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ORIGINAL ARTICLE

Periodicity computation of generalized mathematical biology problems involving delay differential equations

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Abstract In this paper, we consider a low initial population model. Our aim is to study the periodicity computation of this model by using neutral differential equations, which are recognized in various studies including biology. We generalize the neutral Rayleigh equation for the third-order by exploiting the model of fractional calculus, in particular the Riemann–Liouville differential operator. We establish the existence and uniqueness of a periodic computational outcome. The technique depends on the continuation theorem of the coincidence degree theory. Besides, an example is presented to demonstrate the finding.

1. Introduction

Biocomputing is proposed as the procedure of constructing models that use biological materials. The class of neutral differential delay equations is the most popular model in Biocomputing. It was introduced by the famous British mathematical biologist, Lord Rayleigh, as follows:

$$x''(t) + f(x'(t)) + ax(t) = 0. \tag{1}$$

Eq. (1) is extended into a third order by various authors. Abou-El-Ela et al. (2012) discussed a criterion for the existence of periodicity to third order neutral delay differential equation with one deviating argument as below:

$$x'''(t) + ax''(t) + g(x'(t-\tau(t))) + f(x(t-\tau(t))) = p(t). \tag{2}$$

Using the idea of the fractional calculus (see Podlubny, 1999), Eq. (1) is developed (see Ibrahim et al., 2016a,b,c). Recently, Rakkiyappan et al. (2016) presented the periodicity by applying fractional neural network model.

The objective of this work is to give new appropriate conditions for guaranteeing the existence and uniqueness of a periodic solution of fractional differential equation of order 3μ $(0 < \mu < 1)$ with two deviating arguments, taking the form

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$$D^{3\mu}u(t)+\Psi(u'(t))u''(t)+\varphi(u(t))u'(t)+\vartheta_1(t,u(t-\varepsilon_1(t))) + \vartheta_2(t,u(t-\varepsilon_2(t)))=p(t), \quad (3)$$

where $D^{3\mu}$ is the Riemann–Liouville fractional differential operator of order 3μ, $\Psi,\varphi,\varepsilon_1,\varepsilon_2,p:\mathcal{R}\to\mathcal{R}$ and $\vartheta_1,\vartheta_2:\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ are continuous functions ε_1, ε_2 and p are periodic, ϑ_1 and ϑ_2 are periodic in their first argument and $T>0$.

2. Material and methods

For convenience, we let

$$|u|_\kappa=\left(\int_0^T|u(t)|^\kappa dt\right)^{\frac{1}{\kappa}},\ \kappa\geqslant 1,\ |u|_\infty=max_{t\in[0,T]}|u(t)|,$$

$$|p|_\infty=max_{t\in[0,T]}|p(t)| \text{ and } \bar{p}=\frac{1}{T}\int_0^T p(t)dt.$$

Let the following sets

$$X=\{u|u\in C^2(\mathcal{R},\mathcal{R}),u(t+T)=u(t),\quad \text{for all } t\in\mathcal{R}\}$$

and

$$\Upsilon=\{y|y\in C(\mathcal{R},\mathcal{R}),y(t+T)=y(t),\quad \text{for all } t\in\mathcal{R}\}$$

are be two Banach spaces with the norms

$||u_X||=max\{|u|_\infty,|u'|_\infty,|u''|_\infty\}$ and $||y||_\Upsilon=|y|_\infty$.

Outline a linear operator $L:Dom(L)\subset X\to\Upsilon$ by setting

$$Dom(L)=\{u|u\in X,D^{3\mu}u(t)\in C(\mathcal{R},\mathcal{R})\},$$

and for $u\in Dom(L)$,

$$L_u=D^{3\mu}u(t). \quad (4)$$

We as well term a nonlinear operator $\mathcal{N}:X\to\Upsilon$ by setting

$$\mathcal{N}_u=-\Psi(u'(t))u''(t)-\varphi(u(t))u'(t)-\vartheta_1(t,u(t-\varepsilon_1(t))) -\vartheta_2(t,u(t-\varepsilon_2(t)))+p(t). \quad (5)$$

Therefore, we have seen that $KerL=\mathcal{R}$, $dim(KerL)=1$; $Im\ L=\{y|y\in\Upsilon,\int_0^T y(\varsigma)d\varsigma=0\}$ is a subset of Υ and $dim(\Upsilon/ImL)=1$, which implies $diom(Im\ L)=\dim(KerL)$.

So the operator L is a Fredholm operator with index zero. Now we define a nonlinear operator as follows:

$$L_u=\alpha\mathcal{N}_u,\ \alpha\in(0,1);$$

$$D^{3\mu}u(t)+\alpha\{\Psi(u'(t))u''(t)+\varphi(u(t))u'(t) +\vartheta_1(t,u(t-\varepsilon_1(t)))+\vartheta_2(t,u(t-\varepsilon_2(t)))\}=\alpha p(t), \quad (6)$$

where the Riemann–Liouville fractional differential operator is defined as follows:

$$D^\mu u(t)=\frac{1}{\Gamma(1-\mu)}\frac{d}{dt}\int_0^t(t-s)^{-\mu}u(s)ds,\quad 0<t<\infty.$$

We need the following outcome:

Method 2.1 *(Continuation method)* Assume that X and Υ be two Banach spaces. Supposing that $L:Dom(L)\subset X\to\Upsilon$ is a Fredholm operator with index zero and $\mathcal{N}:X\to\Upsilon$ is L-compact on $\overline{\mathcal{F}}$, where $\mathcal{F}$ is an open bounded subset in X. Furthermore, let the next conditions are satisfied:

(a) $Lu\neq\alpha\mathcal{N}u$, for all $u\in\omega\mathcal{F}\cap Dom(L)$, $\alpha\in(0,1)$;
(b) $\mathcal{N}u\notin ImL$, for all $u\in\omega\mathcal{F}\cap KerL$;
(c) The Brower degree $deg\{Q\mathcal{N},\mathcal{F}\cap KerL,0\}\neq 0$.

Then $Lu=\mathcal{N}u$ has at least one solution on $\bar{\mathcal{F}}\cap Dom(L)$. Moreover, we need the following assumptions in the sequel:

(i) Suppose that there exist non-negative constants A_1; A_2; B_1; B_2; C_1 and C_2 such as

$$|\Psi(y)|\leqslant A_1,\ |\Psi(y_1)-\Psi(y_2)|\leqslant A_2|y_1-y_2|$$

For all $y,y_1,y_2\in\mathcal{R}$,

$$|\varphi(u)|\leqslant C_1,\ |\varphi(u_1)-\varphi(u_2)|\leqslant C_2|u_1-u_2|$$

For all $u,u_1,u_2\in\mathcal{R}$ and

$$|\vartheta_\iota(t,\upsilon)-\vartheta_\iota(t,v)|\leqslant B_\iota|\upsilon-v|$$

For all $y,\upsilon,v\in\mathcal{R},\ \iota=1,2.$

(ii) Assume that the subsequent conditions are satisfied:

(H_1) One of the next conditions holds

(1) $(\vartheta_\iota(t,\upsilon)-\vartheta_\iota(t,v))(\upsilon-v)>0$ for all $t,\upsilon,v\in\mathcal{R},\ \upsilon\neq v$, $\iota=1,2$,
(2) $(\vartheta_\iota(t,\upsilon)-\vartheta_\iota(t,v))(\upsilon-v)<0$ for all $t,\upsilon,v\in\mathcal{R},\ \upsilon\neq v$, $\iota=1,2$;

(H_2) There exists d >0 like one of the following conditions holds

(1) $u\{\vartheta_1(t,u)+\vartheta_2(t,u)-\bar{p}\}>0$ for all $t\in\mathcal{R},\ |u|>d$,
(2) $u\{\vartheta_1(t,u)+\vartheta_2(t,u)-\bar{p}\}<0$ for all $t\in\mathcal{R},|u|>d$;

If $u(t)$ is a periodic solution of (6), then

$$|u|_\infty\leqslant d+\frac{1}{2}\sqrt{T}|u'|_2. \quad (7)$$

(iii) Assume that (*i*) and (*ii*) hold such that
(iv)

$$\Gamma(3\mu+1)\left[A_1\frac{T}{2}+C_1\frac{T^2}{4}+(B_1+B_2)\frac{T^3}{8}\right]<1. \quad (8)$$

If $u(t)$ is a periodic solution of (3), then

$$|u''|_\infty\leqslant\frac{[(B_1+B_2)d+M+|p|_\infty]T}{2\left\{\frac{1}{\Gamma(3\mu+1)}-A_1\frac{T}{2}-C_1\frac{T^2}{4}-(B_1+B_2)\frac{T^3}{8}\right\}}=\kappa,$$

$M:=max\{|\vartheta_1(t,0)|+|\vartheta_2(t,0)|:0\leqslant t\leqslant T\}$.

(v) Assume that (*i*)–(*iii*) hold. Also let the next condition holds

$$\Gamma(3\mu+1)\left[A_1\frac{T}{2}+(A_1\kappa+C_1)\frac{T^2}{4}(B_1+B_2)+C_2\kappa\frac{T}{8}\right]<1. \quad (9)$$

3. Results

We impose the periodicity computation of the generalized neutral equation (3) in the following result:

Result 3.1: Assume that (i) - (iv) hold. Then (3) has a unique periodic solution.

Demonstration: Condition (*iv*) implies that (3) has at most one periodic solution. Therefore, it is enough to prove that

(3) has at least one periodic solution. Let the set of periodic solutions of (6) be bounded (by the boundedness of $D^{3\mu}u(t)$).

Let $u(t)$ be T-periodic solution of (6). Multiplying (4) by $D^{3\mu}(t)$ and then by integrating it over $[0; T]$, we get

$$\int_0^T (D^{3\mu}u(t))^2 dt = -\alpha\int_0^T \Psi(u'(t))u''(t)D^{3\mu}u(t)dt - \alpha\int_0^T \varphi(u(t))u'(t)D^{3\mu}u(t)dt - \alpha\int_0^T \vartheta_1(t,u(t-\mathcal{E}_1(t)))D^{3\mu}u(t)dt - \alpha\int_0^T \vartheta_2(t,u(t-\mathcal{E}_2(t)))D^{3\mu}u(t)dt + \alpha\int_0^T p(t)D^{3\mu}u(t)dt.$$

Using the fractional Taylor series (Tarasov, 2016), according to the condition (i), inequality (7) and the Cauchy–Schwarz inequality, we obtain

$$|D^{3\mu}u|_2^2 \leqslant \Gamma(3\mu+1)\left\{A_1\frac{T}{2} + C_1\frac{T^2}{4} + (B_1+B_2)\frac{T^3}{8}\right\}|u^{(3)}(t)|_2^2 + \Gamma(3\mu+1)[(B_1+B_2)d + M|P|_\infty\sqrt{T}]|u^{(3)}(t)|_{2,}$$

Thus, there exists $\mathfrak{M}_0 > 0$ such as $|D^{3\mu}u|_2 < \mathfrak{M}_0$ with the following inequalities:

$$|u''|_\infty \leqslant \frac{1}{2}\sqrt{T}\mathfrak{M}_0,$$

$$|u'|_\infty \leqslant \frac{1}{4}T^{\frac{3}{2}}\mathfrak{M}_0,$$

$$|u|_\infty \leqslant d + \frac{1}{8}T^{\frac{5}{2}}\mathfrak{M}_0.$$

Let $\mathfrak{M} = max\{d+\frac{1}{8}T^{\frac{5}{2}}\mathfrak{M}_0, \frac{1}{4}T^{\frac{3}{2}}\mathfrak{M}_0, \frac{1}{2}\sqrt{T}\mathfrak{M}_0\}$, now we have $\mathcal{F} = \{u|u \in X, ||u|| < M\}$ as a non-empty open bounded subset of X. Thus, condition (a) in Method 2.1 holds. According to $(H_2)(1)$ and $(H_2)(2)$, we aim to study two cases:

Case (i): Let $(H_2)(1)$ hold. Since

$$Q\mathcal{N}u = -\frac{1}{T}\int_0^T \{\Psi(u'(t))u''(t) + \varphi(u(t))u'(t) + \vartheta_1(t,u(t-\varepsilon_1(t))) + \vartheta_2(t,u(t-\varepsilon_2(t))) - \bar{p}\}dt,$$

For any $u \in \omega\mathcal{F}\cap KerL = \omega\mathcal{F}\cap\mathcal{R}$, then u is a constant with $u(t) = \mathfrak{M}$ or $u(t) = -\mathfrak{M}$. Then

$$Q\mathcal{N}(\mathfrak{M}) = -\frac{1}{T}\int_0^T \{\vartheta_1(t,\mathfrak{M}) + \vartheta_2(t,\mathfrak{M}) - \bar{p}\}dt < 0, \quad (10)$$

$$Q\mathcal{N}(-\mathfrak{M}) = -\frac{1}{T}\int_0^T \{\vartheta_1(t,\mathfrak{M}) + \vartheta_2(t,\mathfrak{M}) - \bar{p}\}dt > 0,$$

The condition (b) of Method 2.1 is fulfilled. Moreover, we define a continuous function $H(u,h)$ by setting

$$H(u,h) = -hu + (1-h)Q\mathcal{N}u = -hu - (1-h)\frac{1}{T} \times \int_0^T \left\{\begin{matrix}\Psi(u'(t))u''(t) + \varphi(u(t))u'(t) + \\ \vartheta_1(t,u(t-\varepsilon_1(t))) + \vartheta_1(t,u(t-\varepsilon_1(t))) - \bar{p}\end{matrix}\right\}dt,$$

According to (10) we have $uH(u,h) < 0$ for all $u \in \omega\mathcal{F}\cap KerL$ and $h \in [0,1]$ therefore, $H(u,h)$ is a homotopic transformation. From the homotopy invariance theorem we get

$$deg\{Q\mathcal{N},\mathcal{F}\cap KerL, 0\} = deg\{-u,\mathcal{F}\cap KerL, 0\} \neq 0,$$

Therefore, condition (c) is satisfied.

Case (ii): Let $(H_2)(2)$ hold. Since

$$Q\mathcal{N}u = -\frac{1}{T}\int_0^T \{\Psi(u'(t))u''(t) + \varphi(u(t))u'(t) + \vartheta_1(t,u(t-\varepsilon_1(t))) + \vartheta_2(t,u(t-\varepsilon_2(t))) - \bar{p}\}dt,$$

For any $u \in \omega\mathcal{F}\cap KerL = \omega\mathcal{F}\cap\mathcal{R}, u(t) = \mathfrak{M}$ or $u(t) = -\mathfrak{M}$.

We have

$$\begin{aligned} Q\mathcal{N}(\mathfrak{M}) &= -\tfrac{1}{T}\int_0^T\{\vartheta_1(t,\mathfrak{M}) + \vartheta_2(t,\mathfrak{M}) - \bar{p}\}dt > 0, \\ Q\mathcal{N}(-\mathfrak{M}) &= -\tfrac{1}{T}\int_0^T\{\vartheta_1(t,\mathfrak{M}) + \vartheta_2(t,\mathfrak{M}) - \bar{p}\}dt < 0, \end{aligned} \quad (11)$$

This means that the condition (b) of Method 2.1 is gratified. Define

$$H(u,h) = hu + (1-h)Q\mathcal{N}u = hu - (1-h)\frac{1}{T} \times \int_0^T \left\{\begin{matrix}\Psi(u'(t))u''(t) + \varphi(u(t))u'(t) + \\ \vartheta_1(t,u(t-\varepsilon_1(t))) + \vartheta_1(t,u(t-\varepsilon_1(t))) - \bar{p}\end{matrix}\right\}dt,$$

According to (11) we have $uH(u,h) > 0$ for all $u \in \omega\mathcal{F}\cap KerL$ and $h \in [0,1]$. Hence, $H(u,h)$ is a homotopic transformation. Using the homotopy invariance theorem we find

$$deg\{Q\mathcal{N},\mathcal{F}\cap KerL, 0\} = deg\{u,\mathcal{F}\cap KerL, 0\} \neq 0.$$

Therefore, the condition (c) of Method 2.1 is achieved. Moreover, we conclude that (3) has at least one periodic solution and the solution is unique. This completes our result.

4. Discussion

To discuss our results, we apply the main Result 3.1, to obtain a periodic solution. Let us consider the -periodic solution of the fractional third-order delay differential equation with two deviating arguments

$$D^{3\mu}u(t) + \frac{1}{30}(\sin\ u')u''(t) + \frac{1}{15}(\cos\ u)u'(t) + \vartheta_1(t,u(t-\cos\ 2t)) + \vartheta_2(t,u(t-\sin\ 2t)) = \frac{1}{\pi}\cos\ 2t, \quad (12)$$

where

$$T = \pi,\ \varepsilon_1(t) = \cos\ 2t,\ \varepsilon_2(t) = \sin\ 2t,$$

$$\vartheta_1(t,u) = \frac{1}{90\pi(1+\cos^2 t)}\tan^{-1}u,$$

$$\vartheta_2(t,u) = \frac{1}{90\pi}(1+\sin^2 t)\tan^{-1}u, \text{ and } p(t) = \frac{1}{\pi}\cos\ 2t.$$

By (12) we have

$$A_1 = A_2 = \frac{1}{30},\ B_1 = \frac{1}{90\pi},\ B_2 = \frac{1}{70\pi},\ C_1 = C_2 = \frac{1}{15}$$

Observing that

$$\bar{p} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{\pi}\int_0^\pi \frac{1}{\pi}\cos\ 2t\ dt = 0,\ |p|_\infty = \frac{1}{\pi}$$

Case I. (d $\leqslant$ 1) We let $d = \frac{1}{20}$, (d is an arbitrary small positive constant). Then we attain

$$\frac{[(B_1+B_2)d+M+|p|_\infty]T}{2\left\{\frac{1}{\Gamma(3\mu+1)} - A_1\frac{T}{2} - C_1\frac{T^2}{4} - (B_1+B_2)\frac{T^3}{8}\right\}} = \frac{\left\{\left(\frac{1}{90\pi}+\frac{1}{70\pi}\right)\frac{1}{20}+\frac{1}{\pi}\right\}\pi}{2\left\{\frac{1}{\Gamma(3\mu+1)} - \frac{1}{30}\frac{\pi}{2} - \frac{1}{16}\frac{\pi^2}{4} - \left(\frac{1}{90\pi}+\frac{1}{70\pi}\right)\frac{\pi^3}{8}\right\}} = \kappa < 1,$$

When $\mu = 1/3$, we have $\kappa = 0.519$. In addition, When $\mu = 1/4$, we have $\kappa = 0.45$, while $\kappa > 1$, when $\mu \geqslant 1/2$. In this case, we conclude that $0 < \mu \leqslant 1/3$. Similarly for the following quantity, when $\mu = 1/3$:

$$\Gamma(3\mu+1)\left[A_1\frac{T}{2} + (A_2\kappa + C_1)\frac{T^2}{4} + \left(B_1 + B_2 + C_2\kappa\frac{T}{2}\right)\frac{T^3}{8}\right] = \frac{\Gamma(3\mu+1)}{20}\frac{\pi}{2} + \left(\frac{0.519}{30}+\frac{1}{20}\right)\frac{\Gamma(3\mu+1)\pi^2}{4} + \left(\frac{1}{90\pi}+\frac{1}{70\pi}+\frac{0.519}{16}\frac{\pi}{2}\right)\frac{\Gamma(3\mu+1)\pi^3}{8} = 0.7 < 1.$$

It is understandable that all the assumptions (*ii*)–(*iv*) are satisfied. Thus, by our Result 3.1, Equation (12) has a unique π -periodic solution.

Case II. (d > 1) When $d = 5$, $\mu = 1/3$, we have $\kappa = \frac{1.145}{1.909} = 0.578 < 1$, (in this case there is a periodic solution), while when $d = 35$, $\mu > 1/3$, we have $\kappa = \frac{2.02}{1.909} = 1.06 > 1$ (there is no periodic solution).

5. Conclusion

The advantage of a periodic solution appears when the initial population is low and this situation may appear in the class of fractional differential equations. We generalized a class of neutral third order differential equations by applying the idea of fractional calculus. The fractional differential operator is taken in the sense of Riemann–Liouville calculus. We imposed the periodicity computation of solutions. The method depends on the continuation theory. We showed by example of a population model, that the periodic solution of the fractional differential equation is approximated to the ordinary equation when the fractional power satisfies the inequality $0 < \mu \leqslant 1/3$. For future work, one can consider n- deviating arguments depending on the actions of the population growth.

Competing interest

The authors declare that there are no competing interests.

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