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Research article

Nonrelativistic solutions of Schrödinger equation and thermodynamic properties with the proposed modified Mobius square plus Eckart potential

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ABSTRACT

We obtain solutions of Schrödinger equation for the modified Mobius square plus Eckart (MMPSE) potential via the formula method. Numerical results are reported. In addition, the partition function Z and other thermodynamic properties such as vibrational free energy, F, vibrational internal energy, U, vibrational entropy, S, and vibrational specific heat, C are presented. We also discuss special cases of this potential. Our result is consistent with previous studies in the literature.

1. Introduction

The Schrödinger equation is vital in quantum mechanics and other related fields, and this has given rise to an important debate in science. Recent works have derived solutions of the Schrödinger equation with various potential models [1, 2, 3, 4, 5, 6, 7]. Some applications in inverse quantum scattering theory [8, 9] have also been reported. However, exact solutions of the Schrödinger equation are limited to some cases [10, 11, 12]. Researchers have attempted to solve the Schrödinger and other wave equations using mathematical methods such as Laplace transform method [13, 14, 15], path integral method [16], algebraic method [17], shifted 1/N expansion [18], Nikiforov–Uvarov (NU) Method [19, 20, 21, 22, 23, 24, 25], formula method [26] and many others [27, 28, 29, 30]. In the present paper, we shall focus our attention on the proposed modified Mobius square plus Eckart (MMSPE) potential. This potential is written

$$V(r) = -V_o\left(\frac{A + Be^{-2\alpha r}}{1 - e^{-2\alpha r}}\right)^2 - \frac{V_1 e^{-2\alpha r}}{1 - e^{-2\alpha r}} + \frac{V_2 e^{-2\alpha r}}{\left(1 - e^{-2\alpha r}\right)^2}, \tag{1}$$

where V_0, V_1, V_2 are parameters that describe the depth of potential well, α is the screening parameter, A and B are potential range and molecular bond length, respectively. MMSPE potential is a combination of the modified Mobius square and the Eckart potentials. This potential serves as a realistic model in nuclear physics, molecular physics and chemical physics since it incorporates well known potentials as special cases.

Recently, thermodynamic properties of various systems have been reported. For example, the authors in Ref. [31], solved the Schrödinger equation for the Hua potential and reported its thermodynamic properties. The thermal properties of the Deng-Fan Eckart potential [32] was reported and applied to some diatomic molecules [33]. Similarly, the energy spectra and thermodynamic properties of the HCl, H_2, CO and LiH molecules with the hyperbolic Poschl-Teller potential have been studied [34]. On a similar note, Nath and Roy [35] studied the thermodynamic properties of the same set of molecules under the Deng-Fan potential. Also, Bakhati and co-workers [36], computed the thermodynamic properties of the light gases, O_2 and H_2. Furthermore, thermodynamic properties of screened Kratzer potential in the presence of magnetic and Aharanov-Bohm fields have been reported [37]. Recent works on thermodynamic properties of various systems have been reported [38, 39, 40, 41, 42, 43, 44, 45].

Some interesting reports on the applications of nonlinear Schrodinger equation in various systems can be found in the references [46, 47, 48, 49, 50, 51, 52, 53, 54, 55]. The purpose of this work is to apply the Formula method to solve the Schrödinger equation for MMSPE and then study its thermodynamic properties. Such a study has not been considered before in the literature to the best of our knowledge.

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The remainder of the paper is outlined as follows: Section 2 gives a brief review of the formula method. We present the energy and wave functions of the MMSPE potential in Sec. 3. Sect. 4 gives the thermodynamic properties. Sect. 5 contains the discussions. Finally, conclusion is given in Sec. 6.

2. Formula method

The formula method can be used to solve differential equations. Consider the following second order differential equation which is a parametric generalization of the Nikiforov-Uvarov method [19, 56].

$$\frac{d^2\psi(s)}{ds^2}+\frac{c_1-c_2s}{s\left(1-c_3s\right)}\frac{d\psi(s)}{ds}+\frac{\xi_1s^2+\xi_2s+\xi_3}{s^2\left(1-c_2s\right)^2}\psi(s)=0, \tag{2}$$

where $c_1,\dots,c_n,\xi_1,\xi_2,\xi_3$ are real parametric constants and $\psi(s)$ is the wave function. The advantage of using this method is the simplicity of its application, due to the fact that the energy expression is simple and easy to manipulate and there are not many constant parameters associated with this method. To obtain the energy of the system in consideration by formula method, the following expression is used

$$\left[\frac{c_4^2-c_5^2-\left[\frac{1-2n}{2}-\frac{1}{2c_3^2}\left(c_2-\sqrt{\left(c_3-c_2\right)^2-4\xi_1}\right)\right]^2}{2\left[\frac{1-2n}{2}-\frac{1}{2c_3^2}\left(c_2-\sqrt{\left(c_3-c_2\right)^2-4\xi_1}\right)\right]}\right]^2-c_5^2=0,\ c_3\neq 0, \tag{3}$$

where,

$$\left.\begin{aligned} c_4&=\frac{(1-c_1)+\sqrt{(1-c_1)^2-4\xi_3}}{2}\\ c_5&=\frac{1}{2}+\frac{c_1}{2}-\frac{c_2}{2c_3}+\sqrt{\left(\frac{1}{2}+\frac{c_1}{2}-\frac{c_2}{2c_3}\right)^2-\left(\frac{\xi_1}{c_3^2}+\frac{\xi_2}{c_3}+\xi_3\right)}\end{aligned}\right\}. \tag{4}$$

The wave function can be obtained in terms of generalized hypergeometric functions from

$$\psi(s)=N_ns^{c_4}\left(1-c_3s\right)_2^{c_5}F_1\left(-n,n+2\left(c_4+c_5\right)+\frac{c_2}{c_3}-1;2c_4+c_1;c_3s\right), \tag{5}$$

where N_n is a normalization constant.

For the condition where $c_1=c_2=c_3=1$, Onate et al. [57], derived a simplified energy equation for the formula method given by

$$\sqrt{-\xi_3}\left(1+\sqrt{1-4\left(\xi_1+\xi_2+\xi_3\right)}+2n\right)=\xi_3-\xi_1-n(n+1)-n\sqrt{1-4\left(\xi_1+\xi_2+\xi_3\right)}-\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\left(\xi_1+\xi_2+\xi_3\right)}\right)^2. \tag{6}$$

3. Solutions of radial Schrödinger equation with MMSPE

Given the radial Schrödinger equation of the form [11]

$$\frac{d^2R_{nl}(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}-V(r)\right)-\frac{l(l+1)}{r^2}\right]R_{nl}(r)=0, \tag{7}$$

where n and l are radial and orbital angular momentum quantum numbers, respectively. While E_{nl} is the non-relativistic energy and $V(r)$ is the potential. By inserting Eq. (1) into Eq. (7), one can obtain

$$\frac{d^2R_{nl}(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E_{nl}+V_o\left(\frac{A+Be^{-2\alpha r}}{1-e^{-2\alpha r}}\right)^2+\frac{V_1e^{-2\alpha r}}{1-e^{-2\alpha r}}-\frac{V_2e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}\right)-\frac{l(l+1)}{r^2}\right]R_{nl}(r)=0. \tag{8}$$

Since Eq. (8) has no solution for $l\neq 0$, we therefore use the approximation [58]

$$\frac{1}{r^2}\approx\frac{4\alpha^2e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}. \tag{9}$$

Inserting Eq. (9) into (8) and using $s=e^{-2\alpha r}$, we have

$$\frac{d^2R_{nl}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR_{nl}(s)}{ds}+\frac{\varepsilon s^2+\Omega s+\Lambda}{s^2(1-s)^2}R_{nl}(s)=0, \tag{10}$$

where,

$$\varepsilon=\frac{\mu\left(E_{nl}+V_0B^2-V_1\right)}{2\alpha^2\hbar^2}, \tag{11}$$

$$\Omega=\frac{\mu\left(2ABV_o-2E_{nl}+V_1-V_2\right)}{2\alpha^2\hbar^2}-l(l+1), \tag{12}$$

$$\Lambda=\frac{\mu\left(E_{nl}+V_oA^2\right)}{2\alpha^2\hbar^2}. \tag{13}$$

By comparing Eq. (2) with Eq. (10), it is seen that $c_1 = c_2 = c_3 = 1$. Hence, we derive the constants in (4) as follows.

$$\left.\begin{aligned} c_4 &= \sqrt{-\Lambda} \\ c_5 &= \frac{1}{2} + \frac{1}{2}\sqrt{(2l+1)^2 + \frac{2\mu\left(V_2 - V_o(A+B)^2\right)}{\alpha^2\hbar^2}} \end{aligned}\right\}. \tag{14}$$

Substitution of the constants in Eq. (14) into Eqs. (6) and (5), the energy eigenvalues and corresponding wave function turn out to be

$$E_{nl} = -A^2V_o - \frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\frac{-\mu(V_1 - V_2 + 2V_0A(A+B))}{2\alpha^2\hbar^2} + n(n+1) + \frac{1}{2} + l(l+1) + \left(n + \frac{1}{2}\right)\sqrt{(2l+1)^2 + \frac{2\mu(V_2 - V_0(A+B)^2)}{\alpha^2\hbar^2}}}{1 + 2n + \sqrt{(2l+1)^2 + \frac{2\mu\left(V_2 - V_0(A+B)^2\right)}{\alpha^2\hbar^2}}}\right)^2, \tag{15}$$

and

$$R(s) = N_n s^{\sqrt{-\Lambda}}(1-s)^{\frac{1}{2}(1+\chi)}\,{}_2F_1\left(-n, n + 2\sqrt{-\Lambda} + 1 + \chi; 2\sqrt{-\Lambda} + 1, s\right). \tag{16}$$

where,

$$\chi = \sqrt{(2l+1)^2 + \frac{2\mu\left(V_2 - V_o(A+B)^2\right)}{\alpha^2\hbar^2}} \tag{17}$$

and N_n is a normalization constant.

4. Thermodynamic properties of MMSPE potential

In the study of thermodynamic properties, the partition function is first considered. The partition function is derived using the expression [31].

$$Z(\beta) = \sum_{n=0}^{\upsilon} e^{-\beta E_n}, \tag{18}$$

$$\beta = \frac{1}{k_B T}.$$

Where k_B and β are the Boltzmann constant and temperature parameter respectively, and υ is the upper bound vibrational quantum number. The exact energy of the MMSPE potential is obtained by setting $l = 0$ and simplifying eq. (15) to get

$$E_n = a - q\left(\frac{b}{2(n+\sigma)} + \frac{(n+\sigma)}{2}\right)^2, \tag{19}$$

where,

$$\left.\begin{aligned} &a = -V_0A^2,\ q = \frac{2\alpha^2\hbar^2}{\mu},\ b = V + \sigma(1-\sigma) \\ &V = \frac{V_1 - V_2 + 2V_0A(A+B)}{q} \\ &\sigma = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2\mu\left(V_2 - V_0(A+B)^2\right)}{\alpha^2\hbar^2}} \end{aligned}\right\}. \tag{20}$$

Substituting eq. (19) into (18), we obtain

$$Z(\beta) = \sum_{n=0}^{\upsilon} \exp\left(-\beta\left(a - q\left(\frac{b}{2(n+\sigma)} + \frac{(n+\sigma)}{2}\right)^2\right)\right). \tag{21}$$

To evaluate eq. (21), we apply the semi-classical Poisson summation approach [59]

$$\sum_{n=0}^{\upsilon} f(n) = \frac{1}{2}[f(0) - f(\lambda+1)] + \sum_{-\infty}^{\infty}\int_0^{\upsilon+1} f(x)e^{-i2\pi mx}dx. \tag{22}$$

For approximation of the lowest-order, i.e., involving only terms for which $m = 0$ and discarding terms for which $m \neq 0$, (22) becomes

$$\sum_{n=0}^{\upsilon} f(n) = \frac{1}{2}[f(0) - f(\lambda+1)] + \int_0^{\lambda+1} f(x)dx. \tag{23}$$

Applying eq. (23) in (21), the partition function is obtained as

$$Z(\beta) = \frac{1}{2}\left[e^{-\beta\left(a - qk_1^2\right)} - e^{-\beta\left(a - qk_2^2\right)} + \int_0^{\upsilon} e^{\left(-M_1\beta - \frac{M_2\beta}{\rho^2} - M_3\beta\rho^2\right)}d\rho\right], \tag{24}$$

where,

Table 1. Energy spectrum of the modified Mobius square plus Eckart potential with $A = 0.4,\ B = 0.5,\ V_0 = 1,\ V_1 = 1.25,\ V_2 = 1.5,\ \mu = \hbar = 1$.

n	l	$E(\alpha = 0.01)$ eV	$E(\alpha = 0.03)$ eV	$E(\alpha = 0.05)$ eV	$E(\alpha = 0.07)$ eV
0	0	-0.2347681619	-0.2248013036	-0.2155857222	-0.2071121183
1	0	-0.2249606012	-0.2004282356	-0.1824817981	-0.1702300005
2	0	-0.2160198203	-0.1826495562	-0.1650683228	-0.1600066931
	1	-0.2158748427	-0.1819010646	-0.1641986470	-0.1602013979
3	0	-0.2078969375	-0.1705003036	-0.1600034866	-0.1694492692
	1	-0.2077654819	-0.1700168279	-0.1600613503	-0.1716401639
	2	-0.2075033546	-0.1690908420	-0.1603818716	-0.1765254922
4	0	-0.2005469051	-0.1632115829	-0.1649404636	-0.1941639702
	1	-0.2004282356	-0.1629582682	-0.1657609128	-0.1979470908
	2	-0.2001916339	-0.1624861145	-0.1675563430	-0.2058449528
	3	-0.1998385684	-0.1618617075	-0.1706086212	-0.2184129078
5	0	-0.1939281540	-0.1601646330	-0.1781919842	-0.2312775523
	1	-0.1938215915	-0.1601132880	-0.1796549780	-0.2363831868
	2	-0.1936091597	-0.1600396829	-0.1826967740	-0.2467993600
	3	-0.1932922401	-0.1600000204	-0.1875286614	-0.2628657961
	4	-0.1928728904	-0.1600739311	-0.1944230318	-0.2849550952

$$\left.\begin{aligned} M_1 &= \frac{\alpha^2\hbar^2 b^2}{\mu} - a,\ M_2 = \frac{\alpha^2\hbar^2 b^2}{2\mu},\ M_3 = \frac{\alpha^2\hbar^2}{\mu} \\ k_1 &= \frac{b}{2\sigma} + \frac{\sigma}{2},\ k_2 = \frac{b}{2(\upsilon+\sigma+1)} + \frac{(\upsilon+\sigma+1)}{2} \end{aligned}\right\}. \tag{25}$$

Using a Marple software, we evaluate Eq. (24) to obtain the expression for the partition function as follows

$$Z(\beta) = \frac{1}{2}\left[e^{-\beta\left(a-qk_1^2\right)} - e^{-\beta\left(a-qk_2^2\right)} + e^{\beta M_3\rho^2 - \beta M_1}\sqrt{\beta M_2}\left(\frac{2\upsilon e^{-\frac{\beta M_2}{\upsilon^2}}}{\sqrt{\beta M_2}} + \frac{2\sqrt{\beta M_2}\sqrt{\pi}erf\left(\frac{\sqrt{\beta M_2}}{\upsilon}\right)}{\sqrt{\beta M_2}} - 2\sqrt{\pi}\right)\right]. \tag{26}$$

Using eq. (26), other thermodynamic functions of the system are evaluated as follows:

- *Vibrational Free Energy, F*

$$F = -\frac{1}{\beta}\ln Z(\beta). \tag{27}$$

- *Vibrational mean energy, U*

$$U = -\frac{\partial \ln Z(\beta)}{\partial\beta}. \tag{28}$$

- *Vibrational Entropy, S*

$$S = k\ln Z(\beta) - k\beta\frac{\partial \ln Z(\beta)}{\partial\beta}. \tag{29}$$

- *Vibrational Specific Heat Capacity, C*

$$C = k\beta^2\frac{\partial^2 \ln Z(\beta)}{\partial\beta^2}. \tag{30}$$

5. Discussions

Numerical results of the energy of the MMSPE potential for various values of quantum numbers n, l and some α values are presented in Table 1. It is obvious that the energy increases as n and l increase for $\alpha = 0.01$. A similar trend is observed for $\alpha = 0.03$, where the energy increases as n and l increase, except beyond $n = 5$, $l = 3$, where it is observed to decreases slightly. For $\alpha = 0.05$, the energy increases up to $n = 3$, $l = 0$, but beyond that point, it starts to decrease gradually. A similar trend is observed for $\alpha = 0.07$. This implies that energy of the system loses its direct proportional relationship with the screening parameter, α as it increases.

Fig. 1 displays the variation of the MMSPE potential with the internuclear distance, r, for various values of α. The potential tends to a constant value as r increases for all values of α. Fig. 2(a) is a verification of the observation made in Table 1, as the proportionality of the energy with α is gradually lost as α increases. This could mean that for a particle, say an electron, within the field of the MMSPE potential, at certain energy state the screening from other electrons becomes so significant that the energy required to remove such a particle from the confinement of this potential starts to decrease since the nuclear attraction of the electron in such a state is weak and as such lower amount of energy is required to escape the confinement of the potential. In Fig. 2(b), the energy is observed to decrease as V_0 increases. In Fig. 2(c), the variation of the energy with V_1 is a curve. As is observed, the energy first increases as V_1 increases up to a point between $V_1 = 5$ and $V_1 = 6$, then starts to decrease beyond that point. This entails that as the value of V_1 increases, the direct proportionality with energy is gradually lost for all values of α. As can be observed in Fig. 2(d), the energy increases sharply as V_2 increases, slightly maintaining a constant value before decreasing slowly. All plots of energy were made with the following values of the parameters: $V_0 = 1$, $V_1 = 1.25$, $V_2 = 1.5$, $A = 2$, $B = -3$, $n = 1$, $l = 1$ and $\alpha = 0.01$.

Fig. 3 shows the variation of the vibrational partition function, Z, with respect to various potential parameters and the temperature parameter, β, for various values of the upper bound vibrational quantum number, υ. In Fig. 3(a), Z decreases as β increases, tending to a constant value. This

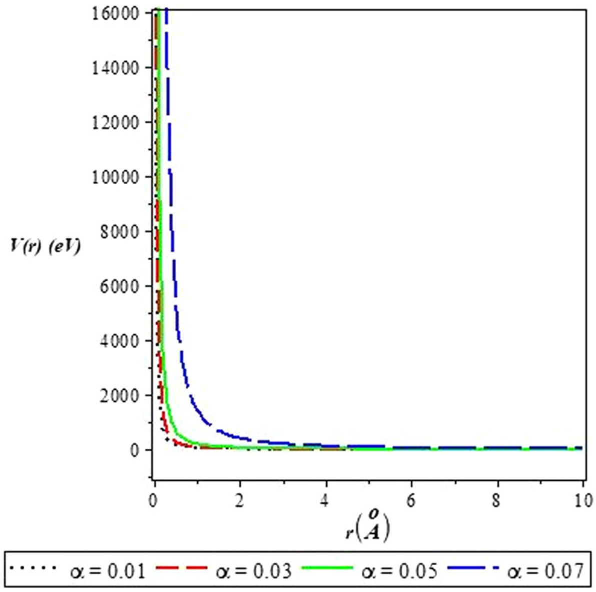

Fig. 1. Shape of the MMSPE potential.

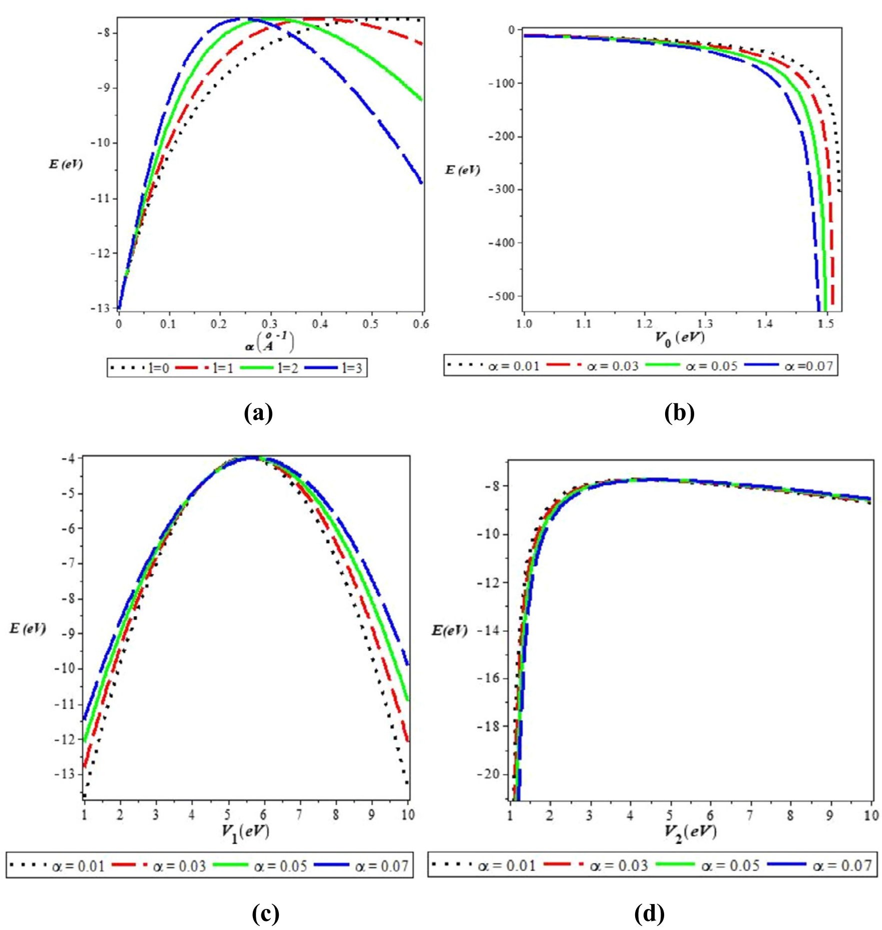

Fig. 2. Variation of the energy of the MMSPE potential with (a) α (b) V_0 (c) V_1 (d) V_2.

implies that the probability of locating a particle confined in the MMSPE potential in an energy state, E_n, increases as β increase. In Fig. 3(b), Z is observed to decrease sharply as V_0 increases up to a point, then increases sharply as V_0 value approaches 1.5. As can be seen in Fig. 3(c), Z forms a curve with peaks at $V_1 = 6$ for all values of υ. The peaks tend to higher values as υ increases. In Fig. 3(d), Z decreases as V_2 increases.

The plots in Fig. 4 show variation of Vibrational free energy F, against various parameters. In Fig. 4(a), F increases as β increases. In Fig. 4(b), F is observed to decrease as V_0 increases, but then increases sharply as V_0 approaches 1.5. In Fig. 4(c), F has peaks which tend to higher values as

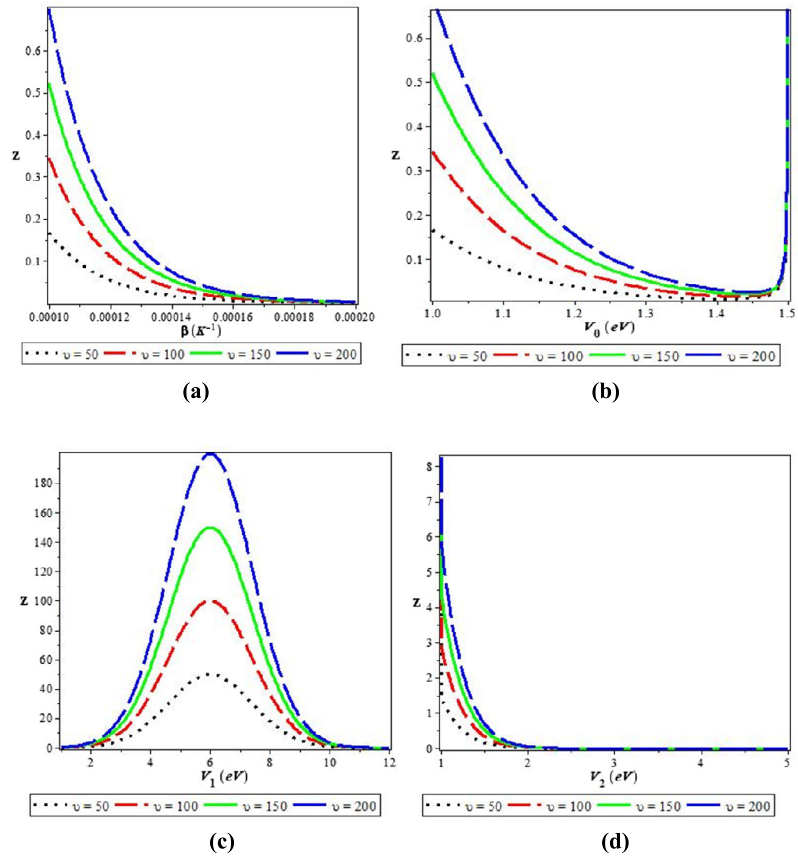

Fig. 3. Variation of the Partition function of the MMSPE potential with (a) β (b) V_0 (c) V_1 (d) V_2.

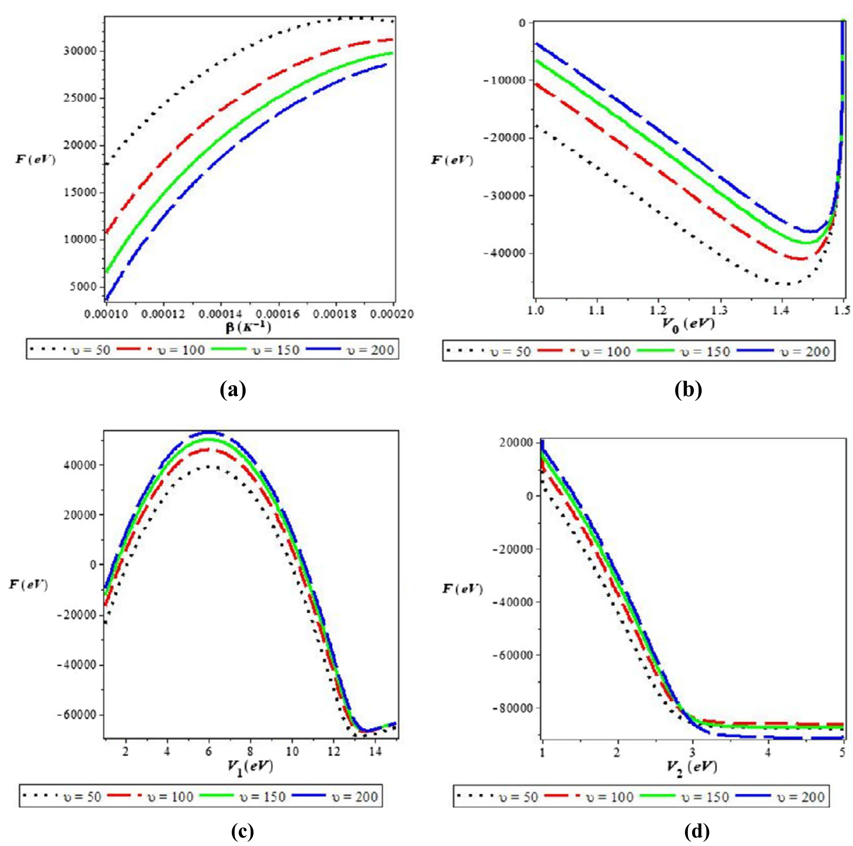

Fig. 4. Variation of the vibrational free energy of the MMSPE potential with (a) β (b) V_0 (c) V_1 (d) V_2.

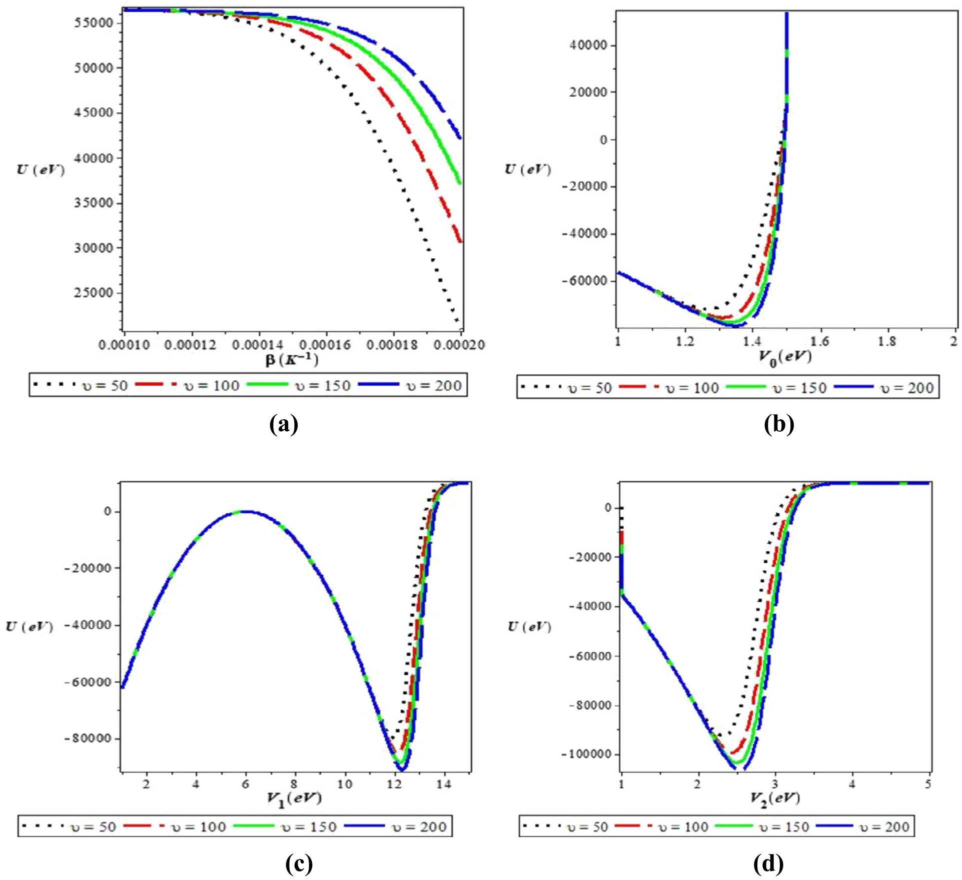

Fig. 5. Variation of the vibrational internal energy of the MMSPE potential with (a) β (b) V_0 (c) V_1 (d) V_2.

υ increases. This trend is similar to that observed in Fig. 3(c) for Z against V_1. In Fig. 4(d), F is seen to decreases as V_2 increases, then tends to a constant value beyond $V_2 = 3$ for values of υ.

The variation of the vibrational internal energy, U, against various parameters are given in Figs. 5(a) to 5(d). In Fig. 5(a), U is observed to decrease as β increases. In Fig. 5(b), the variation of U with V_0 has a similar trend as in Figs. 3(b) and 4(b) for Z and F. The variation of U with V_1 has a peculiar shape with a peak and a trough as observed in Fig. 5(c). The peak is at a similar position as that observed in Fig. 3(c) and in Fig. 4(c), but, maintains an almost constant value for all υ. The trough is observed just beyond $V_1 = 12$. In Fig. 5(d), U is observed to decrease first with increasing V_2, then increases sharply, thus, forming a trough between $V_2 = 2$ and $V_2 = 3$ for all υ.

In Fig. 6(a), the variation of vibrational entropy, S, with β is identical to that for U, as shown in Fig. 5(a). Figs. 6(b), 6(c) and 6(d) show respectively, a reverse trend to that observed in Figs. 5(b), 5(c) and 5(d) for the variation of U with the parameters V_0, V_1 and V_2.

Figs. 7(a-d) give the variation of vibrational specific heat, C, with parameters β, V_0, V_1 and V_2 respectively. In Fig. 7(a), C is observed to increase as β increases. This implies that at lower temperatures, the quantity of heat required to raise the temperature of a unit mass of a particle confined in the MMSPE potential is higher. The trend of C against V_0, V_1 and V_2 as shown in Figs. 7(b-d) are the same as for U in Figs. 5 (b-d). All plots of partition function were made with the following values of the parameters: $V_0 = 1$, $V_1 = 1.25$, $V_2 = 1.5$, $A = 2$, $B = -3$, and $\beta = 0.0001$.

5.1. Special cases

In this part, we assign some values to the parameters V_0, V_1, and V_2, which allows to derive special forms of energy eigenvalues.

Case I: Assume that $V_1 = V_2 = 0$, then the potential in Eq. (1) turns into the modified Mobius square potential

$$V(r) = -V_0\left(\frac{A + Be^{-2\alpha r}}{1 - e^{-2\alpha r}}\right)^2. \tag{31}$$

Here, the energy eigenvalues from Eq. (15) becomes:

$$E = -A^2V_o - \frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\frac{-\mu V_o A(A+B)}{\alpha^2\hbar^2} + n(n+1) + \frac{1}{2} + l(l+1) + \left(n + \frac{1}{2}\right)\sqrt{(2l+1)^2 - \frac{2\mu V_o (A+B)^2}{\alpha^2\hbar^2}}}{1 + 2n + \sqrt{(2l+1)^2 - \frac{2\mu V_o (A+B)^2}{\alpha^2\hbar^2}}}\right)^2. \tag{32}$$

Eq. (32) is consistent with eq. (22) in Ref. [60].

Case II: If $V_0 = 0$, Eq. (1) reduces to Eckart potential

$$V(r) = -\frac{V_1 e^{-2\alpha r}}{1 - e^{-2\alpha r}} + \frac{V_2 e^{-2\alpha r}}{\left(1 - e^{-2\alpha r}\right)^2}, \tag{33}$$

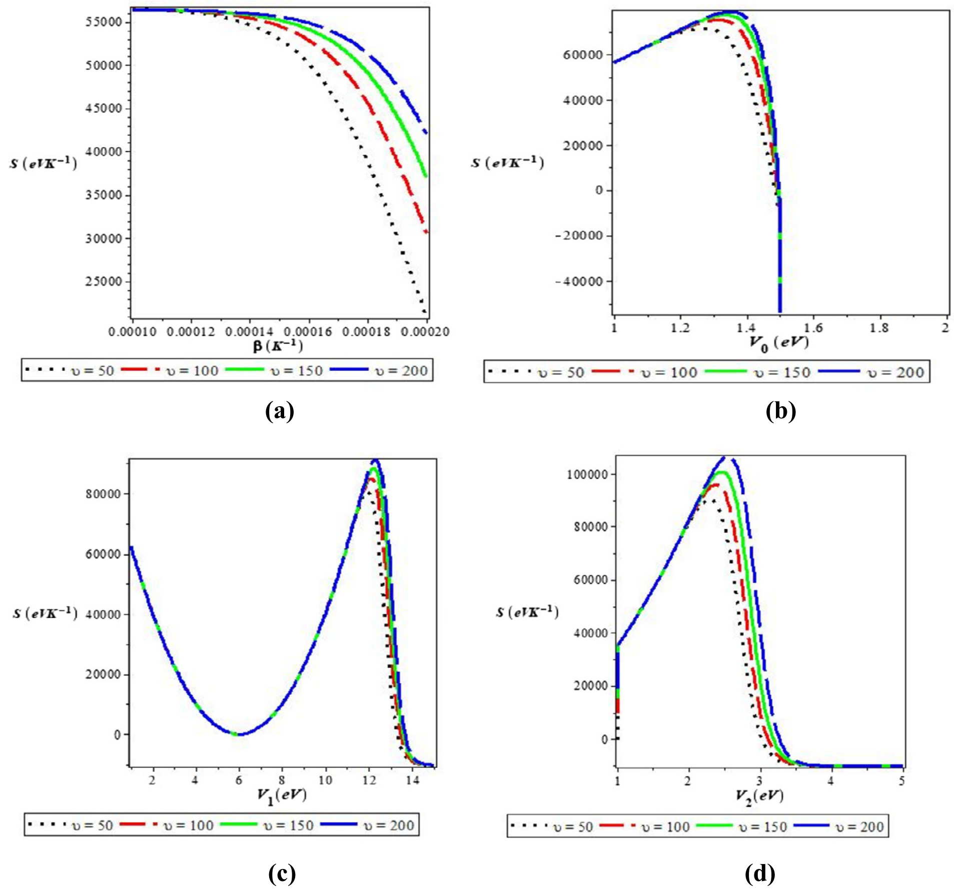

Fig. 6. Variation of the vibrational Entropy of the MMSPE potential with (a) β (b) V_0 (c) V_1 (d) V_2.

with the energy as

$$E=-\frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\frac{\mu(V_2-V_1)}{2\alpha^2\hbar^2}+n(n+1)+\frac{1}{2}+l(l+1)+\left(n+\frac{1}{2}\right)\sqrt{(2l+1)^2+\frac{2\mu V_2}{\alpha^2\hbar^2}}}{1+2n+\sqrt{(2l+1)^2+\frac{2\mu V_2}{\alpha^2\hbar^2}}}\right)^2. \tag{34}$$

By assigning $V_1=\alpha$, $\frac{1}{2\alpha}=a$, $V_2=\beta$, $\hbar=\mu=1$, eq. (34), becomes the expanded form eq. (22) of Ref. [20].
Case III: If $V_0=V_2=0$, Eq. (1) becomes the Hulthen potential of the form

$$V(r)=\frac{V_1e^{-2\alpha r}}{1-e^{-2\alpha r}}, \tag{35}$$

with the energy spectrum from Eq. (15) as

$$E=-\frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\frac{\mu V_1}{2\alpha^2\hbar^2}-(n+l+1)^2}{2(n+l+1)}\right)^2. \tag{36}$$

We note that eq. (36) is consistent with results in Ref. [61], if we adjust $V_1=Z\alpha$, and $\mu\to4\mu$. Similarly, eq. (36) is similar to that reported in Ref. [59], if we set $V_1=Ze^2\delta$.
Case VI: When $V_0=V_1=0$, Eq. (1) yields the Poschl-Teller potential of the form

$$V(r)=\frac{V_2e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}. \tag{37}$$

Here, the energy eq. (15) becomes

$$E=-\frac{2\alpha^2\hbar^2}{\mu}\left(\frac{\frac{\mu V_2}{2\alpha^2\hbar^2}+n(n+1)+\frac{1}{2}+l(l+1)+\left(n+\frac{1}{2}\right)\sqrt{(2l+1)^2+\frac{2\mu V_2}{\alpha^2\hbar^2}}}{1+2n+\sqrt{(2l+1)^2+\frac{2\mu V_2}{\alpha^2\hbar^2}}}\right)^2. \tag{38}$$

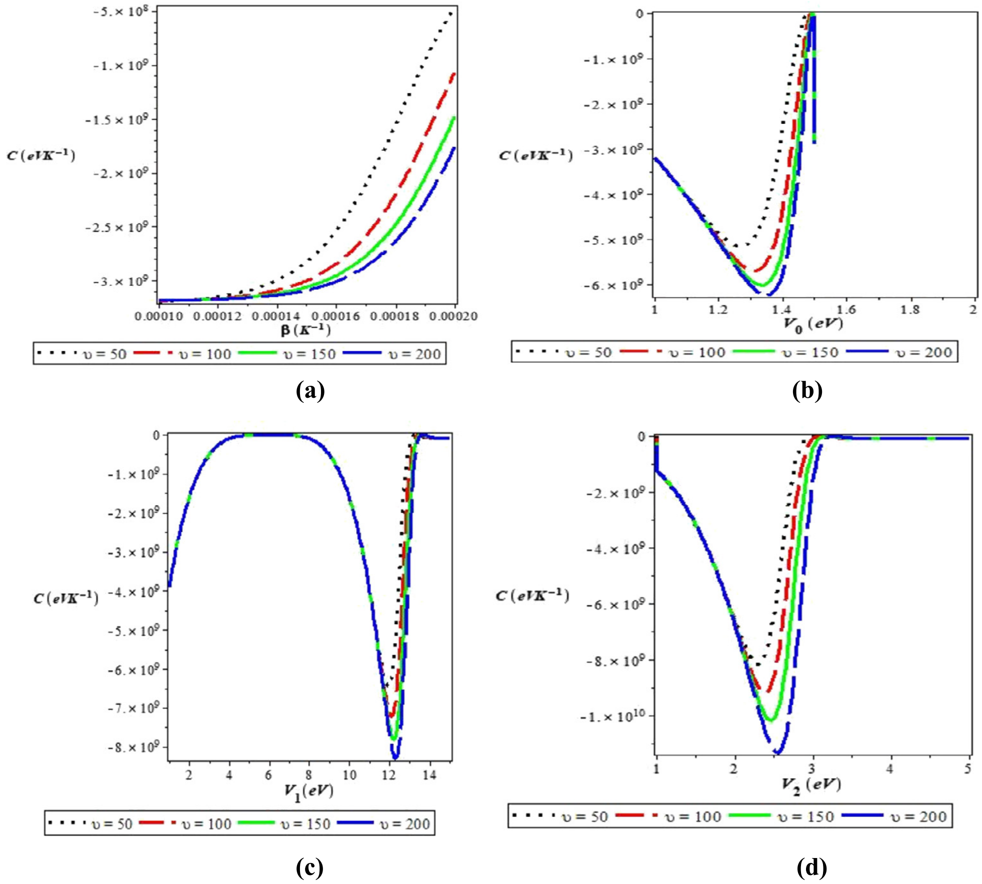

Fig. 7. Variation of the vibrational specific heat capacity of the MMSPE potential with (a) β (b) V_0 (c) V_1 (d) V_2.

6. Conclusions

In this paper, we obtained approximate solutions of Schrödinger equation for the modified Mobius square plus Eckart potential using the formula method. Special cases of this potential are also discussed and their respective energy eigenvalues computed numerically. Furthermore, the partition function and other thermodynamic properties of this potential were investigated. The partition function is found to converge to a constant value as the temperature parameter, β, increases, implying that probability of locating a particle confined in the MMSPE potential in an energy state, E_n, increases then tends to a constant value as β increases. Our results are consistent with reports in literature. The results of this study will be of interest in chemical physics, Nuclear physics, elementary particle physics and molecular physics.

Declarations

Author contribution statement

C.P. Onyenegecha, I.J. Njoku: Conceived and designed the experiments; Wrote the paper. O.K. Echendu, A.I. Opara: Performed the experiments; Wrote the paper. F.C. Eze, E.N. Omoko: Analyzed and interpreted the data; Wrote the paper. F.U. Nwaneho, E. Onyeocha, C.J. Okereke: Contributed reagents, materials, analysis tools or data; Wrote the paper.

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Data availability statement

No data was used for the research described in the article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

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