbb{R}^+$ (maybe expand further), but the results still belong to D_g^*, for a simple example, with $\lambda = 10$ and $x = (x_1, x_2, x_3) = (0.004, 0.0065, 0.0009) \in D_g^*$ then $\lambda \odot_g x = (\lambda.x_1, \lambda.x_2, \lambda.x_3) = (10 \times 0.004, 10 \times 0.0065, 10 \times 0.0009) = (0.04, 0.065, 0.009) = (z_1, z_2, z_3) = z$, we can see that z belongs to D_g^* because $0 \le z_1 + z_2 + z_3 \le 1$.

Theorem 3.9 *Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, and $z = (z_1, z_2, z_3)$ be three PFNs and $\lambda, \beta \in \mathbb{R}^+$, then*

(1) $x \oplus_g y = y \oplus_g x$;
(2) $(x \oplus_g y) \oplus_g z = x \oplus_g (y \oplus_g z)$;
(3) $x \oplus_g \theta = x$;
(4) $\lambda \odot_g (x \oplus_g y) = \lambda \odot_g x \oplus_g \lambda \odot_g y$;
(5) $(\lambda + \beta) \odot_g x = \lambda \odot_g x + \beta \odot_g x$;
(6) $(\lambda.\beta) \odot_g x = \lambda \odot_g (\beta \odot_g x)$;
(7) $1 \odot_g x = x$ *and* $0 \odot_g x = \theta$.

Proof From the addition in Definition 3.5 and scalar multiplication in Definition 3.8, we have:

(1) According to the item (i) of Corollary 3.6, we get

$$
\begin{aligned}
x \oplus_g y &= \left(\frac{x_1 + y_1}{2^{1-p}}, \frac{x_2 + y_2}{2^{1-p}}, \frac{x_3 + y_3}{2^{1-p}}\right) \\
&= \left(\frac{y_1 + x_1}{2^{1-p}}, \frac{y_2 + x_2}{2^{1-p}}, \frac{y_3 + x_3}{2^{1-p}}\right) \\
&= y \oplus_g x.
\end{aligned}
$$

(2) This is the result of the item (ii) of Corollary 3.6.
(3) According to the item (i) of Corollary 3.6 with $x \ne \theta = (0, 0, 0)$. Because in the formula $x \oplus_g \theta$, there is a zero element ($\theta = (0, 0, 0) \in D_g^*$) so $p = 1$ and we get

$$
\begin{aligned}
x \oplus_g \theta &= \left(\frac{x_1 + 0}{2^{1-1}}, \frac{x_2 + 0}{2^{1-1}}, \frac{x_3 + 0}{2^{1-1}}\right) \\
&= (x_1, x_2, x_3) = x.
\end{aligned}
$$

(4) According to the item (i) of Corollary 3.6 and scalar multiplication in Definition 3.8, we get

$$\lambda \odot_g (x \oplus_g y) = \lambda \odot_g \left(\frac{x_1 + y_1}{2^{1-p}}, \frac{x_2 + y_2}{2^{1-p}}, \frac{x_3 + y_3}{2^{1-p}}\right)$$

$$= \left(\lambda\frac{x_1+y_1}{2^{1-p}}, \lambda\frac{x_2+y_2}{2^{1-p}}, \lambda\frac{x_3+y_3}{2^{1-p}}\right)$$
$$= \left(\frac{\lambda x_1+\lambda y_1}{2^{1-p}}, \frac{\lambda x_2+\lambda y_2}{2^{1-p}}, \frac{\lambda x_3+\lambda y_3}{2^{1-p}}\right)$$
$$= \lambda \odot_g x \oplus_g \lambda \odot_g y.$$

(5) According to the scalar multiplication in Definition 3.8, we get

$$\begin{aligned}(\lambda+\beta) \odot_g x &= ((\lambda+\beta)x_1, (\lambda+\beta)x_2, (\lambda+\beta)x_3)\\ &= (\lambda x_1+\beta x_1, \lambda x_2+\beta x_2, \lambda x_3+\beta x_3)\\ &= (\lambda x_1, \lambda x_2, \lambda x_3)+(\beta x_1, \beta x_2, \beta x_3)\\ &= \lambda \odot_g x + \beta \odot_g x.\end{aligned}$$

(6) According to the scalar multiplication in Definition 3.8, we get

$$\begin{aligned}(\lambda\beta) \odot_g x &= ((\lambda\beta)x_1, (\lambda\beta)x_2, (\lambda\beta)x_3)\\ &= (\lambda\beta x_1, \lambda\beta x_2, \lambda\beta x_3)\\ &= \lambda \odot_g (\beta x_1, \beta x_2, \beta x_3)\\ &= \lambda \odot_g (\beta \odot_g x)\end{aligned}$$

(7) According to the scalar multiplication in Definition 3.8, we get $1 \odot_g x = (1.x_1, 1.x_2, 1.x_3) = (x_1, x_2, x_3) = x$ and $0 \odot_g x = (0.x_1, 0.x_2, 0.x_3) = (0, 0, 0) = \theta$.

The theorem has completed proof. □

Theorem 3.10 *Let the set of PFNs D_g^* in Definition 3.1, the geometric addition $\oplus_g$ in Definition 3.5 and the scalar multiplication $\odot_g$ in Definition 3.8. Then, $(D_g^*, \oplus_g, \odot_g)$ is a semi-linear space.*

Proof From Definition 2.6, we see that Theorem 3.10 is a direct result of Theorem 3.9. □

Theorem 3.11 *Let a mapping, denoted by $H_{D_g^*}$, be defined:*

$$H_{D_g^*} : D_g^* \times D_g^* \longrightarrow \mathbb{R}$$
$$(x, y) \longmapsto H_{D_g^*}(x, y) = \sup_{i\in\overline{1,3}} \{|x_i - y_i|\}.$$

Then, $\left(D_g^*, H_{D_g^*}\right)$ is a metric semi-linear space.

Proof First of all, we need to prove that the mapping $H_{D_g^*}$ is a metric on D_g^*. Indeed, for any $x, y \in D_g^*$ with $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, we have

(1)

$$H_{D_g^*}(x, y) = \sup\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\} > 0$$

with $x \neq y$ and if $x = y$,

$$H_{D_g^*}(x, y) = \sup\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\} = \sup\{0, 0, 0\} = 0;$$

(2) $H_{D_g^*}(x, y) = \sup_{i\in\overline{1,3}}\{|x_i - y_i|\} = \sup_{i\in\overline{1,3}}\{|y_i - x_i|\} = H_{D_g^*}(y, x)$;

(3)

$$\begin{aligned}H_{D_g^*}(x, y) &= \sup_{i\in\overline{1,3}}\{|x_i - y_i|\} = \sup_{i\in\overline{1,3}}\{|x_i - z_i + z_i - y_i|\}\\ &\leq \sup_{i\in\overline{1,3}}\{|x_i - z_i| + |z_i - y_i|\}\\ &\leq \sup_{i\in\overline{1,3}}\{|x_i - z_i|\} + \sup_{i\in\overline{1,3}}\{|z_i - y_i|\}\\ &= H_{D_g^*}(x, z) + H_{D_g^*}(z, y) \text{ with}\end{aligned}$$
$$z = (z_1, z_2, z_3) \in D_g^*.$$

From Definition 2.8, we obtain that the mapping $H_{D_g^*}$ is a metric on D_g^*. Thus, $\left(D_g^*, H_{D_g^*}\right)$ is a metric space. □

Definition 3.12 Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two PFNs. We define the order relations between these two PFNs as follows:

Type 1: $x \prec_1 y$ in D_g^* iff it satisfies $x_1 \leq y_1$, $x_2 \leq y_2$, and $x_3 \leq y_3$. And, $D_1 = \left\{(x, y) \in D_g^* \times D_g^* | x \prec_1 y\right\}$.
Type 2: $x \prec_2 y$ in D_g^* iff it satisfies $y_1 \leq x_1$, $x_2 \leq y_2$, and $x_3 \leq y_3$. And, $D_2 = \left\{(x, y) \in D_g^* \times D_g^* | x \prec_2 y\right\}$.
Type 3: $x \prec_3 y$ in D_g^* iff it satisfies $x_1 \leq y_1$, $y_2 \leq x_2$, and $x_3 \leq y_3$. And, $D_3 = \left\{(x, y) \in D_g^* \times D_g^* | x \prec_3 y\right\}$.
Type 4: $x \prec_4 y$ in D_g^* iff it satisfies $x_1 \leq y_1$, $x_2 \leq y_2$, and $y_3 \leq x_3$. And, $D_4 = \left\{(x, y) \in D_g^* \times D_g^* | x \prec_4 y\right\}$.
Type 5: $x \prec_5 y$ in D_g^* iff it satisfies $y_1 \leq x_1$, $y_2 \leq x_2$, and $x_3 \leq y_3$. And, $D_5 = \left\{(x, y) \in D_g^* \times D_g^* | x \prec_5 y\right\}$.
Type 6: $x \prec_6 y$ in D_g^* iff it satisfies $y_1 \leq x_1$, $x_2 \leq y_2$, and $y_3 \leq x_3$. And, $D_6 = \left\{(x, y) \in D_g^* \times D_g^* | x \prec_6 y\right\}$.
Type 7: $x \prec_7 y$ in D_g^* iff it satisfies $x_1 \leq y_1$, $y_2 \leq x_2$, and $y_3 \leq x_3$. And, $D_7 = \left\{(x, y) \in D_g^* \times D_g^* | x \prec_7 y\right\}$.
Type 8: $x \prec_8 y$ in D_g^* iff it satisfies $y_1 \leq x_1$, $y_2 \leq x_2$, and $y_3 \leq x_3$. And, $D_8 = \left\{(x, y) \in D_g^* \times D_g^* | x \prec_8 y\right\}$.

Definition 3.13 Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two PFNs. We define the geometric difference between these two PFNs as follows:

Case 1 If $(x, y) \in D_1$ and $0 \leq \sum_{i=1}^{3} |y_i - x_i| \leq 1$, there correspond a element $y \ominus_{g1} x \in D_g^*$ and $y \ominus_{g1} x = (y_1 - x_1,\ y_2 - x_2,\ y_3 - x_3)$, where geometric difference denoted by $\ominus_{g1}$;
Case 2 If $(x, y) \in D_2$ and $0 \leq \sum_{i=1}^{3} |y_i - x_i| \leq 1$, there correspond a element $y \ominus_{g2} x \in D_g^*$ and $y \ominus_{g2} x = (x_1 - y_1,\ y_2 - x_2,\ y_3 - x_3)$, where geometric difference denoted by $\ominus_{g2}$;
Case 3 If $(x, y) \in D_3$ and $0 \leq \sum_{i=1}^{3} |y_i - x_i| \leq 1$, there correspond a element $y \ominus_{g3} x \in D_g^*$ and $y \ominus_{g3} x = (y_1 - x_1,\ x_2 - y_2,\ y_3 - x_3)$, where geometric difference denoted by $\ominus_{g3}$;
Case 4 If $(x, y) \in D_4$ and $0 \leq \sum_{i=1}^{3} |y_i - x_i| \leq 1$, there correspond a element $y \ominus_{g4} x \in D_g^*$ and $y \ominus_{g4} x = (y_1 - x_1,\ y_2 - x_2,\ x_3 - y_3)$, where geometric difference denoted by $\ominus_{g4}$;
Case 5 If $(x, y) \in D_5$ and $0 \leq \sum_{i=1}^{3} |y_i - x_i| \leq 1$, there correspond a element $y \ominus_{g5} x \in D_g^*$ and $y \ominus_{g5} x = (x_1 - y_1,\ x_2 - y_2,\ y_3 - x_3)$, where geometric difference denoted by $\ominus_{g5}$;
Case 6 If $(x, y) \in D_6$ and $0 \leq \sum_{i=1}^{3} |y_i - x_i| \leq 1$, there correspond a element $y \ominus_{g6} x \in D_g^*$ and $y \ominus_{g6} x = (x_1 - y_1,\ y_2 - x_2,\ x_3 - y_3)$, where geometric difference denoted by $\ominus_{g6}$;
Case 7 If $(x, y) \in D_7$ and $0 \leq \sum_{i=1}^{3} |y_i - x_i| \leq 1$, there correspond a element $y \ominus_{g7} x \in D_g^*$ and $y \ominus_{g7} x = (y_1 - x_1,\ x_2 - y_2,\ x_3 - y_3)$, where geometric difference denoted by $\ominus_{g7}$;
Case 8 If $(x, y) \in D_8$ and $0 \leq \sum_{i=1}^{3} |y_i - x_i| \leq 1$, there correspond a element $y \ominus_{g8} x \in D_g^*$ and $y \ominus_{g8} x = (x_1 - y_1,\ x_2 - y_2,\ x_3 - y_3)$, where geometric difference denoted by $\ominus_{g8}$.

In conclusion, every two PFNs $(x, y) \in \bigcup_{k=1}^{8} D_k$ and there exists a element $y \ominus_{gk} x \in D_g^*$ with $k \in \{1, 2, 3, 4, 5, 6, 7, 8\}$, then we say that there exists a geometric difference $y \ominus_g x$ (with symbol $\ominus_g$).

Definition 3.14 Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be any two PFNs. Then, the geometric difference, denoted by $\ominus_g$, between these two PFNs is a PFN $z = y \ominus_g x$, if it exists, such that

$$z = (|y_1 - x_1|, |y_2 - x_2|, |y_3 - x_3|)$$

Theorem 3.15 *The concepts of geometric difference in Definition 3.13 and in Definition 3.14 are the same.*

Proof Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be any two PFNs. Assume that the geometric difference between these two PFNs in Definition 3.17 exists, this means that there is a PFN $z = y \ominus_g x$ belonging to D_g^* and $z = (|y_1 - x_1|, |y_2 - x_2|, |y_3 - x_3|)$. Because $z \in D_g^*$, we have $0 \leq \sum_{i=1}^{3} |y_i - x_i| \leq 1$ and

Case 1 $z = (y_1 - x_1,\ y_2 - x_2,\ y_3 - x_3)$ if $x_1 \leq y_1$, $x_2 \leq y_2$, and $x_3 \leq y_3$ are equivalent to $(x, y) \in D_1$;
Case 2 $z = (x_1 - y_1,\ y_2 - x_2,\ y_3 - x_3)$ if $y_1 \leq x_1$, $x_2 \leq y_2$, and $x_3 \leq y_3$ are equivalent to $(x, y) \in D_2$;
Case 3 $z = (y_1 - x_1,\ x_2 - y_2,\ y_3 - x_3)$ if $x_1 \leq y_1$, $y_2 \leq x_2$, and $x_3 \leq y_3$ are equivalent to $(x, y) \in D_3$;
Case 4 $z = (y_1 - x_1,\ y_2 - x_2,\ x_3 - y_3)$ if $x_1 \leq y_1$, $x_2 \leq y_2$, and $y_3 \leq x_3$ are equivalent to $(x, y) \in D_4$;
Case 5 $z = (x_1 - y_1,\ x_2 - y_2,\ y_3 - x_3)$ if $y_1 \leq x_1$, $y_2 \leq x_2$, and $x_3 \leq y_3$ are equivalent to $(x, y) \in D_5$;
Case 6 $z = (x_1 - y_1,\ y_2 - x_2,\ x_3 - y_3)$ if $y_1 \leq x_1$, $x_2 \leq y_2$, and $y_3 \leq x_3$ are equivalent to $(x, y) \in D_6$;
Case 7 $z = (y_1 - x_1,\ x_2 - y_2,\ x_3 - y_3)$ if $x_1 \leq y_1$, $y_2 \leq x_2$, and $y_3 \leq x_3$ are equivalent to $(x, y) \in D_7$;
Case 8 $z = (x_1 - y_1,\ x_2 - y_2,\ x_3 - y_3)$ if $y_1 \leq x_1$, $y_2 \leq x_2$, and $y_3 \leq x_3$ are equivalent to $(x, y) \in D_8$.

Therefore, the concept of geometric difference in Definition 3.13 and Definition 3.14 is equivalent. □

Theorem 3.16 *Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two PFNs. Then, we have the following properties:*

(1) $x \ominus_g x = \theta$;
(2) If $z = y \ominus_g x$ exists, it is unique;
(3) If $y \ominus_g x$ exists, then $x \ominus_g y$ exists and $y \ominus_g x = x \ominus_g y$;
(4) If $y \ominus_g x = x \ominus_g y = \theta$, then $y = x$;

Proof For property (1), we have $x \ominus_g x = (|x_1 - x_1|, |x_2 - x_2|, |x_3 - x_3|) = (0, 0, 0) = \theta$. For property (ii), assume that we have two PFNs $h = y \ominus_g x$, $g = y \ominus_g x$ and $h \neq g$, then $(|y_1 - x_1|, |y_2 - x_2|, |y_3 - x_3|) \neq (|y_1 - x_1|, |y_2 - x_2|, |y_3 - x_3|)$. For convenience, we rewrite $|y_i - x_i| \neq |y_i - x_i|$ with $i \in \{1, 2, 3\}$. Case 1, if $x_i \leq y_i$, then

$$|y_i - x_i| \neq |y_i - x_i| \Rightarrow y_i - x_i \neq y_i - x_i \Rightarrow 0 \neq 0 \text{ with } i \in \{1, 2, 3\}.$$

Case 2, if $y_i \leq x_i$, then

$$|y_i - x_i| \neq |y_i - x_i| \Rightarrow x_i - y_i \neq x_i - y_i \Rightarrow 0 \neq 0 \text{ with } i \in \{1, 2, 3\}.$$

both cases 1 and 2 contradict the above assumption. Thus, we get $h = g$. For property (3), suppose that $y \ominus_g x$ exists, we have

$$y \ominus_g x = (|y_1 - x_1|, |y_2 - x_2|, |y_3 - x_3|)$$

$$= (|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|) = x \ominus_g y.$$

To property (iv), suppose that $y \ominus_g x = \theta$, we have

$$\begin{aligned}(|y_1 - x_1|, |y_2 - x_2|, |y_3 - x_3|) &= (0, 0, 0) \Rightarrow |y_i - x_i| \\ &= 0 \Rightarrow y_i \\ &= x_i \text{ with } i \in \{1, 2, 3\}.\end{aligned}$$

Thus, we get $x = y$. □

Definition 3.17 Let x and y be two PFNs with $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then there correspond a element $x \otimes_g y \in D_g^*$ and $x \otimes_g y = (x_1 y_1,\ x_2 y_2,\ x_3 y_3)$, where multiplication of two PFNs is denoted by $\otimes_g$.

Theorem 3.18 *$\otimes_g$ in Definition 3.17 is well defined, i.e., let x and y be two PFNs, then $x \otimes_g y$ also is a PFN.*

Proof To prove this theorem, we need to prove that if $a \in [0, 1]$ and $b \in [0, 1]$, where a and b are two real numbers then $a.b \in [0, 1]$. Indeed, Putting $a = \frac{n}{m}$ and $b = \frac{p}{q}$, which $n,\ m,\ p$, and q are positive integers. Since $a \in [0, 1]$ and $b \in [0, 1]$, we obtain $n \leq m$ and $p \leq q$. Simultaneously, we also have $n.p \leq m.q$ because they are positive integers. Thus, we obtain $a.b = \frac{n}{m}.\frac{p}{q} = \frac{n.p}{m.q} \in [0, 1]$. Hence, let x and y be two PFNs with $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, we have $0 \leq x_1 + x_2 + x_3 \leq 1$, $0 \leq y_1 + y_2 + y_3 \leq 1$, and $0 \leq (x_1 + x_2 + x_3).(y_1 + y_2 + y_3) \leq 1$. With $(x_1 + x_2 + x_3).(y_1 + y_2 + y_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} x_i y_j$, we have

$$0 \leq x_1 y_1 + x_2 y_2 + x_3 y_3 \leq \sum_{i=1}^{3} \sum_{j=1}^{3} x_i y_j \leq 1.$$

Furthermore, since $x_i \in [0, 1]$ and $y_i \in [0, 1]$ with $i \in \{1, 2, 3\}$, we get $x_i y_i \in [0, 1]$. Therefore, $x \otimes_g y = (x_1 y_1,\ x_2 y_2,\ x_3 y_3)$ is a PFN. □

3.2 The geometric picture fuzzy functions

In this subsection, we study the picture fuzzy functions (PFFs), which are the functions related to PFNs, with a real domain. Let

$$\begin{aligned} f : I \subset \mathbb{R} &\longrightarrow D_g^* \\ t &\longmapsto f(t) = (f_1(t), f_2(t), f_3(t)), \end{aligned}$$

where $0 \leq f_1(t) + f_2(t) + f_3(t) \leq 1$. We call $f(t)$ is geometric picture fuzzy functions (GPFFs) in D_g^*. In classical mathematics, the limit of a function or sequence of numbers is a fundamental concept in calculus and analysis that involves the behavior of that function or sequence of numbers near a particular input. It is the main tool for the development of important properties in the theory of calculus such as continuity, differentiable, integrable, etc. In the following, we will present definitions and properties for the limit of GPFFs and the sequence of PFNs in detail.

Definition 3.19 Let $f(t) = (f_1(t), f_2(t), f_3(t))$ be a GPFF for $t \in I \subset \mathbb{R}$. If the limits of component functions exist, i.e., $\lim_{t \to t_0} f_1(t) = L_1$, $\lim_{t \to t_0} f_2(t) = L_2$, and $\lim_{t \to t_0} f_3(t) = L_3$, such that $(L_1, L_2, L_3) \in D_g^*$, then we define the limit of $f(t)$ as follows:

$$\begin{aligned}\lim_{t \to t_0} f(t) &= \left(\lim_{t \to t_0} f_1(t), \lim_{t \to t_0} f_2(t), \quad \lim_{t \to t_0} f_3(t)\right) \\ &= (L_1, L_2, L_3) = L.\end{aligned}$$

Lemma 3.20 *Let $f(t) = (f_1(t), f_2(t), f_3(t))$ be a GPFF for $t \in I \subset \mathbb{R}$. If the limit of $f(t)$ exists for all $t \in I$ and $\lim_{t \to t_0} f(t) = a^*$, where $a^* = (a_1, a_2, a_3)$, then a^* is PFN.*

Proof Suppose that the limit of $f(t)$ exists for all $t \in I$ and $\lim_{t \to t_0} f(t) = a^*$. From Definition 3.19, we have $\lim_{t \to t_0} f_1(t) = a_1$, $\lim_{t \to t_0} f_2(t) = a_2$, and $\lim_{t \to t_0} f_3(t) = a_3$. Because $f(t)$ is a GPFF for all $t \in I$, we get $0 \leq f_1(t) + f_2(t) + f_3(t) \leq 1$ and $0 \leq f_1(t) \leq 1$, $0 \leq f_2(t) \leq 1$, $0 \leq f_3(t) \leq 1$. Putting $g(t) = f_1(t) + f_2(t) + f_3(t)$, then $g(t)$ is a real-valued function and $0 \leq g(t) \leq 1$ for all $t \in I$. Thus, with $t_0 \in I$ we have

$$\begin{aligned}0 \leq \lim_{t \to t_0} g(t) \leq 1 &\Leftrightarrow 0 \leq \lim_{t \to t_0} (f_1(t) + f_2(t) + f_3(t)) \leq 1 \\ &\Leftrightarrow 0 \leq \lim_{t \to t_0} f_1(t) + \lim_{t \to t_0} f_2(t) + \lim_{t \to t_0} f_3(t) \\ &\leq 1 \Rightarrow 0 \leq a_1 + a_2 + a_3 \leq 1\end{aligned}$$

At the same time, since $0 \leq f_1(t) \leq 1$, $0 \leq f_2(t) \leq 1$, and $0 \leq f_3(t) \leq 1$, then $0 \leq \lim_{t \to t_0} f_1(t) \leq 1$, $0 \leq \lim_{t \to t_0} f_2(t) \leq 1$, and $0 \leq \lim_{t \to t_0} f_3(t) \leq 1$ with $t_0 \in I$. Thus, we obtain $0 \leq a_1 \leq 1$, $0 \leq a_2 \leq 1$, and $0 \leq a_3 \leq 1$. Therefore, $a^* = (a_1, a_2, a_3)$ is a PFN. □

Theorem 3.21 *Let $f(t) = (f_1(t), f_2(t), f_3(t))$ be a GPFF with $t \in I \subset \mathbb{R}$ and $L = (L_1, L_2, L_3)$ be a PFN. Then, $\lim_{t \to t_0} f(t) = L$ if and only if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 \leq |t - t_0| \leq \delta$, then*

$$H_{D_g^*}(f(t), L) \leq \varepsilon.$$

Proof Let $f(t) = (f_1(t), f_2(t), f_3(t))$ be a GPFF and $L = (L_1, L_2, L_3)$ be a PFN.

Necessity: Suppose that $\lim_{t \to t_0} f(t) = L$, then $\lim_{t \to t_0} f(t)$ exists, by Definition 3.19 we get

$$\lim_{t \to t_0} f(t) = \left(\lim_{t \to t_0} f_1(t), \lim_{t \to t_0} f_2(t), \lim_{t \to t_0} f_3(t)\right)$$

$$= (L_1, L_2, L_3) = L$$

and $\lim_{t\to t_0} f_1(t) = L_1$, $\lim_{t\to t_0} f_2(t) = L_2$, and $\lim_{t\to t_0} f_3(t) = L_3$. Because these limits of the component functions are limit of real-valued function, by Definition 2.9, for every $\varepsilon > 0$ there exists $\delta_1 > 0$, $\delta_2 > 0$, and $\delta_3 > 0$ such that if $0 \le |t - t_0| \le \delta_1$ then $|f_1(t) - L_1| \le \varepsilon$, if $0 \le |t - t_0| \le \delta_2$ then $|f_2(t) - L_2| \le \varepsilon$, and if $0 \le |t - t_0| \le \delta_3$ then $|f_3(t) - L_3| \le \varepsilon$. Putting $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then if $0 \le |t - t_0| \le \delta$, we have

$$H_{D_g^*}(f(t), L) = \sup_{i\in\overline{1,3}} \{|f_i - L_i|\} \le \sup\{\varepsilon, \varepsilon, \varepsilon\} = \varepsilon.$$

Thus, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 \le |t - t_0| \le \delta$, then $H_{D_g^*}(f(t), L) \le \varepsilon$.

Necessity: Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 \le |t - t_0| \le \delta$, then

$$H_{D_g^*}(f(t), L) = \sup_{i\in\overline{1,3}} \{|f_i - L_i|\} \le \varepsilon.$$

Thus, if $0 \le |t - t_0| \le \delta$, then $|f_1 - L_1| \le \varepsilon$, $|f_2 - L_2| \le \varepsilon$, and $|f_3 - L_3| \le \varepsilon$. By Definition 2.9, we have $\lim_{t\to t_0} f_1(t) = L_1$, $\lim_{t\to t_0} f_2(t) = L_2$, and $\lim_{t\to t_0} f_3(t) = L_3$. And, by Definition 3.19,

$$\lim_{t\to t_0} f(t) = \left(\lim_{t\to t_0} f_1(t), \lim_{t\to t_0} f_2(t), \lim_{t\to t_0} f_3(t)\right) \\ = (L_1, L_2, L_3) = L.$$

□

Definition 3.22 Let $f(n) = (f_1(n), f_2(n), f_3(n))$ be a GPFF with $n \in \mathbb{N}$ and put $x^{(n)} = f(n)$, then we define that a sequence of PFNs, $\{x^{(1)}, x^{(2)}, x^{(3)}, \ldots\}$, is denoted by $\left\{x^{(n)}\right\}_{n\in\mathbb{N}}$ or $\left\{x^{(n)}\right\}$ for short.

Definition 3.23 Let $\left\{x^{(n)}\right\} \subset D_g^*$ be a sequence of PFNs with $n \in \mathbb{N}$ and $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right)$. If the limits of component sequences exist, i.e., $\lim_{n\to+\infty} x_1^{(n)} = x_1^{(0)}$, $\lim_{n\to+\infty} x_2^{(n)} = x_2^{(0)}$, and $\lim_{n\to+\infty} x_3^{(n)} = x_3^{(0)}$, then we define the limit of sequence $\left\{x^{(n)}\right\}$ as follows:

$$\lim_{n\to+\infty} x^{(n)} = \left(\lim_{n\to+\infty} x_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)}\right) \\ = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = x^{(0)}.$$

On the other hands, the sequence $\left\{x^{(n)}\right\}$ is convergent in D_g^* if there exist $x^{(0)} \in D_g^*$ such that $\lim_{n\to+\infty} x^{(n)} = x^{(0)}$.

Corollary 3.24 *Let $\left\{x^{(n)}\right\}$ be a sequence of PFNs with $n \in \mathbb{N}$ and $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right)$. Then, we have:*

(1) *If the limit of sequence $\left\{x^{(n)}\right\}$ exists and $\lim_{n\to+\infty} x^{(n)} = x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, then $x^{(0)}$ is a PFN;*
(2) *$\lim_{n\to+\infty} x^{(n)} = x^{(0)}$ if and only if for every $\varepsilon > 0$ there is a number $N \in \mathbb{N}$ such that if $n > N$, then*

$$H_{D_g^*}\left(x^{(n)}, x^{(0)}\right) \le \varepsilon.$$

Proof The item (1) is a direct result of Lemma 3.20 when we replace $t \in I$ with $n \in \mathbb{N}$ and put $f(n) = x^{(n)}$. For the item (ii), suppose that $\lim_{n\to+\infty} x^{(n)} = x^{(0)}$, then the limit of $\left\{x^{(n)}\right\}$ exists, by Definition 3.23 we obtain $\lim_{n\to+\infty} x^{(n)} = x^{(0)}$, where $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$, and $\lim_{n\to+\infty} x_1^{(n)} = x_1^{(0)}$, $\lim_{n\to+\infty} x_2^{(n)} = x_2^{(0)}$, and $\lim_{n\to+\infty} x_3^{(n)} = x_3^{(0)}$. Because these limits of the component sequences are limit of real numbers sequence, by Definition 2.10, for every $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$, $N_2 \in \mathbb{N}$, and $N_3 \in \mathbb{N}$ such that if $n > N_1$ then $|x_1^{(n)} - x_1^{(0)}| \le \varepsilon$, if $n > N_2$ then $|x_2^{(n)} - x_2^{(0)}| \le \varepsilon$, and if $n > N_1$ then $|x_3^{(n)} - x_3^{(0)}| \le \varepsilon$, Putting $N = \min\{N_1, N_2, N_3\}$, then if $n > N$, we have

$$H_{D_g^*}\left(x^{(n)}, x^{(0)}\right) = \sup_{i\in\overline{1,3}} \left\{\left|x_i^{(n)} - x_i^{(0)}\right|\right\} \le \sup\{\varepsilon, \varepsilon, \varepsilon\} = \varepsilon.$$

Thus, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$, then $H_{D_g^*}\left(x^{(n)}, x^{(0)}\right) \le \varepsilon$.

On the contrary, suppose that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$, then

$$H_{D_g^*}\left(x^{(n)}, x^{(0)}\right) = \sup_{i\in\overline{1,3}} \left\{\left|x_i^{(n)} - x_i^{(0)}\right|\right\} \le \varepsilon.$$

Thus, if $n > N$, then $\left|x_1^{(n)} - x_1^{(0)}\right| \le \varepsilon$, $\left|x_2^{(n)} - x_2^{(0)}\right| \le \varepsilon$, and $\left|x_3^{(n)} - x_3^{(0)}\right| \le \varepsilon$. By Definition 2.10, we have $\lim_{n\to+\infty} x_1^{(n)} = x_1^{(0)}$, $\lim_{n\to+\infty} x_2^{(n)} = x_2^{(0)}$, and $\lim_{n\to+\infty} x_3^{(n)} = x_3^{(0)}$. And, by Definition 3.23,

$$\lim_{n\to+\infty} x^{(n)} = \left(\lim_{n\to+\infty} x_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)}\right) \\ = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = x^{(0)}.$$

□

Theorem 3.25 *Let $\left\{x^{(n)}\right\}$, $\left\{y^{(n)}\right\}$, and $\left\{z^{(n)}\right\}$ be the sequences of PFNs with $n \in \mathbb{N}$, where $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right)$,*

$y^{(n)} = \left(y_1^{(n)}, y_2^{(n)}, y_3^{(n)}\right)$, *and* $z^{(n)} = \left(z_1^{(n)}, z_2^{(n)}, z_3^{(n)}\right)$, *respectively. We have the following properties:*

(1) *If the limit of* $\left\{x^{(n)}\right\}$ *exists, then it is unique;*
(2) *If* $x^{(n)} \prec_i y^{(n)}$ *with* $i \in \{1,2,3,4,5,6,7,8\}$ *and* $n \geq N$, *which N is a fixed positive integer, and* $\lim_{n\to+\infty} x^{(n)} = x^{(0)}$, $\lim_{n\to+\infty} y^{(n)} = y^{(0)}$, *then* $x^{(0)} \prec_i y^{(0)}$.
(3) *If* $x^{(n)} \prec_i y^{(n)} \prec_i z^{(n)}$ *with* $i \in \{1,2,3,4,5,6,7,8\}$ *and* $n \geq N$, *which N is a fixed positive integer, and* $\lim_{n\to+\infty} x^{(n)} = \lim_{n\to+\infty} z^{(n)} = L$, *where* $L = (L_1, L_2, L_3)$, *then* $\lim_{n\to+\infty} y^{(n)} = L$.

Proof For (i), suppose that $x^{(0)}, y^{(0)} \in D_g^*$ and $x^{(0)} \neq y^{(0)}$ are two limits of $\left\{x^{(n)}\right\}$. For every $\varepsilon > 0$, since $\lim_{n\to+\infty} x^{(n)} = x^{(0)}$, there exists $N_1 \in \mathbb{N}$ such that if $\forall n \geq N_1$, then $H_{D_g^*}\left(x^{(n)}, x^{(0)}\right) \leq \varepsilon$, and $\lim_{n\to+\infty} x^{(n)} = y^{(0)}$, there exists $N_2 \in \mathbb{N}$ such that if $\forall n \geq N_2$, then $H_{D_g^*}\left(x^{(n)}, y^{(0)}\right) \leq \varepsilon$. Let $\varepsilon = \frac{1}{2} H_{D_g^*}\left(x^{(0)}, y^{(0)}\right)$ and $N = \max\{N_1, N_2\}$, then $\forall n \geq N$, we obtain

$$\begin{aligned} H_{D_g^*}\left(x^{(0)}, y^{(0)}\right) &\leq H_{D_g^*}\left(x^{(n)}, x^{(0)}\right) \\ &\quad + H_{D_g^*}\left(x^{(n)}, y^{(0)}\right) < 2\varepsilon \\ &= H_{D_g^*}\left(x^{(0)}, y^{(0)}\right). \end{aligned}$$

This is a contradiction, so $x^{(0)} = y^{(0)}$.

For (2), we first consider the case $i = 1$, suppose that $x^{(n)} \prec_1 y^{(n)}$ and $n \geq N$, which N is a fixed positive integer, and $\lim_{n\to+\infty} x^{(n)} = x^{(0)}$, $\lim_{n\to+\infty} y^{(n)} = y^{(0)}$, where $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$ and $y^{(0)} = \left(y_1^{(0)}, y_2^{(0)}, y_3^{(0)}\right)$, respectively. From Definition 3.23, we have

$$\begin{cases} \lim_{n\to+\infty} x_1^{(n)} = x_1^{(0)} \\ \lim_{n\to+\infty} x_2^{(n)} = x_2^{(0)} \\ \lim_{n\to+\infty} x_3^{(n)} = x_3^{(0)} \end{cases} \text{ and } \begin{cases} \lim_{n\to+\infty} y_1^{(n)} = y_1^{(0)} \\ \lim_{n\to+\infty} y_2^{(n)} = y_2^{(0)} \\ \lim_{n\to+\infty} y_3^{(n)} = y_3^{(0)}. \end{cases} \tag{3.4}$$

Because the component sequences $\left\{x_j^{(n)}\right\}$ and $\left\{y_j^{(n)}\right\}$ with $j \in \{1,2,3\}$ are sequences in $[0,1] \subset \mathbb{R}$, so they converge as sequence of real numbers. At the same time, $x^{(n)} \prec_1 y^{(n)}$, i.e., $x_1^{(n)} \leq y_1^{(n)}$, $x_2^{(n)} \leq y_2^{(n)}$, and $x_3^{(n)} \leq y_3^{(n)}$. Therefore, we obtain $x_1^{(0)} \leq y_1^{(0)}$, $x_2^{(0)} \leq y_2^{(0)}$, and $x_3^{(0)} \leq y_3^{(0)}$, this implies that $x^{(0)} \prec_1 y^{(0)}$. In a similar way, we can prove cases $i \in \{2,3,4,5,6,7,8\}$.

For (iii), let us first consider the case $i = 1$, suppose that $x^{(n)} \prec_1 y^{(n)} \prec_1 z^{(n)}$ and $n \geq N$, which N is a fixed positive integer, and $\lim_{n\to+\infty} x^{(n)} = \lim_{n\to+\infty} z^{(n)} = L$, where $L = (L_1, L_2, L_3)$. From Definition 3.23, we have

$$\begin{cases} \lim_{n\to+\infty} x_1^{(n)} = \lim_{n\to+\infty} z_1^{(n)} = L_1 \\ \lim_{n\to+\infty} x_2^{(n)} = \lim_{n\to+\infty} z_2^{(n)} = L_2 \\ \lim_{n\to+\infty} x_3^{(n)} = \lim_{n\to+\infty} z_3^{(n)} = L_3. \end{cases} \tag{3.5}$$

Since the component sequences $\left\{x_j^{(n)}\right\}$, $\left\{y_j^{(n)}\right\}$ and $\left\{z_j^{(n)}\right\}$ with $j \in \{1,2,3\}$ are sequences in $[0,1] \subset \mathbb{R}$, so they converge as sequence of real numbers. Besides, $x^{(n)} \prec_1 y^{(n)} \prec_1 z^{(n)}$, i.e., $x_1^{(n)} \leq y_1^{(n)} \leq z_1^{(n)}$, $x_2^{(n)} \leq y_2^{(n)} \leq z_2^{(n)}$, and $x_3^{(n)} \leq y_3^{(n)} \leq z_3^{(n)}$. Thus, we obtain $\lim_{n\to+\infty} y_1^{(n)} = L_1$, $\lim_{n\to+\infty} y_2^{(n)} = L_2$, and $\lim_{n\to+\infty} y_3^{(n)} = L_3$. From the item (i) of Corollary 3.24, we obtain $L \in D_g^*$ and Definition 3.23 we get $\lim_{n\to+\infty} y^{(n)} = L$. In a similar way, we can prove cases $i \in \{2,3,4,5,6,7,8\}$. □

Theorem 3.26 *Let* $\left\{x^{(n)}\right\}$ *and* $\left\{y^{(n)}\right\}$ *be the sequences of PFNs with* $n \in \mathbb{N}$ *that possess limits as* $n \to +\infty$, *where* $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right)$ *and* $y^{(n)} = \left(y_1^{(n)}, y_2^{(n)}, y_3^{(n)}\right)$, *respectively. Then,*

1. $\lim_{n\to+\infty}\left[x^{(n)} \oplus_g y^{(n)}\right] = \lim_{n\to+\infty} x^{(n)} \oplus_g \lim_{n\to+\infty} y^{(n)}$;
2. $\lim_{n\to+\infty}\left[x^{(n)} \ominus_g y^{(n)}\right] = \lim_{n\to+\infty} x^{(n)} \ominus_g \lim_{n\to+\infty} y^{(n)}$;
3. $\lim_{n\to+\infty}\left[\lambda \odot_g x^{(n)}\right] = \lambda \odot_g \left[\lim_{n\to+\infty} x^{(n)}\right]$;
4. $\lim_{n\to+\infty}\left[x^{(n)} \otimes_g y^{(n)}\right] = \lim_{n\to+\infty} x^{(n)} \otimes_g \lim_{n\to+\infty} y^{(n)}$.

Proof In each item of this theorem, the basic procedure is to use Definition 3.23 and then, analyze the individual component sequences using the limit properties which have already been used to develop the real-valued functions. For (1.), from the item (i) of Corollary 3.6, we get

$$\begin{aligned} &\lim_{n\to+\infty}\left[x^{(n)} \oplus_g y^{(n)}\right] = \lim_{n\to+\infty}\left(\frac{x_1^{(n)} + y_1^{(n)}}{2^{1-p}},\right. \\ &\left.\frac{x_2^{(n)} + y_2^{(n)}}{2^{1-p}}, \frac{x_3^{(n)} + y_3^{(n)}}{2^{1-p}}\right) = \left(\lim_{n\to+\infty} \frac{x_1^{(n)} + y_1^{(n)}}{2^{1-p}},\right. \\ &\left.\lim_{n\to+\infty} \frac{x_2^{(n)} + y_2^{(n)}}{2^{1-p}}, \lim_{n\to+\infty} \frac{x_3^{(n)} + y_3^{(n)}}{2^{1-p}}\right) \\ &= \left(\frac{\lim_{n\to+\infty} x_1^{(n)} + \lim_{n\to+\infty} y_1^{(n)}}{2^{1-p}},\right. \\ &\left.\frac{\lim_{n\to+\infty} x_2^{(n)} + \lim_{n\to+\infty} y_2^{(n)}}{2^{1-p}}, \frac{\lim_{n\to+\infty} x_3^{(n)} + \lim_{n\to+\infty} y_3^{(n)}}{2^{1-p}}\right) \end{aligned}$$

$$= \left(\lim_{n\to+\infty} x_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)}\right) \oplus_g \left(\lim_{n\to+\infty} y_1^{(n)}, \lim_{n\to+\infty} y_2^{(n)}, \lim_{n\to+\infty} y_3^{(n)}\right) = \lim_{n\to+\infty} x^{(n)} \oplus_g \lim_{n\to+\infty} y^{(n)}.$$

For (2.), from Definition 3.13, we get

Case 1 If $\left(x^{(n)}, y^{(n)}\right) \in D_1$ for all $n \in \mathbb{N}$, then $x_1^{(n)} \leq y_1^{(n)}$, $x_2^{(n)} \leq y_2^{(n)}$, and $x_3^{(n)} \leq y_3^{(n)}$ and

$$\lim_{n\to+\infty} \left[x^{(n)} \ominus_g y^{(n)}\right] = \lim_{n\to+\infty} \left(y_1^{(n)} - x_1^{(n)},\ y_2^{(n)} - x_2^{(n)},\ y_3^{(n)} - x_3^{(n)}\right) = \left(\lim_{n\to+\infty} y_1^{(n)} - \lim_{n\to+\infty} x_1^{(n)}, \lim_{n\to+\infty} y_2^{(n)} - \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} y_3^{(n)} - \lim_{n\to+\infty} x_3^{(n)}\right)$$

$$= \left(\lim_{n\to+\infty} x_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)}\right) \ominus_g \left(\lim_{n\to+\infty} y_1^{(n)}, \lim_{n\to+\infty} y_2^{(n)}, \lim_{n\to+\infty} y_3^{(n)}\right) = \lim_{n\to+\infty} x^{(n)} \ominus_g \lim_{n\to+\infty} y^{(n)}.$$

Case 2 If $\left(x^{(n)}, y^{(n)}\right) \in D_2$ for all $n \in \mathbb{N}$, then $y_1^{(n)} \leq x_1^{(n)}$, $x_2^{(n)} \leq y_2^{(n)}$, and $x_3^{(n)} \leq y_3^{(n)}$ and

$$\lim_{n\to+\infty} \left[x^{(n)} \ominus_g y^{(n)}\right] = \lim_{n\to+\infty} \left(x_1^{(n)} - y_1^{(n)},\ y_2^{(n)} - x_2^{(n)},\ y_3^{(n)} - x_3^{(n)}\right)$$

$$= \left(\lim_{n\to+\infty} x_1^{(n)} - \lim_{n\to+\infty} y_1^{(n)}, \lim_{n\to+\infty} y_2^{(n)} - \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} y_3^{(n)} - \lim_{n\to+\infty} x_3^{(n)}\right) = \left(\lim_{n\to+\infty} x_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)}\right) \ominus_g \left(\lim_{n\to+\infty} y_1^{(n)}, \lim_{n\to+\infty} y_2^{(n)}, \lim_{n\to+\infty} y_3^{(n)}\right)$$

$$= \lim_{n\to+\infty} x^{(n)} \ominus_g \lim_{n\to+\infty} y^{(n)}.$$

Case 3, 4, 5, 6, 7, 8 With $\left(x^{(n)}, y^{(n)}\right) \in D_3$, $\left(x^{(n)}, y^{(n)}\right) \in D_4$, $\left(x^{(n)}, y^{(n)}\right) \in D_5$, $\left(x^{(n)}, y^{(n)}\right) \in D_6$, $\left(x^{(n)}, y^{(n)}\right) \in D_7$, and $\left(x^{(n)}, y^{(n)}\right) \in D_8$, respectively. We also demonstrate a similar way.

For (3.), from Definition 3.8, we get

$$\lim_{n\to+\infty} \left[\lambda \odot_g x^{(n)}\right] = \lim_{n\to+\infty} \left(\lambda x_1^{(n)}, \lambda x_2^{(n)}, \lambda x_3^{(n)}\right) = \left(\lim_{n\to+\infty} \lambda x_1^{(n)}, \lim_{n\to+\infty} \lambda x_2^{(n)}, \lim_{n\to+\infty} \lambda x_3^{(n)}\right)$$

$$= \left(\lambda \lim_{n\to+\infty} x_1^{(n)}, \lambda \lim_{n\to+\infty} x_2^{(n)}, \lambda \lim_{n\to+\infty} x_3^{(n)}\right) = \lambda \odot_g \left(\lim_{n\to+\infty} x_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)}\right) = \lambda \odot_g \left[\lim_{n\to+\infty} x^{(n)}\right].$$

For (4.), from Definition 3.17, we get

$$\lim_{n\to+\infty} \left[x^{(n)} \otimes_g y^{(n)}\right] = \lim_{n\to+\infty} \left(x_1^{(n)} y_1^{(n)}, x_2^{(n)} y_2^{(n)}, x_3^{(n)} y_3^{(n)}\right) = \left(\lim_{n\to+\infty} x_1^{(n)} y_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)} y_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)} y_3^{(n)}\right)$$

$$= \left(\lim_{n\to+\infty} x_1^{(n)} . \lim_{n\to+\infty} y_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)} . \lim_{n\to+\infty} y_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)} . \lim_{n\to+\infty} y_3^{(n)}\right)$$

$$= \left(\lim_{n\to+\infty} x_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)}\right) \otimes_g \left(\lim_{n\to+\infty} y_1^{(n)}, \lim_{n\to+\infty} y_2^{(n)}, \lim_{n\to+\infty} y_3^{(n)}\right) = \lim_{n\to+\infty} x^{(n)} \otimes_g \lim_{n\to+\infty} y^{(n)}.$$

□

In classical mathematics, the completeness of a metric space is an important property because if this space is incomplete, then the limit operation will be meaningless and it is not entirely well-behaved metric space. Therefore, we will demonstrate that the metric semi-linear space of PFNs is complete in the following.

Definition 3.27 Let $\left\{x^{(n)}\right\}$ be a sequence of PFNs with $n \in \mathbb{N}$. In the metric space $\left(D_g^*, H_{D_g^*}\right)$, $\left\{x^{(n)}\right\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there is a number $N \in \mathbb{N}$ such that if $m, n > N$, then

$$H_{D_g^*}\left(x^{(m)}, x^{(n)}\right) < \varepsilon.$$

Theorem 3.28 *$\left(D_g^*, H_{D_g^*}\right)$ is a complete metric space.*

Proof Suppose that $\left\{x^{(n)}\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence of PFNs in $\left(D_g^*, H_{D_g^*}\right)$, where $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right)$. Since the component sequences $\left\{x_i^{(n)}\right\}$ with $i \in \{1, 2, 3\}$ are sequences in $[0, 1] \subset \mathbb{R}$ under the absolute-value metric, so we have $|x_i^{(m)} - x_i^{(n)}| \leq H_{D_g^*}\left(x^{(m)}, x^{(n)}\right)$ $\forall m, n \geq 1$. Besides, $\left\{x^{(n)}\right\}$ is a Cauchy sequence, i.e, for every $\varepsilon > 0$ there is a number $N \in \mathbb{N}$ such that if $m, n > N$, then

$$|x_i^{(m)} - x_i^{(n)}| \leq H_{D_g^*}\left(x^{(m)}, x^{(n)}\right) < \varepsilon.$$

Thus, $\left\{x_i^{(n)}\right\}$ with $i \in \{1, 2, 3\}$ are Cauchy sequences in $[0, 1] \subset \mathbb{R}$ that $[0, 1]$ is a complete metric space under the absolute-value metric. So the component sequences $\left\{x_i^{(n)}\right\}$ with $i \in \{1, 2, 3\}$ are convergent in $[0, 1]$, i.e, there exist $x_i^{(0)} \in D_g^*$ such that $\lim_{n\to+\infty} x_i^{(n)} = x_i^{(0)}$. Therefore, by Definition 3.23 we obtain $\lim_{n\to+\infty} x^{(n)} = \left(\lim_{n\to+\infty} x_1^{(n)}, \lim_{n\to+\infty} x_2^{(n)}, \lim_{n\to+\infty} x_3^{(n)}\right) = \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$. At the same time, by the item (i) of Corollary 3.24, we get $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) \in D_g^*$. So $\left(D_g^*, H_{D_g^*}\right)$ is a complete metric space. □

Remark 3.29 For the picture fuzzy set, we have a realistic illustration of the voting problem, which is given by Cuong in (Cuong 2014). His idea was to divide the voters into four groups (including voting in favor, abstaining from voting, voting against, refusal of the voting). For the picture fuzzy number which is the main object in this study, we only need to consider three groups of subjects participating in voting, namely voting for, abstaining, and voting against. To relate the results of this study to real-life such as a matter of voting for something, for example, a new law, a new administrator, a certain choice, and so on. We concretize this problem as follows: Given A and B are two places holding the vote about something in a certain area X, to make this easier to visualize, we assume that A and B are two provinces of country X and these two provinces are organizing people vote to pass or reject a new regulation. Now, for the space D_g^*, without loss of generality, we consider A and B to be two picture fuzzy numbers with $A = (x_1, x_2, x_3)$ and $B = (y_1, y_2, y_3)$. Where, x_1, x_2, x_3 and y_1, y_2, y_3 are the ratio of the number of votes of the three groups: vote for, abstain and vote against of A and B compared to the population of each province, respectively. From the results in this study, the following can be inferred: Firstly, the sum between A and B in the space D_g^* exists. If people in two provinces A and B both participate in the vote, then from Definition 3.5, this means that $m = 2$, $p \neq 0$ and the sum of A and B in the space D_g^* is $A + B = \left(\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2}, \frac{x_3+y_3}{2}\right)$, means the sum of the proportions of the votes of the population that voted in favor of provinces A and B to the population of each province is $\frac{x_1+y_1}{2}$, similarly for the total ratio of abstaining and voting against of A and B. If province A organizes for people to vote and province B does not, that is, B is the zero element in the space D_g^*, then we have $m = 2$ and $p = 1$. Therefore, $A + B = A$, this result is true because province B does not organize people to vote.

Secondly, multiplying a scalar value $k \in \mathbb{R}^+$ by an element A in the space D_g^*, will show the impact of objective or subjective factors on the value of the element A. For example, in voting, province A decided not to hold direct voting due to the appearance of the Covid-19 epidemic. This means that the value of the three groups: voting for, abstaining, and voting against of province A in the space D_g^* is $A = (0, 0, 0)$. in the space D_g^*, we can explain this as follows: suppose that the values of the three groups: voting for, abstaining, voting against of province A is $A = (x_1, x_2, x_3)$, if province A allows people to vote, then obviously $A \neq 0$. However, with $k = 0$ representing the Covid-19 epidemic appears, then we will now have $kA = (kx_1, kx_2, kx_3) = (0x_1, 0.x_2, 0.x_3) = (0, 0, 0)$. Hence, the result of the multiplication scalar kA reflects the fact that the voting results of the three groups: voting for, abstain, and vote against in province A is $(0, 0, 0)$. because this province did not vote.

Finally, from a mathematical point of view, the limit is the value a function approaches when the input variable approaches a certain value. Corresponding to the voting problem, the limit of the voting process will give us a result that the sum of the proportions of the votes of the three groups voting for, abstaining, and voting against compared to the number of voters will not be more than 1. This is true because if the result of the election process is that the sum of votes of the three groups voting for, abstaining, and voting against is greater than 1, then it is clear that this voting process has fraud on the number of votes.

4 Conclusions

In this paper, we establish the concept of the limit and study its properties on the metric semi-linear space of PFNs. Firstly, we propose two new operations, addition and scalar multiplication, to replace two of Wei's operations. We also discuss some limitations of Wei's two operations. These are also the reason that we want to replace them with two new operations with more advantages. The highlight is that the set of PFNs together with these two new operations becomes a semi-linear space. Along with that we also provide some related concepts on this semi-linear space such as metric, order relations between two PFNs, geometric difference, multiplication of two PFNs. Next, we define a type of function whose value is in this semi-linear space. It called the geometry picture fuzzy function and used to give the concept of limit for it and the sequences of PFNs in the metric semi-linear space of PFNs. Finally, to ensure that the limit operations are well-defined over the metric semi-linear space of PFNs, we proved that this space is complete.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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