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Nonstandard finite difference method for solving complex-order fractional Burgers' equations

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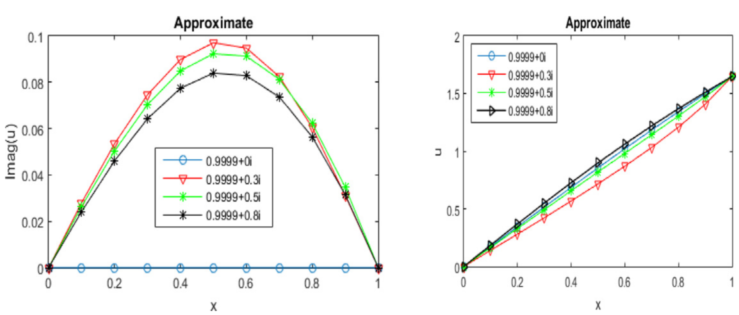
GRAPHICAL ABSTRACT



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ABSTRACT

The aim of this work is to present numerical treatments to a complex order fractional nonlinear one-dimensional problem of Burgers' equations. A new parameter σ_t is presented in order to be consistent with the physical model problem. This parameter characterizes the existence of fractional structures in the equations. A relation between the parameter σ_t and the time derivative complex order is derived. An unconditionally stable numerical scheme using a kind of weighted average nonstandard finite-difference discretization is presented. Stability analysis of this method is studied. Numerical simulations are given to confirm the reliability of the proposed method.

Introduction

It is known that the complex order fractional derivative is a generalization of fractional order derivative and the integer order derivative when the imaginary part of complex order is equal to zero [1]. In recent years, mathematical systems could be depicted suitability and more accurately by employing the fractional order derivative. There are several definitions for derivatives of fractional order. The most common is Caputo its have several applications [3]. More recently, Atangana-Baleanu Caputo sense (ABC) defined a modified Caputo fractional derivative by introducing generalized Mittag–Leffler function as the nonlocal and non-singular kernel [18]. These new type of derivatives have been used in modeling of real life applications in different fields ([4–7]). In order to a better understanding of some mistakes and limitations of the

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fractional classical mathematical models can be seen in the comment of Baleanu in [2]. Recently, in [20] Fernandez proposed the complex analysis approach to Atangana-Baleanu fractional calculus. The integer-order derivatives cannot describe systems with the effects of history memory and hereditary properties of materials and processes as fractional order derivatives and complex order fractional derivative [8–10]. In [10], Pinto and Carvalho presented a new mathematical model for complex order fractional model for HIV infection with drug resistance. They concluded that, the complex order fractional system has many advantages such as its dynamics are rich, moreover, the changes of the complex order derivative value can sheds a new light on the modeling of the intracellular delay. Also, in [22] the complex-order approximation to the forced van der Pol oscillator is proposed.

Burgers' equations can describe the communication between acoustic waves, reaction apparatuses, convection effects, heat conduction, diffusion transports, and modeling of dynamics, for more details see [11–14,16,17]. Several authors have investigated studied Burgers' model for various physical flow problem in fluid dynamics. The structure of Burgers' equation is roughly similar to that of Navier–Stokes equations due to the presence of the nonlinear convection term and the occurrence of the diffusion term with viscosity coefficient. So this equation can be considered as a simplified form of the Navier–Stokes equations. The one dimensional coupled Burgers' equation can be taken as a simple model of sedimentation and evolution of scaled volume of two kinds of particles in fluid, suspensions and colloids under the effect of gravity [15].

In this work, we present applications for the new definition of complex fractional order which given in [20], these applications are Burgers' equation with proportional delays in one-dimensional (1-D) and the coupled Burgers' equations in 1-D. In order to characterize the existence of complex fractional structure in the model, a parameter σ_t is added to the model problem [2]. A relation between σ_t and the complex order derivative $(\mu+\lambda i)$ is derived. Moreover, a numerical scheme is constructed using weighted average nonstandard finite-difference method (WANFDM) ([24–27]) to solve numerically the proposed equations.

To our knowledge the nonstandard finite difference method for solving complex-order fractional Burgers' equations was never explored before.

This paper is organized as follows: In Section 2, we explain some of the required mathematical concepts and preliminaries of complex fractional order derivatives. In Section 3, two complex order fractional Burgers' equations models are introduced and the construction of WANFDM to solve these equations. Moreover, the stability of this scheme is studied in Section 4. Numerical simulations for the proposed equations are given in Section 5. Finally, the conclusions are given in Section 6.

Preliminaries and notations

Let us consider the complex order fractional differentiation equation as follows:

$$ {}^{ABC}D_t^{\mu+\lambda i}y(t) = f(t,y(t)),\ \ 0<t\leqslant T,\quad (\mu+\lambda i)\in\mathbb{C}, \tag{1}$$

$y(0)=y_0.$

The Atangana-Baleanu fractional order derivative in Caputo sense (ABC) given is defined as follows [18]:

$$ {}^{ABC}D_t^{\mu}y(t) = \frac{M(\mu)}{(1-\mu)}\int_0^t E_\mu\left(-\mu\frac{(t-q)^\mu}{(1-\mu)}\right)\dot{y}(q)dq, \tag{2}$$

where, $0<\mu<1$, $M(\mu)=1-\mu+\frac{\mu}{\Gamma(\mu)}$ is normalization function, E_μ is Mittag–Leffler function, where, $E_\mu(Z)=\sum_{n=0}^{\infty}\frac{Z^n}{\Gamma(\mu n+1)}$, $Z\in\mathbb{C}$.

The Atangana-Baleanu complex order fractional derivative in Caputo sense is defined as follows [20]:

$$ {}^{ABC}D_t^{(\mu+\lambda i)}y(t) = \frac{M(\mu+\lambda i)}{2\pi i(1-(\mu+\lambda i))} \times\int_0^t E_{(\mu+\lambda i)}\left(-(\mu+\lambda i)\frac{(t-q)^{(\mu+\lambda i)}}{(1-(\mu+\lambda i))}\right)\dot{y}(q)dq, \tag{3}$$

where, $M(\mu+\lambda i)=1-(\mu+\lambda i)+\frac{(\mu+\lambda i)}{\Gamma(\mu+\lambda i)}$, $\mathrm{Re}(\mu+\lambda i)>0$ and $\Gamma(\mu+\lambda i)$ is the Stirling asymptotic formula of gamma function [21].

Numerical discretization for the ABC complex order derivatives

In this section we aim to construct WANFDM with ABC complex order fractional derivative to obtain the discretization of complex order fractional derivative numerically. Using (3) let $\alpha=(\mu+\lambda i)\in\mathbb{C}$. Then the discretization of complex order fractional derivative is given numerically as follows:

$$ {}^{ABC}D_t^{\alpha}u = \frac{M(\alpha)}{2\pi i(1-\alpha)}\int_0^{t_j}E_\alpha\left(\frac{-\alpha(t-s)^\alpha}{1-\alpha}\right)\frac{du(s)}{ds}ds, \tag{4}$$

$$ {}^{ABC}D_t^{\alpha}u = \frac{M(\alpha))}{2\pi i(1-\alpha)}\sum_{p=0}^{j-1}\int_{t_p}^{t_{p+1}}E_\alpha\left(\frac{-\alpha(t-s)^\alpha}{1-\alpha}\right)\frac{u_i^{j+1-p}-u_i^{j-p}}{\varphi(\triangle t)}ds,$$

$$ {}^{ABC}D_t^{\alpha}u = \frac{M(\alpha)}{2\pi i(1-\alpha)}\sum_{p=0}^{j-1}\frac{u_i^{j+1-p}-u_i^{j-p}}{\varphi(\triangle t)}\int_{t_p}^{t_{p+1}}E_\alpha\left(\frac{-\alpha(t_{j+1}-s)^\alpha}{1-\alpha}\right)ds,$$

$$ {}^{ABC}D_t^{\alpha}u = H\sum_{p=0}^{j-1}\frac{u_i^{j+1-p}-u_i^{j-p}}{\varphi(\triangle t)}\Theta_{p,j}, \tag{5}$$

where

$$H=\frac{M(\alpha)}{2\pi i(1-\alpha)},$$

$$\begin{aligned}\Theta_{p,j} &= \int_{t_p}^{t_{p+1}}E_\alpha\left(\tfrac{-\alpha(t_{j+1}-s)^\alpha}{1-\alpha}\right)ds\\ &=(t_{j+1}-t_{p+1})E_\alpha\left(\tfrac{-\alpha(t_{j+1}-t_{p+1})^\alpha}{1-\alpha}\right)-(t_{j+1}-t_p)E_\alpha\left(\tfrac{-\alpha(t_{j+1}-t_p)^\alpha}{1-\alpha}\right).\end{aligned}$$

Complex order fractional Burgers' equations

In the following, two nonlinear complex order fractional Burgers' models in 1-D are presented as follows:

1-D Burgers' equation

Consider the Burgers' equation in 1-D as follows ([12,23]):

$$u_t(t,x)+\lambda_1 u(t,x)u_x(t,x)+\mu_1 u(t,x)-\rho u_{xx}(t,x)=0, \tag{6}$$

with the initial and the boundary conditions given as follows:

$$u(t_0,x)=g(x),\ L_0\leqslant x\leqslant L,$$

$$u(t,L_0)=u(L,t)=f(t),\quad t>0,$$

where, $\lambda_1,\rho>0$ and μ_1 are constants, $u_t(t,x)$ is the variation term, $u(t,x)$ is the velocity component, ρ is diffusion coefficient, $u(t,x)u_x(t,x)$ is the nonlinear convective term and u_{xx} is the diffusion term, $g(x)$ and $f(t)$ are known functions. t_0 is the initial time.

In the following, the ordinary time derivative will be replaced by the complex order derivative.

$$\frac{d}{dt} \to \frac{d^{\mu+\lambda i}}{dt^{\mu+\lambda i}}. \tag{7}$$

It can be seen that (7) is not quite right, from a physical point of view, because the time derivative operator $\frac{d}{dt}$ has dimension of inverse time T^{-1}, while the fractional complex time derivative operator $\frac{d^{\mu+\lambda i}}{dt^{\mu+\lambda i}}$ has, $T^{-(\mu+\lambda i)}$. Now we introduce σ_t in the following way:

$$\frac{1}{\sigma_t^{1-(\mu+\lambda i)}}\frac{d^{\mu+\lambda i}}{dt^{\mu+\lambda i}} = (T)^{-1}. \tag{8}$$

In the case the expression (7) becomes an ordinary derivative operator $\frac{d}{dt}$ in case $\mu = 1, \lambda = 0$. In this way (7) is dimensionally consistent if and only if the new parameter σ_t , has dimension of time $[\sigma_t] = T$. Put ${}^{ABC}D_t^{(\mu+\lambda i)} = \frac{d^{\mu+\lambda i}}{dt^{(\mu+\lambda i)}}$, Now, we can write a fractional complex differential equation corresponding to the fractional complex order Burgers' equation in the following way:

$$\frac{1}{\sigma_t^{1-(\mu+\lambda i)}}\,{}^{ABC}D_t^{(\mu+\lambda i)}u(t,x) + \lambda_1 u(t,x)u_x(x,t) + \mu_1 u(t,x) - \rho u_{xx}(t,x) = 0, \tag{9}$$

put $\alpha = (\mu + \lambda i)$, then we can write (9) as follows:

$$\frac{1}{\sigma_t^{1-(\mu+\lambda i)}}\,{}^{ABC}D_t^{\alpha}u(t,x) + \lambda_1 u(t,x)u_x(t,x) + \mu_1 u(t,x) - \rho u_{xx}(t,x) = 0. \tag{10}$$

Using Eq. (10), the particular case can be obtained when $\rho = \lambda_1 = 0$,

$$\frac{1}{\sigma_t^{1-(\mu+\lambda i)}}\,{}^{ABC}D_t^{\alpha}u(t,x) + \mu_1 u(t,x) = 0. \tag{11}$$

By using the same steps in [28], the numerical solution of (11) when $\mu = 1,\ \ \lambda = 0$, i.e., $\alpha = 1$ is given as follows:

$$U = U_0 e^{\frac{-1}{\mu_1}t}. \tag{12}$$

In this case the relation between α and σ_t is given by [28]:

$$\alpha = \frac{\sigma_t}{\mu_1}, \quad 0 < \sigma_t \leqslant \frac{1}{\mu_1}.$$

1-D coupled Burgers' equations

Consider the complex order coupled Burgers' equations in 1-D as follows:

$$\begin{aligned}
&\frac{1}{\sigma_t^{1-\alpha}}\,{}^{ABC}D_t^{\alpha}u(t,x) + \lambda_1 u(t,x)u_x(t,x) + \beta_1 \tfrac{\partial}{\partial x}(u(t,x)v(t,x)) = \rho u_{xx}(t,x),\\
&\frac{1}{\sigma_t^{1-\alpha}}\,{}^{ABC}D_t^{\alpha}v(t,x) + \lambda_2 v(t,x)v_x(t,x) + \beta_2 \tfrac{\partial}{\partial x}(u(t,x)v(t,x)) = \rho v_{xx}(t,x),\\
&\quad \alpha \in \mathbb{C},
\end{aligned} \tag{13}$$

with the initial conditions:

$$u(t_0,x) = g_1(x),\ \ v(t_0,x) = g_2(x),\ \ L_0 \leqslant x \leqslant L,$$

and the boundary conditions:

$$u(t,L_0) = u(t,L) = f_1(t),\ \ v(t,L_0) = v(t,L) = f_2(t), \quad t > 0.$$

Where $\lambda_1, \lambda_2, \beta_1$ and β_2 are constants, $u(t,x)$ and $v(t,x)$ are the velocity components, $g_1(x),\ g_2(x)$,

$f_1(t,x)$ and $f_2(t,x)$ are known functions and t_0 is the initial time. This coupled equation found in [15] when $\lambda = 0$.

Construction of WANFDM

In the following, we aim to construct WANFDM in order to obtain the discretization of the model problems.

1-D complex fractional order Burgers' equation

The discretization of 1-D complex fractional order Burgers' Eq. (6) and the nonstandard finite differences approximation can be claimed as follows:

$$\begin{aligned}
&\frac{1}{\sigma_t^{1-\alpha}}H\sum_{p=0}^{j-1}\frac{u_i^{j+1-p}-u_i^{j-p}}{\varphi(\triangle t)}\Theta_{p,j}\\
&\quad+(1-\theta)\left[\lambda_1 u_i^{j+1}\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\psi(\triangle x)} + \mu_1 u_i^{j+1} - \rho\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{\psi(\triangle x)^2}\right]\\
&\quad+\theta\left[\lambda_1 u_i^{j+1}\frac{u_{i+1}^{j-1}-u_i^{j-1}}{\psi(\triangle x)} + \mu_1 u_i^{j-1} - \rho\frac{u_{i+1}^{j-1}-2u_i^{j-1}+u_{i-1}^{j-1}}{\psi(\triangle x)^2}\right] = R.
\end{aligned} \tag{14}$$

Where $(j = 0,\ 1,\ 2,\ldots,\ N,\quad i = 0,\ 1,\ 2,\ldots,\ M)$ and R is the truncation error. Neglecting the truncation error, the resulting computable difference scheme takes the form:

$$\begin{aligned}
&\frac{1}{\sigma_t^{1-\alpha}}H\sum_{p=0}^{j-1}\frac{u_i^{j+1-p}-u_i^{j-p}}{\varphi(\triangle t)}\Theta_{p,j}\\
&\quad+(1-\theta)\left[\lambda_1 u_i^{j+1}\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\psi(\triangle x)} + \mu_1 u_i^{j+1} - \rho\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{\psi(\triangle x)^2}\right]\\
&\quad+\theta\left[\lambda_1 u_i^{j+1}\frac{u_{i+1}^{j-1}-u_i^{j-1}}{\psi(\triangle x)} + \mu_1 u_i^{j-1} - \rho\frac{u_{i+1}^{j-1}-2u_i^{j-1}+u_{i-1}^{j-1}}{\psi(\triangle x)^2}\right] = 0.
\end{aligned} \tag{15}$$

1-D complex fractional order coupled Burgers' equation

The discretization form of 1-D complex fractional order coupled Burgers' Eqs. (13) given as follows:

$$\begin{aligned}
&\frac{1}{\sigma_t^{1-\alpha}}\ H\sum_{p=0}^{j-1}\frac{u_i^{j+1-p}-u_i^{j-p}}{\varphi(\triangle t)}\Theta_{p,j} + (1-\theta)\left[(\lambda_1 u_i^{j+1} + \beta_1 v_i^{j+1})\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\psi(\triangle x)}\right.\\
&\quad\left.+\beta_1 u_i^{j+1}\frac{v_{i+1}^{j+1}-v_i^{j+1}}{\psi(\triangle x)} - \rho\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{\psi(\triangle x)^2}\right] + \theta\left[(\lambda_1 u_i^{j-1} + \beta_1 v_i^{j-1})\right.\\
&\quad\left.\frac{u_{i+1}^{j-1}-u_i^{j-1}}{\psi(\triangle x)} + \beta_1 u_i^{j-1}\frac{v_{i+1}^{j-1}-v_i^{j-1}}{\psi(\triangle x)} - \rho\frac{u_{i+1}^{j-1}-2u_i^{j-1}+u_{i-1}^{j-1}}{\psi(\triangle x)^2}\right] = R_{1,i}^{j},
\end{aligned}$$

where $(j = 0,\ 1,\ 2,\ldots,\ N,\quad i = 0,\ 1,\ 2,\ldots,\ M)$.

$$\begin{aligned}
&\frac{1}{\sigma_t^{1-\alpha}}H\sum_{p=0}^{j-1}\frac{v_i^{j+1-p}-v_i^{j-p}}{\varphi(\triangle t)}\Theta_{p,j} + (1-\theta)\left[(\lambda_2 v_i^{j+1} + \beta_2 u_i^{j+1})\frac{v_{i+1}^{j+1}-v_i^{j+1}}{\psi(\triangle x)}\right.\\
&\quad\left.+\beta_2 v_i^{j+1}\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\psi(\triangle x)} - \rho\frac{v_{i+1}^{j+1}-2v_i^{j+1}+v_{i-1}^{j+1}}{\psi(\triangle x)^2}\right] + \theta\left[(\lambda_2 v_i^{j-1} + \beta_2 u_i^{j-1})\right.\\
&\quad\left.\frac{v_{i+1}^{j-1}-v_i^{j-1}}{\psi(\triangle x)} + \beta_2 v_i^{j-1}\frac{u_{i+1}^{j-1}-u_i^{j-1}}{\psi(\triangle x)} - \rho\frac{v_{i+1}^{j-1}-2v_i^{j-1}+v_{i-1}^{j-1}}{\psi(\triangle x)^2}\right] = R_{2,i}^{j}.
\end{aligned} \tag{16}$$

Where $R_{1,i}^{j}$ and $R_{2,i}^{j}$ are the truncation errors. Neglecting the truncation errors, the resulting computable difference scheme takes the form:

$$\begin{aligned}
&\frac{1}{\sigma_t^{1-\alpha}}H\sum_{p=0}^{j-1}\frac{u_i^{j+1-p}-u_i^{j-p}}{\varphi(\triangle t)}\Theta_{p,j} + (1-\theta)\left[(\lambda_1 u_i^{j+1} + \beta_1 v_i^{j+1})\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\psi(\triangle x)}\right.\\
&\quad\left.+\beta_1 u_i^{j+1}\frac{v_{i+1}^{j+1}-v_i^{j+1}}{\psi(\triangle x)} - \rho\frac{u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1}}{\psi(\triangle x)^2}\right] + \theta\left[(\lambda_1 u_i^{j-1} + \beta_1 v_i^{j-1})\right.\\
&\quad\left.\frac{u_{i+1}^{j-1}-u_i^{j-1}}{\psi(\triangle x)} + \beta_1 u_i^{j-1}\frac{v_{i+1}^{j-1}-v_i^{j-1}}{\psi(\triangle x)} - \rho\frac{u_{i+1}^{j-1}-2u_i^{j-1}+u_{i-1}^{j-1}}{\psi(\triangle x)^2}\right] = 0,
\end{aligned}$$

$$\frac{1}{\sigma_t^{1-\alpha}}H\sum_{p=0}^{j-1}\frac{v_i^{j+1-p}-v_i^{j-p}}{\varphi(\triangle t)}\Theta_{p,j}+(1-\theta)\Big[(\lambda_2 v_i^{j+1}+\beta_2 u_i^{j+1})\frac{v_{i+1}^{j+1}-v_i^{j+1}}{\psi(\triangle x)}$$
$$+\beta_2 v_i^{j+1}\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\psi(\triangle x)}-\rho\frac{v_{i+1}^{j+1}-2v_i^{j+1}+v_{i-1}^{j+1}}{\psi(\triangle x)^2}\Big]+\theta\Big[(\lambda_2 v_i^{j-1}+\beta_2 u_i^{j-1})$$
$$\frac{v_{i+1}^{j-1}-v_i^{j-1}}{\psi(\triangle x)}+\beta_2 v_i^{j-1}\frac{u_{i+1}^{j-1}-u_i^{j-1}}{\psi(\triangle x)}-\rho\frac{v_{i+1}^{j-1}-2v_i^{j-1}+v_{i-1}^{j-1}}{\psi(\triangle x)^2}\Big]=0. \tag{17}$$

Stability analysis for the WANSFDM for solving Burgers' models

Stability analysis for the WANSFDM for solving 1-D Burgers' equation

In the following, we used the idea of Jon von Neumann technique to claim the stability of (15), ([25,26]). This idea will be applied after linearizing (10). Assume that $u_i^j=\xi^j e^{i\gamma q\triangle x}$, where $\gamma=\sqrt{-1}$, the requirement is $|\xi(q)|\leqslant 1$, then (15) will be written as follows:

$$\begin{aligned}&\frac{1}{\sigma_t^{1-\alpha}}H\sum_{p=0}^{j-1}\frac{\xi^{j+1-p}e^{i\gamma q\triangle x}-\xi^{j-p}e^{i\gamma q\triangle x}}{\varphi(\triangle t)}\Theta_{p,j}\\&+(1-\theta)\Big[\mu_1\xi^{j+1}e^{i\gamma q\triangle x}-\frac{\rho}{\psi(\triangle x)^2}(\xi^{j+1}e^{(i+1)\gamma q\triangle x}\\&-2\xi^{j+1}e^{i\gamma q\triangle x}+\xi^{j+1}e^{(i-1)\gamma q\triangle x})\Big]\\&+\theta\Big[\mu_1\xi^{j-1}e^{i\gamma q\triangle x}-\frac{\rho}{\psi(\triangle x)^2}(\xi^{j-1}e^{(i+1)\gamma q\triangle x}\\&-2\xi^{j-1}e^{i\gamma q\triangle x}+\xi^{j-1}e^{(i-1)\gamma q\triangle x})\Big]=0.\end{aligned}\tag{18}$$

Diving by $\xi^j e^{i\gamma q\triangle x}$, put $\eta=\frac{\xi^{j+1}}{\xi^k}$, and using the Euler formula we have:

$$\frac{1}{\sigma_t^{1-\alpha}}H\sum_{p=0}^{j-1}\frac{\xi^{-p}(\eta-1)}{\varphi(\triangle t)}\Theta_{p,j}\quad+\eta(1-\theta)\Big[\mu_1-\frac{2\rho}{\psi(\triangle x)^2}(cos(q\triangle x)-1)\Big]$$
$$+\eta^{-1}\theta\Big[\mu_1-\frac{2\rho}{\psi(\triangle x)^2}(cos(q\triangle x)-1)\Big]=0. \tag{19}$$

Assume

$$\sum_{p=0}^{j-1}(\xi^{-p}\delta)/\varphi(\triangle t)=\sum_{p=0}^{j-1}(\eta^{-p}\delta)/\varphi(\triangle t)=A_o,$$

$$\frac{1}{\sigma_t^{1-\alpha}}HA_o\eta-\frac{1}{\sigma_t^{1-\alpha}}HA_o+B\eta+C\eta^{-1}=0, \tag{20}$$

where, $B=(1-\theta)\Big[\mu_1-\frac{2\rho}{\psi(\triangle x)^2}(cos(q\triangle x)-1)\Big]$ and $C=\theta\Big[\mu_1-\frac{2\rho}{\psi(\triangle x)^2}(cos(q\triangle x)-1)\Big]$.

$$\left(\frac{1}{\sigma_t^{1-\alpha}}HA_o+B\right)\eta^2-\frac{1}{\sigma_t^{1-\alpha}}HA_o\eta+C=0,|\eta|\le 1, \tag{21}$$

$$|\eta_1|=\frac{\left|\frac{1}{\sigma_t^{1-\alpha}}HA_o-\sqrt{(\frac{1}{\sigma_t^{1-\alpha}}HA_o)^2-4(\frac{1}{\sigma_t^{1-\alpha}}HA_o+B)C}\right|}{\left|2(\frac{1}{\sigma_t^{1-\alpha}}HA_o+B)\right|}\le 1,$$

then,

$$\left|\frac{1}{\sigma_t^{1-\alpha}}HA_o-\sqrt{(\frac{1}{\sigma_t^{1-\alpha}}HA_o)^2-4(\frac{1}{\sigma_t^{1-\alpha}}HA_o+B)C}\right|\leqslant|2(HA_o+B)|.$$

$$|\eta_2|=\frac{\left|\frac{1}{\sigma_t^{1-\alpha}}HA_o+\sqrt{(\frac{1}{\sigma_t^{1-\alpha}}HA_o)^2-4(\frac{1}{\sigma_t^{1-\alpha}}HA_o+B)C}\right|}{|2(HA_o+B)|}\le 1,$$

where,

$$\left|\frac{1}{\sigma_t^{1-\alpha}}HA_o+\sqrt{\left(\frac{1}{\sigma_t^{1-\alpha}}HA_o\right)^2-4\left(\frac{HA_o}{\sigma_t^{1-\alpha}}+B\right)C}\right|\leqslant\left|2\left(\frac{1}{\sigma_t^{1-\alpha}}HA_o+B\right)\right|.$$

Stability analysis for the WANSFDM for solving 1-D coupled Burgers' equation

We consider the stability analysis for the WANFDM for solving system (17), we used the kind of Jon von Neumann technique. We will apply this technique after linearized the system (13), we write this system in matrix form as follows:

$${}^{ABC}D_t^{\alpha}X(t,x)=Y\frac{\partial^2}{\partial x^2}(X), \tag{22}$$

where,

$X=\begin{pmatrix}u(t,x)\\v(t,x)\end{pmatrix}$ and $Y=\begin{pmatrix}\rho&0\\0&\rho\end{pmatrix}$.

Then we can write system (22) using WANFDM as follows [24]:

$$\frac{1}{\sigma_t^{1-\alpha}}H\sum_{p=0}^{j-1}\frac{X_i^{j+1-p}-X_i^{j-p}}{\varphi(\triangle t)}\Theta_{p,j}+(1-\theta)\left[-\rho\frac{X_{i+1}^{j+1}-2X_i^{j+1}+X_{i-1}^{j+1}}{\psi(\triangle x)^2}\right]$$
$$+\theta\left[-\rho\frac{X_{i+1}^{j-1}-2X_i^{j-1}+X_{i-1}^{j-1}}{\psi(\triangle x)^2}\right]=0, \tag{23}$$

As in the Jon von Neumann stability we assume that:

$X_i^j=\xi^j\Upsilon e^{i\gamma q\triangle x}$,

where $\gamma=\sqrt{-1},\Upsilon\in\mathbb{R}^{2\times1}$ and $\xi\in\mathbb{R}^{2\times2}$ is the amplification matrix. By substituting into (23) and using the Euler formula, we have:

$$\left(A_1-\frac{1}{\sigma_t^{1-\alpha}}HB_1\right)\xi^2+\frac{1}{\sigma_t^{1-\alpha}}HB_1\xi+C_1I=0, \tag{24}$$

where,

I is the unit matrix, $A_1=(1-\theta)\frac{2\rho}{\psi(\triangle x)^2}(cos(q\triangle x)-1)$, $B_1=\sum_{p=0}^{j-1}(\xi^{-p}\delta)/\varphi(\triangle t)$, and.

$C_1=\theta\frac{2\rho}{\psi(\triangle x)^2}(cos(q\triangle x)-1)$.

The system will be stable as long as $|\xi(q)|\leqslant 1$.

$$|\xi_1|=\frac{\left|-\frac{1}{\sigma_t^{1-\alpha}}HB_1+\sqrt{(\frac{1}{\sigma_t^{1-\alpha}}HB_1)^2-4C_1(A_1-\frac{1}{\sigma_t^{1-\alpha}}HB_1)}\right|}{\left|2(A_1-\frac{1}{\sigma_t^{1-\alpha}}HB_1)\right|}\le 1,$$

where,

$$\left|-\frac{1}{\sigma_t^{1-\alpha}}HB_1+\sqrt{(\frac{1}{\sigma_t^{1-\alpha}}HB_1)^2-4C_1(A_1-\frac{1}{\sigma_t^{1-\alpha}}HB_1)}\right|$$
$$\leqslant\left|2(A_1-\frac{1}{\sigma_t^{1-\alpha}}HB_1)\right|,$$

$$|\xi_2|=\frac{\left|-\frac{1}{\sigma_t^{1-\alpha}}HB_1-\sqrt{(\frac{1}{\sigma_t^{1-\alpha}}HB_1)^2-4C_1(A_1-\frac{1}{\sigma_t^{1-\alpha}}HB_1)}\right|}{\left|2(A_1-\frac{1}{\sigma_t^{1-\alpha}}HB_1)\right|}\le 1,$$

where,

$$\left|-\frac{1}{\sigma_t^{1-\alpha}}HB_1-\sqrt{(\frac{1}{\sigma_t^{1-\alpha}}HB_1)^2-4C_1(A_1-\frac{1}{\sigma_t^{1-\alpha}}HB_1)}\right|$$
$$\leqslant\left|2(A_1-\frac{1}{\sigma_t^{1-\alpha}}HB_1)\right|.$$

Application of WNFDM for complex order derivative

This section deals with the effectiveness and validity of the proposed method for solving the test problem of complex fractional order Burgers' models.

Example 1. The complex order fractional Burgers' equation with proportional delay a, c [19]:

$$\frac{1}{-}\sigma_t^{1-(\mu+\lambda i)}{}^{ABC}D_t^{\mu+\lambda i}u(t,x) - u(ct,ax)u_x(ct,x) + \frac{1}{2}u(t,x) - u_{xx}(t,x) = 0. \quad (25)$$

$x \in [0, L],\ t \in [0, 1],\ a, c \in]0, 1[$

The initial and boundary conditions are given as follows:

$u(0, x) = x.$

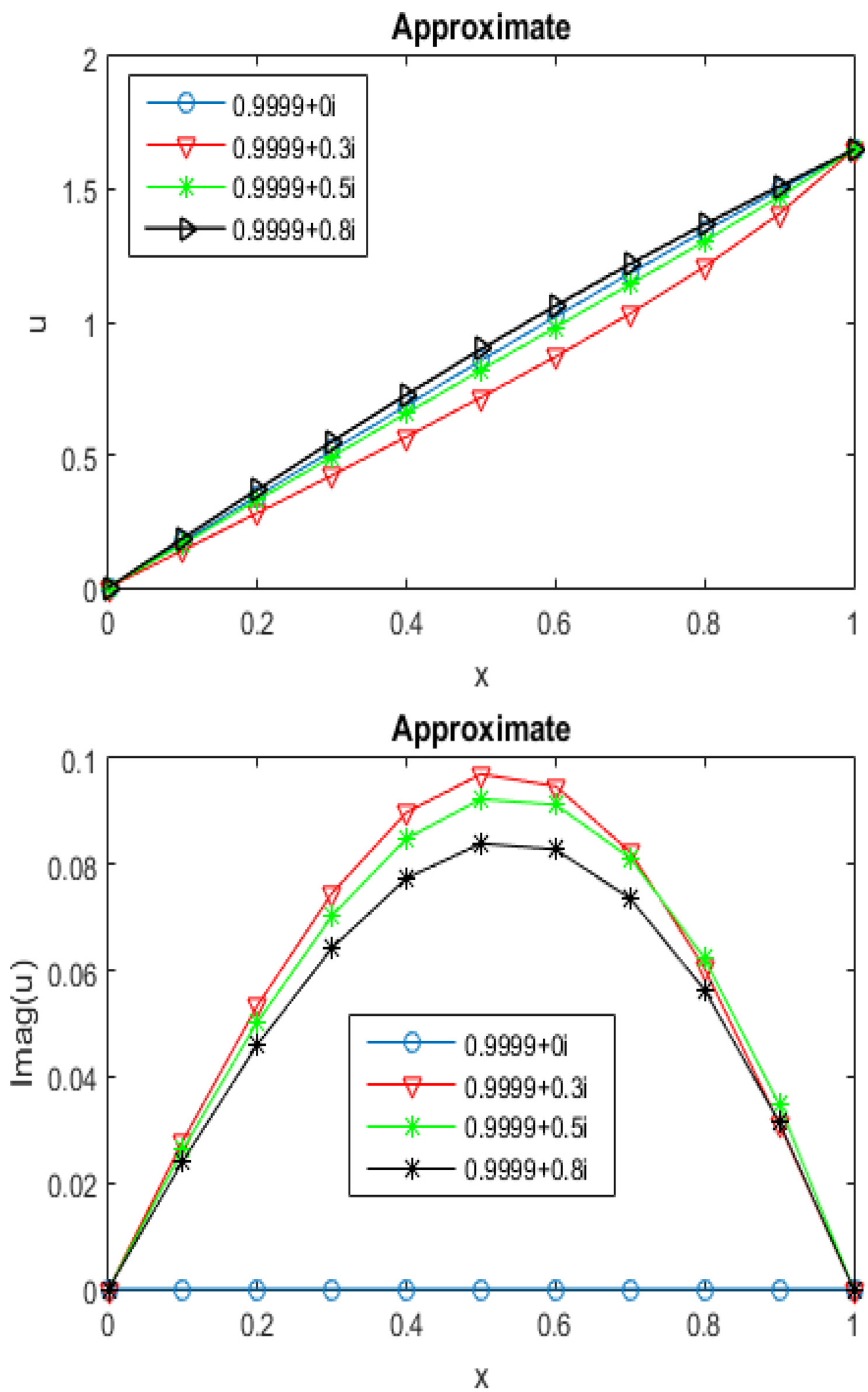

Fig. 1. Numerical simulations for the Example 1 at different values of imaginary part, $\theta = 0$.

$u(t, 0) = u(t, L) = 0.$

The exact solution is $u(t, x) = xe^t$ when $a = c = 0.5$ and $\mathrm{Re}(\alpha) \simeq 1$. Taking $\psi(h) = q(e^h - 1)$ and $\varphi(\triangle t) = p(e^{\triangle t} - 1)$, where $0 < q \le 1, 0 < p \le 1$ and $0 < \sigma_t \leqslant 2$.

The proposed numerical scheme (15), together with the boundary conditions and the initial condition yield a nonlinear algebraic system of $(N+1)(M+1)$ equation with the unknown u_i^j $(j = 0, 1, 2, \ldots, N, \quad i = 0, 1, 2, \ldots, M)$. This system will be solved in this work using Newton's iteration methods. The following are noted:

Fig. 1 shows that the behavior for the solution at different values of imaginary part and the value of real part equal 0.999. We compare the obtained solutions with the solution in the case $\mu = 0.999$ and $\lambda = 0$. Fig. 2 illustrates the behavior of the numerical solution at different values of the real part and the value of

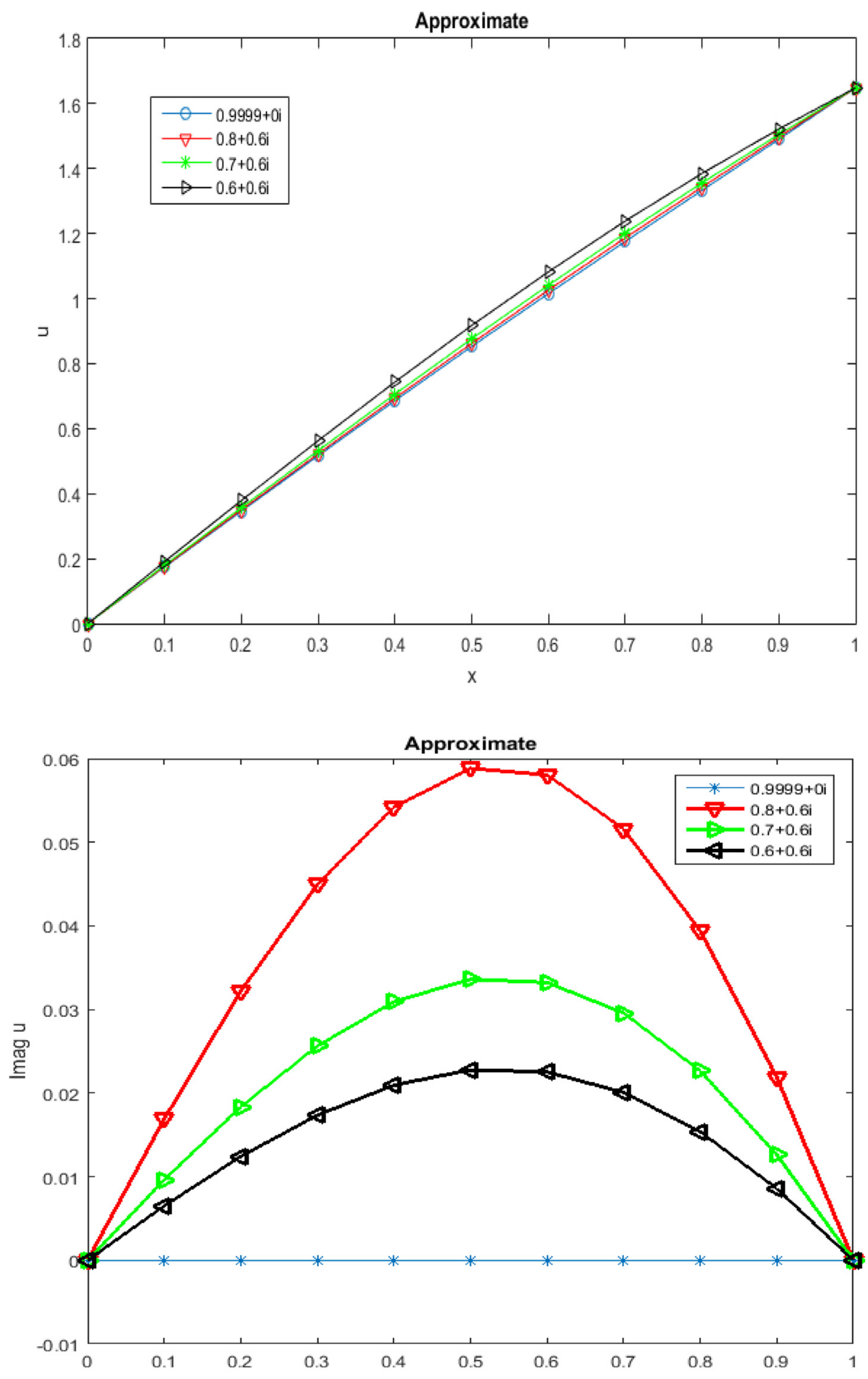

Fig. 2. Numerical simulations for the Example 1 at different values of the real part, $\theta = 0$.

imaginary part equal 0.6. We compare the obtained solutions with the a solution in case $\mu = 0.999$ and $\lambda = 0$. We noted that a new behavior appears that are not seeing in case of integer and fractional order models.

Example 2. Consider the following fractional complex order coupled Burgers' equations in 1-D as follows:

$$\begin{aligned}
&\frac{1}{\sigma_t^{1-(\mu+\lambda i)}} {}^{ABC}D_t^{\mu+\lambda i} u(t,x) + 2u(t,x)u_x(t,x) + \frac{\partial}{\partial x}(u(t,x)v(t,x)) - u_{xx}(t,x) = 0,\\
&\frac{1}{\sigma_t^{1-(\mu+\lambda i)}} {}^{ABC}D_t^{\mu+\lambda i} v(t,x) + 2v(x,t)v_x(t,x) + \frac{\partial}{\partial x}(u(t,x)v(t,x)) - v_{xx}(t,x) = 0,
\end{aligned} \quad (26)$$

with the initial conditions:

$u(t_0,x) = sin(x),\ \ v(t_0,x) = sin(x),\ \ 0 \leqslant x \leqslant \pi,$

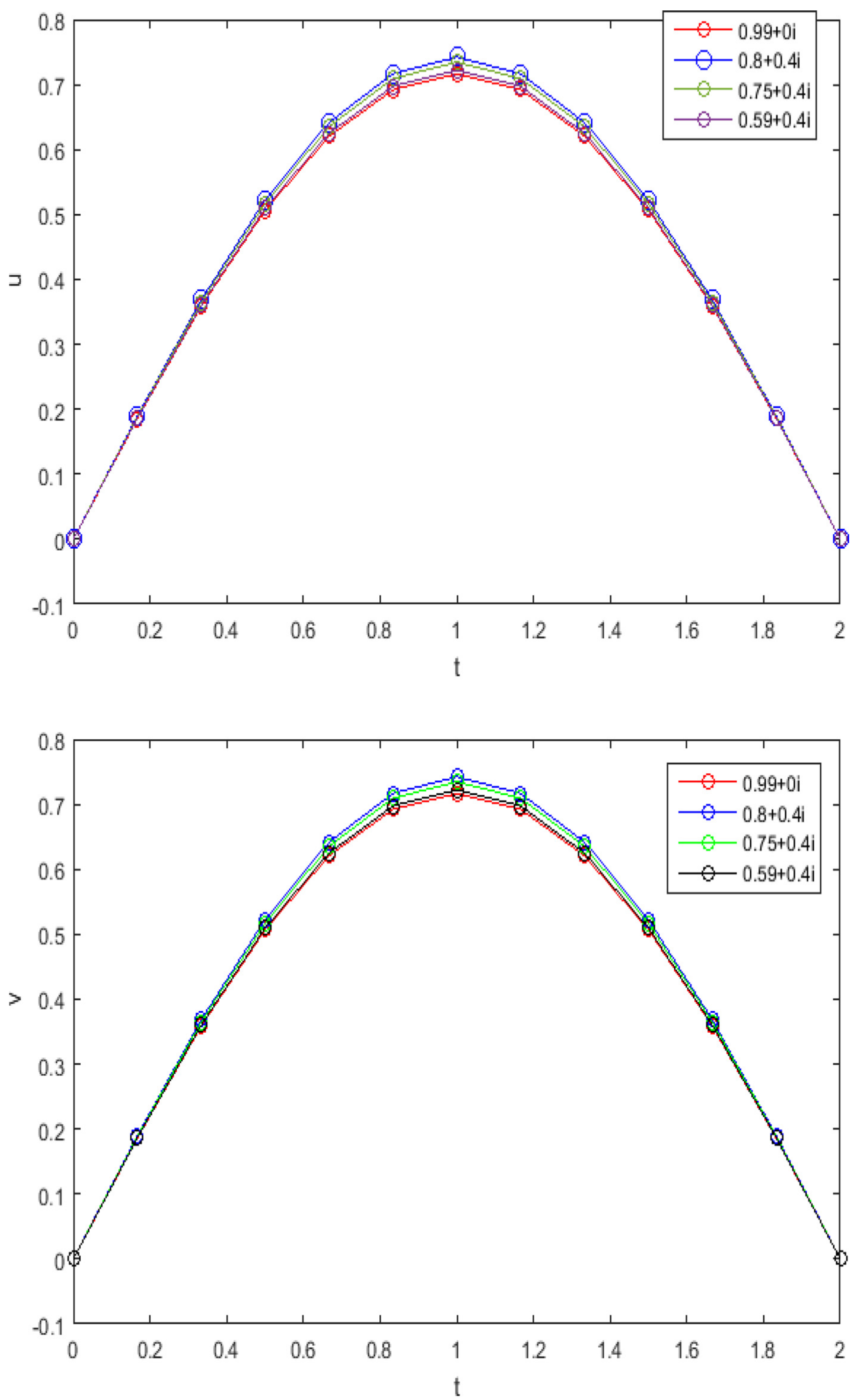

Fig. 3. Numerical simulations for Example 2 at different values of Real part, $\theta = 0.5$.

and the boundary conditions:

$$u(t,0) = u(t,\pi) = 0, \;\; v(0,t) = v(t,\pi) = 0, \quad t > 0.$$

The exact solutions of velocity components are $u(x,t) = e^{-t}sin(x)$ and $v(x,t) = e^{-t}sin(x)$, when $\mathrm{Re}(\alpha) \simeq 1$. Taking $\psi(\triangle x) = q\, sinh(\triangle x)$ and $\varphi(\triangle t) = p\, sinh(\triangle t)$, where $0 < q \le 1$ and $0 < p \le 1$. The proposed numerical scheme (17), together with the boundary conditions and the initial condition construct a nonlinear algebraic system of $(N+1)(M+1)$ equation with the unknown u_i^j, v_j^i, $(j = 0, 1, 2, \ldots, N, \quad i = 0, 1, 2, \ldots, M)$. This system will be solved in this work using Newton's iteration methods. We have the following observations:

Fig. 3 illustrates the behavior of the numerical solution u and v at different values of the real part and the value of imaginary part is equal to 0.4. We compared the obtained solutions with the approximated integer order solution. Fig. 4 shows that the behavior

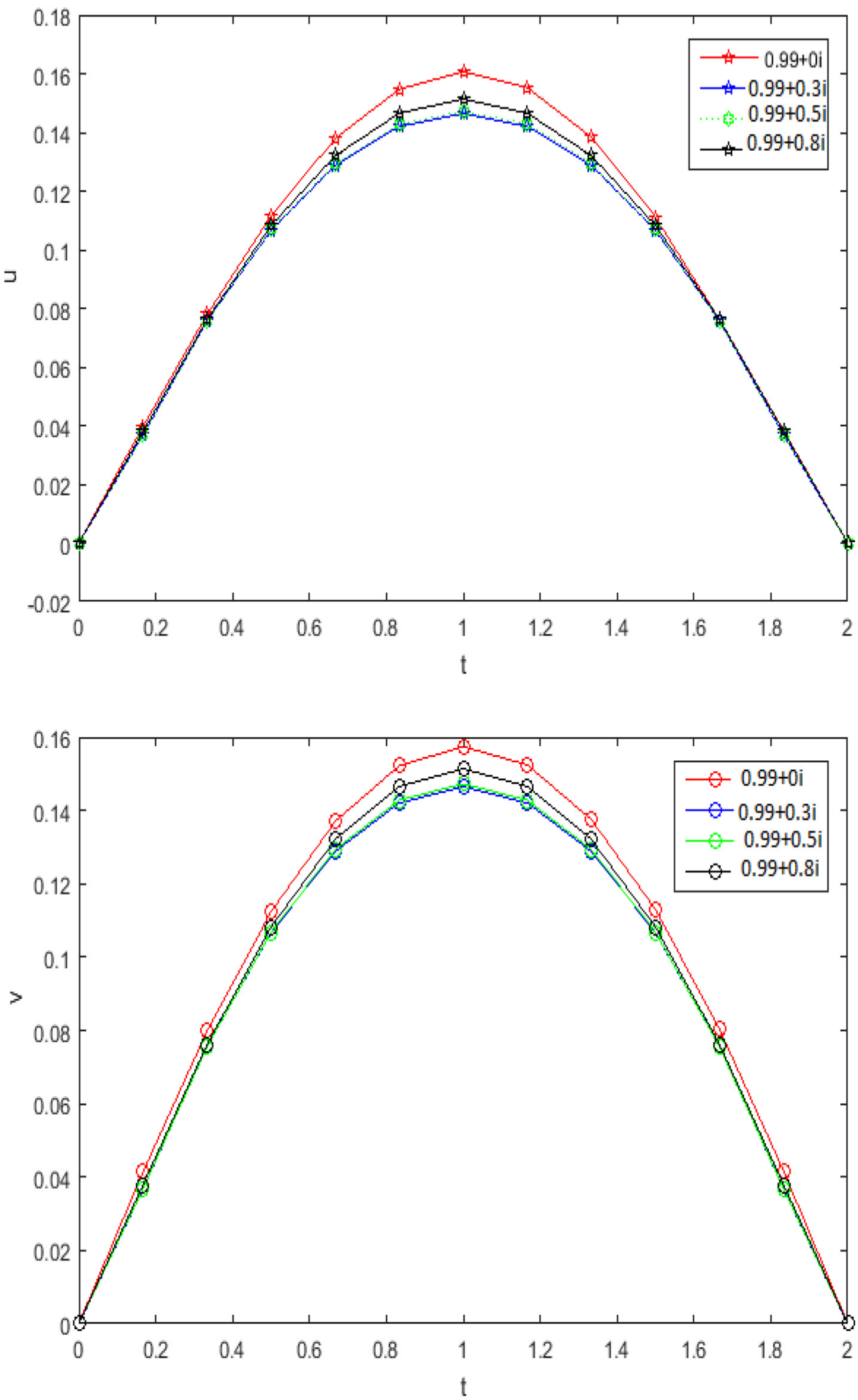

Fig. 4. Numerical simulations for Example 2 at different values of imaginary part, $\theta = 0.5$.

for the solutions u and v at different values of imaginary part and the value of real part equal 0.999. We compared the obtained solutions with the solution in case $\mu = 0.999$ and $\lambda = 0$. Fig. 5 illustrates the behavior of the numerical solution for imaginary part of u and v at different values of real part and the value of imaginary part equal 0.4. We compare the obtained solutions with the solution in case $\mu = 0.999$ and $\lambda = 0$. Fig. 6 illustrates the behavior of the numerical solution for imaginary part of u and v at different values of imaginary part and the value of the real part equal to 0.999. We compare the obtained solutions with the solution in the case $\mu = 0.999$ and $\lambda = 0$. We noted that the complex order is more general than integer and fractional order.

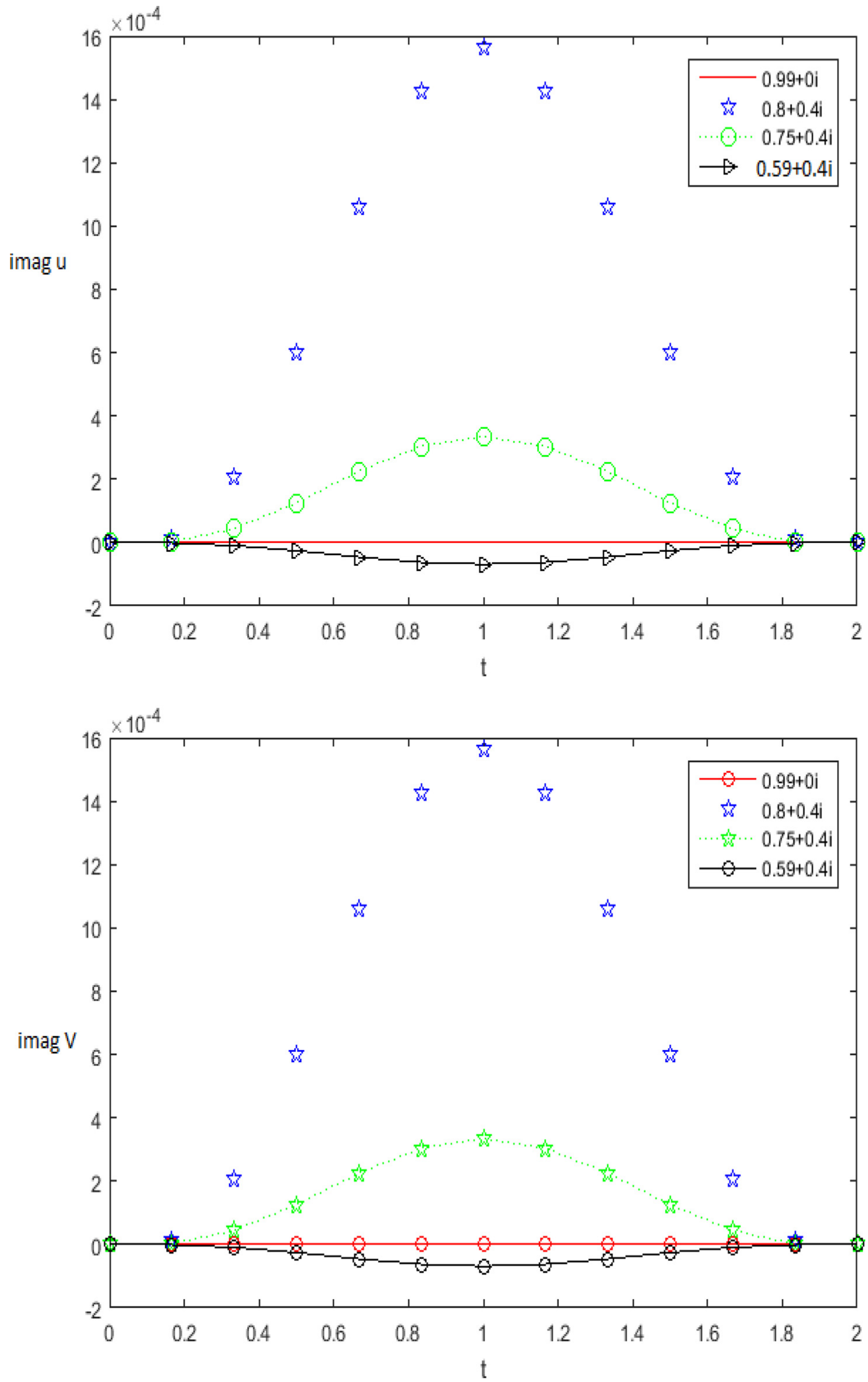

Fig. 5. Numerical simulations for Example 2 at different values of real part, $\theta = 0.5$.

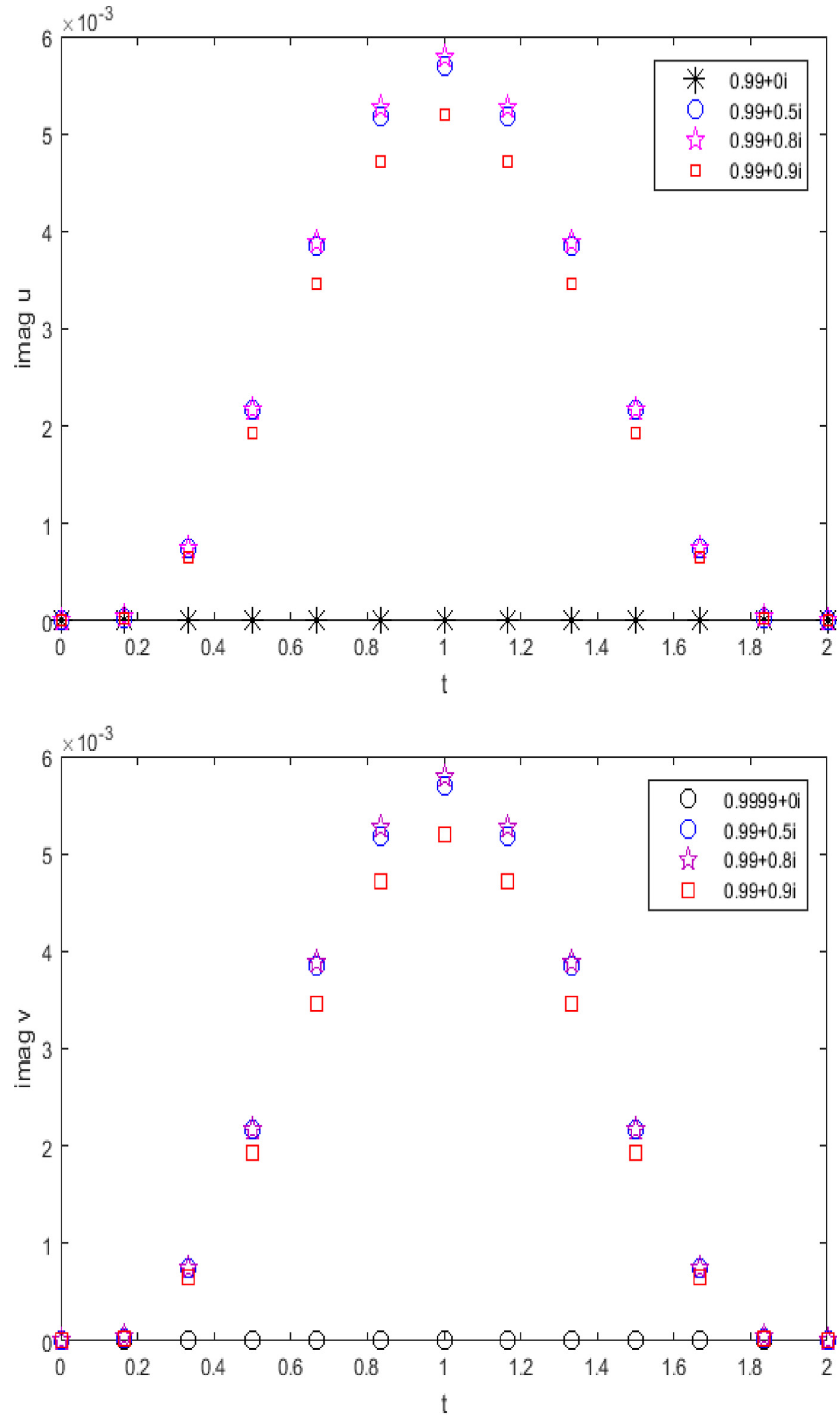

Fig. 6. Numerical simulations for Example 2 at different values of imaginary part, $\theta = 0.5$.

Conclusions

In this work, the numerical treatments for a complex order fractional nonlinear one-dimensional Burgers' equations are presented. It is more suitable and more general to describe these problems than the integer order and fractional order derivatives as we can see from Figs. 1–4. A novel parameter σ_t is given in order to be consistent with the physical equation. A relation between the complex order and σ_t depending on the model is derived for the propose model problem. The numerical simulations for the solutions of complex fractional order Burgers' equations are performed. WANFDM is constructed to study the nonlinear complex order fractional Burgers' equations numerically. This method is based on choosing the weight factor theta. The main advantage of this method is it can be explicit or implicit with large stability regions using the idea of the weighed step introduced by the nonstandard finite difference method. Finally, we suggest that the complex order fractional derivative provides best results and could be more useful for the researchers and scientists. All results in this work were obtained by using MATLAB (R2013a), on a computer machine with intel(R) core i5.

Declaration of Competing Interest

No conflict of interest exists regarding the publications of this work.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

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