Mathematical Musings on the External Anatomy of the Novel Coronavirus*

Part 2: Chasing After Quasi-Symmetry

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What is the shape of the novel coronavirus which has turned our world upside down? Even though it looks dull, unattractive, and even disgusting under a microscope, creative artists have attributed to it bright colors, made it look pretty, and depicted it as a thing of beauty. What can a mathematician contribute to this effort? We take a purist's point of view by imposing on it a quasi-symmetry and then deriving some consequences. In an idealistic world, far removed from reality but still constrained by the rules of mathematics, anyone can enjoy this ethereal beauty of the mind's creation, beckoning others to join in the pleasure.

Our musings are split into four parts. We fondly hope while readers wait for the future parts to appear, they will indulge in their own musings, tell others about them, and propagate the good virus of mathematical thinking.

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Gist of Part 1[1]

The general shape of the n-Cov is a sphere with various proteins protruding out of the sphere. Therefore, we mused on the properties of a sphere—volume and surface area of caps and pedestals, and geodesics between two points—presenting easy proofs and computational formulas. Having seen a plethora of 2D and 3D representations of the external anatomy of the n-Cov, we posed the challenge of determining which depiction is closer to the truth and which has taken an artistic license to deviate from the truth.

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4. How Many S-Proteins and Where?

[1]To read Part 1 of the article, see: *Resonance*, Vol.27, No.4, pp.609–622, https://www.ias.ac.in/article/fulltext/reso/027/04/0609-0622.

Faced with the dilemma of not knowing the number of S-Proteins for sure, in this part, we create a mathematical model that will support our choice of the number of S-Proteins, as well as locate them on the sphere. Construction of our model proceeds by first inverting the problem to a new one—an oft-used tool in the mathematician's toolbox.

Keywords
Great circle, Platonic solids, duality, spherical geometry, Girard's theorem, Hamiltonian cycle, intermediate value theorem.

4.1 The Inverse Problem

Convex Hull Problem: Given all points on the surface of a spherical n-CoV through which the S-proteins emerge; what shape is formed by taking their convex hull?

Inverting the problem often facilitates finding a solution.

When we attempt to impose symmetry on the above-mentioned convex hull, we achieve a stark reduction in the number of viable answers to the original question of how many S-proteins there are on the n-CoV. A demand for perfect symmetry requires that the convex hull be a *regular* polyhedron, which is defined as a 3D object bounded by identical regular polygonal faces (plane regions bounded by (at least 3) sides of equal length that make all angles equal) such that equal number of faces (at least 3) meet at each vertex. Such a three-dimensional object is called a Platonic solid in honor of the Greek philosopher Plato (428/427 or 424/423–348/347 BCE), who theorized that these objects represent elements of nature: earth, water, fire, air, and ether. Henceforth, when we talk about a polyhedron in this article, we assume that it is regular. We state and prove an ancient gem of a mathematical result. See *Wikipedia* [2].

A demand for perfect symmetry reduces the number of options.

"The five Platonic solids correspond to the five elements of nature."
– Plato

Theorem 2.1. *There are exactly five Platonic solids as shown in Figure 1.*

Proof. A Platonic solid must satisfy three properties:

(a) On a regular n-gonal face, each angle measures 180^o $circ(n-2)/n$ (with $n \geq 3$);

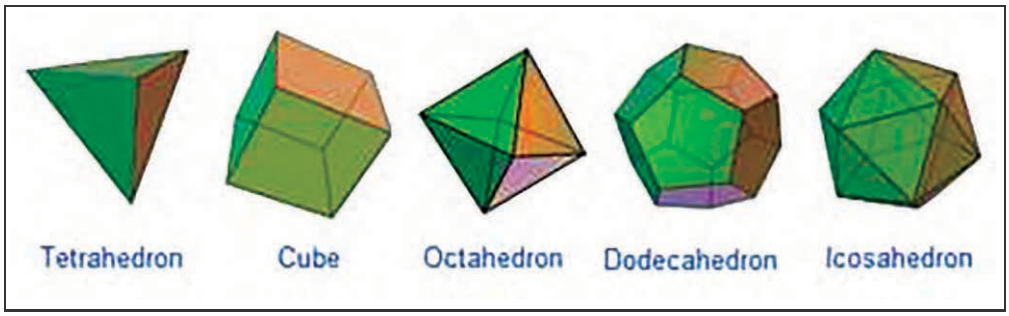

Figure 1. The five Platonic solids.

n	Each angle	m	At each vertex meet	Total angle	Name of solid	Number of faces	Number of vertices	Number of edges
3	60°	3	3 triangles	180°	Tetrahedron	4	4	6
3	60°	4	4 triangles	240°	Octahedron	8	6	12
3	60°	5	5 triangles	300°	Icosahedron	20	12	30
4	90°	3	3 squares	270°	Hexahedron	6	8	12
5	108°	3	3 pentagons	324°	Dodecahedron	12	20	30

Table 1. Enlisting all (n, m) pairs that lead to formation of Platonic solids.

(b) At each vertex there meet exactly m faces (with $m \geq 3$); and

(c) Total angle at each vertex is strictly less than 360^o.

We document in *Table* 1 the solutions one by one, starting from small values of n and small values of m. We identify five solids (shown in *Figure* 1) and list their number of faces, number of vertices, and number of edges.

A systematic search yields five solutions, and only five solutions.

For each $n \leq 5$, the next higher value of m satisfying Properties (a) and (b), fails to satisfy Property (c). For $n \geq 6$, even the smallest value of $m = 3$ fails to satisfy Property (c). This completes the proof of the theorem. □

4.2 Duality of Platonic Solids

Any Platonic solid can be enclosed in a circumscribing sphere. A fascinating property of these five Platonic solids is duality: Given any Platonic solid, locate the centers of all its faces, which are regular polygons. Then the convex hull of these face-centers lives strictly inside the given Platonic solid and is also a Platonic solid, called the dual of the given Platonic solid. In fact, a tetrahedron is its own dual, a hexahedron (cube) and an octahedron are duals of each other, and so are a dodecahedron and an icosahedron.

Duals are like dance partners.

Clearly, none of the five Platonic solids have enough vertices to account for all the S-proteins we see on the diagrams in *Figure* 1 in Part 1. Therefore, we must settle for something less than perfect symmetry. We conjecture that the S-proteins are not only located at the vertices but also at the face-centers (or rather at their radial projections or their shadows on the sphere when a point source of light is placed at the center of the circumscribing sphere). Then, for the five Platonic solids, there would be altogether 8, 14, and 32 locations, as one can see by adding the second last and the third last columns of *Table* 1. None of these values are large enough to model the majority of the depictions of n-CoV we have come across. We need more locations for S-proteins!

To increase the locations of S-proteins, we will allow vertices, face-centers, and maybe some more points.

Consequently, we also conjecture that S-proteins are located at the mid-points of the edges of the Platonic solids (or rather at their shadows on the circumscribing sphere under a radial projection). If both conjectures hold, then adding the last three columns of *Table* 1, we note that for a tetrahedron, there would be 14 locations; for an octahedron or a hexahedron, 26 locations; and for a dodecahedron or an icosahedron, 62 locations. This last option appears to be promising since it fits within the range of various estimates of the number of S-proteins obtained from *Figure* 1 in Part 1. Although the convex hull of these 62 points on the sphere is not a Platonic solid, this is what we propose as a model for a 'quasi-symmetric' distribution of locations for the S-proteins. Alternatively, the reader can work with an octahedron and impose a

"It cannot be," says Theorem 1.

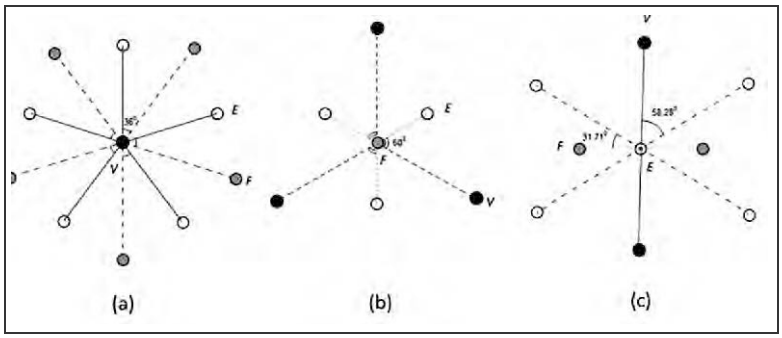

Figure 2. When viewed directly from above an S-protein, located at **(a)** a vertex (V), **(b)** a face-center (F), and **(c)** an edge-center (E) of an icosahedron, two/three sets of near-neighbor S-proteins appear at regular/quasi-regular angular intervals.

third conjecture to include more points for S-proteins.

We will consider the locations of M- and E-Proteins in Part 4 of the series.

4.3 Neighbors of S-Proteins

Looking carefully at the 3D diagrams in *Figure* 1 of Part 1, if we focus on any one S-protein and then identify all its near-neighbor S-proteins, taking a 360° panoramic view, though they may not be equally far, we notice an over-abundance of spherical hexagons and frequent appearances of spherical pentagons formed by the near-neighbors. Indeed, our proposed model mimics this feature and exhibits a few more features, as depicted in *Figure* 2. The actual distances alluded to in these features will be revealed towards the end of Subsection 6.3 in Part 3. So will be the angles mentioned in *Figure* 2(c). Here we simply remark on their relative magnitudes. All line segments in *Figure* 2 represent the corresponding tangents to the geodesics between the points.

Just as hexagons and pentagons are seen on a soccer ball.

Recall that a geodesic is the shortest path along a great circle.

(a) Each of the 12 locations originally at the vertices of the icosahedron (or at the face centers of the dodecahedron) has five near-neighbors on the edge-centers (solid lines). Once you remove these near neighbors, there is a different set of five near-neighbors at the face-centers (along dotted lines) of the icosahedron (or at the vertices of the dodecahedron). Moreover, these two sets of neighbors alternate and constitute regular angular intervals of 36°.

(b) Likewise, each of the 20 locations originally at the face-centers of the icosahedron (or at the vertices of the dodecahedron) have three very-near-neighbors on the edge centers (gray lines) and three near-enough-neighbors (along dotted lines) at the vertices of the icosahedron (or at the face centers of the dodecahedron); and the two types keep alternating as we turn around at regular angular intervals of 60^o.

(c) Finally, each of the 30 locations originally at the edge-centers of the icosahedron (or of the dodecahedron) has a pair of very-near-neighbors on the face-centers of the icosahedron (gray lines), and the next pair of near-neighbors (along solid lines) are orthogonal to the first pair. Once these four neighbors are removed, there is still another batch of four neighbors that form a rectangle whose sides are parallel to the two pairs just removed (though not in the same plane). These four points, when matched with the nearest member of the first pair, subtend an angle of 31.71^o at the edge-center, and when matched with the second pair, subtend an angle of 58.78^o at the edge-center. One can also say that the second pair and the third quadruplet together form a quasi-regular hexagon with one pair of opposite sides equal, and the other four equal sides are slightly smaller than these two. We leave it to the reader to figure out the interior angles of this hexagon. The hexagon exhibits vertical- and horizontal reflection symmetry (and hence a 180° rotation symmetry).

We are imagining a lot. But not as much as some stargazers do!

Readers, please figure out the interior angles.

These features of our adopted model align well with the 3D depictions in *Figure* 1 in Part 1, and thereby they provide a strong mutual support. Affirmed by this evidence, let us proceed to study our model shape of the n-CoV in more detail.

5. Our Model for the Shape of the n-CoV

This is our model for the number and locations of the S-proteins.

In subsection 4.2, we have proposed as a model for the S-proteins on the n-CoV, a radial projection of an icosahedron and its dual dodecahedron. Note that the shadows of the edges of each polyhedron bisect the shadows of the edges of the other orthogonally. The S-proteins protrude from the shadows of the 12 vertices of

the icosahedron, 20 vertices of the dodecahedron, and the 30 bisection points of the edges of these two polyhedra. Thus, our model accommodates 62 S-proteins in total.

For all computations henceforth, we start with an icosahedron, circumscribe it with a sphere, and then inscribe in it the dual dodecahedron. We evaluate the parameters of our proposed model and describe its mathematical properties in three subsections. We leave it to the reader to repeat the computations starting with a dodecahedron, superposing a sphere, and then inscribing its dual icosahedron.

5.1 Circumscribe an Icosahedron by a Sphere

Consider a unit icosahedron; that is, all its edges measure 1 unit. Circumscribe a sphere around it; that is, consider the smallest sphere inside which the unit icosahedron fits tightly, as shown in *Figure* 3. A natural question is how large is the sphere? Suffices it to find the circumradius R. Indeed, the following result holds.

A natural question deserves to be asked and answered.

Theorem 2.2. *The radius of the sphere circumscribing the unit icosahedron (whose all sides are of unit length) measures*

$$R = \sqrt{\frac{5 + \sqrt{5}}{8}} = 0.9510565$$

In particular, R is constructible using only the Euclidean geometric tools of straightedge and compass.

Proof. Refer to *Figure* 4, which shows selected elements of the icosahedron and its circumscribing sphere. Without explicitly stating it, we shall use the Pythagorean theorem repeatedly.

Imagine that one of the vertices of the unit icosahedron is at the North Pole N. Its five nearest neighboring vertices A, B, C, D, E, form a regular pentagon (which lives entirely inside the icosahedron, except for the five boundary points on the sphere, and hence is not visible). These five neighboring vertices are co-planer, which is parallel to the equator and intersect the circumscribing

With a mathematician's eyes, we can see through the opaque icosahedron.

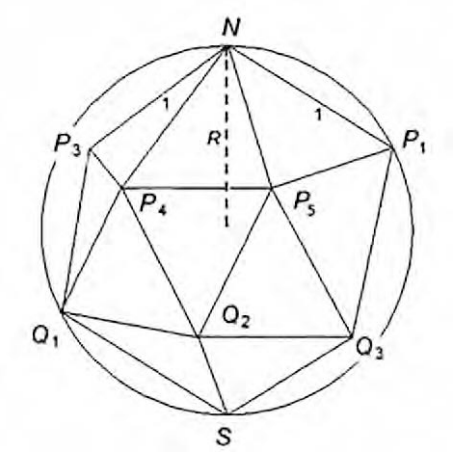

Figure 3. A transparent sphere circumscribing an opaque unit icosahedron, or an opaque icosahedron inscribed in a transparent sphere.

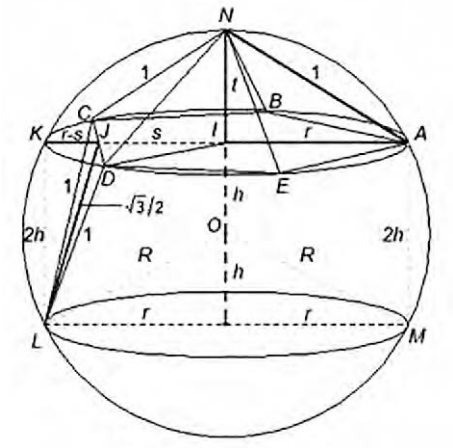

Figure 4. When a sphere circumscribes a unit icosahedron, the radius of the sphere is found by calculating several intermediate quantities.

sphere to form a circle circumscribing the pentagon, having center I and radius r. The midpoint of each side of the pentagon is s units away from I, and the pentagon itself is t units away from the North Pole N and h units away from the center O of the sphere of radius R. We sequentially compute the quantities r, s, t, h, R.

Join vertex D to the center I of the circle circumscribing pentagon $ABCDE$, and join the midpoint J of side CD to I. From the right

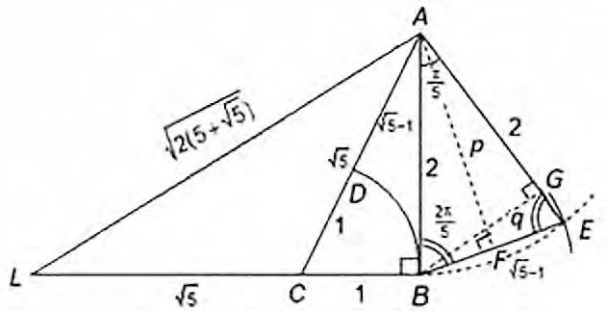

Figure 5. To draw an angle measuring $\pi/5$ and compute $sin(\pi/5)$. In fact, one-fourth of AL is the radius R of the sphere that circumscribes a unit icosahedron.

triangle IJD so formed, we note that

$$\sin\left(\frac{\pi}{5}\right) = \frac{1/2}{r}. \qquad (1)$$

In *Figure* 5, starting from a right triangle ABC whose legs measure 1 unit and 2 units respectively, we demonstrate how to draw an angle BAE measuring $\pi/5$ using straightedge and compasses, and then construct q as the perpendicular from B to AE. Hence, we evaluate $\sin(\pi/5) = q/2$. Next, using (1), we can construct $r = 1/q$ and evaluate it.

In *Figure* 5, note that the area of triangle ABE, with side lengths $2, 2, \sqrt{5} - 1$, can be calculated using two different pairs of base and altitude. As a result, $2q = (\sqrt{5} - 1)p$. However, from right triangle AFB, we have

$$p = \sqrt{4 - \left(\frac{\sqrt{5}-1}{2}\right)^2} = \frac{1}{2}\sqrt{16 - \left(\sqrt{5}-1\right)^2} = \sqrt{\frac{5+\sqrt{5}}{2}}.$$

Hence, $q = \frac{\sqrt{5}-1}{2}p = \frac{\sqrt{5}-1}{2}\sqrt{\frac{5+\sqrt{5}}{2}} = \sqrt{\frac{5-\sqrt{5}}{2}}$. Therefore, from right triangle AGB, we evaluate

$$\sin\left(\frac{\pi}{5}\right) = \frac{q}{2} = \sqrt{\frac{5-\sqrt{5}}{8}} = 0.587785$$

and from (1), we get $r = 1/q = \sqrt{\frac{2}{5-\sqrt{5}}} = \sqrt{\frac{5+\sqrt{5}}{10}} = 0.850651$.

Glory to the Pythagorean theorem!

Returning to *Figure* 4, from right triangle IJD, we evaluate IJ as

$$s = \sqrt{r^2 - 1/4} = \frac{1}{2}\sqrt{\frac{3+\sqrt{5}}{5-\sqrt{5}}} = \sqrt{\frac{5+2\sqrt{5}}{20}} = 0.688191,$$

from right triangle NIA, we evaluate NI as

$$t = \sqrt{1 - r^2} = \sqrt{\frac{3-\sqrt{5}}{5-\sqrt{5}}} = \sqrt{\frac{5-\sqrt{5}}{10}} = 0.525731$$

and from right triangle JKL, we evaluate KL as

$$2h = \sqrt{\frac{3}{4} - (r-s)^2} = \sqrt{\frac{2}{5-\sqrt{5}}} = \sqrt{\frac{5+\sqrt{5}}{10}} = 0.850651.$$

Finally, we can find the radius of the circumscribing sphere as

$$R = NI{+}IO = t{+}h = \frac{\sqrt{3-\sqrt{5}} + \sqrt{1/2}}{\sqrt{5-\sqrt{5}}} = \sqrt{\frac{5+\sqrt{5}}{8}} = 0.951057.$$

Next, in *Figure* 5, we can extend BC to L until $CL = CA = \sqrt{5}$, and construct $AL = \sqrt{2(5+\sqrt{5})} = 4R$ using only Euclidean geometric tools. The proof is therefore complete. □

The proof of Theorem 2.2 reveals several fascinating properties:

(1) Algebra shows that in *Figure* 5, $AL = \sqrt{(2^2 + (\sqrt{5}+1)^2)} = 4R$, is twice the diameter of the circumscribing sphere. Also, one can verify that $2Rt = 1$; that is, t, the depth of the spherical cap, is the reciprocal of the diameter of the circumscribing sphere.

The depth of the cap equals the reciprocal of the sphere's diameter.

(2) Rectangle $AKLM$ is exactly twice as wide as it is tall, since $2h = r$. That is, the planar cut through the five neighboring vertices of the North Pole is parallel to the planar cut through the five neighboring vertices of the South Pole; and the distance between these two cutting planes equals the radius of their cross-sectional circles.

Distance between cutting planes equals the radius.

(3) Since $2h = r$, from the right triangle AKL, we have $5r^2 = 4R^2$; or equivalently, $h = r/2 = R/\sqrt{5}$. That is, the area of the cross-sectional circle of the spherical cap is 4/5 of the area of the great

The cross-sectional circle is 4/5 the great circle.

circle. Likewise, the area of the curved surface of the spherical cap is $2\pi Rt = \pi$, which is $R^{-2} = 2(1 - 1/\sqrt{5}) = 1.105573$ times the area of the great circle.

(4) The volume of the spherical cap is $\pi(2R+h)(R-h)^2/3 = \pi(R-t/3)t^2$, which is $1/2 - 7\sqrt{5}/50 = 0.1869505$ times the volume of the circumscribing sphere.

Property (2) is a pleasantly surprising result, which mathematicians like to call beautiful because of its unexpected simplicity. *A priori* there was no hint for it!

Beauty emerges from unexpected simplicity.

5.2 An Application

Here is its application to *Art and Design*: Suppose that two sculptors are given a sphere and asked to slice out the largest icosahedron out of it. They can split their workload evenly between themselves: One sculptor will cut out the cap, and the other will cut out another identical cap from the bottom side of the pedestal using a parallel cut, leaving a slab in between. (In fact, they can do so simultaneously: They should tilt the NS line to become horizontal; then, the cuts will be parallel vertical planes. Moreover, cutting vertically is more efficient than cutting horizontally.) The important thing to remember is that the slab should be as thick as the radius of the cutting circle.

The slab should be as thick as the radius of the cutting circle.

Thereafter, each sculptor can work with one cap each, and they can work together on the slab. They each will draw regular pentagons on the two cut surfaces, being extremely careful that the pentagons on the opposite planar faces of the slab are rotated by exactly 180° (also 36° or 108° will do). Then they will make five planar cuts on each cap using the pentagon and the top of the cap, and they will make ten planar cuts on the middle portion, going from an edge of the pentagon on one face of the slab to the vertex on the opposite face, each cutting five times. When they are done, they can reassemble the three pieces into a unit icosahedron. Incidentally, a total of 22 planar cuts are involved. Can the reader do with fewer?

Can you do it with fewer than 22 cuts?

What to Expect in Future Parts?

We shall build our model for the n-Cov as the superposition of an inscribed icosahedron and an inscribed dodecahedron within a sphere. Then we shall study the properties of spherical triangles. These are essential steps towards settling the challenge posed in Part 1, namely which diagrams within *Figure* 1 in Part 1 are closer to truth and which are far-fetched. Stay tuned.

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Suggested Reading

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