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Spectral collapse in anisotropic two-photon Rabi model

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In this communication, based upon a squeezed-state trial wave function, we have performed a simple variational study of the spectral collapse in the anisotropic two-photon Rabi model. Our analysis indicates that the light-matter interaction and the spin-flipping (together with the anisotropy) effectively constitute two competing impacts upon the radiation mode. Whilst the former tries to decrease the radiation mode frequency, the latter may counteract or reinforce it. The light-matter interaction appears to dominate the frequency modulation as its coupling strengths go beyond the critical values, leading to the emergence of the spectral collapse. However, at the critical couplings the dominance of the light-matter interaction is not complete, and incomplete spectral collapse appears. Accordingly, at the critical couplings the eigenenergy spectrum comprises both a set of discrete energy levels and a continuous energy spectrum. The discrete eigenenergy spectrum can be derived via a simple one-to-one mapping to the bound state problem of a particle of variable effective mass in a finite potential well, and the number of bound states available is determined by the energy difference between the two atomic levels. Each of these eigenenergies has a twofold degeneracy corresponding to the spin degree of freedom.

In recent years the quantum two-photon Rabi model has been attracting much attention in the literature because it exhibits a counter-intuitive feature, commonly known as the "spectral collapse" , which occurs when the light-matter coupling strength ϵ goes beyond a critical value ϵ_c[1–11] . The spectral collapse was first demonstrated by Ng et al. in 1998[12,13] by means of exact numerical diagonalization, and was shown to be generated by a fundamental change of the system due to the intensification of the light-matter interaction. As pointed out by the authors, in the special case of vanishing energy difference ω_0 between the two atomic levels, the system behaves as a simple harmonic oscillator for $\epsilon < \epsilon_c$ and as an inverted harmonic potential barrier for $\epsilon > \epsilon_c$. Thus, in the subcritical regime the system has a discrete set of eigenenergies whereas the discrete energy spectrum turns into a continuous one as ϵ goes beyond the critical value ϵ_c. In addition, such a structural change of the eigenenergy spectrum persists for $\omega_0 \neq 0$ because this spin-flip term is a bounded operator. Since then, a number of theoretical studies on the two-photon Rabi model have confirmed the results and observations of Ng et al., via both analytical and numerical investigations[14–23] . Recently, via an elementary quantum mechanics approach, Lo[24] has also demonstrated explicitly that at the critical coupling the eigenenergy spectrum consists of both a set of discrete energy levels and a continuous energy spectrum, indicating an incomplete spectral collapse at the critical coupling. Moreover, recent advancement in the state-of-the-art quantum technology has dramatically made the applications of the two-photon Rabi model feasible in the strong coupling regime[1,2,4,11].

Stimulated by such a success, some people have started to pay special attention to some generalizations of the two-photon Rabi model. The simplest generalization is the anisotropic two-photon Rabi model, in which the rotating and counter-rotating terms have different coupling parameters. By means of a variation of Braak's G-function method based upon Bogoliubov rotation of the underlying $su(1,1)$ Lie algebra[25], Cui et al.[26] have performed an exact analysis of its spectral properties and found condensation of discrete energy levels, i.e. spectral collapse, when the critical values of light-matter couplings are breached. However, it should be noted that the G-function analysis can only approach the critical coupling limits asymptotically. It is thus the aim of this paper to scrutinize the spectral collapse in the anisotropic two-photon Rabi model, particularly the characteristic behaviour of the eigenenergy spectrum at the critical couplings.

The structure of this paper is organized as follows. In "Squeezed-state trial wave function" section a simple variational study based upon a squeezed-state trial wave function is performed to demonstrate rigorously how the spectral collapse in the anisotropic two-photon Rabi model occurs as its coupling strengths go beyond the critical values. A simple physical picture providing intuitive insights for the underlying physics is presented. In "Incomplete spectral collapse" section the eigenenergy spectrum of the system at the critical couplings is investigated. The system has an incomplete spectral collapse, implying that the eigenenergy spectrum comprises

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both a set of discrete energy levels and a continuous energy spectrum. The discrete eigenenergy spectrum can be derived via a simple one-to-one mapping to the bound state problem of a particle of variable effective mass in a finite potential well. The depth of the potential well is proportional to the energy difference between the two atomic levels so that the extent of the spectral collapse (or the number of bound states available) can be monitored straightforwardly. The final section conludes the paper.

Squeezed-state trial wave function

As in Cui et al.[26], the Hamiltonian of the anisotrpic two-photon Rabi model is given by ($\hbar = 1$)

$$\begin{aligned} H =& \omega a^\dagger a + \Delta\sigma_z + g\left(\sigma_+ a^2 + \sigma_- a^{\dagger 2}\right) + \lambda g\left(\sigma_+ a^{\dagger 2} + \sigma_- a^2\right) \\ =& \omega a^\dagger a + \omega_0 S_z + 2g_1\left(a^{\dagger 2} + a^2\right)S_x + i2g_2\left(a^2 - a^{\dagger 2}\right)S_y, \end{aligned} \tag{1}$$

for $\sigma_\pm = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right)$, $\vec{S} = \frac{1}{2}\vec{\sigma}$, $\omega_0 = 2\Delta$, $g_1 = \frac{1}{2}g(1+\lambda)$ and $g_2 = \frac{1}{2}g(1-\lambda)$. Here the radiation mode of frequency ω is described by the bosonic operators $\{a, a^\dagger\}$, the two atomic levels separated by an energy difference ω_0 are represented by the spin-half operators $\{S_x, S_y, S_z\}$, the light-matter coupling strength is measured by the positive parameter g, and λ is the anisotropy parameter. Without loss of generality, we may assume that both g_1 and g_2 are non-negative definite, i.e. $-1 \le \lambda \le 1$. In addition, under the unitary transformation $U = \exp\left\{i\frac{1}{2}\pi S_z\right\}\exp\left\{i\frac{1}{4}\pi a^\dagger a\right\}$, the Hamiltonian H becomes

$$\mathcal{H} = U^\dagger H U = \omega a^\dagger a + \omega_0 S_z + 2g_2\left(a^{\dagger 2} + a^2\right)S_x + i2g_1\left(a^2 - a^{\dagger 2}\right)S_y. \tag{2}$$

Evidently, $\mathcal{H}$ bears a striking resemblance to H, except that g_1 and g_2 are interchanged. In the following we shall investigate the critical values of the coupling parameters g_1 and g_2. For simplicity, we set the energy unit in such a way that $\omega = 1$.

To begin with, we try to examine the critical value of g_1 under the assumption that $g_2 < g_1$. By applying the unitary transformation

$$R = \exp\left\{-\frac{i\pi}{2}\left(S_z - \frac{1}{2}\right)\right\}a^\dagger a \tag{3}$$

to the Hamiltonian H, we obtain

$$\begin{aligned} \tilde{H} =& R^\dagger H R \\ =& \left[\omega_0 - 2g_2\left(a^2 - a^{\dagger 2}\right)\right]\left\{\cos\left(\frac{\pi}{2}a^\dagger a\right)S_z + \sin\left(\frac{\pi}{2}a^\dagger a\right)S_y\right\} + a^\dagger a \\ &+ g_1\left(a^{\dagger 2} + a^2\right). \end{aligned} \tag{4}$$

It is not difficult to realize that within the subspace of even number states of $a^\dagger a$ the transformed Hamiltonian $\tilde{H}$ becomes

$$\tilde{H}_e = \left[\omega_0 - 2g_2\left(a^2 - a^{\dagger 2}\right)\right]\cos\left(\frac{\pi}{2}a^\dagger a\right)S_z + a^\dagger a + g_1\left(a^{\dagger 2} + a^2\right), \tag{5}$$

whereas within the subspace of odd number states we have

$$\tilde{H}_o = \left[\omega_0 - 2g_2\left(a^2 - a^{\dagger 2}\right)\right]\sin\left(\frac{\pi}{2}a^\dagger a\right)S_y + a^\dagger a + g_1\left(a^{\dagger 2} + a^2\right). \tag{6}$$

In both cases the spin degree of freedom and the boson mode are decoupled. Thus, the eigenstates of each subspace are simply given by the product states $|M\rangle|\phi_n\rangle$, where $|M\rangle$ is an eigenstate of the spin operator S_z or S_y, and $|\phi_n\rangle$ the n-th eigenstate of the one-body bosonic Hamiltonian

$$\bar{H} = M\left[\omega_0 - 2g_2\left(a^2 - a^{\dagger 2}\right)\right]F\left(a^\dagger a\right) + a^\dagger a + g_1\left(a^{\dagger 2} + a^2\right), \tag{7}$$

where $M = \pm\frac{1}{2}$ and

$$F\left(a^\dagger a\right) = \begin{cases} \cos\left(\frac{\pi}{2}a^\dagger a\right) & \text{for the subspace of even number states} \\ \sin\left(\frac{\pi}{2}a^\dagger a\right) & \text{for the subspace of odd number states} \end{cases}. \tag{8}$$

As a result, the Hilbert space of H is divided into four different subspaces, each of which is specified by the spin quantum number and the parity.

Likewise, in the special case of $\omega_0 = g_2 = 0$ the one-body Hamiltonian $\bar{H}$ can be diagonalized by the unitary squeezing transformation $T = \exp\left\{-\frac{1}{4}\tanh^{-1}\left(2g_1\right)\left(a^{\dagger 2} - a^2\right)\right\}$ for $g_1 < \frac{1}{2}$ as follows[12]:

$$\mathcal{H} = T^\dagger \bar{H} T = \tilde{\omega}\left(a^\dagger a + \frac{1}{2}\right) - \frac{1}{2}, \tag{9}$$

where $\tilde{\omega} = \sqrt{1 - 4g_1^2} < 1$, implying that the transformed radiation mode has a lower frequency, and that the light-matter interaction has the effect of generating a redshift to the radiation mode frequency. On the other hand, there is no unitary transformation which can diagonalize $\bar{H}$ for $g_1 > \frac{1}{2}$. Indeed, as pointed out by Ng et al.[12,13], the system represents a simple harmonic oscillator for $g_1 < \frac{1}{2}$, becomes a free particle at the critical coupling, *i.e.* $g_1 = \frac{1}{2}$, and finally turns into an inverted harmonic potential barrier for $g_1 > \frac{1}{2}$. Such an abrupt structural change of the system thus leads to the collapse of a set of discrete eigenenergy levels into a continuum energy spectrum in this special case.

Now we perform a variational study of the ground state in each of the four subspaces. In view of the aforementioned unitary squeezing transformation, a natural candidate for the variational wave function is the squeezed-state trial wave function:

$$|G\rangle = \exp\left\{\frac{1}{4}\xi\left(a^{\dagger 2} - a^2\right)\right\}|\Psi\rangle \tag{10}$$

for some real variational parameter ξ, where

$$|\Psi\rangle = \begin{cases} |0\rangle & \text{for the subspace of even number states} \\ |1\rangle & \text{for the subspace of odd number states} \end{cases}. \tag{11}$$

With respect to this trial wave function, the expectation value of $\bar{H}$ can be calculated readily:

$$\begin{aligned} E &= \langle G|\bar{H}|G\rangle \\ &= M\omega_0 \mathrm{sech}^{2k}(\xi)\left\{1 + \frac{8kg_2}{\omega_0}\tanh(\xi)\right\} + 2k\left[\cosh(\xi) + 2g_1\sinh(\xi)\right] - \frac{1}{2} \\ &= M\omega_0 \mathrm{sech}^{2k}(\xi)\left\{1 + \frac{8kg_2}{\omega_0}\tanh(\xi)\right\} + k\left[\left(1+2g_1\right)e^{\xi} + \left(1-2g_1\right)e^{-\xi}\right] - \frac{1}{2}, \end{aligned} \tag{12}$$

where

$$k = \begin{cases} \frac{1}{4} & \text{for the subspace of even number states} \\ \frac{3}{4} & \text{for the subspace of odd number states} \end{cases}. \tag{13}$$

Apparently, for $g_1 < \frac{1}{2}$ the second term of E, *i.e.* the term with the square brackets, is positive definite for all values of ξ so that E approaches infinity as $\xi \to \pm\infty$ for

$$\left|M\omega_0 \mathrm{sech}^{2k}(\xi)\left\{1 + \frac{8kg_2}{\omega_0}\tanh(\xi)\right\}\right| < \frac{\omega_0}{2}\left(1 + \frac{8kg_2}{\omega_0}\right),$$

regardless of k. Thus, a minimum value of E is guaranteed for each subspace. For $\omega_0 = g_2 = 0$, the minimum appears at $\xi = -\tanh^{-1}(2g_1) \equiv \xi_0$, giving the exact energy $E = 2\tilde{\omega}k - \frac{1}{2}$ for $\tilde{\omega} = \sqrt{1 - 4g_1^2}$[12,13]. It is obvious that the light-matter interaction tries to decrease the radiatioon mode frequency. Then, once the spin-flip term is turned on, the minimum moves above ξ_0 for $M = -\frac{1}{2}$ and below ξ_0 for $M = \frac{1}{2}$, regardless of k. That is, the spin-flipping counteracts the light-matter interaction for $M = -\frac{1}{2}$ whilst reinforcing it for $M = \frac{1}{2}$. Similar observations can be made for a finite g_2. On the other hand, for $g_1 > \frac{1}{2}$ the term with the square brackets decreases monotonically as ξ approaches $-\infty$, indicating that E is not bounded below. This is in agreement with the numerical diagonalization of $\bar{H}$ using the basis states in each subspace, which does not give any converged results at all, indicating the non-existence of bound states.

Finally, we examine the critical value of g_2 under the assumption that $g_1 < g_2$. Since $\mathcal{H}$ closely resembles H, except that g_1 and g_2 are interchanged, the aforementioned analysis can be straightforwardly applied to $\mathcal{H}$. Beyond question, the same conclusion is reached; for $g_2 < \frac{1}{2}$ a normalisable ground state exists in each subspace whilst for $g_2 > \frac{1}{2}$ there is no bound state at all. Accordingly, the critical values of g_1 and g_2 are both equal to $\frac{1}{2}$, implying that a discrete eigenenergy spectrum exists for the confined region specified by $g_1 < \frac{1}{2}$ and $g_2 < \frac{1}{2}$ only.

Incomplete spectral collapse

In this section we apply Lo's approach[24] to determine the eigenstates of the one-body Hamiltonian $\bar{H}$ in both subspaces at the critical couplings. First of all, we examine the case that $g_2 < g_1$ and $g_1 = \frac{1}{2}$. In terms of the "position" and "momentum" operators of the boson mode:

$$x = \frac{1}{\sqrt{2}}\left(a + a^{\dagger}\right) \qquad \text{and} \qquad p = \frac{1}{i\sqrt{2}}\left(a - a^{\dagger}\right), \tag{14}$$

we rewrite $\bar{H}$ as

$$\bar{H} = i^{\nu}\sqrt{i}M\omega_0\left[1 - i\frac{2g_2}{\omega_0}(xp + px)\right]\exp\left(-\frac{i\pi}{2}H_0\right) + \left(H_0 - \frac{1}{2}\right) - \frac{1}{2}\left(p^2 - x^2\right), \tag{15}$$

where

$$H_0 = \frac{p^2}{2} + \frac{x^2}{2} = a^{\dagger}a + \frac{1}{2} \tag{16}$$

is the Hamiltonian of a quantum simple harmonic oscillator of unit mass and frequency, and

$$\nu = \begin{cases} 0 & \text{for the subspace of even number states} \\ 1 & \text{for the subspace of odd number states} \end{cases}. \tag{17}$$

In the coordinate space the eigenvalue equation of $\bar{H}$ reads

$$\begin{aligned} E\phi(x) &= \left\{ i^{\nu}\sqrt{i}M\omega_0\left[1 - i\frac{2g_2}{\omega_0}(xp + px)\right]\exp\left(-\frac{i\pi}{2}H_0\right)\phi(x) + x^2 - \frac{1}{2}\right\}\phi(x) \\ &= \left(x^2 - \frac{1}{2}\right)\phi(x) + i^{\nu}M\omega_0\left[1 - \frac{2g_2}{\omega_0}\left(x\frac{d}{dx} + \frac{d}{dx}x\right)\right]\tilde{\phi}(x), \end{aligned} \tag{18}$$

where E denotes the eigenenergy and

$$\tilde{\phi}(p) = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\exp\{-ipx\}\phi(x) \tag{19}$$

is the Fourier transform of $\phi(x)$. Here we have made use of the fact that

$$\exp(-itH_0)\phi(x) = \int_{-\infty}^{\infty} dyK(x, t; y)\phi(y), \tag{20}$$

where $K(x, t; y)$ is the propagator of H_0 defined by

$$K(x, t; y) = \frac{1}{\sqrt{i2\pi\sin(t)}}\exp\left\{-\frac{(x^2 + y^2)\cos(t) - 2xy}{i2\sin(t)}\right\}. \tag{21}$$

Similarly, in the momentum space the eigenvalue equation of $\bar{H}$ is given by

$$E\tilde{\phi}(p) = \left(-\frac{d^2}{dp^2} - \frac{1}{2}\right)\tilde{\phi}(p) + (-i)^{\nu}M\omega_0\left[1 + \frac{2g_2}{\omega_0}\left(p\frac{d}{dp} + \frac{d}{dp}p\right)\right]\phi(p), \tag{22}$$

Eliminating $\phi(x)$ from Eqs. (18) and (22) yields

$$\begin{aligned} \left(E + \frac{1}{2}\right)\tilde{\phi}(p) = &-\frac{d^2\tilde{\phi}(p)}{dp^2} + \frac{1}{4}\left\{\omega_0 + 2g_2\left(p\frac{d}{dp} + \frac{d}{dp}p\right)\right\}\frac{1}{E + \frac{1}{2} - p^2} \\ &\times\left\{\omega_0 - 2g_2\left(p\frac{d}{dp} + \frac{d}{dp}p\right)\right\}\tilde{\phi}(p), \end{aligned} \tag{23}$$

for $M^2 = 1/4$ and $i^{\nu}(-i)^{\nu} = 1$. The disappearance of both the parameter ν and spin eigenvalue M in Eq. (23) implies that the even-parity and odd-parity eigenstates of $\tilde{H}$ simply correspond to the even-parity and odd-parity solutions of the eigenvalue equation, respectively, and that each eigenstate is doubly degenerate. For $E + \frac{1}{2} < 0$, we introduce the parameter $\kappa = \sqrt{|E + \frac{1}{2}|}$ and define a new variable $q = p/\kappa$ such that Eq. (23) can be expressed as

$$\begin{aligned} -\kappa^4\tilde{\phi}(q) &= -\frac{d^2\tilde{\phi}(q)}{dq^2} - \left\{\frac{\omega_0}{2} + g_2\left(q\frac{d}{dq} + \frac{d}{dq}q\right)\right\}\frac{1}{1 + q^2} \\ &\quad\times\left\{\frac{\omega_0}{2} - g_2\left(q\frac{d}{dq} + \frac{d}{dq}q\right)\right\}\tilde{\phi}(q) \\ &= \left\{-\frac{d}{dq}\left[\frac{1}{2m(q)}\right]\frac{d}{dq} + V(q)\right\}\tilde{\phi}(q), \end{aligned} \tag{24}$$

where

$$m(q) = \frac{1 + q^2}{2[1 + (1 - 4g_2^2)q^2]} \tag{25}$$

$$V(q) = -\frac{(\omega_0/2)^2 - g_2^2}{(1 + q^2)^2}\left\{1 + \frac{(\omega_0/2) - 3g_2}{(\omega_0/2) + g_2}q^2\right\}. \tag{26}$$

Accordingly, we have obtained the time-independent Schrödinger equation for a particle of variable effective mass $m(q)$ in the finite potential well $V(q)$[27,28]. Since $g_2 < \frac{1}{2}$ in this case, the variable effective mass $m(q)$ is positive definite. Here $V(q)$ describes a simple finite potential well for $g_2 < \frac{1}{2}\omega_0$ whilst it behaves like a finite double-well potential for $g_2 > \frac{1}{2}\omega_0$. It should be noted that in the absence of anisotropy, *i.e.* $g_2 = 0$, the variable effective mass $m(q)$ is reduced to a constant and the finite potential well $V(q)$ becomes a "Lorentzian function" potential well[24]. Likewise, as shown in Lo[29], we may perform a suitable unitary transformation to reduce Eq. (24) to the time-independent Schrödinger equation for a particle of unit mass in a finite potential well (see "Appendix").

On the other hand, for $E+\frac{1}{2}>0$, in terms of the parameter $k=\sqrt{E+\frac{1}{2}}$ and the new variable $\bar{q}=p/k$, Eq. (23) becomes

$$k^4\tilde{\phi}(\bar{q})=\left\{-\frac{d}{d\bar{q}}\left[\frac{1}{2\bar{m}(\bar{q})}\right]\frac{d}{d\bar{q}}+\bar{V}(\bar{q})\right\}\tilde{\phi}(\bar{q}), \tag{27}$$

where

$$\bar{m}(\bar{q})=\frac{1-\bar{q}^2}{2\left[1-(1-4g_2^2)\bar{q}^2\right]} \tag{28}$$

$$\bar{V}(\bar{q})=\frac{(\omega_0/2)^2-g_2^2}{(1-\bar{q}^2)^2}\left\{1-\frac{(\omega_0/2)-3g_2}{(\omega_0/2)+g_2}\bar{q}^2\right\}. \tag{29}$$

Obviously, this is the time-independent Schrödinger equation of the scattering state problem associated with a particle of variable effective mass $\bar{m}(\bar{q})$ in the presence of the potential barrier $\bar{V}(\bar{q})$ that is singular at $\bar{q}=\pm 1$.

Finally, the aforementioned analysis can be readily applied to the case that $g_1<g_2$ and $g_2=\frac{1}{2}$ by simply replacing g_2 by g_1 in Eq. (23) and the subsequent equations because $\mathcal{H}$ can be derived from H by interchanging g_1 and g_2. Beyond question, the same conclusion can be drawn.

Conclusion

Based upon a squeezed-state trial wave function, we have performed a simple variational study of the spectral collapse in the anisotropic two-photon Rabi model. Our analysis indicates that the light-matter interaction and the spin-flipping (together with the anisotropy) effectively constitute two competing impacts upon the radiation mode. Whilst the former tries to decrease the radiation mode frequency, the latter may counteract or reinforce it. The light-matter interaction appears to dominate the frequency modulation as its coupling strengths go beyond the critical values, namely $g_1>\frac{1}{2}$ and/or $g_2>\frac{1}{2}$, leading to the emergence of the spectral collapse. However, at the critical couplings the dominance of the light-matter interaction is not complete, and incomplete spectral collapse appears. Accordingly, at the critical couplings the eigenenergy spectrum comprises both a set of discrete energy levels and a continuous energy spectrum. The discrete eigenenergy spectrum can be derived via a simple one-to-one mapping to the bound state problem of a particle of variable effective mass in a finite potential well, and the number of bound states available is determined by the energy difference between the two atomic levels. Each of these eigenenergies has a twofold degeneracy corresponding to the spin degree of freedom.

Appendix

In order to faciliate a better understanding of the eigenenergy spectrum, we introduce the unitary transformation[29]

$$U=\exp\left\{\frac{i}{2}\left[\left(\frac{1}{i}\frac{d}{dq}\right)f(q)+f(q)\left(\frac{1}{i}\frac{d}{dq}\right)\right]\right\} \tag{30}$$

for some function $f(q)$, which transforms q and $\frac{1}{i}\frac{d}{dq}$ as follows:

$$U^\dagger qU=q+F(q) \tag{31}$$

$$U^\dagger\left(\frac{1}{i}\frac{d}{dq}\right)U=\frac{1}{2}\left\{\left(\frac{1}{i}\frac{d}{dq}\right)G(q)+G(q)\left(\frac{1}{i}\frac{d}{dq}\right)\right\}. \tag{32}$$

Here

$$F(q)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}f_n(q),\quad f_{n+1}(q)=f(q)\frac{df_n(q)}{dq},\quad f_1(q)=f(q) \tag{33}$$

$$G(q)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}g_n(q),\quad g_{n+1}(q)=f^2(q)\frac{d}{dq}\left\{\frac{g_n(q)}{f(q)}\right\},\quad g_0(q)=1. \tag{34}$$

Obviously, in order that the commutation relation between q and $\frac{1}{i}\frac{d}{dq}$ are preserved under the unitary transformation U, *i. e.* $\left[U^\dagger qU, U^\dagger\left(\frac{1}{i}\frac{d}{dq}\right)U\right]=\left[q,\frac{1}{i}\frac{d}{dq}\right]=i$, we must require

$$\frac{d}{dq}\{q+F(q)\}=\frac{1}{G(q)}. \tag{35}$$

Then, applying the unitary transformation U to Eq. (24) gives

$$-\kappa^4\tilde{\phi}(q) = \Bigg\{-\frac{d}{dq}\left(\frac{G(q)^2}{2m(\xi(q))}\right)\frac{d}{dq} - \frac{1}{8}\left[\frac{d}{dq}, \frac{2G(q)}{m(\xi(q))}\frac{dG(q)}{dq}\right] + \frac{1}{8m(\xi(q))}\left(\frac{dG(q)}{dq}\right)^2 + V(m(\xi(q)))\Bigg\}\tilde{\phi}(q), \tag{36}$$

where $\xi(q) = q + F(q)$ and $\bar{\phi}(q) = U^\dagger\phi(q)$. By setting $G(q) = \sqrt{m(\xi(q))}$, we obtain

$$\left[\frac{d}{dq}, \frac{2G(q)}{m(\xi(q))}\frac{dG(q)}{dq}\right] = \frac{d}{dq}\left\{\frac{2G(q)}{m(\xi(q))}\frac{dG(q)}{dq}\right\} = \frac{d^2 \ln m(\xi(q))}{dq^2} \tag{37}$$

and

$$\frac{1}{m(\xi(q))}\left(\frac{dG(q)}{dq}\right)^2 = \frac{1}{4}\left\{\frac{d\ln m(\xi(q))}{dq}\right\}^2. \tag{38}$$

As a result, Eq. (36) is reduced to

$$-\kappa^4\tilde{\phi}(q) = \left\{-\frac{1}{2}\frac{d^2}{dq^2} + \tilde{V}(\xi(q))\right\}\tilde{\phi}(q), \tag{39}$$

where

$$\begin{aligned}\bar{V}(\xi(q)) =& V(\xi(q)) - \frac{1}{8}\left(\frac{d^2\ln m(\xi(q))}{dq^2}\right) + \frac{1}{32}\left(\frac{d\ln m((q))}{dq}\right)^2 \\ =& V(\xi(q)) - \frac{4g_2^2}{\left(1+\xi(q)^2\right)^3}\left\{\frac{1-g_2^2\xi(q)^2}{1+(1-4g_2^2)\xi(q)^2} - \frac{3}{2}\xi(q)^2\right\}.\end{aligned} \tag{40}$$

By taking a close look at the $\bar{V}(\xi)$, we can readily recognise that it represents a symmetric finite potential well in ξ with its minimum value occurring at $\xi = 0$: $\bar{V}(\xi = 0) = -\{(\omega_0/2)^2 + 3g_2^2\}$, and it vanishes asymptotically as $\xi \to \pm\infty$. In addition, Eq. (35) yields

$$q(\xi) = \int_0^\xi d\eta\sqrt{m(\eta)}\,, \tag{41}$$

indicating that q is a monotonically increasing function of ξ. Consequently, we have succeeded in transforming Eq. (24) into the time-independent Schrödinger equation of the bound state problem associated with a particle of unit mass moving in the finite potential well $\bar{V}(\xi(q))$, and this system has a set of discrete bound state eigenenergy levels.

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Competing interests
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Additional information
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