



Parametrized topological complexity of poset-stratified spaces

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Abstract

In this paper, parametrized motion planning algorithms for a fiberwise space $X \rightarrow P$ over a poset P are studied. Such an algorithm assigns paths in a space X decomposed into subspaces with the index set P , that do not cross the boundaries of the separated regions. We compute the parametrized topological complexity of $X \rightarrow P$, which is one less than the minimal number of local parametrized motion planning algorithms used for designing non-cross-border robot motions in X .

Keywords Parametrized topological complexity · Poset-stratified space · Fiberwise space · Robot motion planning

Mathematics Subject Classification 55M30 · 06A07

1 Introduction

The robotic motion planning problem considers how robots move from an initial point to a final point. The central theme in the motion planning problem is to assign a path that connects x and y to each pair (x, y) of points in the space.

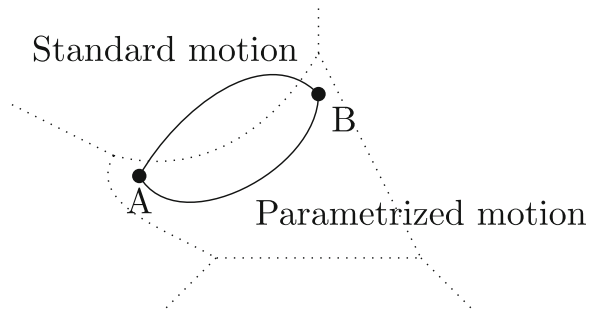
Farber introduced a numerical invariant $\text{TC}(X)$ (Farber 2003), called the *topological complexity* of a space X , which indicates the complexity of the design of motion planning algorithms in X . The equality $\text{TC}(X) = n$ implies that we need at least $n + 1$ local motion planning algorithms to move robots in X .

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Fig. 1 Standard and parametrized motion



In contrast, various efficient motion planning algorithms such as symmetric motion (Farber and Grant 2007; Basabe et al. 2014), monoidal (reserved) motion (Iwase and Sakai 2010, 2012), equivariant motion (Colman and Grant 2012; Dranishnikov 2015), and directed motion planning algorithms (Goubault et al. 2020; Borat and Grant 2020) have been developed. Recently, Cohen, Farber, and Weinberger introduced parametrized motion planning algorithms for fibrations to study collision-free motion planning (Cohen et al. 2021), [CFW]. The original definition of parametrized topological complexity $\text{TC}(\pi)$ of a fibration $\pi: E \rightarrow B$ was defined as the sectional category of the associated fibration $\Pi: E_B^I \rightarrow E_B^2$, $\Pi(\gamma) = (\gamma(0), \gamma(1))$. Here, $E_B^2 = E \times_B E$ is the fiberwise product over B , and E_B^I consists of paths $\gamma: I = [0, 1] \rightarrow E$ such that $\pi \circ \gamma$ is constant, i.e., γ maps into the fiber $\pi^{-1}(b)$ for some $b \in B$. In other words, $\text{TC}(\pi)$ is defined as one less than the smallest number of open sets covering E_B^2 with local homotopy sections of Π .

A more general setting for fiberwise spaces (not necessarily fibrations) was considered by García-Calines [Gar]. The parametrized topological complexity $\text{TC}(\pi)$ in his sense agrees with the one given by Cohen, Farber, and Weinberger when π is a fibration.

In this study, we focus on parametrized motion planning algorithms for a fiberwise space over a poset regarded as a T_0 -Alexandroff space. Such a fiberwise space $\pi: X \rightarrow P$ is called a *stratified space* over P , and $e_p = \pi^{-1}(p)$ is called a *stratum* of $p \in P$. Typical examples of poset-stratified spaces include simplicial complexes or, more generally, (normal) CW complexes with the face posets. A parametrized motion planning algorithm for a poset-stratified space $\pi: X \rightarrow P$ assigns a path $I \rightarrow X$ in a stratum e_p to each pair (x, y) of points in e_p .

This algorithm effectively works for motion planning in a local area. For example, when we go on domestic travel in a country, a parametrized motion planning algorithm on the Earth (decomposed into countries) proposes a route in the country that does not cross the border, while a standard motion planning algorithm may suggest a route through a different country. In recent years, the spread of COVID-19 has imposed severe restrictions on cross-border travel. Parametrized motion planning algorithms on poset-stratified spaces can contribute to the design of intra-country routes for regional tourism.

In this study, we compute several examples of TC for poset-stratified spaces. We show that $\text{TC}(\pi) = 0$ for the stratified space $\pi: X \rightarrow \mathcal{P}(X)$ associated with a simpli-

cial complex, or more generally, a regular CW complex X with the face poset $\mathcal{P}(X)$. Furthermore, the parametrized topological complexity of a couple of fundamental stratifications on the cone and the suspension of a space is considered. As a result, $TC(X \rightarrow \mathcal{P}(X)) = \infty$ for some familiar CW complexes, such as; sphere S^n , bouquet $B_k = \vee_k S^1$, torus $T^n = \prod^n S^1$, and real (complex) projective space $\mathbb{R}P^n$ ($\mathbb{C}P^n$) with the canonical (minimal) cell decomposition (Example 3.7).

This is caused by the definition of $TC(\pi : E \rightarrow B)$ using open sets that cover the fiberwise product E_B^2 . We can not construct a parametrized motion planning algorithm on an open neighborhood of a 0-cell in the above case of non-regular CW complexes. In order to consider algorithms on more flexible regions, we compute the generalized version $TC_g(\pi)$ of $TC(\pi)$ using arbitrary sets that separate E_B^2 . For example, $TC_g(\pi : X \rightarrow \mathcal{P}(X))$ becomes finite for any finite CW-complex X , unlike the case of the non-generalized version TC .

The paper is organized as follows. Section 2 recalls the idea of parametrized topological complexity based on the papers (Cohen et al. 2021)[CFW, Gar]. Furthermore, we review a reconstruction method for stratified spaces from their combinatorial data (Tamaki 2018) to compute TC and TC_g .

In Sect. 3, we compute TC for poset-stratified spaces including simplicial complexes, regular CW complexes, cones, and suspensions. We show that $TC(X \rightarrow \mathcal{P}(X)) = \infty$ for some non-regular CW complexes X .

In Sect. 4, we compute TC_g for the poset-stratified spaces given in Sect. 3.

2 Preliminaries

This section briefly reviews the definitions and properties on parametrized topological complexity and stratified spaces. We deal only with path-connected spaces in this paper.

2.1 Parametrized topological complexity

First we review the definition and properties on parametrized topological complexity based on prior papers (Cohen et al. 2021) [CFW, Gar].

For a fiberwise space $\pi : E \rightarrow B$, we consider the subspace

$$E_B^I = \{\gamma : I \rightarrow E \mid \pi \circ \gamma = c\}$$

of the path space of E , where c is the constant path at a point in B . For the fiberwise product

$$E_B^2 = E \times_B E = \{(x, y) \in E \times E \mid \pi(x) = \pi(y)\},$$

we have $\Pi : E_B^I \rightarrow E_B^2$ given by $\Pi(\gamma) = (\gamma(0), \gamma(1))$. For a subspace U of E_B^2 , a continuous (strict) local section $U \rightarrow E_B^I$ of Π is called a *parametrized motion planning algorithm* on U .

The original idea of parametrized topological complexity was defined as the sectional category of the associated map Π for fibrations (Cohen et al. 2021)[CFW].

Definition 2.1 Let $p: E \rightarrow B$ be a fiberwise space. The *sectional category* $\text{secat}(p)$ of p is the minimal number n such that B is covered by $n+1$ open subsets U_0, \dots, U_n , where each U_i admits a homotopy local section of p . That is, we have $s_i: U_i \rightarrow E$ such that $p \circ s_i$ is homotopic to the inclusion $U_i \hookrightarrow B$. If no such number exists, we set $\text{secat}(p) = \infty$.

If $p: E \rightarrow B$ is a (Hurewicz) fibration, the sectional category above agrees with one less than the minimal number of open sets covering B with strict local sections of p .

Example 2.2 Several topological invariants are expressed as sectional categories.

- (1) For a space X with a base point x_0 , the based path space $PX = \{\gamma: I \rightarrow X \mid \gamma(0) = x_0\}$ is equipped with a fibration $\text{ev}_1: PX \rightarrow X$ given by $\text{ev}_1(\gamma) = \gamma(1)$. The sectional category $\text{secat}(\text{ev}_1)$ agrees with the LS(Lusternik-Schnirelmann) category $\text{cat}(X)$ originally defined as the minimal number n such that X is covered by $n+1$ categorical open sets. Here, a subset A of X is *categorical* if the inclusion $A \hookrightarrow X$ is null homotopic.
- (2) For a space X , the free path space $X^I = \{\gamma: I \rightarrow X\}$ is equipped with a fibration $\text{ev}: X^I \rightarrow X^2 = X \times X$ given by $\text{ev}(\gamma) = (\gamma(0), \gamma(1))$. The topological complexity $\text{TC}(X)$ is defined as the sectional category $\text{secat}(\text{ev})$ (Farber 2003).

The parametrized topological complexity of a fibration π is defined as $\text{secat}(\Pi)$ in Cohen et al. (2021). It should be noted that the associated map Π always becomes a fibration if π is a fibration.

García-Calines considered the topological complexity for general fiberwise spaces including non-fibrations [Gar].

Definition 2.3 Let $\pi: E \rightarrow B$ be a fiberwise space. The *parametrized topological complexity* $\text{TC}(\pi)$ is the minimal number n such that E_B^2 is covered by $n+1$ open subsets U_0, \dots, U_n , where each U_i admits a parametrized motion planning algorithm. If no such number exists, we set $\text{TC}(\pi) = \infty$.

Remark 2.4 Our $\text{TC}(\pi)$ in Definition 2.3 agrees with the one given by García-Calines [Gar] for fiberwise spaces which are not necessarily fibrations. When π is a fibration, our $\text{TC}(\pi)$ also agrees with the one given by Cohen et al. (2021) [CFW]. Moreover, when the base space $B = *$ consists of a single point, the parametrized topological complexity $\text{TC}(\pi)$ agrees with the standard topological complexity $\text{TC}(E)$ of the total space introduced in Farber (2003).

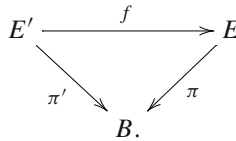
The fundamental properties of the parametrized topological complexity were compiled in Cohen et al. (2021) [Gar].

Proposition 2.5 (Corollary 15 of [Gar]) *Let $\pi: E \rightarrow B$ be a fiberwise space and let $f: B' \rightarrow B$ be a continuous map. For the pullback $f^*(\pi): B' \times_B E \rightarrow B'$ of π along f , we have $\text{TC}(f^*(\pi)) \leq \text{TC}(\pi)$.*

A special case of the above proposition is the following corollary.

Corollary 2.6 *Let $\pi : E \rightarrow B$ be a fiberwise space. For a subspace $B' \subset B$ and the restriction $\pi' = \pi|_{\pi^{-1}(B')} : \pi^{-1}(B') \rightarrow B'$, we have $\text{TC}(\pi') \leq \text{TC}(\pi)$. In particular, we have $\text{TC}(\pi^{-1}(b)) \leq \text{TC}(\pi)$ for each $b \in B$.*

The next property is the homotopy invariance of $\text{TC}(\pi)$. We consider the following commutative diagram with a map f between fiberwise spaces:



Even if π' and π are not fibrations, the next proposition holds by the same argument in the proof of (Cohen et al. 2021 Proposition 5.2).

Proposition 2.7 *Let $\pi : E \rightarrow B$ and $\pi' : E' \rightarrow B$ be fiberwise spaces, and let $f : E' \rightarrow E$ be a homotopy equivalence over B (satisfying $\pi \circ f = \pi'$). If we have a map $g : E \rightarrow E'$ of f over B with a fiberwise homotopy $g \circ f \simeq_B \text{id}_{E'}$, then $\text{TC}(\pi') \leq \text{TC}(\pi)$.*

Corollary 2.8 *If fiberwise spaces $\pi : E \rightarrow B$ and $\pi' : E' \rightarrow B$ are fiberwise homotopy equivalent, then $\text{TC}(\pi') = \text{TC}(\pi)$.*

The topological complexity $\text{TC}(X) = 0$ if and only if X is contractible. A similar property of the parametrized topological complexity was studied in Cohen et al. (2021) [Gar] for fibrations or fiberwise pointed spaces.

Proposition 2.9 (Proposition 4.5 of Cohen et al. (2021)) *Let $\pi : E \rightarrow B$ be a fibration, and let E_B^2 have the homotopy type of a CW complex. The parametrized topological complexity $\text{TC}(\pi) = 0$ if and only if $\text{TC}(X) = 0$ for the fiber X of π .*

Proposition 2.10 (Corollary 12 of [Gar]) *Let $\pi : E \rightarrow B$ be a fiberwise pointed space. The parametrized topological complexity $\text{TC}(\pi) = 0$ if and only if E is fiberwise contractible.*

Unfortunately, a poset-stratified space is neither a fibration nor a fiberwise pointed space in general. For a general fiberwise space $\pi : E \rightarrow B$, a condition equivalent to $\text{TC}(\pi) = 0$ can be described as follows:

Proposition 2.11 *Let $\pi : E \rightarrow B$ be a fiberwise space. The parametrized topological complexity $\text{TC}(\pi) = 0$ if and only if the diagonal $\Delta(E) = \{(e, e) \in E^2\}$ is a fiberwise deformation retract of E_B^2 over B .*

Proof We assume that $\text{TC}(\pi) = 0$. We have a global section $s : E_B^2 \rightarrow E_B^1$ of Π . A fiberwise homotopy $H : E_B^2 \times I \rightarrow E_B^2$ defined by $H(x, y, t) = (x, s(x, y)(1 - t))$ presents a fiberwise deformation retraction $H_1 : E_B^2 \rightarrow \Delta(E)$ over B . Conversely, let

$H : E_B^2 \times I \rightarrow E_B^2$ be a fiberwise homotopy associated with a deformation retraction over B that satisfies $H(x, y, 0) = (x, y)$ and $H(x, y, 1) \in \Delta(E)$. We have a section $s : E_B^2 \rightarrow E_B^I$ of Π , defined as

$$s(x, y)(t) = \begin{cases} H_1(x, y, 2t) & 0 \leq t \leq \frac{1}{2}, \\ H_2(x, y, 2 - 2t) & \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $H(x, y, t) = (H_1(x, y, t), H_2(x, y, t))$. Hence, $\text{TC}(\pi) = 0$. □

2.2 Poset-stratified spaces

This subsection reviews the definition and properties on poset-stratified spaces.

A poset-stratified space is roughly a space decomposed into subspaces (called strata) with the index poset P such that the inclusion relation on the closures of strata corresponds to the partial order on P . Detailed observations on decompositions and poset-stratified spaces can be found in Tamaki and Tanaka (2019), Yokura (2020).

A poset P can be regarded as a T_0 -Alexandroff space whose open sets are closed under infinite intersection. Open sets of P are filters (upper sets) of P , that is, subsets closed under the upper order. Conversely, a T_0 -Alexandroff space X can be regarded as a poset with the partial order $x \leq y$ defined by $x \in O_y$, where O_y is the minimal open neighborhood of y (the intersection of all open sets including y). From this perspective, we identify T_0 -Alexandroff spaces with posets.

We focus on fiberwise spaces $\pi : X \rightarrow P$ over posets P . The following definition of poset-stratified spaces is essentially based on Tamaki and Tanaka (2019).

Definition 2.12 A stratified space over a poset P is an open surjective continuous map $\pi : X \rightarrow P$ such that each stratum $e_p = \pi^{-1}(p)$ is connected and locally closed.

Remark 2.13 Our stratified space $\pi : X \rightarrow P$ is required to be an open map because of the compatibility of the orders. Let $\pi : X \rightarrow P$ be a fiberwise space over a poset P . The map π is further an open map if and only if it satisfies the following condition: $e_p \subset \bar{e}_q$ if and only if $p \leq q$ for any $p, q \in P$ (Remark 2.2 Tamaki 2018).

A CW complex X has a natural map $\pi : X \rightarrow \mathcal{P}(X)$ to the face poset $\mathcal{P}(X)$ given by $\pi(x) = e$ if $x \in e$. Here, the face poset $\mathcal{P}(X)$ consists of (open) cells of X with the relation $e \leq e'$ if, and only if, $e \subset \bar{e}'$. This map π is not always continuous; however, the normality (the axiom of the frontier) makes π continuous. Recall that a CW complex is normal if each pair of cells e_p, e_q satisfying $e_p \cap \bar{e}_q \neq \emptyset$ implies $e_p \subset \bar{e}_q$. It should be noted that the above term “normal” is a different concept from a space satisfying Axiom T4.

Proposition 2.14 (Corollary 3.7 of Tamaki and Tanaka (2019)) *If X is a normal CW complex, then the canonical map $X \rightarrow \mathcal{P}(X)$ to the face poset is a stratified space.*

When we deal with stratified spaces with infinite strata, the CW condition is a useful property in homotopy theory, as is the case with cell complexes.

Definition 2.15 A stratified space $\pi : X \rightarrow P$ is *CW* if it satisfies the following two conditions:

- (1) The boundary ∂e_p of a stratum e_p is covered by a finite number of strata.
- (2) The space X has the weak topology with respect to the closures of strata $\{\bar{e}_p \mid p \in P\}$.

A stratified space $\pi : X \rightarrow P$ is called *locally finite* if every point $x \in X$ has an open neighborhood U intersecting with a finite number of strata.

Lemma 2.16 (Proposition 2.21 of Tamaki (2018)) *Any locally finite stratified space $\pi : X \rightarrow P$ is CW.*

2.3 Stellar stratified spaces and cylindrical structures

We present an overview of the paper (Tamaki 2018) about a reconstruction method of stellar stratified spaces by the face categories. This reconstruction method plays a central role in computing the parametrized topological complexity of poset-stratified spaces in this paper.

A stellar stratified space is a generalized idea of CW complex introduced in Tamaki (2018), Tamaki and Tanaka (2019). A CW complex is constructed by gluing disks along the boundaries. On the other hand, a stellar stratified space is constructed by attaching star-shaped cells.

Let S be a space. The cone $CS = S \times I/S \times \{1\}$ is expressed as the join $S \star \{\mathbf{v}\}$, where \mathbf{v} is the top vertex $[s, 1]$. An element $x \in CS$ is denoted by $(1 - t)y + t\mathbf{v}$ for some $y \in S$ and $0 \leq t \leq 1$.

Definition 2.17 Let S be a space. A subset $D \subset CS$ is an *aster* if for any $x \in D$, the line segment between \mathbf{v} and x is contained in D . That is, if x is described as $x = (1 - t)y + t\mathbf{v}$, then $(1 - s)y + s\mathbf{v} \in D$ for any $t \leq s \leq 1$. The boundary ∂D of an aster D denotes the intersection $D \cap S$. An aster D is called *thin* if $D = \partial D \star \{v\}$.

Definition 2.18 Let $\pi : X \rightarrow P$ be a stratified space. A *characteristic map* of a stratum e_p is a continuous map $\varphi_p : D_p \rightarrow \bar{e}_p$ from an aster $D_p \subset CS_p$ for some space S_p that satisfies the following conditions:

- (1) φ_p is a quotient map.
- (2) $(\varphi_p)|_{\text{Int}(D_p)} : \text{Int}(D_p) \rightarrow e_p$ is a homeomorphism.

A stratum e_p is called *thin* if the domain of the characteristic map $D_p \rightarrow \bar{e}_p$ is a thin aster.

A *stellar stratified space* X is a stratified space $X \rightarrow P$ with a family of characteristic maps $\{\varphi_p\}_{p \in P}$ such that the boundary $\partial e_p = \bar{e}_p - e_p$ of each stratum e_p is covered by the strata indexed by $P_{<p} = \{q \in P \mid q < p\}$. A stellar stratified space is called a *stellar complex* if all of the strata are thin.

Definition 2.19 Let $\pi : X \rightarrow P$ be a stellar stratified space. A stratum e_p is *regular* if the characteristic map $\varphi_p : D_p \rightarrow \bar{e}_p$ is a homeomorphism. When all of the strata are regular, π is called *regular*.

For a stratified space $\pi : X \rightarrow P$ and a subposet $Q \subset P$, we consider the stratified subspace $\pi_Q = \pi|_{\pi^{-1}(Q)} : \pi^{-1}(Q) \rightarrow Q$. Even if π admits a stellar structure, the restriction may not present a stellar structure on π_Q . This is because the restriction does not preserve quotient maps in general (see Tamaki 2018 Section 6). However, the restriction preserves regular stellar structures because the restriction of a homeomorphism is again a homeomorphism onto its image (and a quotient map).

Lemma 2.20 *Let $\pi : X \rightarrow P$ be a regular stellar stratified space, and let $Q \subset P$ be a subposet. The restriction $\pi_Q : \pi^{-1}(Q) \rightarrow Q$ is again a regular stellar stratified space.*

Proof The assumption ensures that the characteristic map $\varphi_p : D_p \rightarrow \bar{e}_p$ of π is a homeomorphism for each $p \in P$. For $q \in Q \subset P$, let \bar{e}_q^Q denote the closure of e_q in $\pi^{-1}(Q)$ and D_q^Q denote the inverse image $\varphi_q^{-1}(\bar{e}_q^Q) \subset D_q$. Note that D_q^Q is again an aster because it is obtained by removing a part of the boundary ∂D_p from D_p . We have a homeomorphism

$$\varphi_q|_{D_q^Q} : D_q^Q \longrightarrow \bar{e}_q^Q$$

for each $q \in Q$. It provides a regular stellar structure on π_Q . □

A typical example of stellar stratified space is a cell complex.

Example 2.21 A cell complex X is a special case of stellar complex. An n -cell e is equipped with a characteristic map $\varphi : D^n \rightarrow \bar{e}$, and an n -disk D^n can be regarded as a thin aster $D^n = S^{n-1} \star \{0\}$ with the boundary $\partial D^n = S^{n-1}$.

For a poset P , the nerve semi-simplicial set $\mathcal{N}P$ consists of totally ordered subsets in P :

$$\mathcal{N}_n P = \{p_0 < \dots < p_n \mid p_i \in P\}$$

with the face maps deleting elements. The geometric realization of $\mathcal{N}P$ is denoted by $\mathcal{B}P$, and is called the *classifying space* or *order complex* of P . This is a special case of the classifying space of a loop-free top-enriched category in Definition 2.28.

Any point in $\mathcal{B}P$ is uniquely expressed as a pair of $a \in \text{Int}(\Delta^n)$ and a totally ordered subset $p_0 < p_1 < \dots < p_n$ in P for some $n \geq 0$. The classifying space $\mathcal{B}P$ is equipped with a natural continuous map $\tau : \mathcal{B}P \rightarrow P$ defined by $\tau(a, p_0 < \dots < p_n) = p_n$. We can naturally consider $\mathcal{B}P$ as a stratified space over P by τ .

Definition 2.22 A poset P is *locally finite* if both $P_{\leq p} = \{q \in P \mid q \leq p\}$ and $P_{\geq p} = \{q \in P \mid q \geq p\}$ are finite for all $p \in P$.

Lemma 2.23 *If P is a locally finite poset, then $\tau : \mathcal{B}P \rightarrow P$ is a locally finite stratified space.*

Proof Any point $x \in \mathcal{B}P$ belongs to a unique open simplex indexed by a totally ordered subset $p_0 < \dots < p_n$ in P . The open neighborhood $x \in U = \tau^{-1}(P_{\geq p_n})$ consists of a finite number of strata. Thus, τ is locally finite. □

The classifying space \mathcal{BP} has a natural stellar structure. More generally, stellar structures on the classifying spaces of loop-free top-enriched categories have been considered in Tamaki and Tanaka (2019).

Example 2.24 (Section 4.1 in Tamaki and Tanaka (2019)) The stratified space $\tau: \mathcal{BP} \rightarrow P$ over a locally finite poset P admits a stellar structure as follows: A stratum e_p consists of open simplices indexed by a totally ordered subset with the maximal element p . The classifying space $\mathcal{B}(P_{\leq p})$ can be expressed as $\mathcal{B}(P_{< q}) \star \{p\}$, and we have a natural homeomorphism $\mathcal{B}(P_{\leq p}) \cong \bar{e}_p$. Lemmas 2.23 and 2.16 imply that τ is a regular CW stellar stratified space.

It is well known that a regular CW complex X is homeomorphic to the classifying space $\mathcal{BP}(X)$ of the face poset. However, the face poset is not sufficient to recover the topology or homotopy type of a non-regular CW complex. We need more informative structures than posets to recover the original non-regular CW complexes.

A category enriched by topological spaces is referred to as a *top-enriched category* in this study. A top-enriched category T consists of a set of objects T_0 and a space of morphisms $T(x, y)$ for each pair of objects x, y with a continuous composition. T is called *loop-free* (or *acyclic*) if it satisfies the following two conditions:

- (1) $T(x, x)$ consists of only the single identity morphism id_x for each object x .
- (2) $T(x, y) = \emptyset$ if $T(y, x) \neq \emptyset$ for $x \neq y$.

A poset is a special case of loop-free top-enriched category with at most one morphism between two objects. For a loop-free top-enriched category T , we have the underlying poset $\mathcal{P}(T)$ defined as $\mathcal{P}(T) = T_0$ with the partial order $x \leq y$ given by $T(x, y) \neq \emptyset$. Furthermore, we have a natural functor $\rho_T: T \rightarrow \mathcal{P}(T)$ preserving the objects.

Definition 2.25 Let $\pi: X \rightarrow P$ be a stellar stratified space with characteristic maps $\{\varphi_p: D_p \rightarrow \bar{e}_p\}$. The *naive face category* $\mathcal{F}(X)$ is a loop-free top-enriched category defined as follows: The set of objects $\mathcal{F}(X)_0 = P$ and the space of morphisms $\mathcal{F}(X)(p, q)$ consists of continuous maps $D_p \rightarrow D_q$ that are compatible with the characteristic maps φ_p and φ_q for $p < q$. We set $\mathcal{F}(X)(p, p) = \{\text{id}_{D_p}\}$ and the composition is given by the composition of maps.

A map $f \in \mathcal{F}(X)(p, q)$ makes the following diagram commute for $p < q$:

$$\begin{array}{ccc}
 D_p & \xrightarrow{f} & D_q \\
 \varphi_p \downarrow & & \downarrow \varphi_q \\
 \bar{e}_p & \hookrightarrow & \bar{e}_q.
 \end{array}$$

The bottom inclusion maps into the boundary ∂e_q ; thus, f also maps into ∂D_q . Therefore, $\mathcal{F}(X)(p, q)$ is the subspace of the mapping space $\text{Map}(D_p, \partial D_q)$ with the compact open topology for $p < q$.

Definition 2.26 Let $\pi: X \rightarrow P$ be a stellar stratified space with characteristic maps $\{\varphi_p: D_p \rightarrow \bar{e}_p\}$. A *cylindrical structure* on π consists of a loop-free top-enriched

category $\mathcal{C}(X)$ (called the *face category*) with a continuous functor $b: \mathcal{C}(X) \rightarrow \mathcal{F}(X)$ satisfying the following conditions:

- (1) $\mathcal{C}(X)_0 = P$ and b preserves the set of objects, that is, $b(p) = p$ for all $p \in P$.
- (2) Let $\tilde{b}_{p,q}: \mathcal{C}(X)(p, q) \times D_p \rightarrow D_q$ denote the adjoint map to $b_{p,q}$. The restriction of $\tilde{b}_{p,q}$ to $\mathcal{C}(X)(p, q) \times \text{Int}(D_p)$ is a homeomorphism onto its image $e'_p = \tilde{b}_{p,q}(\mathcal{C}(X)(p, q) \times \text{Int}(D_p)) \subset \partial D_q$ for $p < q$.
- (3) The boundary of D_q is decomposed

$$\partial D_q = \bigcup_{p < q} e'_p$$

as a stratified space over $P_{<q}$ for each q .

A stellar stratified space with a cylindrical structure is called a *cylindrically normal stellar stratified space*.

Example 2.27 A regular stellar stratified space $\pi: X \rightarrow P$ has a natural cylindrical structure with $\mathcal{C}(X) = \mathcal{F}(X) = P$ because a map $f: D_p \rightarrow D_q$ compatible with the characteristic maps φ_p and φ_q is uniquely expressed as $\varphi_q^{-1} \circ \varphi_p$.

The classifying space construction of posets can be naturally extended to top-enriched categories (more generally, topological categories, which are internal categories in topological spaces). See (Segal 1968) for the construction of the classifying spaces for topological categories.

Definition 2.28 For a loop-free top-enriched category T , the classifying space $\mathcal{B}T$ is the geometric realization of the semi-simplicial space $\mathcal{N}T$ defined by

$$\mathcal{N}_k T = \prod_{x_0, \dots, x_k} T(x_{k-1}, x_k) \times T(x_{k-2}, x_{k-1}) \times \dots \times T(x_0, x_1).$$

For a cylindrical structure on a stellar stratified space $\pi: X \rightarrow \mathcal{P}(X)$ with face category $\mathcal{C}(X)$, the classifying space $\mathcal{B}\mathcal{C}(X)$ has a wealth of topological (homotopical) information about X . The following theorem was proved by Tamaki (2018). See also (Furuse et al. 2015; Tamaki and Tanaka 2019).

Theorem 2.29 (Theorem 5.16 in Tamaki (2018)) *Let $\pi: X \rightarrow P$ be a cylindrically normal CW stellar stratified space with face category $\mathcal{C}(X)$. Then, we have a natural embedding $\iota: \mathcal{B}\mathcal{C}(X) \rightarrow X$ over P . Here, we regard $\mathcal{B}\mathcal{C}(X)$ as a stratified space over P by $\tau \circ \mathcal{B}(\rho): \mathcal{B}\mathcal{C}(X) \rightarrow P$ for the natural functor $\rho: \mathcal{C}(X) \rightarrow P = \mathcal{P}(\mathcal{C}(X))$. Furthermore, if π is a stellar complex, then ι is a homeomorphism.*

The embedding $\iota: \mathcal{B}\mathcal{C}(X) \rightarrow X$ is constructed by gluing maps

$$\mathcal{C}(X)(p_{k-1}, p_k) \times \dots \times \mathcal{C}(X)(p_0, p_1) \times \Delta^k \longrightarrow D_{p_k} \xrightarrow{\varphi_{p_k}} X$$

defined inductively on totally ordered subsets $p_0 < \dots < p_k$ in P (see the proof of Theorem 5.16 in Tamaki (2018)). Hence, ι is a map over P .

Moreover, if π is regular, ι embeds $\mathcal{BC}(X)$ into X as a deformation retract. Tamaki proved it in more general setting than regular stellar stratified space (Corollary 5.19 in Tamaki (2018)). The idea of the construction of a deformation retraction and a homotopy is essentially the same as ι . A deformation retraction $X \rightarrow \iota(\mathcal{BC}(X))$ and a homotopy $X \times I \rightarrow X$ can be taken as maps over P because these are constructed by homotopies $D_p \times I \rightarrow D_p$ for $p \in P$ (see the proof of Theorem 2.50 in Furuse et al. (2015)).

Theorem 2.30 (Theorem 2.50 in Furuse et al. (2015)) *Let $\pi : X \rightarrow P$ be a regular CW stellar stratified space with face category $\mathcal{C}(X)$. The map ι embeds $\mathcal{BC}(X)$ into X as a fiberwise deformation retract over P .*

Many examples of cylindrically normal stellar stratified spaces were introduced in Tamaki (2018), Tamaki and Tanaka (2019). In particular, the face categories associated with natural cylindrical structures on some familiar CW complexes are described as follows. These face categories are used in Sect. 4 for the calculation of the generalized version of TC.

Example 2.31 (Tamaki (2018)) The following CW complexes admit cylindrical structures:

- (1) A sphere $S^n = e^{(0)} \cup e^{(n)}$; the face category $\mathcal{C}(S^n)$ has morphisms

$$\mathcal{C}(S^n)(e^{(0)}, e^{(n)}) = S^{n-1}.$$

- (2) A bouquet $B_k = \vee_k S^1 = e^{(0)} \cup e_1^{(1)} \cup \dots \cup e_k^{(1)}$; the face category $\mathcal{C}(B_k)$ has morphisms $\mathcal{C}(B_k)(e^{(0)}, e_i^{(1)}) = S^0$ for each i .
- (3) A torus $T^n = \prod_n S^1$ with the product cell structure of $S^1 = e^{(0)} \cup e^{(1)}$; the face category $\mathcal{C}(T^n) = \mathcal{C}(S^1)^n$ is the product of copies of the face category given in (1).
- (4) A real projective space $\mathbb{R}P^n = e^{(0)} \cup e^{(1)} \cup \dots \cup e^{(n)}$; the face category $\mathcal{C}(\mathbb{R}P^n)$ has morphisms $\mathcal{C}(\mathbb{R}P^n)(e^{(i)}, e^{(j)}) = S^0$ for $i < j$ with the composition given by the multiplication on $\mathbb{Z}_2 = S^0$.
- (5) A complex projective space $\mathbb{C}P^n = e^{(0)} \cup e^{(2)} \cup \dots \cup e^{(2n)}$; the face category $\mathcal{C}(\mathbb{C}P^n)$ has morphisms $\mathcal{C}(\mathbb{C}P^n)(e^{(2i)}, e^{(2j)}) = S^1$ for $i < j$ with the composition given by the multiplication on $U(1) = S^1$.

3 Parametrized topological complexity of poset-stratified spaces

This section is devoted to the computation of the parametrized topological complexity for simplicial complexes, regular CW complexes, cones, and suspensions.

3.1 Simplicial complexes and regular CW complexes

A typical example of poset-stratified space is a normal CW complex X with the face poset $\mathcal{P}(X)$. We will show the equality $TC(X \rightarrow \mathcal{P}(X)) = 0$ for a locally finite regular CW complex X .

First, we consider the simple case of simplicial complexes. For a convex set X , we have the *linear motion planning algorithm* $L: X^2 \rightarrow X^I$ given by $L(x, y)(t) = (1 - t)x + ty$ for $t \in I$.

Proposition 3.1 *For any simplicial complex K , we have $\text{TC}(K \rightarrow \mathcal{P}(K)) = 0$.*

Proof For a point (x, y) in $K^2_{\mathcal{P}(K)}$, both x and y are contained in the same simplex. We have a global section $s: K^2_{\mathcal{P}(K)} \rightarrow K^I_{\mathcal{P}(K)}$ of Π defined as $s(x, y) = L(x, y)$. Hence, $\text{TC}(K \rightarrow \mathcal{P}(K)) = 0$. \square

Recall that the classifying space $\mathcal{B}P$ of a poset P is considered a stratified space over P by $\tau: \mathcal{B}P \rightarrow P$. Let us compute $\text{TC}(\tau)$. Note that the face poset of $\mathcal{B}P$ is not P (but the barycentric subdivision of P), and τ is different from the canonical map $\mathcal{B}P \rightarrow \mathcal{P}(\mathcal{B}P)$. Each stratum e_p of τ consists of open simplices indexed by totally ordered sequences that have the maximal element p . This is not convex in general; thus, the linear motion planning algorithm does not work.

Furthermore, we notice that e_p is contractible to p , and a contraction presents a motion planning algorithm on e_p . However, this algorithm only works continuously for each e_p (cannot be extended globally).

Proposition 3.2 *For a locally finite poset P , we have $\text{TC}(\tau: \mathcal{B}P \rightarrow P) = 0$.*

Proof We have a homeomorphism $\varphi: \mathcal{B}(P^2) \rightarrow (\mathcal{B}P)^2$ over P^2 , which is induced from the projections $P^2 \rightarrow P$. By Example 2.24, $\tau: \mathcal{B}(P^2) \rightarrow P^2$ has a regular stellar structure. Consider the following pullback diagram:

$$\begin{array}{ccc} \mathcal{B}_\Delta(P^2) & \longrightarrow & \mathcal{B}(P^2) \\ \downarrow & & \downarrow \\ P & \xrightarrow{\Delta} & P^2. \end{array}$$

The restriction of φ induced the following homeomorphism over P :

$$\varphi|_{\mathcal{B}_\Delta(P^2)}: \mathcal{B}_\Delta(P^2) \longrightarrow (\mathcal{B}P)^2_P.$$

Lemma 2.20 guarantees that $\mathcal{B}_\Delta(P^2)$ has a regular stellar structure. Hence, Example 2.27 and Theorem 2.30 imply that $\iota = \Delta: \mathcal{B}P \rightarrow \mathcal{B}_\Delta(P^2) = (\mathcal{B}P)^2_P$ is a fiberwise deformation retract of $(\mathcal{B}P)^2_P$ over P . Proposition 2.11 ensures that $\text{TC}(\tau) = 0$. \square

Next, we focus on regular CW complexes. Our aim is to show the equality $\text{TC}(X \rightarrow \mathcal{P}(X)) = 0$ for a locally finite regular CW complex X with the face poset $\mathcal{P}(X)$. Before discussing the general case, we observe the case of canonical regular CW decompositions on spheres.

Example 3.3 For an n -sphere S^n ($n \geq 1$), we have the canonical regular cell decomposition with $(2n + 2)$ cells

$$\left\{ e_+^{(0)}, e_-^{(0)}, \dots, e_+^{(n)}, e_-^{(n)} \right\},$$

where

$$e_{+(-)}^{(k)} = \{(x_0, \dots, x_k, 0, \dots, 0) \in S^n \mid x_k > 0(< 0)\}$$

denotes the k -dimensional upper(lower) hemisphere. The face poset $R = \mathcal{P}(S^n)$ consists of $2n + 2$ points, and the fiberwise product $(S^n)_R^2$ consists of pairs of points lying in the same cell. According to Farber’s computation of $\text{TC}(S^n)$ (Farber 2003), the shortest arc provides a motion planning algorithm s on $U = S^n \times S^n - \{(x, -x)\}$. The restriction of s to $(S^n)_R^2 \subset U$ maps into $(S^n)_R^I$. Thus, $\text{TC}(S^n \rightarrow R) = 0$ for any $n \geq 1$, while the usual topological complexity

$$\text{TC}(S^n) = \begin{cases} 1 & \text{for } n \text{ odd,} \\ 2 & \text{for } n \text{ even.} \end{cases}$$

Theorem 3.4 *For a locally finite regular CW complex X , we have $\text{TC}(X \rightarrow \mathcal{P}(X)) = 0$.*

Proof The product X^2 is a regular CW complex because of the local finiteness. Similarly to the discussions in Theorem 3.2, the fiberwise product X_P^2 is a regular CW stellar stratified space over the face poset $P = \mathcal{P}(X)$, and the diagonal $\Delta: X \cong \mathcal{B}P \rightarrow X_P^2$ is a fiberwise deformation retract of X_P^2 . Our desired result follows from Proposition 2.11. □

3.2 One-point stratification on cone and suspension

The cone CX and the suspension ΣX of a space X admit stratifications $CX = \{\mathbf{v}\} \cup (CX - \{\mathbf{v}\})$ and $\Sigma X = \{\mathbf{v}\} \cup (\Sigma X - \{\mathbf{v}\})$, where \mathbf{v} is the top vertex. The stratifications are quite simple, however, we will show that TC becomes infinite when X is not contractible. As a result, $\text{TC}(X \rightarrow \mathcal{P}(X))$ also becomes infinite for some non-regular CW complex X (see Example 3.7).

Let J denote the poset $\{0 < 1\}$. A fiberwise space $X \rightarrow J$ corresponds to choosing an open set (or closed set) in X . We consider the stratified space $\pi_J: CX \rightarrow J$ on the cone $CX = X \star \{\mathbf{v}\}$, where $\pi_J(\mathbf{v}) = 0$ for the top vertex \mathbf{v} and $\pi_J(\hat{C}X) = 1$ for $\hat{C}X = CX - \{\mathbf{v}\}$.

Theorem 3.5 *For $\pi_J: CX \rightarrow J$, we have*

$$\text{TC}(\pi_J) = \begin{cases} 0 & \text{if } X \text{ is contractible,} \\ \infty & \text{otherwise.} \end{cases}$$

Proof First, we assume that X is contractible. A contraction on X implies that CX and I are fiberwise homotopy equivalent over J , where we regard I as a fiberwise space $\pi_I: I \rightarrow J$ given by $\pi_I(1) = 0$ and $\pi_I[0, 1) = 1$. Corollary 2.8 ensures the equality $\text{TC}(\pi_J) = \text{TC}(\pi_I) = 0$ because I has the linear motion planning algorithm that is parametrized with respect to π_I .

Next, we consider a non-contractible space X . If we assume that $\text{TC}(\pi_J) < \infty$, we have an open set $(\mathbf{v}, \mathbf{v}) \in U \subset (CX)^2_J$ with a parametrized motion planning algorithm $s: U \rightarrow (CX)^1_J$. For a sufficiently small $\varepsilon > 0$, we have a neighborhood of \mathbf{v} as

$$V = \{(1 - t)x + t\mathbf{v} \in CX \mid 1 - \varepsilon \leq t \leq 1, x \in X\}$$

such that $(\mathbf{v}, \mathbf{v}) \in V^2_J \subset U$. Here, we regard V as a fiberwise subspace $\pi_V: V \rightarrow J$ of π_J . We have a deformation retraction $r: CX \rightarrow V$ over J given by $r((1 - t)x + t\mathbf{v}) = (1 - \varepsilon)x + \varepsilon\mathbf{v}$ for $t < 1 - \varepsilon$, and the motion planning algorithm s on U provides a global parametrized motion planning algorithm $s'(x, y) = r \circ s(x, y)$ on V . π_J and π_V are fiberwise homotopy equivalent by the fiberwise deformation retraction r . Hence, Corollary 2.8 ensures the equality $\text{TC}(\pi_J) = \text{TC}(\pi_V) = 0$. However, the assumption implies $\text{TC}(X) > 0$. Corollary 2.6 shows the following inequalities:

$$\text{TC}(\pi_J) \geq \text{TC}(\pi_J^{-1}(1)) = \text{TC}(\hat{C}X) = \text{TC}(X) > 0.$$

This is a contradiction; thus, $\text{TC}(\pi_J) = \infty$. □

We also consider the suspension $\Sigma X = X \times [-1, 1]/\sim$, where $(x, t) \sim (y, s)$ if, and only if, either $t = s = 1$ or $t = s = -1$. Let us consider $\pi_J: \Sigma X \rightarrow J$ given by $\pi_J(\mathbf{v}) = 0$ for the top vertex $\mathbf{v} = [x, 1]$ and $\pi_J(\hat{\Sigma}X) = 1$ for $\hat{\Sigma}X = \Sigma X - \{\mathbf{v}\}$.

Theorem 3.6 For $\pi_J: \Sigma X \rightarrow J$, we have

$$\text{TC}(\pi_J) = \begin{cases} 0 & \text{if } X \text{ is contractible,} \\ \infty & \text{otherwise.} \end{cases}$$

Proof When X is contractible, we can show the equality $\text{TC}(\pi_J) = 0$ by the same argument in the proof of Theorem 3.5. For a non-contractible space X , we assume that $\text{TC}(\pi_J) < \infty$. By the same argument in the proof of Theorem 3.5, we have a small neighborhood V of \mathbf{v} as

$$V = \{(1 - t)x + t\mathbf{v} \in \Sigma X \mid 1 - \varepsilon \leq t \leq 1, x \in X\},$$

with a parametrized motion planning algorithm $s: V^2_J \rightarrow (\Sigma X)^1_J$. Moreover, we can choose a small $\varepsilon > 0$ such that the path $s(y, z)$ never passes through the bottom vertex $\mathbf{w} = [x, -1]$ for any $(y, z) \in V^2_J$ because s is continuous and $s(\mathbf{v}, \mathbf{v})$ is the constant path at \mathbf{v} . Hence, s maps into $(\Sigma X - \{\mathbf{w}\})^1_J$. A deformation retraction $\Sigma X - \{\mathbf{w}\} \rightarrow V$ and s present a parametrized motion planning algorithm on V . The equality $\text{TC}(\pi_V: V \rightarrow J) = 0$ leads to the same contradiction as the proof of Theorem 3.5. Hence, $\text{TC}(\pi_J: \Sigma X \rightarrow J) = \infty$. □

Example 3.7 We compute the parametrized topological complexity of some CW complexes given in Example 2.31.

- (1) $\text{TC}(S^n \rightarrow \{e^{(0)} < e^{(n)}\}) = \infty$ by Theorem 3.6.

(2) $TC(B_k \rightarrow \mathcal{P}(B_k)) = \infty$ because Corollary 2.6 shows

$$TC(B_k \rightarrow \mathcal{P}(B_k)) \geq TC(S^1 \rightarrow \{e^{(0)} < e^{(1)}\}) = \infty.$$

(3) $TC(T^n \rightarrow \mathcal{P}(T^n)) = \infty$ because Corollary 2.6 shows

$$TC(T^n \rightarrow \mathcal{P}(T^n)) = TC(T^n \rightarrow J^n) \geq TC(S^1 \rightarrow \{e^{(0)} < e^{(1)}\}) = \infty.$$

(4) $TC(\mathbb{R}P^n \rightarrow \mathcal{P}(\mathbb{R}P^n)) = \infty$ because Corollary 2.6 shows

$$TC(\mathbb{R}P^n \rightarrow \mathcal{P}(\mathbb{R}P^n)) \geq TC(S^1 \rightarrow \{e^{(0)} < e^{(1)}\}) = \infty.$$

(5) $TC(\mathbb{C}P^n \rightarrow \mathcal{P}(\mathbb{C}P^n)) = \infty$ because Corollary 2.6 shows

$$TC(\mathbb{C}P^n \rightarrow \mathcal{P}(\mathbb{C}P^n)) \geq TC(S^2 \rightarrow \{e^{(0)} < e^{(2)}\}) = \infty.$$

The CW complexes X given in the above example have infinite $TC(X \rightarrow \mathcal{P}(X))$. This is because TC uses open sets that cover the fiberwise product. If we use arbitrary subspaces instead of open sets with parametrized motion planning algorithms, we can consider the generalized version TC_g of TC, and obtain a different result from Example 3.7 (see Example 4.20).

Remark 3.8 Several properties of parametrized topological complexity in Cohen et al. (2021) may not hold for non-fibrations by Theorems 3.5 and 3.6.

- (1) For a fibration $\pi : E \rightarrow B$, the inequality $TC(\pi) \leq \text{cat}(E_B^2)$ holds, as mentioned in (Section 7 Cohen et al. 2021). However, it is not true for general fiberwise spaces. For a non-contractible space X and the stratified space $\pi_J : CX \rightarrow J$, the fiberwise product $(CX)_J^2$ is contractible to (\mathbf{v}, \mathbf{v}) by the contraction induced from the natural linear contraction on CX to \mathbf{v} . Thus, $\text{cat}((CX)_J^2) = 0$, whereas $TC(\pi_J) = \infty$.
- (2) Proposition 2.9 is not true for general fiberwise spaces. For a circle S^1 with the minimal cell decomposition $S^1 \rightarrow J$, the fiberwise product

$$(S^1)_J^2 = T^2 - (e^{(0)} \times e^{(1)} \cup e^{(1)} \times e^{(0)})$$

is a torus with two open 1-cells removed, and is homotopy equivalent to $S^1 \vee S^1 \vee S^1$. Each stratum is a contractible open cell; however, $TC(S^1 \rightarrow J) = \infty$.

3.3 Equatorial stratification on cone and suspension

With another stratification on the cone CX over J , we have $\pi_E : CX \rightarrow J$, defined by $\pi_E(X) = 0$ for $X = X \times \{0\}$ and $\pi_E(CX_+) = 1$ for $CX_+ = CX - X$. In this subsection, we will show that $TC(\pi_E)$ equals to $TC(X)$ or $TC(X) + 1$.

Lemma 3.9 *We have $TC(X) \leq TC(\pi_E) \leq TC(X) + 1$.*

Proof It is obvious that $TC(X) = TC(\pi_E^{-1}(0)) \leq TC(\pi_E)$ by Corollary 2.6. We show the other inequality $TC(\pi_E) \leq TC(X) + 1$. Let U be an open subset of X^2 with a motion planning algorithm $s : U \rightarrow X^I$. We consider an open set

$$\tilde{U} = \{((1 - t)x + t\mathbf{v}, (1 - u)y + u\mathbf{v}) \in (CX)_J^2 \mid (x, y) \in U, t, u \in [0, 1]\}$$

and a parametrized motion planning algorithm $\tilde{s} : \tilde{U} \rightarrow (CX)_J^I$, given by

$$\tilde{s}(((1 - t)x + t\mathbf{v}, (1 - u)y + u\mathbf{v}))(r) = (1 - L(t, u)(r))s(x, y)(r) + L(t, u)(r)\mathbf{v},$$

where $L(t, u)(r) = (1 - r)t + ru$ for $r \in I$.

In contrast, the open set $CX_+^2 = CX_+ \times CX_+ \subset (CX)_J^2$ admits a motion planning algorithm given by a contraction on CX_+ . This is a parametrized motion planning algorithm $CX_+^2 \rightarrow (CX)_J^I$ because it only works in CX_+ .

If $TC(X) = m$ with open sets U_0, U_1, \dots, U_m covering X^2 , where each U_i admits a motion planning algorithm, then $\tilde{U}_0, \dots, \tilde{U}_m$, and CX_+^2 constitute an open cover of $(CX)_J^2$ with parametrized motion planning algorithms. Hence, $TC(\pi_E) \leq m + 1 = TC(X) + 1$. \square

A natural question to ask at this point is whether $TC(\pi_E) = TC(X)$ or $TC(\pi_E) = TC(X) + 1$. We have not completely solved the problem, but some cases show $TC(\pi_E) = TC(X)$.

Proposition 3.10 *If X is contractible, then $TC(\pi_E) = TC(X) = 0$.*

Proof The cone CX is fiberwise homotopy equivalent to $C\{0\} = I$ over J . The interval I admits the linear motion planning algorithm, which is a parametrized motion planning algorithm. Hence, $TC(\pi_E) = TC(I \rightarrow J) = 0$. \square

For example, in the case of non-contractible space $X = S^n$ shows $TC(\pi_E) = TC(X)$. In this case, the cone $CS^n = D^{n+1}$ is convex, and we can extend motion planning algorithms in S^n to parametrized ones in D^{n+1} using linear combinations.

Proposition 3.11 *If the cone CX is homeomorphic to a convex set in \mathbb{R}^n , then we have $TC(\pi_E) = TC(X)$.*

Proof It is sufficient to show the inequality $TC(\pi_E) \leq TC(X)$. Assume that $TC(X) = m$ with open sets U_0, \dots, U_m covering X^2 , where each U_i admits a motion planning algorithm. It should be noted that CX and the product $(CX)^2$ are metrizable. Furthermore, X^2 is normal (satisfying Axiom T4). Thus, we can take an open set $V \subset U_0$ such that $\bar{V} \subset U_0$ and V, U_1, \dots, U_m cover X^2 . We extend V to an open set in $(CX)_J^2$ as follows:

$$W = \{((1 - t)x + t\mathbf{v}, (1 - u)y + u\mathbf{v}) \in (CX)_J^2 \mid (x, y) \in V, t, u \in [0, 1/2]\}.$$

We have a separating function $f : (CX)_J^2 \rightarrow I$ satisfying $f(\bar{W}) = 0$ and $f(\tilde{U}_0^c) = 1$, where \tilde{U}_0^c denotes the complement of \tilde{U}_0 with respect to $(CX)_J^2$. Recall that the open

set

$$\tilde{U}_0 = \{((1 - t)x + t\mathbf{v}, (1 - u)y + u\mathbf{v}) \in (CX)_J^2 \mid (x, y) \in U_0, t, u \in [0, 1)\}$$

admits a parametrized motion planning algorithm $\tilde{s}: \tilde{U}_0 \rightarrow (CX)_J^I$ in the proof of Lemma 3.9. Furthermore, the contractible space CX also has a motion planning algorithm (not necessarily parametrized) $h: (CX)^2 \rightarrow (CX)^I$. We can construct the following motion planning algorithm:

$$\gamma: (CX)_J^2 \rightarrow (CX)^I$$

given by $\gamma(a) = (1 - f(a))\tilde{s}(a) + f(a)h(a)$. The restriction of γ to $W \cup CX_+^2$ is a parametrized motion planning algorithm. The open sets $\tilde{U}_1, \dots, \tilde{U}_m$, and $W \cup CX_+^2$ cover $(CX)_J^2$ with parametrized motion planning algorithms. Hence, $TC(\pi_E) \leq m = TC(X)$. □

The next computation immediately follows from Proposition 3.11.

Example 3.12 Consider the stratified space $\pi_E: D^n \rightarrow J$ for an n -disk $D^n = CS^{n-1}$ ($n \geq 2$). Then we have

$$TC(\pi_E) = TC(S^{n-1}) = \begin{cases} 1 & \text{for } n \text{ even,} \\ 2 & \text{for } n \text{ odd.} \end{cases}$$

For the general case, we leave it as a conjecture.

Conjecture 3.13 $TC(\pi_E: CX \rightarrow J) = TC(X)$ for any space X .

In contrast, the suspension ΣX is separated into three strata: the upper open cone CX_+ , equator X , and lower open cone CX_- . This is a stratified space $\pi_E: \Sigma X \rightarrow E$ over the poset $E = \{-1 > 0 < 1\}$ defined by $\pi_E(CX_+) = 1$, $\pi_E(X) = 0$, and $\pi_E(CX_-) = -1$. Note that the previous stratified space $\pi_E: CX \rightarrow J$ is a stratified subspace of $\pi_E: \Sigma X \rightarrow E$.

Proposition 3.14 $TC(\pi_E: \Sigma X \rightarrow E) = TC(\pi_E: CX \rightarrow J)$ for any space X .

Proof It is obvious that

$$TC(\pi_E: \Sigma X \rightarrow E) \geq TC(\pi_E: \pi_E^{-1}(J) \rightarrow J) = TC(\pi_E: CX \rightarrow J).$$

We will show the converse inequality. Let U be an open set in $(CX)_J^2$ with a parametrized motion planning algorithm $s: U \rightarrow (CX)_E^I$. For a point $a = [x, t] \in \Sigma X$, let $-a \in \Sigma X$ denote the vertically symmetrical point $[x, -t]$. We consider the open sets

$$-U = \{(a, b) \in (\Sigma X)_E^2 \mid (-a, -b) \in U\},$$

and $\tilde{U} = U \cup (-U)$. The parametrized motion planning algorithm s on U can be extended to \tilde{s} on \tilde{U} as follows: $\tilde{s}(a, b)(t) = -s(-a, -b)(t)$ for $(a, b) \in -U$, and $t \in I$. Hence, $\text{TC}(\pi_E: \Sigma X \rightarrow E) \leq \text{TC}(\pi_E: CX \rightarrow J)$. \square

Example 3.15 Consider the stratified space $\pi_E: S^n \rightarrow E$ for an n -sphere $S^n = \Sigma S^{n-1}$ ($n \geq 2$). Then we have that

$$\text{TC}(\pi_E) = \text{TC}(S^{n-1}) = \begin{cases} 1 & \text{for } n \text{ even,} \\ 2 & \text{for } n \text{ odd,} \end{cases}$$

by Proposition 3.14 and Example 3.12.

4 Generalized version of parametrized topological complexity

Example 3.7 suggests that it is impossible to construct parametrized motion planning algorithms on open sets covering the fiberwise product of some CW complexes. However, we can separate the fiberwise product into a finite number of subspaces (not necessarily open sets) with parametrized motion planning algorithms for finite CW complexes. From this perspective, we can consider another version of parametrized topological complexity.

4.1 Generalized parametrized topological complexity

We briefly review the generalized version of parametrized topological complexity. This concept was considered for fibrations in Cohen et al. (2021).

Definition 4.1 For a fiberwise space $\pi: E \rightarrow B$, the *generalized parametrized topological complexity* $\text{TC}_g(\pi)$ is defined as the minimal number n such that the fiberwise product E_B^2 admits a partition into $n + 1$ subsets

$$E_B^2 = U_0 \sqcup U_1 \sqcup \dots \sqcup U_n, \quad (U_i \cap U_j = \emptyset, i \neq j).$$

where each U_i admits a parametrized motion planning algorithm. In particular, when $B = *$ consists of a single point, $\text{TC}_g(\pi) = \text{TC}_g(E)$ is called the *generalized topological complexity* of E .

Clearly, the inequality $\text{TC}_g(\pi) \leq \text{TC}(\pi)$ always holds for any fiberwise space π . The converse inequality also holds for nice fiberwise spaces.

Theorem 4.2 (Proposition 4.7 in Cohen et al. (2021), Corollary 2.8 in García-Calines (2019)) *For a locally trivial fibration $\pi: E \rightarrow B$ between metrizable separable ANR spaces E and B , we have $\text{TC}_g(\pi) = \text{TC}(\pi)$. In particular, $\text{TC}_g(X) = \text{TC}(X)$ for a space X having the homotopy type of a CW complex.*

Unfortunately, a poset-stratified space $X \rightarrow P$ is not a fibration, and the base poset P is not an ANR space in general. The above equality fails for $X \rightarrow \mathcal{P}(X)$, which

is associated with the CW complexes X with the face poset $\mathcal{P}(X)$ given in Examples 3.7 and 4.20.

Remark 4.3 For a fiberwise space $\pi : E \rightarrow B$, $\text{TC}(\pi) = 0$ indicates that there exists a global section of $\Pi : E_B^1 \rightarrow E_B^2$. Hence, $\text{TC}(\pi) = 0$ if and only if $\text{TC}_g(\pi) = 0$.

A similar equality to Theorem 4.2 holds for LS category and its generalization. The generalized LS category $\text{cat}_g(X)$ is defined as the minimal number n such that X is separated into $n + 1$ categorical subspaces.

Theorem 4.4 (Corollary 2.10 in García-Calines (2019)) $\text{cat}_g(X) = \text{cat}(X)$ for a space X having the homotopy type of a CW complex.

As seen in Section 2.1, some fundamental properties of $\text{TC}(\pi)$ also hold for $\text{TC}_g(\pi)$ because they do not depend on open sets.

Proposition 4.5 Let $\pi : E \rightarrow B$ be a fiberwise space.

- (1) $\text{TC}_g(f^*\pi) \leq \text{TC}_g(\pi)$ for the pull-back $f^*\pi : E \times_B X \rightarrow X$ for a map $f : X \rightarrow B$. In particular, $\text{TC}_g(\pi|_{\pi^{-1}(A)}) \leq \text{TC}(\pi)$ for $A \subset B$.
- (2) $\text{TC}_g(\pi') \leq \text{TC}_g(\pi)$ for a fiberwise space $\pi' : E' \rightarrow B$ with fiberwise maps $f : E' \rightarrow E$ and $g : E \rightarrow E'$ satisfying $g \circ f \simeq_B \text{id}_{E'}$. In particular, $\text{TC}_g(\pi') = \text{TC}_g(\pi)$ if π and π' are fiberwise homotopy equivalent.

4.2 Generalized parametrized topological complexity of poset-stratified spaces

We will deal with the computation of $\text{TC}_g(\pi)$ for poset-stratified spaces $\pi : X \rightarrow P$ given in Sect. 3.

The inequality $\text{TC}_g(\pi) \leq \text{TC}(\pi)$ implies that $\text{TC}_g(X \rightarrow \mathcal{P}(X)) = 0$ for a locally finite regular CW complex X with the face poset $\mathcal{P}(X)$. Furthermore, $\text{TC}_g(\tau : BP \rightarrow P) = 0$ for a locally finite poset P .

Let us recall the stratifications $\pi_J : CX \rightarrow J$ and $\pi_J : \Sigma X \rightarrow J$ respectively on the cone CX and the suspension ΣX given in Sect. 3.2.

Theorem 4.6 For $\pi_J : CX \rightarrow J$, we have $\text{TC}_g(\pi_J) = \text{TC}_g(X)$.

Proof It is obvious that $\text{TC}_g(\pi_J) \geq \text{TC}_g(\pi_J^{-1}(1)) = \text{TC}_g(\hat{C}X) = \text{TC}_g(X)$. We will show the converse inequality. Let $\text{TC}_g(X) = m$ with subspaces U_0, \dots, U_m separating X^2 with motion planning algorithms $s_i : U_i \rightarrow X^I$. Recall the proof of Lemma 3.9. We extend U_i to a subspace \tilde{U}_i in $(\hat{C}X)^2$ and s_i to a motion planning algorithm $\tilde{s}_i : \tilde{U}_i \rightarrow (\hat{C}X)^I$. Furthermore, $V = \tilde{U}_0 \cup \{(\mathbf{v}, \mathbf{v})\}$ admits a parametrized motion planning algorithm $s : V \rightarrow (CX)_J^I$ given by

$$s((1 - t)x + t\mathbf{v}, (1 - u)y + u\mathbf{v})(r) = (1 - L(t, u)(r))s_0(x, y)(r) + L(t, u)(r)\mathbf{v},$$

because $s(\mathbf{v}, \mathbf{v})(r) = \mathbf{v}$ for any $r \in I$. Hence, $(CX)_J^2$ is separated into $m + 1$ subspaces

$$(CX)_J^2 = \{(\mathbf{v}, \mathbf{v})\} \sqcup (\hat{C}X)^2 = V \sqcup \tilde{U}_1 \sqcup \dots \sqcup \tilde{U}_m$$

with parametrized motion planning algorithms, and $\text{TC}_g(\pi_J) \leq m = \text{TC}_g(X)$. □

If P is a finite poset, then X is separated into finite strata and $X_P^2 = \sqcup_p (e_p)^2$. The next lemma follows immediately from this fact.

Lemma 4.7 *For a stratified space $\pi : X \rightarrow P$ over a finite poset P , we have*

$$\text{TC}_g(\pi) \leq \sum_{p \in P} (\text{TC}_g(e_p) + 1) - 1.$$

Theorem 4.8 *For $\pi_J : \Sigma X \rightarrow J$, we have*

$$\text{TC}(\pi_J) = \begin{cases} 0 & \text{if } X \text{ is contractible,} \\ 1 & \text{otherwise.} \end{cases}$$

Proof The contractible case can be shown by the same argument in the proof of Theorem 3.6. For a non-contractible space X , assume that $\text{TC}_g(\pi_J) = 0$. Remark 4.3 implies $\text{TC}(\pi_J) = 0$; however, this is a contradiction by Theorem 3.6. Therefore, $\text{TC}_g(\pi_J) > 0$. Moreover, ΣX is separated into two contractible strata $\{\mathbf{v}\}$ and $\hat{\Sigma}X = \Sigma X - \{\mathbf{v}\}$; therefore, $\text{TC}_g(\pi_J) \leq 1$ by Lemma 4.7. Hence, $\text{TC}_g(\pi_J) = 1$. \square

Next, we consider TC_g for the stratifications $\pi_E : CX \rightarrow J$ and $\pi_E : \Sigma X \rightarrow E$ given in Sec. 3.3.

Lemma 4.9 *For $\pi_E : CX \rightarrow J$, we have $\text{TC}_g(X) \leq \text{TC}_g(\pi_E) \leq \text{TC}_g(X) + 1$.*

Proof The essential argument here is the same given in Lemma 3.9; however, the proof is simpler because we do not need open sets. We have

$$\text{TC}_g(X) = \text{TC}_g(\pi_E^{-1}(0)) \leq \text{TC}_g(\pi_E).$$

Therefore, we will show the inequality $\text{TC}_g(\pi_E) \leq \text{TC}_g(X) + 1$. Let $\text{TC}_g(X) = m$ and let X^2 be separated into $m + 1$ subspaces U_0, \dots, U_m with motion planning algorithms. The open cone CX_+ is contractible and $\text{TC}_g(CX_+) = 0$. Lemma 4.7 provides

$$\text{TC}_g(\pi_E) \leq (\text{TC}_g(X) + 1) + (\text{TC}_g(CX_+) + 1) - 1 = m + 1 = \text{TC}_g(X) + 1.$$

\square

We have the following conjecture similar to Conjecture 3.13.

Conjecture 4.10 $\text{TC}_g(\pi_E : CX \rightarrow J) = \text{TC}_g(X)$ for any space X .

If Conjecture 3.13 is true, the above conjecture is also true for spaces having the homotopy type of a CW complex as the following result asserts.

Proposition 4.11 *Let X be a space having the homotopy type of a CW complex. If $\text{TC}(\pi_E : CX \rightarrow J) = \text{TC}(X)$, then we have that $\text{TC}_g(\pi_E : CX \rightarrow J) = \text{TC}_g(X)$.*

Proof Theorem 4.2 provides the following inequalities:

$$TC(X) = TC_g(X) \leq TC_g(\pi_E) \leq TC(\pi_E).$$

Thus, $TC(\pi_E) = TC(X)$ implies $TC_g(\pi_E) = TC_g(X)$. □

The following equality follows from the argument in the proof of Proposition 3.14.

Proposition 4.12 $TC_g(\pi_E: \Sigma X \rightarrow E) = TC_g(\pi_E: CX \rightarrow J)$ for any space X .

4.3 Generalized parametrized topological complexity of CW-complexes

In this subsection, we will compute $TC_g(X \rightarrow \mathcal{P}(X))$ for the non-regular CW complexes X given in Example 3.7. While $TC(X \rightarrow \mathcal{P}(X)) = \infty$, the calculation shows that $TC_g(X \rightarrow \mathcal{P}(X)) = \text{cat}(X)$ in this case.

A CW complex X is separated into contractible open cells (strata). Lemma 4.7 implies the following proposition.

Proposition 4.13 For a finite CW complex X with the face poset $\mathcal{P}(X)$, we have $TC_g(X \rightarrow \mathcal{P}(X)) \leq \mathcal{P}(X)^\sharp - 1$, where $\mathcal{P}(X)^\sharp$ stands for the cardinal of $\mathcal{P}(X)$.

Proposition 4.14 For a finite-dimensional CW complex X with the face poset $\mathcal{P}(X)$, we have $TC_g(X \rightarrow \mathcal{P}(X)) \leq \dim X$.

Proof We consider the subset

$$U_n = \bigcup_{\dim e=n} (e \times e) \subset X_{\mathcal{P}(X)}^2.$$

Each e is contractible; hence, we have a section $U_n \rightarrow X_{\mathcal{P}(X)}^I$ of Π . Thus, we have

$$X_{\mathcal{P}(X)}^2 = U_0 \sqcup U_1 \sqcup \dots \sqcup U_{\dim X}$$

and $TC_g(X \rightarrow \mathcal{P}(X)) \leq \dim X$. □

Using the results given above we are able to compute TC_g of some CW complexes given in Example 3.7.

Example 4.15 The minimal cell decomposition on a sphere S^n consists of two cells. Proposition 4.13 provides $TC_g(S^n \rightarrow \mathcal{P}(S^n)) \leq 1$. Moreover, $TC(S^n \rightarrow \mathcal{P}(S^n)) = \infty$ in Example 3.7 implies that there is no global section of Π . Hence, $TC_g(S^n \rightarrow \mathcal{P}(S^n)) = 1$.

Example 4.16 For a bouquet $B_k = \vee_k S^1$ with the cell decomposition given in Example 3.7, Proposition 4.14 provides $TC_g(B_k \rightarrow \mathcal{P}(B_k)) \leq 1$. Moreover, $TC(B_k \rightarrow \mathcal{P}(B_k)) = \infty$ in Example 3.7 implies that there is no global section of Π . Hence, $TC_g(B_k \rightarrow \mathcal{P}(B_k)) = 1$.

We now focus on lower bounds of TC_g . For a fiberwise pointed space $\pi : E \rightarrow B$ with a section $s : B \rightarrow E$, García-Calines presented a fiberwise LS lower bound of $TC(\pi)$. We can similarly show the generalized version by the same argument in the proof of Proposition 13 in [Gar].

A subspace U of E is called *fiberwise categorical* over B if the inclusion $U \hookrightarrow E$ is fiberwise homotopic to $s \circ p|_U$. The *fiberwise LS category* $cat^B(E)$ is the smallest number n such that E is covered by $n + 1$ fiberwise categorical open subsets over B . Similarly, the *generalized fiberwise LS category* $cat_g^B(E)$ is the smallest number n such that E is separated into $n + 1$ fiberwise categorical subspaces over B .

Proposition 4.17 (Proposition 13 of [Gar]) *For a fiberwise pointed space $\pi : E \rightarrow B$, we have $cat^B(E) \leq TC(\pi)$ and $cat_g^B(E) \leq TC_g(\pi)$.*

Proposition 4.17 provides a lower bound of TC and TC_g respectively for fiberwise pointed space. However, a poset-stratified space is not a fiberwise pointed space, that is, it does not admit a section in general.

Lemma 4.18 *Let P be a finite connected poset and let X be a T_1 space. Any continuous map $P \rightarrow X$ must be constant.*

Proof Let $f : P \rightarrow X$ be a continuous map. For a comparable pair $p < q$ in P , we assume that $f(p) \neq f(q)$ in X . We can take an open set $f(p) \in U$ such that $f(q) \notin U$. The open set $V = f^{-1}(U)$ includes p while $q \notin V$. However, the minimal open neighborhood $P_{\geq p}$ of p includes q . Hence, V must include q . This contradiction implies that $f(p) = f(q)$ for any comparable pair p, q . If P is a finite connected poset, then any two points p, q in P are connected by comparable pairs: $p = p_0, p_1, p_2, \dots, p_n = q$, such that $p_i \leq p_{i+1}$ or $p_i \geq p_{i+1}$ for each i . Thus, $f(p) = f(q)$ for any p, q , and f is constant. \square

The above lemma suggests that the fiberwise space $X \rightarrow \mathcal{P}(X)$ associated with a connected normal finite CW complex X is not pointed, except when X is a single point. It is difficult to construct a section of a continuous map $X \rightarrow P$ for a poset P and a Hausdorff space X . However, it may be possible for the classifying space $\mathcal{B}P$ instead of P .

Theorem 4.19 *Let X be a cylindrically normal CW complex with face category $C = \mathcal{C}(X)$. If the canonical functor $\rho : C \rightarrow P$ to the face poset $P = \mathcal{P}(X)$ has a section and $\mathcal{B}P$ is contractible, then*

- (1) $cat(X) \leq TC(\pi)$,
- (2) $cat(X) = cat_g(X) \leq TC_g(\pi)$,

for the stratified space $\pi : X \rightarrow P$.

Proof (1) Our aim is to show the following inequalities:

$$TC(\pi) \geq TC(\tau^*(\pi)) \geq cat^{\mathcal{B}P}(X \times_P \mathcal{B}P) \geq cat(X \times_P \mathcal{B}P) \geq cat(X).$$

Let us focus on each of the inequalities.

- (i) First we consider the pullback $\tau^*(\pi): X \times_P \mathcal{B}P \rightarrow \mathcal{B}P$ of π along the natural map $\tau: \mathcal{B}P \rightarrow P$. Proposition 2.5 ensures $\text{TC}(\tau^*(\pi)) \leq \text{TC}(\pi)$.
- (ii) Let $s: P \rightarrow C$ denote a section of $\rho: C \rightarrow P$ and $\iota: \mathcal{B}C \rightarrow X$ denote the natural homeomorphism in Theorem 2.29. We notice that $\tau^*(\pi)$ is a fiberwise pointed space over $\mathcal{B}P$ because we have a section $\mathcal{B}P \rightarrow X \times_P \mathcal{B}P$ of $\tau^*(\pi)$ sending a to $(\iota(\mathcal{B}s(a)), a)$. Hence, $\text{cat}^{\mathcal{B}P}(X \times_P \mathcal{B}P) \leq \text{TC}(\tau^*(\pi))$ by Proposition 4.17.
- (iii) Fiberwise categorical subsets in a space over the contractible classifying space $\mathcal{B}P$ are categorical subsets in the standard sense. Thus, we have $\text{cat}(X \times_P \mathcal{B}P) \leq \text{cat}^{\mathcal{B}P}(X \times_P \mathcal{B}P)$.
- (iv) The first projection $X \times_P \mathcal{B}P \rightarrow X$ has a section $X \rightarrow X \times_P \mathcal{B}P$ sending x to $(x, \mathcal{B}\rho(\iota^{-1}(x)))$. This implies $\text{cat}(X) \leq \text{cat}(X \times_P \mathcal{B}P)$.

(2) Similarly, we can show $\text{cat}_g(X) \leq \text{TC}_g(\pi)$. Theorem 4.4 shows the equality $\text{cat}_g(X) = \text{cat}(X)$. Thus, $\text{cat}(X) \leq \text{TC}_g(\pi)$. □

Now, using all the machinery we have developed above, we are able to compute TC_g for the CW complexes given in Example 3.7.

Example 4.20 Let us recall the cylindrically normal CW complexes X and their face categories $\mathcal{C}(X)$ given in Example 2.31. The canonical functor $\rho: \mathcal{C}(X) \rightarrow \mathcal{P}(X)$ admits a section and $\mathcal{B}P$ is contractible in each example. Hence, Theorem 4.19 and Propositions 4.13, 4.14 provide

$$\text{cat}(X) \leq \text{TC}_g(\pi) \leq \min\{\mathcal{P}(X)^\sharp - 1, \dim X\}.$$

These inequalities determine TC_g of the following CW complexes:

- (1) The canonical functor $\mathcal{C}(S^n) \rightarrow \mathcal{P}(S^n)$ admits a section by choosing a point in S^{n-1} , and $\mathcal{B}(\mathcal{P}(S^n)) \cong I$ is contractible. Hence, $\text{TC}_g(S^n \rightarrow \mathcal{P}(S^n)) = 1$, as shown in Example 4.15.
- (2) The canonical functor $\mathcal{C}(B_k) \rightarrow \mathcal{P}(B_k)$ admits a section by choosing a point S^0 , and $\mathcal{B}(\mathcal{P}(B_k)) \cong \vee_k I$ is contractible. Hence, $\text{TC}_g(B_k \rightarrow \mathcal{P}(B_k)) = 1$, as shown in Example 4.16.
- (3) The canonical functor $\mathcal{C}(T^n) \rightarrow \mathcal{P}(T^n)$ admits a section as the product of sections of $\mathcal{C}(S^1) \rightarrow \mathcal{P}(S^1)$, and $\mathcal{B}(\mathcal{P}(T^n)) \cong I^n$ is contractible. Hence, $\text{TC}_g(T^n \rightarrow \mathcal{P}(T^n)) = n$.
- (4) The canonical functor $\mathcal{C}(\mathbb{R}P^n) \rightarrow \mathcal{P}(\mathbb{R}P^n)$ has a section given by the unit element in \mathbb{Z}_2 , and $\mathcal{B}(\mathcal{P}(\mathbb{R}P^n)) \cong \Delta^n$ is contractible. Hence, $\text{TC}_g(\mathbb{R}P^n \rightarrow \mathcal{P}(\mathbb{R}P^n)) = n$.
- (5) The canonical functor $\mathcal{C}(\mathbb{C}P^n) \rightarrow \mathcal{P}(\mathbb{C}P^n)$ has a section given by the unit element in $U(1)$, and $\mathcal{B}(\mathcal{P}(\mathbb{C}P^n)) \cong \Delta^n$ is contractible. Hence, $\text{TC}_g(\mathbb{C}P^n \rightarrow \mathcal{P}(\mathbb{C}P^n)) = n$.

Conclusion and future work

We have computed the (generalized) parametrized topological complexity of various poset-stratified spaces compiled in the following Table 1.

Table 1 TC and TC_g for poset-stratified spaces

$\pi: E \rightarrow B$	$TC(\pi)$	$TC_g(\pi)$
$\tau: \mathcal{B}P \rightarrow P$	0	0
$\pi_J: CX \rightarrow J$	$\begin{cases} 0 & (X \simeq *) \\ \infty & (X \not\simeq *) \end{cases}$	$TC_g(X)$
$\pi_J: \Sigma X \rightarrow J$	$\begin{cases} 0 & (X \simeq *) \\ \infty & (X \not\simeq *) \end{cases}$	$\begin{cases} 0 & (X \simeq *) \\ 1 & (X \not\simeq *) \end{cases}$
$\pi_E: CX \rightarrow J \pi_E: \Sigma X \rightarrow E$ Regular CW $X \rightarrow \mathcal{P}(X)$	$TC(X) \leq TC(\pi_E) \leq TC(X) + 1$	$TC_g(X) \leq TC_g(\pi_E) \leq TC_g(X) + 1$
Non-regular CW $S^n \rightarrow \mathcal{P}(S^n) B_k \rightarrow \mathcal{P}(B_k)$	0	0
Non-regular CW $T^n \rightarrow \mathcal{P}(T^n) \mathbb{R}P^n \rightarrow \mathcal{P}(\mathbb{R}P^n) \mathbb{C}P^n \rightarrow \mathcal{P}(\mathbb{C}P^n)$	∞	1
	∞	n

Examples 3.7 and 4.20 display $\text{TC}(X \rightarrow \mathcal{P}(X))$ and $\text{TC}_g(X \rightarrow \mathcal{P}(X))$ for some non-regular CW complexes X with the face posets $\mathcal{P}(X)$, and exhibit that TC_g is far different from TC in this case. On the other hand, $\text{TC}(X \rightarrow \mathcal{P}(X)) = \text{TC}_g(X \rightarrow \mathcal{P}(X)) = 0$ for any locally finite regular CW complex X (Theorem 3.4). Hence, $\text{TC}(X \rightarrow \mathcal{P}(X))$ and $\text{TC}_g(X \rightarrow \mathcal{P}(X))$ strongly depend on the cell decomposition on X .

A natural question to ask is whether there is a non-regular CW complex X satisfying $\text{TC}(X \rightarrow \mathcal{P}(X)) = 0$ or not. If $\text{TC}(X \rightarrow \mathcal{P}(X)) = 0$ is equivalent to the regularity of X , we can say that the (generalized) parametrized topological complexity for CW complexes measure the difference from regularity.

Furthermore, in order to better understand the characteristics of TC and TC_g for non-regular CW complexes with the face posets, we need more computational examples. As seen in Example 3.7, $\text{TC}(X \rightarrow \mathcal{P}(X)) = \infty$ for various non-regular CW complexes. Moreover, all examples in Example 4.20 show that $\text{TC}_g(X \rightarrow \mathcal{P}(X)) = \text{cat}(X)$. It may be interesting to find non-regular CW complexes X with $\text{TC}(X \rightarrow \mathcal{P}(X)) < \infty$, or $\text{TC}_g(X \rightarrow \mathcal{P}(X)) \neq \text{cat}(X)$.

Declarations

Conflict of interest The author states that there is no conflict of interest.

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