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Symmetry-Like Relation of Relative Entropy Measure of Quantum Coherence

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Abstract: Quantum coherence is an important physical resource in quantum information science, and also as one of the most fundamental and striking features in quantum physics. To quantify coherence, two proper measures were introduced in the literature, the one is the relative entropy of coherence $C_r(\rho) = S(\rho_{\text{diag}}) - S(\rho)$ and the other is the ℓ_1-norm of coherence $C_{\ell_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|$. In this paper, we obtain a symmetry-like relation of relative entropy measure $C_r(\rho^{A_1 A_2 \cdots A_n})$ of coherence for an n-partite quantum states $\rho^{A_1 A_2 \cdots A_n}$, which gives lower and upper bounds for $C_r(\rho)$. As application of our inequalities, we conclude that when each reduced states ρ^{A_i} is pure, $\rho^{A_1 \cdots A_n}$ is incoherent if and only if the reduced states ρ^{A_i} and $\text{tr}_{A_i} \rho^{A_1 \cdots A_n} (i = 1, 2, \ldots, n)$ are all incoherent. Meanwhile, we discuss the conjecture that $C_r(\rho) \leq C_{\ell_1}(\rho)$ for any state ρ, which was proved to be valid for any mixed qubit state and any pure state, and open for a general state. We observe that every mixture η of a state ρ satisfying the conjecture with any incoherent state σ also satisfies the conjecture. We also observe that when the von Neumann entropy is defined by the natural logarithm ln instead of $\log_2$, the reduced relative entropy measure of coherence $\bar{C}_r(\rho) = -\rho_{\text{diag}} \ln \rho_{\text{diag}} + \rho \ln \rho$ satisfies the inequality $\bar{C}_r(\rho) \leq C_{\ell_1}(\rho)$ for any state ρ.

Keywords: quantum coherence; measure; lower bound; upper bound

1. Introduction

Quantum computing utilizes the superposition and entanglement of quantum states to operate and process information. Its most significant advantage lies in the parallelism of operations [1–3]. To achieve efficient parallel computing in quantum computers, quantum coherence is essentially used. Quantum coherence arising from quantum superposition plays a central role in quantum mechanics and so becomes an important physical resource in quantum information and quantum computation [4]. It also plays an important role in a wide variety of research fields, such as quantum biology [5–10], nanoscale physics [11,12], and quantum metrology [13,14].

In 2014, Baumgratz et al. [15] proposed a framework to quantify coherence. In their seminal work, conditions that a suitable measure of coherence should satisfy have been put forward, including nonnegativity, the monotonicity under incoherent completely positive and trace preserving operations, the monotonicity under selective incoherent operations on average and the convexity under mixing of states. By introducing such a rigorous theoretical framework, a mass of properties and operations of quantification of coherence were discussed. Moreover, based on that framework, many coherence measures have been found, such as ℓ_1-norm of coherence and relative entropy of coherence [15], fidelity and trace norm distances for quantifying coherence [16], robustness of coherence [17], geometric measure of coherence [18], coherence of formation [19], relative quantum coherence [20], measuring coherence with entanglement concurrence [21], trace distance measure of coherence [22–24].

In addition, some research related to quantum coherence have been developed, including quantum coherence and quantum correlations [25–30], an uncertainly-like relation about coherence [31], distribution of quantum coherence in multipartite systems [32], quantum coherence over the noisy quantum channels [33], maximally coherent mixed states [34], ordering states with coherence measures [35], coherence and path information [36], complementarity relations for quantum coherence [37], converting coherence to quantum correlations [38] and logarithmic coherence [39], quantum coherence and geometric quantum discord [40]. Recently, Guo and Cao [41] discussed the question of creating quantum correlation from a coherent state via incoherent quantum operations and obtained explicit interrelations among incoherent operations (IOs), maximally incoherent operations, genuinely incoherent operations and coherence breaking operations.

In this paper, we discuss some inequalities on the measures of quantum coherence. The organization of this paper is as follows: In Section 2, we recall the framework of coherence measure and basic properties of quantum coherence. In Section 3, we establish lower and upper bounds for the relative entropy measure of coherence in a multipartite system. In Section 4, we discuss the relation between $C_r(\rho)$ and $C_{\ell_1}(\rho)$. In Section 5, we give our conclusions obtained in this paper.

2. Preliminaries

In this section, we give a review of some fundamental notions about quantification of coherence, such as incoherence states, incoherence operations, and measures of coherence.

Let H be a d-dimensional Hilbert space, whose elements are denoted by the Dirac notations $|\psi\rangle, |x\rangle$ and so on, and let $B(H)$ be the C^*-algebra consisting of all bounded linear operators on H. The adjoint operator of an operator T in $B(H)$ is denoted by $T^\dagger$. The identity operator on H is denoted by I_H, or simply, I. We use $D(H)$ to denote the set of all density operators (positive and trace-1 operators) on H, whose elements are said to be the states of the quantum system S described by H. Fixed an orthonormal basis (ONB) $e = \{|e_i\rangle\}_{i=1}^d$ for H, a state ρ of S is said to be *incoherent* with respect to (w.r.t.) the basis e if $\langle e_i|\rho|e_j\rangle = 0 (i \neq j)$. Otherwise, it is said to be *coherent* w.r.t. e. Let $\mathcal{I}(e)$ be the set of all states of S that are incoherent w.r.t. e, that is,

$$\mathcal{I}(e) = \left\{\rho \in D(H) : \langle e_i|\rho|e_j\rangle = 0 (i \neq j)\right\}.$$

For every $\rho \in D(H)$, we define

$$\rho_{e\text{-diag}} = \sum_{i=1}^{d} \langle e_i|\rho|e_i\rangle |e_i\rangle\langle e_i|.$$

Clearly, $\rho_{e\text{-diag}} \in \mathcal{I}(e)$. By definition, a state ρ is incoherent w.r.t e if and only if $\rho = \rho_{e\text{-diag}}$, i.e., it has a diagonal matrix representation w.r.t. e, i.e.,

$$\rho = \sum_{i=1}^{d} \lambda_i |e_i\rangle\langle e_i| \equiv \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{pmatrix},$$

where $\lambda_i \geq 0$ are eigenvalues of ρ and $\sum_{i=1}^{d} \lambda_i = \mathrm{tr}\rho = 1$; it is coherent w.r.t. e if and only if it can not be written as a diagonal matrix under this basis.

According to [42], a linear map $\mathcal{E}$ on the C^*-algebra $B(H)$ is a completely positive and trace preserving (CPTP) map if and only if there exists a set of operators $K_1, \ldots, K_m$ in $B(H)$ (called Kraus operators of $\mathcal{E}$) with $\sum_{n=1}^{m} K_n^\dagger K_n = I_H$ such that

$$\mathcal{E}(T) = \sum_{n=1}^{m} K_n T K_n{}^\dagger, \ \ \forall T \in B(H).$$

A CPTP map $\mathcal{E}$ on $B(H)$ is said to be an *e-incoherent operation (IO)* if it has Kraus operators $K_1, \ldots, K_m$ such that for all $n = 1, 2, \ldots, m$, it holds that

$$K_n \rho K_n{}^\dagger \in \text{tr}\left(K_n \rho K_n{}^\dagger\right) \mathcal{I}(e), \ \ \forall \rho \in \mathcal{I}(e).$$

In this case, we call $\{K_n\}_{n=1}^m$ a set of e-incoherent Kraus operators of $\mathcal{E}$.

In order to measure coherence, Baumgratz et al. [15] presented the following four defining conditions for a coherence measure C^e:

(A_1) $C^e(\rho) \geq 0$, $\forall \rho \in D(H)$; and $C^e(\rho) = 0$ if and only if $\rho \in \mathcal{I}(e)$.

(A_2) $C^e(\rho) \geq C^e(\mathcal{E}(\rho))$ for any e-incoherent operation $\mathcal{E}$ and any state $\rho \in D(H)$.

(A_3) $C^e(\rho) \geq \sum_n p_n C^e(\rho_n)$ for any e-incoherent operation $\mathcal{E}$ with a set of e-incoherent Kraus operators $\{K_n\}$ and any state $\rho \in D(H)$ where $\rho_n = p_n{}^{-1} K_n \rho K_n^\dagger$ with $p_n = \text{tr}(K_n \rho K_n^\dagger) \neq 0$.

(A_4) $\sum_i p_i C^e(\rho_i) \geq C^e(\sum_i p_i \rho_i)$ for any ensemble $\{p_i, \rho_i\}$.

It was proved in [15] that the relative entropy $C_r^e(\rho)$ and the ℓ_1-norm measure $C_{\ell_1}^e(\rho)$ of coherence satisfy these defining conditions, which are defined as follows:

$$C_r^e(\rho) = S(\rho_{e\text{-diag}}) - S(\rho), \tag{1}$$

where $S(\rho) = -\text{tr}(\rho \log \rho)$ is the von Neumann entropy, and

$$C_{\ell_1}^e(\rho) = \sum_{i \neq j} |\langle e_i|\rho|e_j\rangle|. \tag{2}$$

Notably, for a bipartite quantum system AB, the reference basis for $H_{AB} = H_A \otimes H_B$ can be taken as a local basis:

$$e_{AB} := e_A \otimes e_B = \{|e_i\rangle \otimes |f_k\rangle | i = 1, 2, \ldots, d_A, k = 1, 2, \ldots, d_B\},$$

where $e_A = \{|e_i\rangle\}_{i=1}^{d_A}$ and $e_B = \{|f_k\rangle\}_{k=1}^{d_B}$ are the orthonormal bases for H_A and H_B, respectively. In this case, every ρ^{AB} of AB has the following representation:

$$\rho^{AB} = \sum_{i,j=1}^{d_A} \sum_{k,l=1}^{d_B} \rho_{i,j,k,l} |e_i\rangle\langle e_j| \otimes |f_k\rangle\langle f_l|. \tag{3}$$

Put

$$\rho^{AB}_{e_{AB}\text{-diag}} = \sum_{i=1}^{d_A} \sum_{k=1}^{d_B} \rho_{i,i,k,k} |e_i\rangle\langle e_i| \otimes |f_k\rangle\langle f_k|. \tag{4}$$

Thus, a state ρ^{AB} of the system AB is incoherent w.r.t. e_{AB} if and only if $\rho^{AB} = \rho^{AB}_{e_{AB}\text{-diag}}$, i.e.,

$$\rho_{i,j,k,l} := \left\langle e_i | f_k | \rho^{AB} | e_j | f_l \right\rangle = 0((i,k) \neq (j,l)).$$

Moreover, let $\rho^A := \text{tr}_B(\rho^{AB})$ and $\rho^B := \text{tr}_A(\rho^{AB})$. Then from Equations (3) and (4), we get that

$$\rho^A_{e_A\text{-diag}} = \text{tr}_B(\rho^{AB}_{e_{AB}\text{-diag}}), \rho^B_{e_B\text{-diag}} = \text{tr}_A(\rho^{AB}_{e_{AB}\text{-diag}}). \tag{5}$$

In next section, we derive some inequalities, which give lower and upper bounds for the relative entropy of coherence of multi-partite states.

3. Lower and Upper Bounds for the Relative Entropy of Coherence

Xi et al. [30] proved that for any bipartite quantum state ρ^{AB}, the relative entropy of coherence obeys some uncertainty-like relation by using the properties of relative entropy, which reads

$$C_r^{e_{AB}}(\rho^{AB}) \geq C_r^{e_A}(\rho^A) + C_r^{e_B}(\rho^B), \tag{6}$$

where $\rho^A = \mathrm{tr}_B\rho^{AB}$, $\rho^B = \mathrm{tr}_A\rho^{AB}$.

Afterwards, Liu et al. [31] proved that any tripartite pure state ρ^{ABC} satisfies

$$C_r^{e_{ABC}}(\rho^{ABC}) \geq C_r^{e_{AB}}(\rho^{AB}) + C_r^{e_{AC}}(\rho^{AC}), \tag{7}$$

where $e_{ABC} := e_A \otimes e_B \otimes e_C, e_{AB} := e_A \otimes e_B, e_{AC} := e_A \otimes e_C$, $\rho^{AB} = \mathrm{tr}_C\rho^{ABC}$ and $\rho^{AC} = \mathrm{tr}_B\rho^{ABC}$, provided that

$$\lambda S(\rho^{AB}_{e\text{-diag}}) \leq S(\rho^{AB}), (1-\lambda)S(\rho^{AC}_{e\text{-diag}}) \leq S(\rho^{AC}) \tag{8}$$

for some $0 \leq \lambda \leq 1$. Combining Equations (6) and (7), the following inequality was derived in [31]:

$$C_r^{e_{ABC}}(\rho^{ABC}) \geq \frac{4}{3}\left(C_r^{e_A}(\rho^A) + C_r^{e_B}(\rho^B) + C_r^{e_C}(\rho^C)\right) \tag{9}$$

for a pure state ρ^{ABC} satisfying the condition (8).

The aim of this section is to establish lower and upper bounds of $C_r(\rho^{A_1A_2\cdots A_n})$ for a general n-partite state $\rho^{A_1A_2\cdots A_n}$. To do this, we use ρ^X_{diag} and $C_r(\rho^X)$ to denote $\rho^X_{e_X\text{-diag}}$ and $C_r^{e_X}(\rho^X)$, respectively.

First, for a bipartite ρ^{AB} of the system AB, we know from Equation (5) and the subadditivity of von Neumann entropy that

$$S(\rho^{AB}_{\mathrm{diag}}) \leq S(\mathrm{tr}_B\rho^{AB}_{\mathrm{diag}}) + S(\mathrm{tr}_A\rho^{AB}_{\mathrm{diag}}) = S(\rho^A_{\mathrm{diag}}) + S(\rho^B_{\mathrm{diag}})$$

and so

$$\begin{aligned}
& C_r(\rho^{AB}) - C_r(\rho^A) - C_r(\rho^B) \\
= \ & S(\rho^{AB}_{\mathrm{diag}}) - S(\rho^{AB}) - S(\rho^A_{\mathrm{diag}}) + S(\rho^A) - S(\rho^B_{\mathrm{diag}}) + S(\rho^B) \\
\leq \ & S(\rho^A_{\mathrm{diag}}) + S(\rho^B_{\mathrm{diag}}) - S(\rho^{AB}) - S(\rho^A_{\mathrm{diag}}) + S(\rho^A) - S(\rho^B_{\mathrm{diag}}) + S(\rho^B) \\
= \ & S(\rho^A) + S(\rho^B) - S(\rho^{AB}).
\end{aligned}$$

Thus,

$$C_r(\rho^{AB}) \leq C_r(\rho^A) + C_r(\rho^B) + S(\rho^A) + S(\rho^B) - S(\rho^{AB}).$$

Combing this with Equation (6), we have

$$\frac{1}{2}\left(2C_r(\rho^A) + 2C_r(\rho^B)\right) \leq C_r(\rho^{AB}) \leq C_r(\rho^A) + C_r(\rho^B) + S(\rho^A) + S(\rho^B) - S(\rho^{AB}). \tag{10}$$

Second, for a tripartite quantum state ρ^{ABC}, according to the super-additivity inequality (6), we have

$$C_r(\rho^{ABC}) \geq C_r(\rho^A) + C_r(\rho^{BC}),$$
$$C_r(\rho^{ABC}) \geq C_r(\rho^B) + C_r(\rho^{AC}),$$
$$C_r(\rho^{ABC}) \geq C_r(\rho^C) + C_r(\rho^{AB}).$$

By finding the sums of two sides of the inequalities above, we obtain

$$C_r(\rho^{ABC}) \geq \frac{1}{3}\left(C_r(\rho^{AB}) + C_r(\rho^{BC}) + C_r(\rho^{AC}) + C_r(\rho^{A}) + C_r(\rho^{B}) + C_r(\rho^{C})\right). \tag{11}$$

On the other hand, using definition (1) yields that

$$\begin{aligned}
& C_r(\rho^{ABC}) - C_r(\rho^{AB}) - C_r(\rho^{AC}) - C_r(\rho^{BC}) \\
= \; & S(\rho^{ABC}_{\text{diag}}) - S(\rho^{ABC}) - S(\rho^{AB}_{\text{diag}}) + S(\rho^{AB}) \\
& - S(\rho^{BC}_{\text{diag}}) + S(\rho^{BC}) - S(\rho^{AC}_{\text{diag}}) + S(\rho^{AC}) \\
= \; & \left[S(\rho^{ABC}_{\text{diag}}) + S(\rho^{B}_{\text{diag}}) - S(\rho^{AB}_{\text{diag}}) - S(\rho^{BC}_{\text{diag}})\right] \\
& + \left[S(\rho^{AC}) - S(\rho^{AC}_{\text{diag}})\right] + \left[S(\rho^{AB}) + S(\rho^{BC}) - S(\rho^{ABC}) - S(\rho^{B}_{\text{diag}})\right] \\
\leq \; & S(\rho^{A}) + S(\rho^{B}) + S(\rho^{B}) + S(\rho^{C}) - S(\rho^{B}_{\text{diag}}) - S(\rho^{ABC}) \\
\leq \; & S(\rho^{A}) + S(\rho^{B}) + S(\rho^{C}) - S(\rho^{ABC}),
\end{aligned}$$

since $S(\rho^{ABC}_{\text{diag}}) + S(\rho^{B}_{\text{diag}}) - S(\rho^{AB}_{\text{diag}}) - S(\rho^{BC}_{\text{diag}}) \leq 0$ (strong subadditivity) and $S(\rho^{AC}) - S(\rho^{AC}_{\text{diag}}) \leq 0$. This shows that

$$C_r(\rho^{ABC}) \leq C_r(\rho^{AB}) + C_r(\rho^{AC}) + C_r(\rho^{BC}) + S(\rho^{A}) + S(\rho^{B}) + S(\rho^{C}) - S(\rho^{ABC}). \tag{12}$$

Combining Equations (11) and (12) gives

$$\begin{aligned}
& \frac{1}{3}\left(C_r(\rho^{AB}) + C_r(\rho^{AC}) + C_r(\rho^{BC}) + C_r(\rho^{A}) + C_r(\rho^{B}) + C_r(\rho^{C})\right) \\
& \leq C_r(\rho^{ABC}) \\
& \leq C_r(\rho^{AB}) + C_r(\rho^{AC}) + C_r(\rho^{BC}) + S(\rho^{A}) + S(\rho^{B}) + S(\rho^{C}) - S(\rho^{ABC}).
\end{aligned} \tag{13}$$

As a generalization of inequalities (10) and (13), we can prove the following inequalities (14) for any n-partite state $\rho^{A_1\cdots A_n}$ of the system $H_{A_1A_2\cdots A_n} = H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_n}$, which give lower and upper bounds for the relative entropy of coherence. To do this, we let $e_{A_k} = \{|e^k_{i_k}\rangle\}_{i_k=1}^{d_k}$ be an orthogonal basis for the Hilbert space $H_{A_k} (k = 1, 2, \ldots, n)$, and let

$$e_{A_1A_2\cdots A_n} = \{|e^1_{i_1}\rangle|e^2_{i_2}\rangle \cdots |e^n_{i_n}\rangle : \; 1 \leq i_k \leq d_k (k = 1, 2, \ldots, n)\},$$

which is an orthogonal basis for the Hilbert space $H_{A_1A_2\cdots A_n}$. Thus,

$$e_{A_2\cdots A_n} = \{|e^2_{i_2}\rangle|e^3_{i_3}\rangle \cdots |e^n_{i_n}\rangle : \; 1 \leq i_k \leq d_k (k = 2, 3, \ldots, n)\}$$

becomes an orthogonal basis for the Hilbert space $H_{A_2A_3\cdots A_n} = H_{A_2} \otimes H_{A_3} \otimes \cdots \otimes H_{A_n}$. With these notations, we have the following.

Theorem 1. *For any state $\rho^{A_1\cdots A_n}$ of the system $H_{A_1A_2\cdots A_n} = H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_n}$, it holds that*

$$\frac{1}{n}\sum_{i=1}^{n}\left[C_r(\text{tr}_{A_i}\rho^{A_1\cdots A_n}) + C_r(\rho^{A_i})\right] \leq C_r(\rho^{A_1\cdots A_n}) \leq \sum_{i=1}^{n}\left[C_r(\text{tr}_{A_i}\rho^{A_1\cdots A_n}) + S(\rho^{A_i})\right] - S(\rho^{A_1\cdots A_n}), \tag{14}$$

where ρ^{A_i} denotes the reduced state of $\rho^{A_1\cdots A_n}$ on the subsystem A_i.

Proof. To prove that the first inequality in Equation (14) holds, we know from Equation (6) that

$$C_r(\rho^{A_1\cdots A_n}) \geq C_r(\rho^{A_1}) + C_r(\text{tr}_{A_1}\rho^{A_1\cdots A_n}),$$

$$C_r(\rho^{A_1\cdots A_n}) \geq C_r(\rho^{A_2}) + C_r(\mathrm{tr}_{A_2}\rho^{A_1\cdots A_n}),$$
$$\vdots$$
$$C_r(\rho^{A_1\cdots A_n}) \geq C_r(\rho^{A_n}) + C_r(\mathrm{tr}_{A_n}\rho^{A_1\cdots A_n}),$$

and consequently,

$$C_r(\rho^{A_1\cdots A_n}) \geq \frac{1}{n}\left(\sum_{i=1}^{n} C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n}) + \sum_{i=1}^{n} C_r(\rho^{A_i})\right).$$

Next, let us prove that the second inequality in (14) holds by using mathematical induction. Firstly, we know from Equation (10) that the desired inequality holds for $n = 2$ and any bipartite state. Secondly, we assume the second inequality in (14) holds for $n = N - 1$ and any $N - 1$-partite state. Then for any N-partite state $\rho^{A_1\cdots A_N}$, we have

$$\begin{aligned} C_r(\rho^{A_1\cdots A_N}) &= C_r(\rho^{A_1\cdots A_{N-2}(A_{N-1}A_N)}) \\ &\leq \sum_{i=1}^{N-2} C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_{N-2}(A_{N-1}A_N)}) + C_r(\mathrm{tr}_{A_{N-1}A_N}\rho^{A_1\cdots A_N}) \\ &\quad + \sum_{i=1}^{N-2} S(\rho^{A_i}) + S(\rho^{A_{N-1}A_N}) - S(\rho^{A_1\cdots A_{N-2}(A_{N-1}A_N)}). \end{aligned}$$

By using Equation (6), we know that $C_r(\mathrm{tr}_X\eta) \leq C_r(\eta)$. Thus,

$$C_r(\mathrm{tr}_{A_{N-1}A_N}\rho^{A_1\cdots A_N}) \leq C_r(\mathrm{tr}_{A_N}\rho^{A_1\cdots A_N}) \leq C_r(\mathrm{tr}_{A_{N-1}}\rho^{A_1\cdots A_N}) + C_r(\mathrm{tr}_{A_N}\rho^{A_1\cdots A_N}).$$

Combining the fact that

$$S(\rho^{A_{N-1}A_N}) \leq S(\rho^{A_{N-1}}) + S(\rho^{A_N}), S(\rho^{A_1\cdots A_{N-2}(A_{N-1}A_N)}) = S(\rho^{A_1\cdots A_N}),$$

we get that

$$\begin{aligned} C_r(\rho^{A_1\cdots A_N}) &\leq \sum_{i=1}^{N-2} C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_{N-2}(A_{N-1}A_N)}) + C_r(\mathrm{tr}_{A_{N-1}}\rho^{A_1\cdots A_N}) \\ &\quad + C_r(\mathrm{tr}_{A_N}\rho^{A_1\cdots A_N}) + \sum_{i=1}^{N-2} S(\rho^{A_i}) + S(\rho^{A_{N-1}}) + S(\rho^{A_N}) - S(\rho^{A_1\cdots A_N}) \\ &= \sum_{i=1}^{N} C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_N}) + \sum_{i=1}^{N} S(\rho^{A_i}) - S(\rho^{A_1\cdots A_N}). \end{aligned}$$

Thus, the validity of the second inequality in Equation (14) is proved. The proof is completed. □

As immediate application of Theorem 1, we have the following corollaries.

Corollary 1. *Let $\rho^{A_1\cdots A_n}$ be a state of the system $H_{A_1A_2\cdots A_n} = H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_n}$. If $\rho^{A_1\cdots A_n}$ is incoherent, then the reduced states ρ^{A_i} and $\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n} (i = 1, 2, \ldots, n)$ are all incoherent. The converse is true if each reduced states ρ^{A_i} is pure.*

Corollary 2. *Let $\rho^{A_1\cdots A_n}$ be a state of the system $H_{A_1A_2\cdots A_n} = H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_n}$ such that the reduced states $\rho^{A_i} (i = 1, 2, \ldots, n)$ are pure and incoherent. Then*

$$\frac{1}{n}\sum_{i=1}^{n} C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n}) \leq C_r(\rho^{A_1\cdots A_n}) \leq \sum_{i=1}^{n} C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n}). \tag{15}$$

It is remarkable that the equalities in Equation (14) may hold in some cases. For example, when $d_1 = d_2 = \cdots = d_n = d$ and

$$|\psi_{A_1\cdots A_n}\rangle = \left(\frac{1}{\sqrt{d}}\right)^n \sum_{i_1,i_2,\cdots,i_n=1}^{d} |e^1_{i_1}\rangle|e^2_{i_2}\rangle\cdots|e^n_{i_n}\rangle, \tag{16}$$

the maximally coherent state

$$\rho^{A_1A_2\cdots A_n} = |\psi_{A_1\cdots A_n}\rangle\langle\psi_{A_1\cdots A_n}|$$

satisfies

$$\frac{1}{n}\sum_{i=1}^{n}\left[C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n}) + C_r(\rho^{A_i})\right] = C_r(\rho^{A_1\cdots A_n}) = n\log_2 d,$$

due to the fact that $C_r(\rho^{A_j}) = \log_2 d$ for $j = 1, 2, \ldots, n$, and

$$C_r(\rho^{A_1\cdots A_n}) = -\frac{1}{d^n}\log_2\frac{1}{d^n}\times d^n = n\log_2 d,\ C_r(\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n}) = (n-1)\log_2 d.$$

Moreover, the second inequality in Equation (14) also becomes equality when $n = 2$. This shows that the inequalities in Equation (14) are tight and can not be improved.

4. The Relation between $C_r(\rho)$ and $C_{\ell_1}(\rho)$

In this section, we discuss the relation between $C_r(\rho)$ and $C_{\ell_1}(\rho)$. Rana et al. found that the inequality

$$C_r(\rho) \leq C_{\ell_1}(\rho) \tag{17}$$

holds for any mixed qubit state ([39], Proposition 1) and any pure state ([39], Proposition 3). Moreover, they conjectured that the inequality (17) holds for all states ρ. It was also proved ([39], Proposition 6) that inequality (17) holds for any state ρ of the form $\rho = p|\psi\rangle\langle\psi| + (1-p)\delta (0 \leq p \leq 1)$ provided that δ is an incoherent state w.r.t. the reference basis. As an extension of this result, we have the following.

Proposition 1. *Let ρ be a state of S satisfying Equation (17) and let σ be any incoherent state of S. Then every mixture $\eta := p\rho + (1-p)\sigma (0 \leq p \leq 1)$ of ρ and σ satisfies (17).*

Proof. The convexity of C_r implies that

$$\begin{aligned} C_r(\eta) &\leq pC_r(\rho) + (1-p)C_r(\sigma) \\ &= pC_r(\rho) \\ &\leq pC_{\ell_1}(\rho) \\ &= C_{\ell_1}(p\rho + (1-p)\sigma) \\ &= C_{\ell_1}(\eta). \end{aligned}$$

The proof is completed. □

Rana et al. proved in ([22], Proposition 6) that for arbitrary state ρ of a d-dimensional system, it holds that

$$C_r(\rho) \leq C_{\ell_1}(\rho)\log_2 d \tag{18}$$

and derived in ([39], Equation (10)) that

$$C_r(\rho) \leq \begin{cases} C_{\ell_1}(\rho), & \text{if } C_{\ell_1}(\rho) \geq 1; \\ C_{\ell_1}(\rho)\log_2 \mathrm{e}, & \text{if } C_{\ell_1}(\rho) < 1. \end{cases} \tag{19}$$

So, $C_r(\rho) \leq C_{\ell_1}(\rho)\log_2 \mathrm{e}$ for all ρ. Thus, if we redefine the von Neumann entropy as $\bar{S}(\rho) = -\mathrm{tr}(\rho\ln\rho)$, then the resulted relative entropy of coherence reads

$$\bar{C}_r(\rho) = \bar{S}(\rho_{\mathrm{diag}}) - \bar{S}(\rho) = \frac{1}{\log_2 \mathrm{e}} C_r(\rho). \tag{20}$$

This leads to the following inequality:

$$\bar{C}_r(\rho) \leq C_{\ell_1}(\rho), \ \ \forall \rho \in D(H). \tag{21}$$

5. Conclusions

In this paper, we have established lower and upper bounds for relative entropy of coherence $C_r(\rho^{A_1A_2\cdots A_n})$ for an n-partite quantum states $\rho^{A_1A_2\cdots A_n}$. As application of our inequalities, we have found that when each reduced states ρ^{A_i} is pure, $\rho^{A_1\cdots A_n}$ is incoherent if and only if the reduced states ρ^{A_i} and $\mathrm{tr}_{A_i}\rho^{A_1\cdots A_n}(i = 1, 2, \ldots, n)$ are all incoherent. Moreover, we have discussed the conjecture that $C_r(\rho) \leq C_{\ell_1}(\rho)$ for any state ρ and observed that every mixture η of a state ρ satisfying the conjecture with any incoherent state σ also satisfies the conjecture. We have also proved that when the von Neumann entropy is defined by the natural logarithm ln instead of $\log_2$, the reduced relative entropy measure of coherence $\bar{C}_r(\rho) = -\rho_{\mathrm{diag}}\ln\rho_{\mathrm{diag}} + \rho\ln\rho$ satisfies the inequality $\bar{C}_r(\rho) \leq C_{\ell_1}(\rho)$ for any state ρ.

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