



Research article

α -Power transformed transformed power function distribution with applications

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ARTICLE INFO

Keywords:

Power function distribution
Transformation
Estimation
Data analysis
Goodness-of-fit

ABSTRACT

By standard transformation of a random variable, we obtained a partially bounded one-parameter version of the bounded three-parameter power function distribution by Saran and Pandey (2004) which we called the Transformed Power Function (TPF) distribution and based on an alpha-power transformation method due to Mahdavi and Kundu (2017) we generalized the TPF distribution as the α -Power Transformed Transformed Power Function (α PTTPF) distribution. Some of the properties of the α PTTPF distribution are given, and we approached the parameter estimation by three methods, namely: maximum likelihood, ordinary least-squares, and weighted least-squares, but after comparing the results from a simulation study, we settled for the maximum likelihood. The new distribution is suitable for modeling data with either decreasing or upside-down bathtub hazard rates. Three real data-sets are used to demonstrate the usefulness of the new model.

1. Introduction

In probability theory, the solutions of a vast majority of problems are sometimes obtained by applying limit theorems thus, these solutions are only approximate. The unavailability of an explicit formula for the remainder terms makes the estimation of errors a daunting task. Consequently, it is important to focus on methods that can result in more precise solutions and functional transformation of random variables form the basis of such methods (Bol'shev, 1959).

Transformation of random variables plays a vital role in probability and distribution theory and they have been used in a wide range of application areas, for instance in finance, biological and medical sciences, reliability engineering, actuarial science, and demography just to name a few. Some of the works on random variable transformation include Dallas (1976); Atchison and Shen (1980); Al-Dayian (2004); Rosco et al. (2011); Jones (2014); Sharma et al. (2016); Kelmansky and Ricci (2017); Barco et al. (2017); Mazucheli et al. (2018a,b, 2019), and more recently Komal and Vikas (2020); just to mention a few.

In many practical situations, some experiments result in an outcome that takes on values within a predefined domain. For instance, during medical diagnostics or examination, patients are often asked to rate

their pain from 0-10, say; where 0 indicate no pain and 10 indicates severe pain hence, the distribution of patients pain, in this case, is a random variable that takes on values in $[0, 10]$. The shapes of the probability density/mass function for bounded outcomes are usually flexible and can include unimodal, U-shape, and J-shape. Any attempt to model bounded support variables by unbounded distributions may not be convenient (Bottai et al., 2010). However, bounded random variables on $[0, 1]$ for example, can be difficult to analyze due to the dependency of the variance on the mean and due to the high skewness in such distributions, meaning that the assumption of normal distribution can yield questionable results (Rocke, 1993) but, transforming the bounded random variable to either partially bounded random variable i.e., $[0, +\infty)$ or unbounded random variable $(-\infty, +\infty)$ could be a reasonable approach to circumventing these problems.

Recently, Mahdavi and Kundu (2017) introduced a method for generating distributions referred to as the α -power transformation (α PT) method and they illustrated this method with the exponential distribution. Following Mahdavi and Kundu (2017), so many other distributions have been extended for instance; Nassar et al. (2017) extended the Weibull distribution, Dey et al. (2017) and Nadarajah and Okorie (2018) extended the generalized exponential, Dey et al. (2019)

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Received 19 April 2021; Received in revised form 16 July 2021; Accepted 17 September 2021

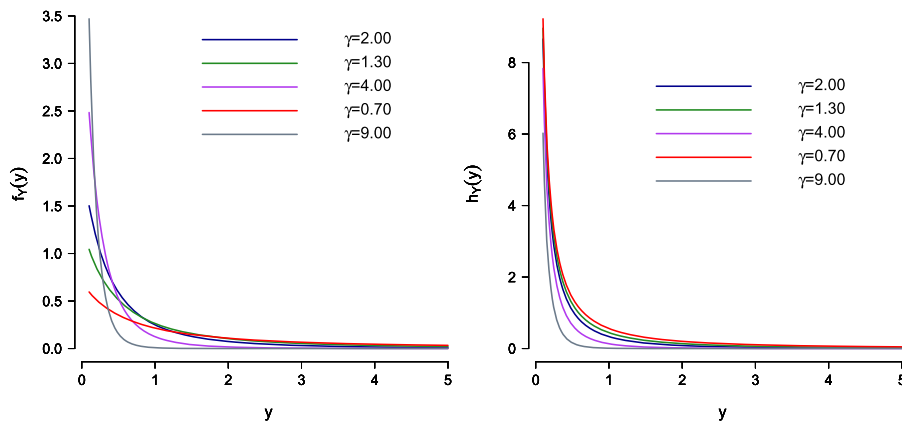


Fig. 1. Plots of the PDF (left) and hazard rate function (right) of the TPF distribution for selected values of γ .

extended the inverse Lindley, Basheer (2019) extended the inverse Weibull, Ihtisham et al. (2019) extended the Pareto, ZeinEldin et al. (2020) extended the inverse Lomax distribution, and so on. We are motivated to introduce the alpha-power transformed transformed power function (α PTTPF) distribution because (i) it can model upside-down bathtub hazard rates; (ii) it can be suitable for fitting right-skewed, decreasing, and unimodal data which may not be suited by other popular distributions and it can be applied in a variety of problems such as medical/biological studies and reliability/survival analysis; and (iii) three real data examples show that it compares well with other competing distributions.

The remaining part of this paper is in the following order. Section 2 introduces the TPF distribution and the α PTTPF distribution with some of their basic properties, Section 3 present the estimation of the parameters of the α PTTPF distribution through the method of maximum likelihood estimation, Section 4 illustrates the applicability of the α PTTPF distribution using three real data-sets, and Section 5 contains the conclusion.

2. The new distribution

Let X denote the power function (PF) random variable with cumulative distribution function (CDF) defined as

$$F_X(x) = 1 - \left[\frac{\beta - x}{\beta - \alpha} \right]^\gamma \tag{1}$$

and probability density function (PDF) defined by

$$f_X(x) = \frac{\gamma}{\beta - \alpha} \left[\frac{\beta - x}{\beta - \alpha} \right]^{\gamma-1}$$

for $\alpha < x < \beta$ and $\gamma > 0$, (Saran and Pandey, 2004). We define a new random variable Y by the transformation $Y = \frac{X-\alpha}{\beta-X}$ and using Equation (1), we propose a new distribution with support on the entire positive real line (i.e., $0 \leq y < +\infty$). The CDF of the new random variable is given by

$$F_Y(y) = 1 - \left[\frac{1}{1+y} \right]^\gamma \tag{2}$$

and the PDF is given by

$$f_Y(y) = \gamma \left[\frac{1}{1+y} \right]^{\gamma+1} \tag{3}$$

for $y \geq 0$ and $\gamma > 0$.

The transformed distribution is called the Transformed Power Function (TPF) distribution. The hazard rate function is a very important measure in survival analysis and life-testing studies. Using Equations (2) and (3) we define the hazard rate function of the TPF distribution as

$$h_Y(y) = \frac{f_Y(y)}{1 - F_Y(y)} = \gamma \left[\frac{1}{1+y} \right]^{\gamma+1} \left(1 - \left[\frac{1}{1+y} \right]^\gamma \right)^{-1},$$

for $y \geq 0$ and $\gamma > 0$. Some plots of the PDF and hazard rate of the TPF distribution for selected values of γ are given in Fig. 1. From Fig. 1, we can see that the PDF and the hazard rate function of the TPF distribution are both monotonic decreasing regardless of the value of γ .

Proposition 1. The TPF satisfies the two basic properties of a probability distribution namely $\int_0^\infty f_Y(y)dy = 1$ and $f_Y(y) \geq 0$.

Proof. By using Equation (3) we proceed as

$$\int_0^{+\infty} f_Y(y)dy = \int_0^{+\infty} \gamma \left[\frac{1}{1+y} \right]^{\gamma+1} dy \tag{4}$$

and substituting $x = \frac{1}{1+y}$ in Equation (4) and simplifying gives

$$\int_0^{+\infty} f_Y(y)dy = \gamma \int_0^1 x^{\gamma-1} dx = 1.$$

Again, for all possible values of γ and $y \geq 0$ it is straightforward to see from Equation (3) that $f_Y(y) \geq 0$. Alternatively, it is not difficult to see from Equation (2) that $\lim_{y \rightarrow 0} F_Y(y) = 0$ and $\lim_{y \rightarrow +\infty} F_Y(y) = 1$. \square

The k -th order moment of the TPF distribution is obtained as $E^*(y^k) = \gamma B(\gamma - k, k + 1)$, $\gamma > k$ where $B(\cdot, \cdot)$ is the beta function defined in Subsection 2.2 and its Q -th quantile function¹ is obtained as $y^*(Q) = [1 - Q]^{-\frac{1}{\gamma}} - 1$, $Q \in [0, 1)$. The PDF and the hazard rate function of the TPF distribution have only decreasing shape characteristics.

Proposition 2. Suppose the random variable $Y \sim$ TPF with CDF in Equation (2) and PDF in Equation (3) we identify some of the special cases of the TPF distribution as follows. Suppose we define a new random variable X by the transformation:

- $X = \frac{Y}{1+Y}$, then X follows the power function distribution (Saran and Pandey, 2003) which is a special case of the Kumaraswamy distribution (Kumaraswamy, 1980) with CDF $F_X(x) = 1 - [1 - x]^\gamma$ and PDF $f_X(x) = \gamma [1 - x]^{\gamma-1}$, for $0 < x < 1$ and $\gamma > 0$.

¹ If Q follows the uniform distribution i.e., $Q \sim U(0,1)$ then, given γ we have that $Y = y^*(U)$ follows the TPF distribution with parameter γ . This method of random number generation is widely known as the inverse transformation method of random number generation.

- $X = -\log(Y)$, then X follows the generalized logistic distribution (Johnson et al., 1995) with CDF $F_X(x) = \left[\frac{1}{1+\exp(-x)}\right]^\gamma$ and PDF $f_X(x) = \gamma \left[\frac{1}{1+\exp(-x)}\right]^{\gamma+1} \exp(-x)$, for $-\infty < x < \infty$ and $\gamma > 0$.
- $X = -\log\left(\frac{1}{1+Y}\right)$, then X follows the exponential distribution with CDF $F_X(x) = 1 - \exp(-\gamma x)$ and PDF $f_X(x) = \gamma \exp(-\gamma x)$, for $x > 0$ and $\gamma > 0$.

Now, on the estimation of the parameter (γ) of the TPF distribution, we suppose that y_1, y_2, \dots, y_n are random observations from the TPF distribution and we outline the procedures for the maximum likelihood estimation (MLE) and the Markov chain Monte Carlo (MCMC) estimation methods for estimating γ as follows.

For the MLE we have the log-likelihood function as

$$\mathcal{L}^*(\Theta|y) = n \log(\gamma) - [\gamma + 1] \sum_{i=1}^n \log[1 + y_i]$$

and the corresponding MLE of γ is obtained as

$$\hat{\gamma} = \frac{n}{\sum_{i=1}^n \log[1 + y_i]}$$

The unique existence of the MLE of γ is not in doubt because, $\frac{d^2 \mathcal{L}^*(\Theta|y)}{d\gamma^2} = -\frac{n}{\gamma^2} < 0$ suggest log-concavity.

The Gibbs sampler is a popular MCMC algorithm for estimating the unknown model parameter(s) in the Bayesian statistics paradigm. For the Gibbs sampler, we first give the likelihood for the TPF distribution as

$$L(\gamma|y) = \prod_{i=1}^n \gamma \left[\frac{1}{1+y_i}\right]^{\gamma+1} \propto \gamma^n \exp\left(\gamma \sum_{i=1}^n \log\left[\frac{1}{1+y_i}\right]\right),$$

using the gamma prior with hyperparameters τ and ψ i.e., $\pi(\gamma) = \text{Gamma}(\tau, \psi)$ we obtain the posterior likelihood as

$$\pi(\gamma|y) = L(\gamma|y)\pi(\gamma) \propto \gamma^{n+\tau-1} \exp\left(-\gamma \left\{\psi - \sum_{i=1}^n \log\left[\frac{1}{1+y_i}\right]\right\}\right)$$

and, the conditional posterior distribution of γ is given by

$$\gamma|y \sim \text{Gamma}\left(n + \tau, \psi - \sum_{i=1}^n \log\left[\frac{1}{1+y_i}\right]\right).$$

The Gibbs sampler algorithm for estimating γ in R (R Core Team, 2020) can be outlined as follows:

- 1 simulate y from the TPF distribution for a specific value of γ ,
- 2 choose initial values for the priors (τ and ψ), any value > 0 will do,
- 3 for $i = 1, 2, \dots, n(1100)$,
 - simulate $\gamma^{(i)}$ from the conditional $\gamma|y$,
- 4 100 burn-in; i.e., discard the first $k(100)$ iterations and estimate summary statistics of the posterior distribution using $\gamma^{k+1}, \gamma^{k+2}, \dots, \gamma^n$.

Table 1 presents a simulation experiment to compare the MLE and MCMC methods of parameter estimation for the TPF distribution. The MLE algorithm is fully described in Section 3.1.

It could be seen from Table 1 that both the MLE and the MCMC methods give an accurate estimate for γ but, the MLE method appears to give more precise estimates especially with increasing sample size, unlike the MCMC method which did not indicate any seriousness to changes in the sample size. However, the standard error (SE) of $\hat{\gamma}$ become smaller as n get larger for both the MLE and MCMC methods.

Table 1. MLE and MCMC parameter estimation methods for the TP distribution.

γ	n	MLE		MCMC	
		$\hat{\gamma}$	SE $_{\hat{\gamma}}$	$\hat{\gamma}$	SE $_{\hat{\gamma}}$
0.25	30	0.26009	0.04805	0.25307	0.04548
	50	0.25472	0.03720	0.25647	0.03739
	75	0.25417	0.02937	0.23452	0.02698
	100	0.25160	0.02500	0.27347	0.02653
	150	0.25093	0.02095	0.27887	0.02257
	200	0.25161	0.01745	0.24598	0.01683
5.00	250	0.25198	0.01647	0.26940	0.01698
	300	0.25108	0.01466	0.25173	0.01479
	350	0.25092	0.01332	0.25426	0.01331
	30	5.17197	0.93683	4.88212	0.87526
	50	5.11989	0.73714	5.64727	0.81271
	75	5.07675	0.58260	4.33008	0.50435
100	100	5.05179	0.51502	5.61408	0.57155
	150	5.02923	0.41354	5.06442	0.39882
	200	4.99931	0.36522	5.46621	0.37699
	250	4.99914	0.31189	4.90500	0.29521
	300	5.02176	0.28865	5.02149	0.28618
	350	5.01734	0.27260	4.99621	0.26218

2.1. α -Power transformed transformed power function distribution

Definition. A random variable Y is said to follow an α -Power Transformed Distribution, denoted by $Y \sim \alpha\text{PT}(\alpha, \gamma)$, if its PDF and CDF are given by

$$f_Y(y)_{\alpha\text{PT}} = \begin{cases} \frac{\log(\alpha) f_Y(y, \gamma) \alpha^{F_Y(y, \gamma)}}{\alpha - 1}, & \text{if } \alpha \in (0, +\infty) \setminus \{1\}, \\ f_Y(y, \gamma), & \text{if } \alpha = 1 \end{cases} \quad (5)$$

and

$$F_Y(y)_{\alpha\text{PT}} = \begin{cases} \frac{\alpha^{F_Y(y, \gamma)} - 1}{\alpha - 1}, & \text{if } \alpha \in (0, +\infty) \setminus \{1\}, \\ F_Y(y, \gamma), & \text{if } \alpha = 1 \end{cases} \quad (6)$$

respectively, where $F_Y(y, \gamma)$ and $f_Y(y, \gamma)$ denote the CDF and PDF of the baseline distribution, respectively.

We now move on to introduce the α -Power Transformed Transformed Power Function distribution denoted by αPTTPF by using Equations (2), (3), (5) and (6). A random variable is said to follow the $\alpha\text{PTTPF}(\alpha, \gamma)$ distribution if its PDF and CDF are given by

$$f_Y(y)_{\alpha\text{PTTPF}} = \begin{cases} \frac{\log(\alpha) \gamma \left[\frac{1}{1+y}\right]^{\gamma+1} \alpha^{-\left[\frac{1}{1+y}\right]^\gamma}}{\alpha - 1}, & \text{if } \gamma > 0, \alpha \in (0, +\infty) \setminus \{1\}, \\ \gamma \left[\frac{1}{1+y}\right]^{\gamma+1}, & \text{if } \gamma > 0, \alpha = 1 \end{cases} \quad (7)$$

and

$$F_Y(y)_{\alpha\text{PTTPF}} = \begin{cases} \frac{\alpha^{-\left[\frac{1}{1+y}\right]^\gamma} - 1}{\alpha - 1}, & \text{if } \gamma > 0, \alpha \in (0, +\infty) \setminus \{1\}, \\ 1 - \left[\frac{1}{1+y}\right]^\gamma, & \text{if } \gamma > 0, \alpha = 1 \end{cases} \quad (8)$$

respectively.

If the random variable Y follows the $\alpha\text{PTTPF}(\alpha, \gamma)$ distribution then, its hazard rate can be defined by using Equations (7) and (8) as

$$h_Y(y)_{\alpha\text{PTTPF}} = \frac{f_Y(y)_{\alpha\text{PTTPF}}}{1 - F_Y(y)_{\alpha\text{PTTPF}}}$$

$$= \begin{cases} \frac{\log(\alpha)\gamma \left[\frac{1}{1+y}\right]^{\gamma+1}}{\alpha \left[\frac{1}{1+y}\right]^{\gamma} - 1}, & \text{if } \gamma > 0, \alpha \in (0, +\infty) \setminus \{1\}, \\ \frac{\gamma}{1+y}, & \text{if } \gamma > 0, \alpha = 1 \end{cases} \quad (9)$$

The hazard rate function of a nonrepairable functional system is defined as the instantaneous rate of failure for the survivor up-to time y during the next instant of time. Note that if the random variable Y denote the failure time of certain operating device/system following the α PTTF distribution that $1 - F_Y(y)$ (reliability/survival function) gives the probability that the system will not fail until time y is passed.

Next we highlight some limit properties of the PDF and hazard rate function. When $y \rightarrow 0$, we have $\lim_{y \rightarrow 0} f_Y(y) = \frac{\log(\alpha)\gamma}{\alpha - 1}$ and when $y \rightarrow +\infty$, we have $\lim_{y \rightarrow +\infty} f_Y(y) \sim \frac{\alpha \log(\alpha)\gamma}{\alpha - 1} y^{-\gamma-1}$ but, for given values of α and γ we observe that $\lim_{y \rightarrow +\infty} f_Y(y) = 0$. Similarly, for the hazard rate function; when $y \rightarrow 0$, we have $\lim_{y \rightarrow 0} h_Y(y) = \frac{\log(\alpha)\gamma}{\alpha - 1}$ and when $y \rightarrow +\infty$, we have $\lim_{y \rightarrow +\infty} h_Y(y) \sim \frac{\gamma+1}{1+y}$ but, for given values of α and γ we get $\lim_{y \rightarrow +\infty} h_Y(y) = \frac{\gamma+1}{1+y}$. We found that the convergence of the upper tail of the PDF of the α PTTF distribution does not rely on the value of the parameters but, the convergence of the upper tail of its hazard rate function depends only on the value of γ .

Proposition 3. *In what follows, we discuss the shape features of the α PTTF distribution relating to its PDF and hazard rate function. We show that the α PTTF distribution can either have a monotonic decreasing or unimodal/upside-down-bathtub shape depending on the value of α . Hence, we call α the shape parameter and γ the scale parameter. In the sequel we state.*

Proof. Using Equation (7) we can proceed as

$$\frac{d \log f_Y(y)}{dy} = -\frac{\gamma+1}{1+y} + \gamma \left[\frac{1}{1+y} \right]^{\gamma+1} \log(\alpha). \quad (10)$$

Setting Equation (10) to zero and solving for y gives the critical value $y_0 \in (0, \infty)$ of the PDF in Equation (7) indicating that the PDF could increase on $(0, y_0)$ and decrease on (y_0, ∞) also, it is easy to verify that $\frac{d^2 \log f_Y(y_0)}{dy^2} < 0$ hence, the PDF has a local extremum (maximum) at y_0 and y_0 is referred to as the mode of the α PTTF distribution and it is explicitly given by

$$\text{mode} = \left[\frac{\gamma \log(\alpha)}{\gamma+1} \right]^{\frac{1}{\gamma}} - 1; \alpha > 1.$$

It is clear that Equation (10) is < 0 when $\alpha \leq 1$; hence, we can easily deduce that the PDF of the α PTTF is decreasing when $\alpha \leq 1$ and unimodal when $\alpha > 1$ and it implies that the shape characteristic for the PDF of the α PTTF is governed by α .

The proof for the shape of the hazard rate function is trivial and analogous to that of the PDF so we have omitted it. By using Equation (9) we obtained the same result in Equation (10) and consequently, the same conclusion applies to the hazard rate function. Specifically, the hazard rate function of the α PTTF is decreasing when $\alpha \leq 1$ and upside-down-bathtub when $\alpha > 1$ and it means that the shape behavior of the hazard rate function of the α PTTF is controlled by α . \square

We show the plots of the PDF and the hazard rate function of the α PTTF distribution for some parameter combinations in Fig. 2. The additional flexibility resulting from the extra parameter α , such as the unimodal shape behavior of the PDF and the upside-down-bathtub characteristics of the hazard rate function of the α PTTF distribution in Fig. 2 are the improved features of this distribution compared to the baseline distribution (TPF distribution).

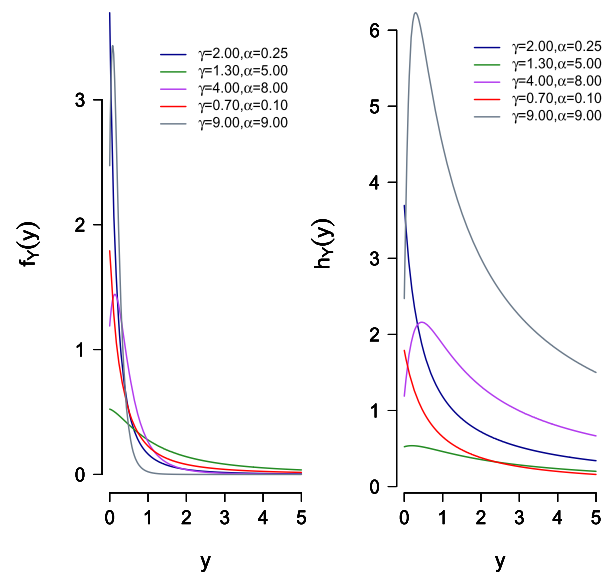


Fig. 2. Plots of the PDF (left) and hazard rate function (right) of the α PTTF distribution for selected values of α and γ .

Theorem 2.1.

1. The limiting distribution of the sample minima $(Y_{1:n})_{n \in N^*}$ is the Weibull distribution with shape parameter 1 and the CDF is $F_Y^*(Y) = 1 - \exp(-y)$; $y > 0$ i.e., $\exists (a_n)_{n \in N^*}$ and $(b_n)_{n \in N^*}$ s.t. $\lim_{n \rightarrow +\infty} P(b_n(X_{1:n} - a_n) \leq y) = F_Y^*(y)$.
2. The limiting distribution of the sample maxima $(Y_{n:n})_{n \in N^*}$ is the Fréchet distribution with CDF $F_Y^{**}(Y) = 1 - \exp(-y^{-\gamma})$; $\gamma, y > 0$ i.e., $\exists (a_n)_{n \in N^*}$ and $(b_n)_{n \in N^*}$ s.t. $\lim_{n \rightarrow +\infty} P(b_n(X_{1:n} - a_n) \leq y) = F_Y^{**}(y)$.

Proof. 1. Recall that, $\lim_{y \rightarrow 0} f_Y(y) = \frac{\log(\alpha)\gamma}{\alpha - 1}$ and $y(0) = 0$ (see Equation (15)). Thus by using the l'Hospital's rule we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{F_Y(y(0) + \epsilon y)}{F_Y(y(0) + \epsilon)} &= \lim_{\epsilon \rightarrow 0} \frac{F_Y(\epsilon y)}{F_Y(\epsilon)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{y f_Y(\epsilon y)}{f_Y(\epsilon)} \\ &= y. \end{aligned}$$

Following from Arnold et al., 1992 (Theorem 8.3.6).

2. Also, recall that, $\lim_{y \rightarrow +\infty} f_Y(y) \sim \frac{\alpha \log(\alpha)\gamma}{\alpha - 1} y^{-\gamma-1}$. Again by using the l'Hospital's rule we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{1 - F_Y(ty)}{1 - F_Y(y)} &= \lim_{t \rightarrow +\infty} \frac{y f_Y(ty)}{f_Y(y)} \\ &= y^{-\gamma}. \end{aligned}$$

Following from Leadbetter et al., 2012 (Theorem 1.6.2 and Corollary 1.6.3). \square

2.2. Technical lemmas

We need the following Lemmas for the derivation of some of the properties of the α PTTF distribution.

Lemma 1. *We have the exponential series*

$$a^x = \sum_{k=0}^{+\infty} \frac{(x \log a)^k}{k!}.$$

Lemma 2. We have

$$\int_0^1 x^{\mu-1}(1-x^\lambda)^{\nu-1} dx = \frac{1}{\lambda} B\left(\frac{\mu}{\lambda}, \nu\right)$$

for; $\mu > 0, \nu > 0$ and $\lambda > 0,$

where $B(a, b)$ denote the beta function defined by

$$B(a, b) = \int_0^1 y^{a-1}(1-y)^{b-1} dy$$

The proof of Lemma 1 and Lemma 2 are immediate from Equation 1.211.2 in page 26 and Equation 3.251.1 in page 324 of Gradshteyn and Ryzhik (2007); respectively.

The properties derived in Subsections 2.3, 2.5, and 2.6 involve the beta function, which is an integral, and the exponential series, which is an infinite series. In general, both the beta function and the exponential series can only be computed numerically. However, built-in routines for computing the beta function and the infinite series are available in most mathematical and statistical packages, for example, MAPLE and MATHEMATICA. Hence, the expressions derived in Sections 2.3, 2.5, and 2.6 can be convenient.

2.3. $k - th$ Order ordinary moment

The mean and other higher-order moments can be used to describe the distribution of any random variable fairly well. Even the celebrated Central Limit Theorem which forms the basis for inferential statistics rely on the moments, just to mention a few importance of moments in probability and statistics. In this section, we derive the moments of the α PTTPF distribution.

Theorem 2.2. The $k - th$ order moment of a positive random variable $Y,$ say is defined by $E(Y^k) = \int_0^{+\infty} y f_Y(y) dy$ hence, if Y follows the α PTTPF(α, γ) distribution its $k - th$ is given by

$$E(y^k) = \frac{\log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \sum_{t=0}^k \frac{(-1)^t \log(\alpha)^s \Gamma(k+1)}{\Gamma(s+1)\Gamma(t+1)\Gamma(k-t+1)} B\left(\frac{t+\gamma-k}{\gamma}, s+1\right),$$

$\gamma > k.$

Proof. The $k - th$ order ordinary moment of the α PTTPF distribution can be obtained by using Equation (7) as follows

$$E(y^k) = \int_0^{+\infty} y^k f_Y(y) dy$$

$$= \int_0^{+\infty} y^k \frac{\log(\alpha)}{\alpha - 1} \gamma \left[\frac{1}{1+y}\right]^{\gamma+1} \alpha^{1-\left[\frac{1}{1+y}\right]^\gamma} dy \tag{11}$$

by Lemma 1, Equation (11) proceed as

$$E(y^k) = \frac{\gamma \log(\alpha)}{\alpha - 1} \int_0^{+\infty} y^k \left[\frac{1}{1+y}\right]^{\gamma+1} \sum_{s=0}^{+\infty} \frac{1}{s!} \left[\left(1 - \left[\frac{1}{1+y}\right]^\gamma\right) \log(\alpha) \right]^s dy$$

$$= \frac{\gamma \log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \frac{\log(\alpha)^s}{s!} \int_0^{+\infty} y^k \left[\frac{1}{1+y}\right]^{\gamma+1} \left[1 - \left(\frac{1}{1+y}\right)^\gamma\right]^s dy \tag{12}$$

substituting $x = \frac{1}{1+y}$ in Equation (12) and simplifying we have

$$E(y^k) = \frac{\gamma \log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \frac{\log(\alpha)^s}{s!} \int_0^1 x^{\gamma-k-1} [1-x]^k [1-x^\gamma]^s dx$$

$$= \frac{\gamma \log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \frac{\log(\alpha)^s}{s!} \int_0^1 \sum_{t=0}^k (-1)^t \binom{k}{t} x^{t+\gamma-k-1} [1-x^\gamma]^s dx$$

$$= \frac{\gamma \log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \sum_{t=0}^k (-1)^t \binom{k}{t} \frac{\log(\alpha)^s}{s!} \int_0^1 x^{t+\gamma-k-1} [1-x^\gamma]^s dx \tag{13}$$

Equation (13) can be expressed in terms of Lemma 2 as

$$E(y^k) = \frac{\log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \sum_{t=0}^k \frac{(-1)^t \log(\alpha)^s \Gamma(k+1)}{\Gamma(s+1)\Gamma(t+1)\Gamma(k-t+1)} B\left(\frac{t+\gamma-k}{\gamma}, s+1\right),$$

$\gamma > k. \quad \square$

Corollary 2.2.1. It follows that the first four ordinary moments are:

$$E(y) = \frac{\log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \sum_{t=0}^1 \frac{(-1)^t \log(\alpha)^s \Gamma(2)}{\Gamma(s+1)\Gamma(t+1)\Gamma(2-t)} B\left(\frac{t+\gamma-1}{\gamma}, s+1\right), \gamma > 1$$

$$E(y^2) = \frac{\log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \sum_{t=0}^2 \frac{(-1)^t \log(\alpha)^s \Gamma(3)}{\Gamma(s+1)\Gamma(t+1)\Gamma(3-t)} B\left(\frac{t+\gamma-2}{\gamma}, s+1\right), \gamma > 2$$

$$E(y^3) = \frac{\log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \sum_{t=0}^3 \frac{(-1)^t \log(\alpha)^s \Gamma(4)}{\Gamma(s+1)\Gamma(t+1)\Gamma(4-t)} B\left(\frac{t+\gamma-3}{\gamma}, s+1\right), \gamma > 3$$

$$E(y^4) = \frac{\log(\alpha)}{\alpha - 1} \sum_{s=0}^{+\infty} \sum_{t=0}^4 \frac{(-1)^t \log(\alpha)^s \Gamma(5)}{\Gamma(s+1)\Gamma(t+1)\Gamma(5-t)} B\left(\frac{t+\gamma-4}{\gamma}, s+1\right), \gamma > 4.$$

The variance $V(Y)$ of the α PTTPF distribution is given by

$$V(Y) = E(Y^2) - [E(Y)]^2.$$

Corollary 2.2.2. The $E(y^k) < +\infty$ therefore, we can calculate the degree of asymmetry and tailedness of the α PTTPF distribution by computing the skewness denoted by S_{kw} and kurtosis denoted by k_{ut} based on the first four ordinary moments as follows

$$S_{kw} = \frac{E(Y^3) - 3E(Y)E(Y^2) + 2[E(Y)]^3}{[V(Y)]^{\frac{3}{2}}}$$

and

$$k_{ut} = \frac{E(Y^4) - 4E(Y)E(Y^3) + 6[E(Y)]^2E(Y^2) - 3E(Y^4)}{[V(Y)]^2}$$

respectively.

Table 2 gives the numerical values of the mean, variance, skewness, and kurtosis of the α PTTPF distribution for some selected values of γ and $\alpha.$ The computations were done in MAPLE. The MAPLE code for calculating the $k - th$ order ordinary moment of the α PTTPF distribution for any choice of α and γ is available in the Appendix.

We can observe from Table 2 that the mean and variance of the α PTTPF distribution increases as α increases but, decreases with respect to the increase in $\gamma.$ The skewness decreases with increase in α and $\gamma.$ The kurtosis increases as α and γ increases.

Another important property of any distribution is the moment generating function (MGF) because apart from generating the moments, the MGF can as well be used to characterize the distribution (see for example; Epps et al., 1982; McCullagh, 1994, and Magableh and Matalgah, 2009).

Theorem 2.3. The MGF of a positive random variable $Y,$ say is defined by

$$M_Y(z) = E(\exp[zy]) = \int_0^{+\infty} \exp(zy) f_Y(y) dy \tag{14}$$

and expressing Equation (14) as a power series gives

Table 2. Mean, variance, skewness, and kurtosis of the α PTTPF distribution for some values of γ and α .

γ	α	Mean	Variance	Skewness	Kurtosis
5.0	0.5	0.2045	0.0810	5.1159	-228.9251
	2.5	0.3171	0.1367	4.1996	-180.7323
	5.0	0.3705	0.1609	3.9679	-172.8521
	15.0	0.4549	0.1961	3.7404	-168.9705
8.5	0.5	0.1100	0.0185	3.3711	-94.9293
	2.5	0.1673	0.0295	2.6726	-83.3519
	5.0	0.1941	0.0338	5.5097	-89.5808
	15.0	0.2359	0.0396	2.2919	-86.5364
15.0	0.5	0.0592	0.0048	2.7974	-74.8795
	2.5	0.0891	0.0073	2.1624	-69.0335
	5.0	0.1029	0.0083	1.9856	-70.2697
	15.0	0.1243	0.0095	1.7992	-75.2450
30.0	0.5	0.0287	0.0010	2.5171	-67.3976
	2.5	0.0428	0.0016	1.9105	-63.8663
	5.0	0.0494	0.0018	1.7380	-65.7400
	15.0	0.0594	0.0020	1.5531	-71.5853

$$M_Y(z) = \sum_{k=0}^{+\infty} \frac{z^k E(y^k)}{\Gamma(k+1)}.$$

It follows that if Y is distributed according to the α PTTPF(α, γ) distribution then its MGF is given by

$$M_Y(z) = \frac{\log(\alpha)}{\alpha - 1} \sum_{k=0}^{+\infty} \sum_{s=0}^{+\infty} \sum_{t=0}^k \frac{(-1)^t z^k \log(\alpha)^s \Gamma(k+1)}{\Gamma(k+1)\Gamma(s+1)\Gamma(t+1)\Gamma(k-t+1)} \times B\left(\frac{t+\gamma-k}{\gamma}, s+1\right), \gamma > k.$$

The proof of Theorem 2.3 is quite straightforward and analogous to that of Theorem 2.2 so we skip it.

2.4. The $Q - th$ quantile function

Given the CDF $F_Y(y)$ of a random variable Y , the quantile function denoted by $y(Q)$ gives the threshold value y below which random samples from the CDF would fall Q percent of the time. The quantile function is defined as $y(Q) = \inf\{y \in \mathbb{R}^+ : Q \leq F_Y(y)\}; Q \in (0, 1)$.

Theorem 2.4. If Y follows the α PTTPF(α, γ) distribution, then its $Q - th$ quantile function is given by

$$y(Q) = \left[1 - \frac{1}{\log(\alpha)} \log[Q(\alpha - 1) + 1]\right]^{-\frac{1}{\gamma}} - 1; Q \in [0, 1]. \tag{15}$$

Corollary 2.4.1. The median (second quartile) of the α PTTPF can be obtained by evaluating Equation (15) at $Q = 0.5$ which is

$$\text{median} = \left[1 - \frac{1}{\log(\alpha)} \log[0.5(\alpha + 1)]\right]^{-\frac{1}{\gamma}} - 1.$$

Similarly, we can calculate the remaining fractiles (e.g., the first quartile, third quartile, and the $r - th$ percentile/decile) of the α PTTPF by appropriately substituting into Equation (14).

If Q follows the uniform distribution; particularly, $Q \sim U(0, 1)$ then, $Y = y(Q)$ follows the α PTTPF distribution. Hence, random number generation from the α PTTPF distribution can be straight-forward.

2.5. Inequality measures

The two most popular inequality measures are the Lorenz and Bonferroni curves. The curves are graphical representations for the distribution of income/wealth. They were originally developed for measuring inequality in an economics context. The Lorenz and Bonferroni

curves have since gained widespread application in other fields including medicine, demography and population studies, reliability studies, and actuarial science.

Theorem 2.5. If the random variable Y follows the α PTTPF(α, γ) distribution then its Lorenz and Bonferroni curves are given by

$$L(r) = \frac{1}{\mu} \int_0^p y(Q) dQ$$

and

$$B(r) = \frac{1}{r\mu} \int_0^p y(Q) dQ$$

respectively, where $\mu = E(Y)$, $r \in [0, 1]$, $p = y(r)$ and

$$\int_0^p y(Q) dQ = \frac{\log(\alpha)}{\alpha - 1} \times \sum_{\ell=0}^{+\infty} \frac{[\log(\alpha)]^\ell}{\ell!} B\left(\frac{1}{\log(\alpha)} \log[p(\alpha - 1) + 1]; \ell + 1, 1 - \frac{1}{\gamma}\right) - p, \gamma > 1.$$

Proof. By using Equation (15) we have;

$$\int_0^p y(Q) dQ = \int_0^p \left[1 - \frac{1}{\log(\alpha)} \log[Q(\alpha - 1) + 1]\right]^{-\frac{1}{\gamma}} - 1 dQ \tag{16}$$

substituting $x = \frac{1}{\log(\alpha)} \log[Q(\alpha - 1) + 1]$ into Equation (16) gives

$$\begin{aligned} \int_0^p y(Q) dQ &= \int_0^{\frac{1}{\log(\alpha)} \log[p(\alpha-1)+1]} [1-x]^{-\frac{1}{\gamma}} \frac{\alpha^x \log(\alpha)}{\alpha-1} dx - p \\ &= \frac{\log(\alpha)}{\alpha-1} \sum_{\ell=0}^{+\infty} \frac{[\log(\alpha)]^\ell}{\ell!} \int_0^{\frac{1}{\log(\alpha)} \log[p(\alpha-1)+1]} x^\ell [1-x]^{-\frac{1}{\gamma}} dx - p \\ &= \frac{\log(\alpha)}{\alpha-1} \sum_{\ell=0}^{+\infty} \frac{[\log(\alpha)]^\ell}{\ell!} \times B\left(\frac{1}{\log(\alpha)} \log[p(\alpha-1)+1]; \ell+1, 1-\frac{1}{\gamma}\right) - p, \gamma > 1, \end{aligned}$$

where $B(z, a, b)$ denote the incomplete beta function defined by $B(z, a, b) = \int_0^z y^{a-1}(1-y)^{b-1} dy$. Equation (16) is referred to as the first order incomplete moment of the α PTTPF distribution. \square

2.6. Entropy

Entropies quantify the uncertainty or randomness of a system. In information theory, the Rényi entropy (Rényi, 1961) generalizes other entropies namely, the Hartley entropy, the Shannon entropy, the collision entropy, and the min-entropy. The Rényi entropy is used in ecology and statistics as an index of diversity. The Rényi entropy is also useful in quantum information.

Theorem 2.6. Let the Rényi entropy be defined as

$$H_\delta(y) = \frac{1}{1-\delta} \log \left[\int_0^{+\infty} f_Y^\delta(y) dy \right]; \delta > 0 \setminus \{1\}. \tag{17}$$

If the random variable Y follows the α PTTPF(α, γ) distribution then its Rényi entropy measure is given by

$$H_\delta(y) = \frac{1}{1-\delta} \left\{ (\delta-1)\log(\gamma) + \delta \left[\log^2(\alpha) - \log(\alpha-1) \right] + \log \left[\sum_{s=0}^{+\infty} \frac{[\delta \log(\alpha)]^s}{s!} B \left(\frac{\delta[\gamma+1]-1}{\gamma}, s+1 \right) \right] \right\}.$$

Proof. By using Equation (7) and Lemma 1 we proceed as

$$\begin{aligned} & \int_0^{+\infty} f_Y^\delta(y) dy \\ &= \int_0^{+\infty} \left\{ \frac{\log(\alpha)\gamma \left[\frac{1}{1+y} \right]^{\gamma+1} \alpha^{1-\left[\frac{1}{1+y} \right]^\gamma}}{\alpha-1} \right\}^\delta \\ &= \left[\frac{\gamma \log(\alpha)}{\alpha-1} \right]^\delta \sum_{s=0}^{+\infty} \frac{[\delta \log(\alpha)]^s}{s!} \int_0^1 \left[\frac{1}{1+y} \right]^{\delta[\gamma+1]} \left[1 - \left(\frac{1}{1+y} \right)^\gamma \right]^s dy \quad (18) \end{aligned}$$

substituting $x = \frac{1}{1+y}$ into Equation (18), simplifying and using Lemma 2 gives

$$\begin{aligned} \int_0^{+\infty} f_Y^\delta(y) dy &= \left[\frac{\gamma \log(\alpha)}{\alpha-1} \right]^\delta \sum_{s=0}^{+\infty} \frac{[\delta \log(\alpha)]^s}{s!} \int_0^1 x^{\delta[\gamma+1]-2} [1-x^\gamma]^s dx \\ &= \gamma^{\delta-1} \left[\frac{\log(\alpha)}{\alpha-1} \right]^\delta \sum_{s=0}^{+\infty} \frac{[\delta \log(\alpha)]^s}{s!} B \left(\frac{\delta[\gamma+1]-1}{\gamma}, s+1 \right) \quad (19) \end{aligned}$$

substituting Equation (19) into Equation (17) gives the required result which is

$$H_\delta(y) = \frac{1}{1-\delta} \left\{ (\delta-1)\log(\gamma) + \delta \left[\log^2(\alpha) - \log(\alpha-1) \right] + \log \left[\sum_{s=0}^{+\infty} \frac{[\delta \log(\alpha)]^s}{s!} B \left(\frac{\delta[\gamma+1]-1}{\gamma}, s+1 \right) \right] \right\}. \quad \square$$

It is important to mention that the limit for $\delta \rightarrow 1$ of $H_\delta(y)$ is the Shannon entropy.

3. Estimation methods

Here we propose different estimators for the unknown parameters of the α PTTPF(α, γ) distribution. We discuss the maximum likelihood, least squares and weighted least-squares estimation methods and compare their performances in a numerical study.

3.1. Maximum likelihood estimation

Let Y follow the α PTTPF(α, γ) distribution with parameter $\Theta = (\alpha, \gamma)'$. The log-likelihood function $\mathcal{L}(\Theta|y)$ based on a random sample of size n (i.e., $y_1, y_2, y_3, \dots, y_n$) from the α PTTPF distribution is given by

$$\begin{aligned} \mathcal{L}(\Theta|y) &= -\log(\alpha) \sum_{i=1}^n \frac{1}{(1+y_i)^\gamma} - (\gamma+1) \sum_{i=1}^n \log(1+y_i) - n\log(\alpha-1) \\ &\quad + n\log(\alpha) + n\log(\log(\alpha)) + n\log(\gamma). \quad (20) \end{aligned}$$

The partial derivative of Equation (20) with respect to α is

$$\frac{\partial \mathcal{L}(\Theta|y)}{\partial \alpha} = \frac{n\alpha - n\log(\alpha) - n}{\alpha(\alpha-1)\log(\alpha)} - \frac{1}{\alpha} \sum_{i=1}^n \frac{1}{(1+y_i)^\gamma} \quad (21)$$

and the partial derivative of Equation (20) with respect to γ is

$$\frac{\partial \mathcal{L}(\Theta|y)}{\partial \gamma} = \log(\alpha) \sum_{i=1}^n \frac{\log(1+y_i)}{(1+y_i)^\gamma} - \sum_{i=1}^n \log(1+y_i) + \frac{n}{\gamma} \quad (22)$$

By setting Equations (21) and (22) to zero we obtain two nonlinear equations whose solutions are the maximum likelihood estimates of α and γ , denoted by $\hat{\alpha}$ and $\hat{\gamma}$. However, the simultaneous equations do

not have any closed-form analytical solution but, we can conveniently approach them numerically for instance, by using the Newton-type algorithm and carrying out a minimization of $\mathcal{L}(\Theta|y)$. There are many available packages in the statistical and mathematical software for solving minimization problems, for instance, the `nlm` and the `optim` functions in R runs the Newton-type algorithm and can be used to obtain $\hat{\Theta}$.

For confidence interval estimation and hypothesis testing on the distribution parameters, we need the information matrix. The expected Fisher information matrix is very complicated but, can be approximated by the total observed Fisher information (Hessian) matrix written as

$$J_n(\hat{\Theta}|y) \approx \left(-\frac{\partial^2 \mathcal{L}(\Theta|y)}{\partial \theta_i \partial \theta_j} \right) \Big|_{2 \times 2}; \quad i, j = 1, 2 \quad (23)$$

The entries of $J_n(\hat{\Theta}|y)$ are listed as:

$$\frac{\partial^2 \mathcal{L}(\Theta|y)}{\partial \alpha \partial \gamma} = \frac{1}{\alpha} \sum_{i=1}^n \frac{\log(1+y_i)}{(1+y_i)^\gamma},$$

$$\frac{\partial^2 \mathcal{L}(\Theta|y)}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \frac{n}{\alpha^2 \log^2(\alpha)} - \frac{n}{\alpha^2 \log(\alpha)} + \frac{n}{(\alpha-1)^2} + \frac{1}{\alpha^2} \sum_{i=1}^n \frac{1}{(1+y_i)^\gamma},$$

and

$$\frac{\partial^2 \mathcal{L}(\Theta|y)}{\partial \gamma^2} = -\frac{n}{\gamma^2} - \log(\alpha) \sum_{i=1}^n \frac{\log^2(1+y_i)}{(1+y_i)^\gamma}.$$

The diagonal entries of the inverse of the observed information matrix denoted by $J_n^{-1}(\hat{\Theta}|y)$ give the variance of $\hat{\Theta}$ while the off-diagonal entries are the covariances and the diagonal entries of $\sqrt{J_n^{-1}(\hat{\Theta}|y)}$ give the standard errors SE's of $\hat{\Theta}$. For a given set of observations, the matrix in Equation (23) can be obtained after the convergence of the Newton-Raphson procedure via the `nlm` or `optim` routine in the R software.

3.2. Least-squares estimation

Suppose we denote the order statistics of size n from the α PTTPF distribution by $y_{(1)}, y_{(2)}, y_{(3)}, \dots, y_{(n)}$, the expectation of the empirical CDF is defined by

$$E[F_n(y_{(i)})] = \frac{i}{n+1}; \quad i = 1, 2, 3, \dots, n.$$

Given the CDF of the α PTTPF distribution in Equation (6), the least-square estimates (LSEs), say $\hat{\alpha}$ and $\hat{\gamma}$, of α and γ are obtained by minimizing

$$LSE(\Theta|y) = \sum_{i=1}^n \left[F_Y(y_{(i)}) - \frac{i}{n+1} \right]^2. \quad (24)$$

3.3. Weighted least-squares estimation

Suppose we denote the order statistics of size n from the α PTTPF distribution by $y_{(1)}, y_{(2)}, y_{(3)}, \dots, y_{(n)}$, the variance of the empirical CDF is defined by

$$V[F_n(y_{(i)})] = \frac{i(n-i+1)}{(n+2)(n+1)^2}; \quad i = 1, 2, 3, \dots, n.$$

Given the CDF of the α PTTPF distribution in Equation (6), the weighted least-square estimates (WLSEs), say $\hat{\alpha}$ and $\hat{\gamma}$, of α and γ are obtained by minimizing

$$WLSE(\Theta|y) = \sum_{i=1}^n \frac{\left[F_Y(y_{(i)}) - \frac{i}{n+1} \right]^2}{V[F_n(y_{(i)})]}. \quad (25)$$

Equations (24) and (25) can be optimized directly by some well-known programming software such as R (`nlm` and `optim` routines), to numerically optimize, LSE and WLSE functions. Readers who are interested in the components of the Hessian matrix for the ordinary least-squares and weighted least-squares estimation methods can request them from the authors.

Table 3. Simulation results for the three parameter estimation methods: MLE, LSE and WLSE.

$\gamma = 2.00, \alpha = 0.25$									
n	Method ^{Rank}	$\hat{\gamma}$	$\hat{\alpha}$	$SE_{\hat{\gamma}}$	$SE_{\hat{\alpha}}$	$Bias_{\hat{\gamma}}$	$Bias_{\hat{\alpha}}$	$MSE_{\hat{\gamma}}$	$MSE_{\hat{\alpha}}$
50	MLE ¹	2.2876	0.6382	0.9118	0.9484	0.2876	0.3882	0.9140	1.0500
	LSE ³	2.2042	0.6568	1.0879	1.2918	0.2042	0.4068	1.2250	1.8340
	WLSE ²	2.1370	0.5995	1.0499	1.0432	0.1370	0.3495	1.1208	1.2103
75	MLE ¹	2.1951	0.4729	0.7909	0.5214	0.1951	0.2229	0.6635	0.3215
	LSE ³	2.1121	0.4955	0.9159	0.7284	0.1121	0.2455	0.8512	0.5908
	WLSE ²	2.0985	0.4768	0.8887	0.6681	0.0985	0.2268	0.7993	0.4977
100	MLE ¹	2.1366	0.4062	0.6985	0.4073	0.1366	0.1562	0.5064	0.1903
	LSE ³	2.0735	0.4439	0.8380	0.5771	0.0735	0.1939	0.7074	0.3705
	WLSE ²	2.0693	0.4199	0.7970	0.5129	0.0693	0.1699	0.6398	0.2919
150	MLE ¹	2.0484	0.3488	0.5908	0.2988	0.0484	0.0988	0.3513	0.0990
	LSE ³	2.0600	0.3823	0.7309	0.4249	0.0600	0.1323	0.5377	0.1980
	WLSE ²	2.0182	0.3632	0.6710	0.3664	0.0182	0.1132	0.4505	0.1470
200	MLE ¹	2.0370	0.3250	0.5306	0.2391	0.0370	0.0750	0.2829	0.0628
	LSE ³	2.0226	0.3468	0.6435	0.3434	0.0226	0.0968	0.4145	0.1273
	WLSE ²	2.0190	0.3332	0.6102	0.3020	0.0190	0.0832	0.3726	0.0981
250	MLE ¹	2.0227	0.3078	0.4932	0.2075	0.0227	0.0578	0.2437	0.0464
	LSE ³	2.0169	0.3252	0.6142	0.2961	0.0169	0.0752	0.3775	0.0933
	WLSE ²	2.0049	0.3162	0.5530	0.2559	0.0049	0.0662	0.3057	0.0698
300	MLE ¹	2.0265	0.2945	0.4396	0.1798	0.0265	0.0445	0.1939	0.0343
	LSE ³	2.0005	0.3155	0.5861	0.2746	0.0005	0.0655	0.3434	0.0797
	WLSE ²	2.0048	0.3020	0.5133	0.2286	0.0048	0.0520	0.2634	0.0549
350	MLE ¹	2.0193	0.2905	0.4143	0.1688	0.0193	0.0405	0.1720	0.0301
	LSE ³	1.9953	0.3053	0.5501	0.2421	-0.0047	0.0553	0.3025	0.0617
	WLSE ²	1.9889	0.2964	0.4817	0.2103	-0.0111	0.0464	0.2321	0.0464

3.4. Simulation results

In this section, we use the MLE, LSE and WLSE methods to estimate the parameters of the α PTTPF distribution and we assess and compare the performance of the MLE, LSE and WLSE through a Monte-Carlo simulation. The simulation involves different sample sizes $n = 50, 75, 100, 150, 200, 250, 300,$ and $350,$ different parameter values, and 5000 replications and the simulation exercise was performed in R software. Table 3 shows the values of the mean estimates, mean standard errors (SEs), mean biases, and mean square errors (MSE) for α and γ for different sample sizes. Comparing the values in Tables 3, 4, 5, 6, and 7 we can see that the MLE frequently approximates the true parameter values not only for large n but, for relatively small sample sizes better than the LSE and WLSE. We tried a wide range of initial guesses for the parameters and there were no observable issues with the convergence and computational time so, any set of initial values is bound to give a similar result in all cases. In general, we note that the SEs, biases, and MSEs decreases as n increases for all the estimation methods. For all the cases, the MLE gave the smallest MSE than the LSE and WLSE methods hence, we adjudge the MLE to be more efficient and consistent for the α PTTPF distribution regardless of how small or large the sample size is followed by the WLSE and the LSE methods. The Monte-Carlo algorithm is outlined as follows.

- (i) For specific set of parameter values Θ , simulate a random sample of size n from the α PTTPF distribution by using the inverse transformation method of random number generation.
- (ii) Estimate the parameters of the new distribution by the methods of MLE, LSE and WLSE.
- (iii) Perform 5000 replications of steps (i)-(ii).
- (iv) For each of the two parameters of the α PTTPF distribution compute the mean, standard error, bias and mean square error of the 1000 parameter estimates in (iii). The analytical expressions for these statistics are:

$$\bar{\Theta} = \frac{1}{5000} \sum_{i=1}^{5000} \hat{\theta}_i,$$

$$SE_{\bar{\Theta}} = \sqrt{\frac{1}{5000} \sum_{i=1}^{5000} (\hat{\theta}_i - \bar{\Theta})^2},$$

$$Bias_{\bar{\Theta}} = \frac{1}{5000} \sum_{i=1}^{5000} (\hat{\theta}_i - \theta), \text{ and}$$

$$MSE_{\bar{\Theta}} = \frac{1}{5000} \sum_{i=1}^{5000} (\hat{\theta}_i - \theta)^2,$$

respectively. Where $\hat{\theta}_i$ is the MLE, LSE and WLSE of α or γ at the $i - th$ iteration for a particular sample size n , $\bar{\Theta}$ corresponds to the mean which involve all the parameter estimates i.e., $\hat{\alpha}_{i's}$ and $\hat{\gamma}_{i's}$ and θ denote the actual values of the parameters α and γ .

The MLE method is used to estimate the parameters of the α PTTPF distribution for several real datasets in the next section.

4. Applications

On one hand, the validity of every statistical analysis depends heavily on the choice of the fitted model and on another hand, the choice of the model depends on the features of the observed data so, in data analysis that involves distribution modeling, there is a graphical tool that can be used to resolve this decision problem and it is called the scaled Total Time on Test (TTT) plot (Aarset, 1987). The scaled TTT plot is constructed by plotting i/n against $T(i/n)$ where,

$$T(i/n) = \frac{[\sum_{j=1}^i G_{j:n} + (n - i)G_{i:n}]}{\sum_{j=1}^n G_{j:n}}; i = 1, \dots, n; G_{1:n} = 1, \dots, n.$$

Table 4. Simulation results for the three parameter estimation methods: MLE, LSE and WLSE.

$\gamma = 0.70, \alpha = 0.10$									
n	Method ^{Rank}	$\hat{\gamma}$	$\hat{\alpha}$	$SE_{\hat{\gamma}}$	$SE_{\hat{\alpha}}$	$Bias_{\hat{\gamma}}$	$Bias_{\hat{\alpha}}$	$MSE_{\hat{\gamma}}$	$MSE_{\hat{\alpha}}$
50	MLE ¹	0.9211	0.4006	0.4275	0.6086	0.2211	0.3006	0.2316	0.4606
	LSE ³	0.8531	0.4092	0.4597	0.9230	0.1531	0.3092	0.2347	0.9474
	WLSE ²	0.8219	0.4019	0.4476	0.8960	0.1219	0.3019	0.2151	0.8939
75	MLE ¹	0.8501	0.3041	0.3605	0.4191	0.1501	0.2041	0.1524	0.2173
	LSE ³	0.8438	0.3508	0.4146	0.6498	0.1438	0.2508	0.1926	0.4851
	WLSE ²	0.8121	0.3114	0.4001	0.5525	0.1121	0.2114	0.1727	0.3499
100	MLE ¹	0.8102	0.2482	0.3260	0.2987	0.1102	0.1482	0.1184	0.1112
	LSE ³	0.8334	0.3127	0.3849	0.5075	0.1334	0.2127	0.1660	0.3028
	WLSE ²	0.8077	0.2808	0.3639	0.4359	0.1077	0.1808	0.1440	0.2227
150	MLE ¹	0.7763	0.1947	0.2807	0.2069	0.0763	0.0947	0.0846	0.0518
	LSE ³	0.8139	0.2545	0.3292	0.3617	0.1139	0.1545	0.1213	0.1546
	WLSE ²	0.7936	0.2278	0.3213	0.3101	0.0936	0.1278	0.1120	0.1125
200	MLE ¹	0.7522	0.1703	0.2517	0.1675	0.0522	0.0703	0.0660	0.0330
	LSE ³	0.8026	0.2294	0.3111	0.2882	0.1026	0.1294	0.1073	0.0998
	WLSE ²	0.7792	0.2012	0.2793	0.2490	0.0792	0.1012	0.0842	0.0722
250	MLE ¹	0.7411	0.1548	0.2339	0.1396	0.0411	0.0548	0.0564	0.0225
	LSE ³	0.7946	0.2136	0.2899	0.2489	0.0946	0.1136	0.0930	0.0748
	WLSE ²	0.7676	0.1895	0.2695	0.2165	0.0676	0.0895	0.0772	0.0549
300	MLE ¹	0.7274	0.1445	0.2204	0.1239	0.0274	0.0445	0.0493	0.0173
	LSE ³	0.7920	0.1993	0.2699	0.2223	0.0920	0.0993	0.0813	0.0593
	WLSE ²	0.7625	0.1760	0.2515	0.1843	0.0625	0.0760	0.0672	0.0397
350	MLE ¹	0.7186	0.1380	0.2066	0.1138	0.0186	0.0380	0.0430	0.0144
	LSE ³	0.7852	0.1903	0.2623	0.2035	0.0852	0.0903	0.0761	0.0496
	WLSE ²	0.7536	0.1656	0.2367	0.1670	0.0536	0.0656	0.0589	0.0322

Table 5. Simulation results for the three parameter estimation methods: MLE, LSE and WLSE.

$\gamma = 3.00, \alpha = 1.50$									
n	Method ^{Rank}	$\hat{\gamma}$	$\hat{\alpha}$	$SE_{\hat{\gamma}}$	$SE_{\hat{\alpha}}$	$Bias_{\hat{\gamma}}$	$Bias_{\hat{\alpha}}$	$MSE_{\hat{\gamma}}$	$MSE_{\hat{\alpha}}$
50	MLE ¹	3.1382	2.5119	0.7935	2.7153	0.1382	1.0119	0.6486	8.3955
	LSE ³	2.8680	2.2244	0.9790	3.4082	-0.1320	0.7244	0.9756	12.1382
	WLSE ²	2.9490	2.3298	0.8989	3.0440	-0.0510	0.8298	0.8105	9.9527
75	MLE ¹	3.0903	2.1715	0.6312	1.8116	0.0903	0.6715	0.4066	3.7324
	LSE ³	2.9019	1.9345	0.8168	2.0273	-0.0981	0.4345	0.6766	4.2980
	WLSE ²	2.9789	1.9580	0.7215	1.7685	-0.0211	0.4580	0.5209	3.3369
100	MLE ¹	3.0594	1.9211	0.5410	1.3307	0.0594	0.4211	0.2961	1.9478
	LSE ³	2.9467	1.8170	0.6845	1.5616	-0.0533	0.3170	0.4713	2.5387
	WLSE ²	2.9939	1.8572	0.6083	1.4698	-0.0061	0.3572	0.3700	2.2874
150	MLE ¹	3.0447	1.7948	0.4376	0.9977	0.0447	0.2948	0.1935	1.0822
	LSE ³	2.9395	1.7063	0.5491	1.1039	-0.0605	0.2063	0.3051	1.2609
	WLSE ²	2.9946	1.7489	0.4882	1.0770	-0.0054	0.2489	0.2384	1.2216
200	MLE ¹	3.0362	1.6978	0.3801	0.7950	0.0362	0.1978	0.1458	0.6711
	LSE ³	2.9683	1.6390	0.4781	0.9233	-0.0317	0.1390	0.2295	0.8717
	WLSE ²	3.0042	1.6815	0.4155	0.8576	0.0042	0.1815	0.1726	0.7683
250	MLE ¹	3.0361	1.6672	0.3337	0.7256	0.0361	0.1672	0.1126	0.5543
	LSE ³	2.9775	1.6107	0.4293	0.7920	-0.0225	0.1107	0.1847	0.6394
	WLSE ²	2.9981	1.6503	0.3711	0.7476	-0.0019	0.1503	0.1377	0.5814
300	MLE ¹	3.0215	1.6369	0.3094	0.6153	0.0215	0.1369	0.0962	0.3973
	LSE ³	2.9796	1.5930	0.3820	0.7356	-0.0204	0.0930	0.1463	0.5497
	WLSE ²	2.9983	1.6096	0.3375	0.6689	-0.0017	0.1096	0.1139	0.4593
350	MLE ¹	3.0159	1.6047	0.2828	0.5648	0.0159	0.1047	0.0802	0.3299
	LSE ³	2.9787	1.5622	0.3547	0.6418	-0.0213	0.0622	0.1262	0.4156
	WLSE ²	3.0061	1.5924	0.3137	0.5957	0.0061	0.0924	0.0984	0.3633

The scaled TTT plot is used to examine the behavior of the empirical hazard rate function. If the scaled TTT plot is linear, it indicates a constant hazard rate, if it is convex, it indicates a decreasing hazard

rate, if it is concave it indicates an increasing hazard rate if it is convex then concave it indicates a bathtub hazard rate, and if it is concave then convex it indicates an upside-down bathtub hazard rate.

Table 6. Simulation results for the three parameter estimation methods: MLE, LSE and WLSE.

$\gamma = 0.50, \alpha = 0.50$									
n	Method ^{Rank}	$\hat{\gamma}$	$\hat{\alpha}$	$SE_{\hat{\gamma}}$	$SE_{\hat{\alpha}}$	$Bias_{\hat{\gamma}}$	$Bias_{\hat{\alpha}}$	$MSE_{\hat{\gamma}}$	$MSE_{\hat{\alpha}}$
50	MLE ¹	0.5377	0.9791	0.1854	1.1631	0.0377	0.4791	0.0358	1.5820
	LSE ³	0.4900	0.9683	0.2186	1.8274	-0.0100	0.4683	0.0479	3.5581
	WLSE ²	0.4953	0.9408	0.2105	1.4706	-0.0047	0.4408	0.0443	2.3566
75	MLE ¹	0.5236	0.8017	0.1544	0.8189	0.0236	0.3017	0.0244	0.7614
	LSE ³	0.4862	0.7390	0.1940	0.9810	-0.0138	0.2390	0.0378	1.0193
	WLSE ²	0.4943	0.7537	0.1775	0.8660	-0.0057	0.2537	0.0315	0.8141
100	MLE ¹	0.5133	0.7042	0.1398	0.5829	0.0133	0.2042	0.0197	0.3814
	LSE ³	0.4878	0.6746	0.1736	0.7745	-0.0122	0.1746	0.0303	0.6301
	WLSE ²	0.4919	0.7058	0.1602	0.6940	-0.0081	0.2058	0.0257	0.5239
150	MLE ¹	0.5109	0.6373	0.1139	0.4355	0.0109	0.1373	0.0131	0.2085
	LSE ³	0.4874	0.6304	0.1483	0.5650	-0.0126	0.1304	0.0221	0.3361
	WLSE ²	0.4947	0.6126	0.1312	0.4872	-0.0053	0.1126	0.0172	0.2500
200	MLE ¹	0.5100	0.6010	0.0974	0.3458	0.0100	0.1010	0.0096	0.1298
	LSE ³	0.4871	0.5759	0.1327	0.4491	-0.0129	0.0759	0.0178	0.2074
	WLSE ²	0.4974	0.6098	0.1140	0.4293	-0.0026	0.1098	0.0130	0.1963
250	MLE ¹	0.5060	0.5784	0.0855	0.3032	0.0060	0.0784	0.0074	0.0981
	LSE ³	0.4859	0.5645	0.1209	0.3917	-0.0141	0.0645	0.0148	0.1576
	WLSE ²	0.5008	0.5817	0.1024	0.3533	0.0008	0.0817	0.0105	0.1315
300	MLE ¹	0.5061	0.5661	0.0789	0.2672	0.0061	0.0661	0.0063	0.0758
	LSE ³	0.4884	0.5571	0.1103	0.3614	-0.0116	0.0571	0.0123	0.1338
	WLSE ²	0.4980	0.5661	0.0907	0.3198	-0.0020	0.0661	0.0082	0.1066
350	MLE ¹	0.5053	0.5479	0.0724	0.2453	0.0053	0.0479	0.0053	0.0624
	LSE ³	0.4919	0.5445	0.1050	0.3296	-0.0081	0.0445	0.0111	0.1106
	WLSE ²	0.4990	0.5537	0.0843	0.2883	-0.0010	0.0537	0.0071	0.0860

Table 7. Simulation results for the three parameter estimation methods: MLE, LSE and WLSE.

$\gamma = 3.00, \alpha = 0.70$									
n	Method ^{Rank}	$\hat{\gamma}$	$\hat{\alpha}$	$SE_{\hat{\gamma}}$	$SE_{\hat{\alpha}}$	$Bias_{\hat{\gamma}}$	$Bias_{\hat{\alpha}}$	$MSE_{\hat{\gamma}}$	$MSE_{\hat{\alpha}}$
50	MLE ¹	3.2101	1.2587	0.9916	1.4452	0.2101	0.5587	1.0271	2.4003
	LSE ³	2.9171	1.1803	1.2486	2.2667	-0.0829	0.4803	1.5655	5.3675
	WLSE ²	2.9689	1.1776	1.1720	1.7432	-0.0311	0.4776	1.3744	3.2663
75	MLE ¹	3.1053	1.0455	0.8456	0.9508	0.1053	0.3455	0.7260	1.0232
	LSE ³	2.8839	0.9883	1.0531	1.2456	-0.1161	0.2883	1.1223	1.6343
	WLSE ²	2.9469	0.9939	0.9769	1.0544	-0.0531	0.2939	0.9570	1.1978
100	MLE ¹	3.0851	0.9588	0.7273	0.7492	0.0851	0.2588	0.5361	0.6282
	LSE ³	2.9070	0.8873	0.9201	0.8432	-0.0930	0.1873	0.8550	0.7459
	WLSE ²	2.9893	0.9291	0.8413	0.8731	-0.0107	0.2291	0.7078	0.8145
150	MLE ¹	3.0668	0.8535	0.5858	0.5286	0.0668	0.1535	0.3476	0.3029
	LSE ³	2.9338	0.8331	0.7696	0.6807	-0.0662	0.1331	0.5966	0.4810
	WLSE ²	2.9972	0.8377	0.6674	0.6055	-0.0028	0.1377	0.4453	0.3855
200	MLE ¹	3.0518	0.8136	0.4969	0.4439	0.0518	0.1136	0.2495	0.2099
	LSE ³	2.9422	0.7983	0.6786	0.5559	-0.0578	0.0983	0.4637	0.3186
	WLSE ²	3.0007	0.8006	0.5874	0.4874	0.0007	0.1006	0.3450	0.2476
250	MLE ¹	3.0326	0.7941	0.4422	0.3823	0.0326	0.0941	0.1966	0.1550
	LSE ³	2.9395	0.7656	0.6159	0.4768	-0.0605	0.0656	0.3829	0.2316
	WLSE ²	2.9918	0.7827	0.5015	0.4352	-0.0082	0.0827	0.2515	0.1962
300	MLE ¹	3.0324	0.7813	0.4072	0.3451	0.0324	0.0813	0.1668	0.1257
	LSE ³	2.9580	0.7618	0.5619	0.4346	-0.0420	0.0618	0.3174	0.1927
	WLSE ²	3.0088	0.7609	0.4631	0.3781	0.0088	0.0609	0.2145	0.1466
350	MLE ¹	3.0325	0.7650	0.3669	0.3108	0.0325	0.0650	0.1356	0.1008
	LSE ³	2.9508	0.7553	0.5187	0.4038	-0.0492	0.0553	0.2715	0.1660
	WLSE ²	3.0053	0.7655	0.4202	0.3536	0.0053	0.0655	0.1766	0.1293

We consider the data representing the times to breakdown of a certain type of electronic insulating material subjected to constant-voltage stress. The data made its first appearance in Nelson (1970)

and has been used by Tiku and Akkaya (2004) and more recently by Nik et al. (2019). Again, we consider the survival times data of 33 patients, in weeks who are suffering from acute Myelogenous Leukemia

Table 8. Summary statistics for the Breakdown, Myelogenous Leukemia and Vinyl data.

Statistics	Breakdown data	Myelogenous Leukemia data	Vinyl data
	values	values	values
nobs	15.000	33.000	34.000
Minimum	0.350	1.000	0.100
Maximum	25.500	156.000	8.000
1st Quartile	1.340	4.000	0.500
3rd Quartile	3.830	65.000	2.475
Mean	4.606	40.879	1.879
Median	2.580	22.000	1.150
Variance	43.961	2181.172	3.813
Skewness	2.190	1.112	1.533
Kurtosis	3.857	-0.064	1.715

(Feigl and Zelen, 1965), the data was recently used by Sen et al. (2019) and Al-Mofleh et al. (2020). Also, we used the Vinyl chloride data from clean upgradient groundwater monitoring wells in (µg/L) (Bhaumik et al., 2009) and recently the data have been used by AL-Fattah et al. (2017). Some of the basic descriptive statistics of the three data-sets are presented in Table 8 and their skewness measures indicate that they are right-skewed. The scaled TTT plots in Figs. 3, 4 and 5 indicate that the

three data-sets have a decreasing hazard rate so, they can be suitably modeled by the αPTTPF distribution.

For the three data sets, we compare the fit of the αPTTPF distribution with the following competing one-parameter, two-parameter, and three-parameter distributions:

1. exponential distribution

$$F_Y(y) = 1 - \exp(-\lambda y); y \geq 0, \lambda > 0 \text{ and}$$

2. Rayleigh distribution

$$F_Y(y) = 1 - \exp(-y^2/[2\sigma^2]); y \geq 0, \sigma > 0,$$

3. Lindley distribution due to Lindley (1958)

$$F_Y(y) = 1 - \left(1 + \frac{\theta y}{\theta + 1}\right) \exp(-\theta y); y > 0, \theta > 0$$

4. generalized exponential (GE) distribution due to Gupta and Kundu (2001)

$$F_Y(y) = [1 - \exp(-\lambda y)]^\alpha; y > 0, \alpha > 0, \lambda > 0,$$

5. Weibull distribution

$$F_Y(y) = 1 - \exp\left\{-\left(\frac{y}{\beta}\right)^\alpha\right\}; y > 0, \alpha > 0, \beta > 0,$$

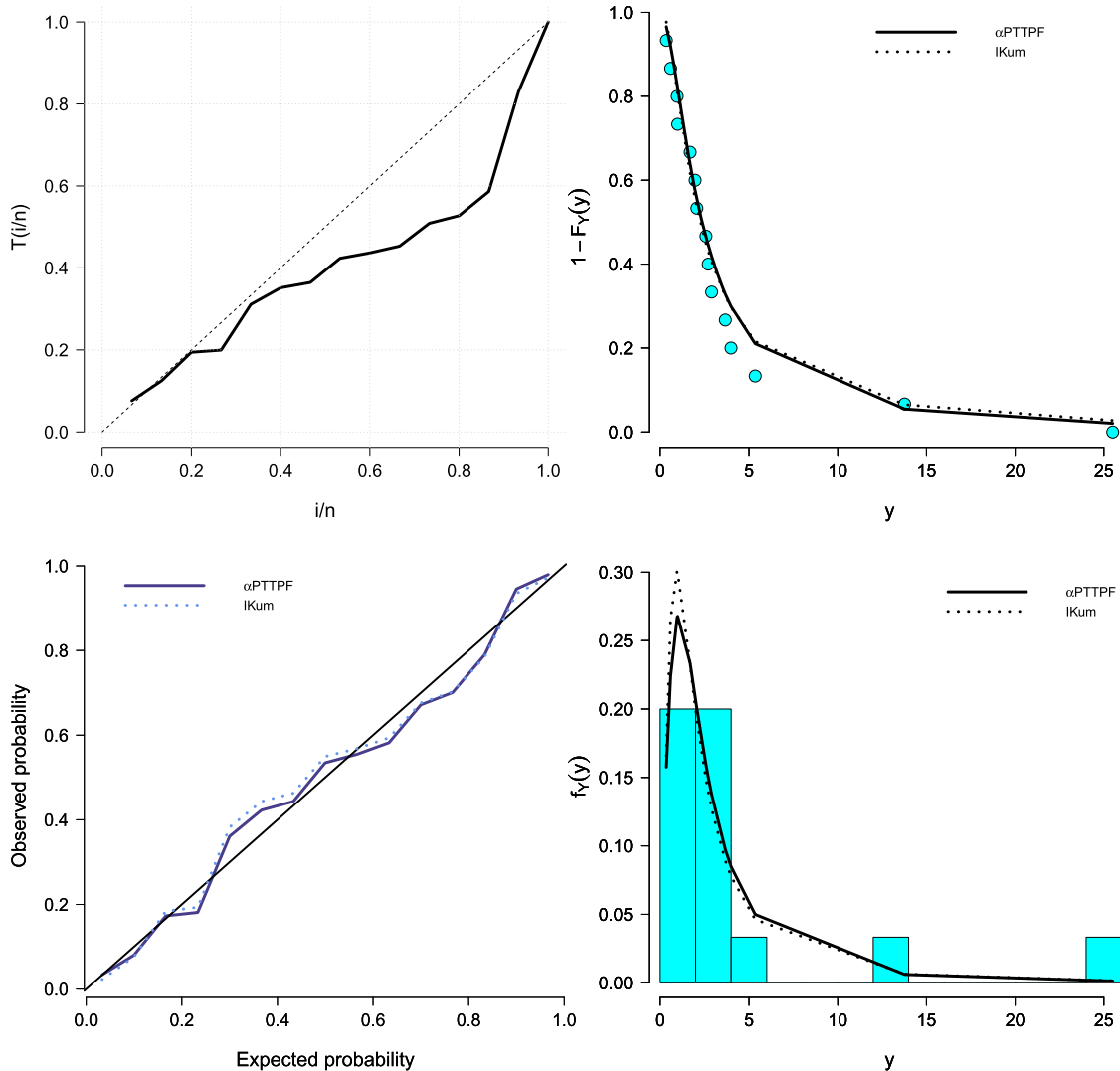


Fig. 3. Scaled TTT plot (top-left), plot of the estimated survival function (top-right), P-P plot (bottom-left) and PDF plot (bottom-right) of the fitted αPTTPF distribution and the IKum distribution superimposed on the empirical survival function for the times to breakdown data.

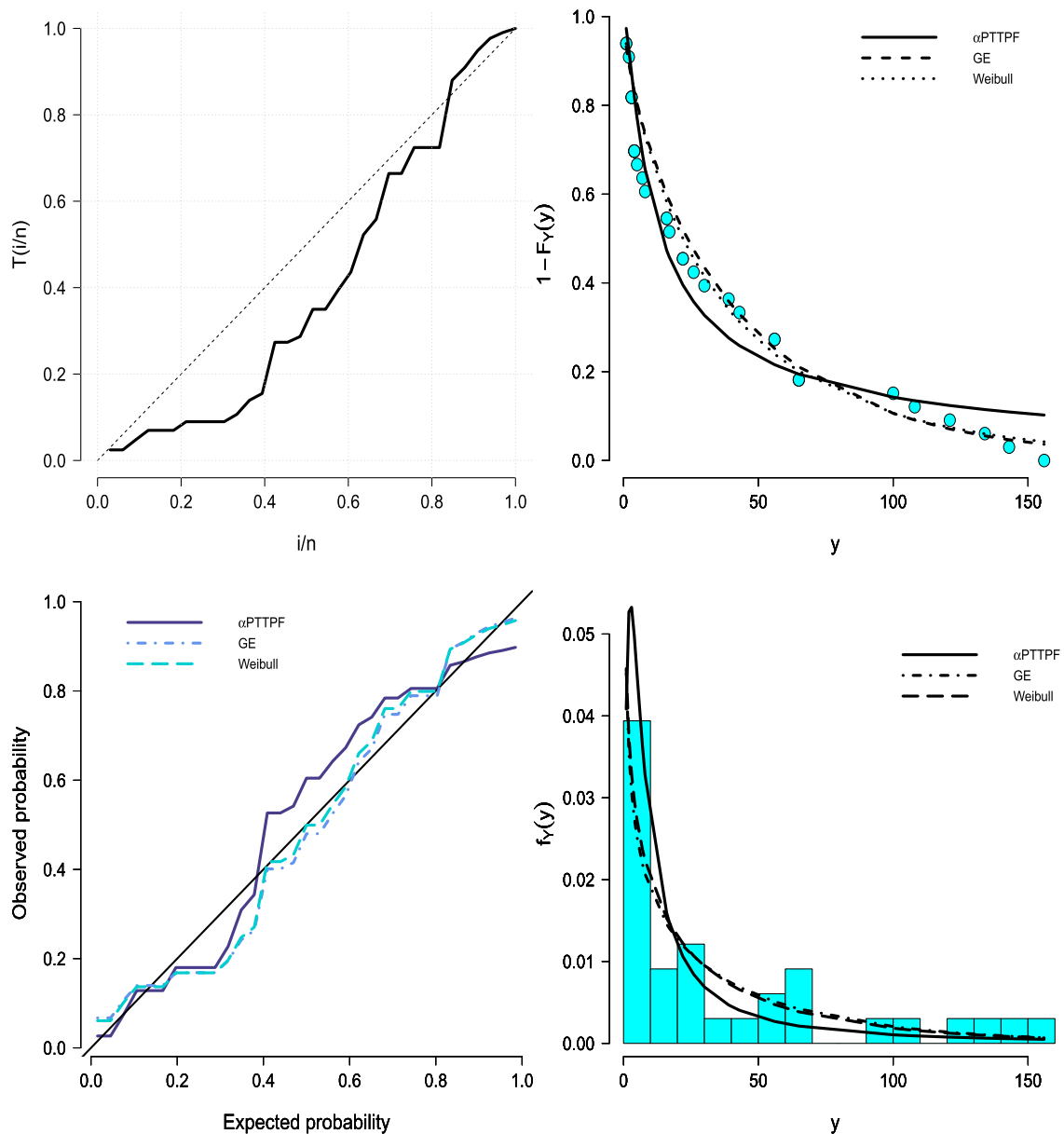


Fig. 4. Scaled TTT plot (top-left), plot of the estimated survival function (top-right), P-P plot (bottom-left) and PDF plot (bottom-right) of the fitted α PTTPF distribution, GE distribution and the Weibull distribution superimposed on the empirical survival function for the Myelogenous Leukemia data.

6. Inverted Kumaraswamy (IKum) distribution due to AL-Fattah et al. (2017)

$$F_Y(y) = [1 - (1 + y)^{-a}]^b; \quad y > 0, \quad a > 0, \quad b > 0.$$

including the three-parameter PF distribution in Equation (1) and the one-parameter TPF distribution in Equation (2).

The goodness-of-fit for the fitted distributions can be checked by comparing the values of their Kolmogorov-Smirnov K-S (see; Kolmogorov, 1933; Smirnov, 1939; Scheffé, 1943; and Wolfowitz, 1949), log-likelihood ($-\mathcal{L}$), AIC, BIC, and the AICc goodness-of-fit statistics. The distribution with the smallest goodness-of-fit statistics and largest K-S p -value is the best for the specific data. The analytical expressions of the goodness-of-fit measures are:

i. Kolmogorov-Smirnov K-S criterion

$$K-S = \max_x \left\{ \frac{i}{n} - \hat{F}(x_{(i)}), \hat{F}(x_{(i)}) - \frac{i-1}{n} \right\},$$

ii. Akaike information criterion (AIC) due to Akaike (1974)

$$AIC = -2\hat{\mathcal{L}} + 2k,$$

iii. Bayes information criterion (BIC) due to Schwarz (1978)

$$BIC = -2\hat{\mathcal{L}} + k \log(n), \text{ and}$$

iv. AIC with a correction (AICc) due to Hurvich and Tsai (1989)

$$AICc = AIC + \frac{2k(k+1)}{n-k-1},$$

where, $\hat{\mathcal{L}}$, k , n , and $\hat{F}(\cdot)$ corresponds to the estimate of the model maximized log-likelihood function, number of parameters in the distribution, the sample size of the fitted data, and the estimated distribution function under the ordered data, respectively.

Tables 9 and 10 give the estimated parameters and model adequacy measures, respectively, for the fitted distributions under the times to breakdown data. Tables 11 and 12 give the estimated parameters and

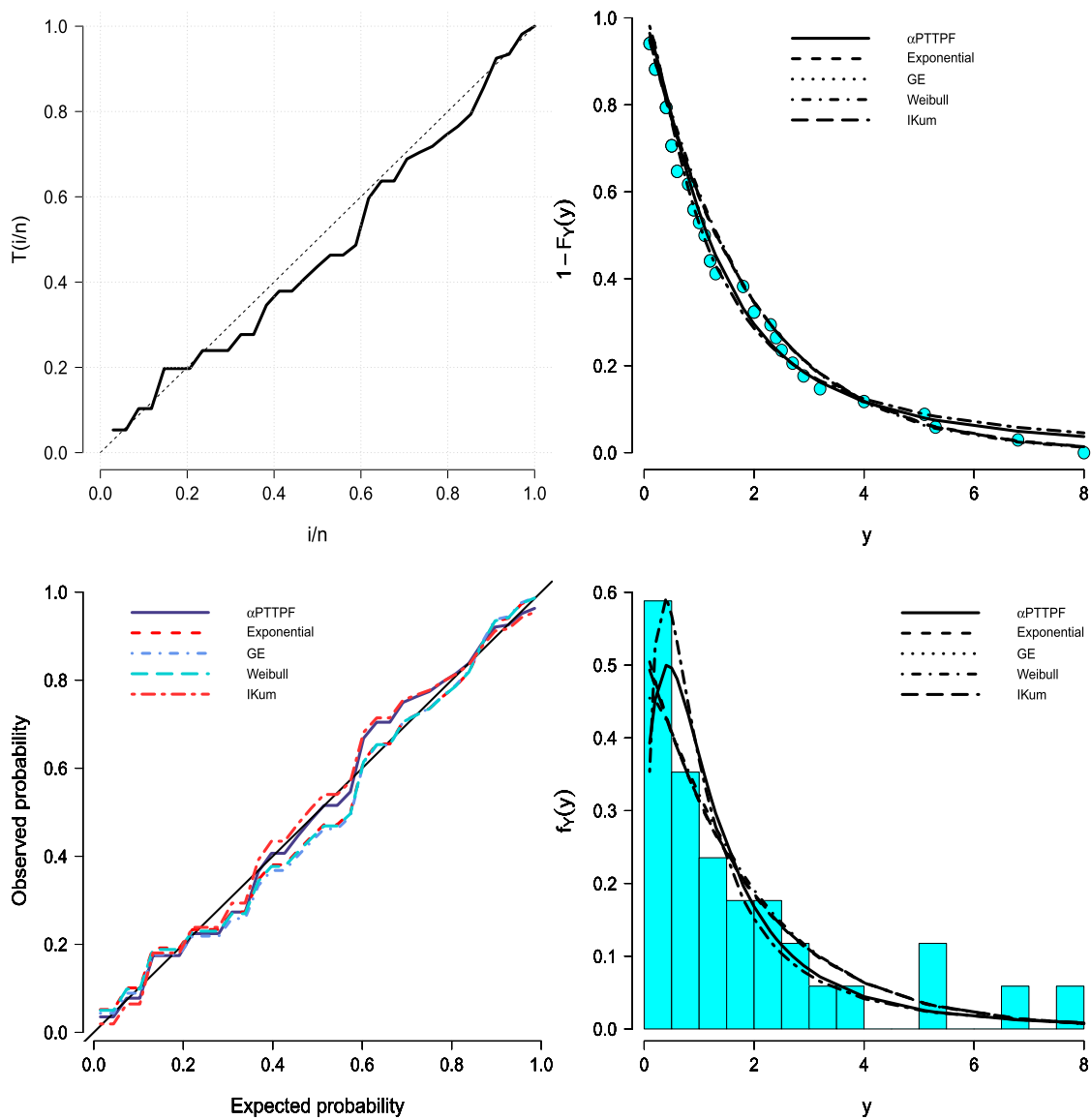


Fig. 5. Scaled TTT plot (top-left), plot of the estimated survival function (top-right), P-P plot (bottom-left) and PDF plot (bottom-right) of the fitted α PTTPF distribution, exponential distribution, GE distribution, Weibull distribution, and the IKum distribution superimposed on the empirical survival function for the Vinyl data.

Table 9. Parameter estimates, standard errors (in bracket), and the log-likelihood of the fitted models for the times to breakdown data.

Models	MLEs Parameters [SEs]	$-\widehat{\mathcal{L}}(\hat{\Theta})$
TPF	$\hat{\gamma} : 0.741148$ [0.1913894]	39.73218
α PTTPF	$\hat{\gamma} : 1.700704; \hat{\alpha} : 231.372679$ [0.3798952] [544.58394]	35.85630
Exponential	$\hat{\lambda} : 0.2171077$ [0.05608278]	37.91040
Rayleigh	$\hat{\sigma} : 5.578764$ [0.7203948]	53.03535
Lindley	$\hat{\theta} : 0.375004$ [0.06979497]	39.87157
GE	$\hat{\alpha} : 0.9647410; \hat{\lambda} : 0.2118104$ [0.3391151] [0.07567362]	37.90516
Weibull	$\hat{\alpha} : 0.8891483; \hat{\beta} : 4.2919303$ [0.1635264] [1.3253863]	37.69143
PF	$\hat{\alpha} : 0.35000; \hat{\beta} : 25.62058; \hat{\gamma} : 14.00000$ [0.09256921] [0.03558536] [0.97282675]	102.0281
IKum	$\hat{a} : 1.493040; \hat{b} : 3.710243$ [0.3835747] [1.6542428]	35.73476

Table 10. Goodness-of-fit of the fitted models for the times to breakdown data.

Models	Information criteria			K-S	
	AIC	BIC	AICc	statistic	p-value
TPF	81.46436	82.17241	81.77205	0.2593783	0.2227788
α PTTPF	75.71261	77.12871	76.71261	0.0992207	0.9950155
Exponential	77.82079	78.52884	78.12849	0.2205215	0.4007175
Rayleigh	108.0707	108.7787	108.3784	0.5743245	3.246922×10^{-5}
Lindley	81.74314	82.45119	82.05083	0.2703022	0.1853775
GE	79.81031	81.22641	80.81031	0.2181023	0.4141493
Weibull	79.38287	80.79897	80.38287	0.1917234	0.5748330
PF	210.0562	212.1803	212.2380	0.2711433	0.1827104
IKum	75.46951	78.52223	75.85661	0.1157700	0.9739000

model adequacy measures, respectively, for the fitted distributions under the Myelogenous Leukemia data. Tables 13 and 14 give the estimated parameters and model adequacy measures, respectively, for the fitted distributions under the Vinyl data.

For the time to breakdown data, we notice that all except the Rayleigh distribution indicate a reasonably good fit, however, the

Table 11. Parameter estimates, standard errors (in bracket), and the log-likelihood of the fitted models for Myelogenous Leukemia data.

Models	MLEs Parameters (SEs)	$-\widehat{\mathcal{L}}(\Theta)$
TPF	$\hat{\gamma} : 0.3371556$ [0.05870865]	166.7555
α PTTPF	$\hat{\gamma} : 0.8013214; \hat{\alpha} : 489.3889393$ [0.1138455] [846.399760]	155.9585
Exponential	$\hat{\lambda} : 0.0244627$ [0.0042758]	155.4502
Rayleigh	$\hat{\sigma} : 43.50947$ [3.787957]	188.6356
Lindley	$\hat{\theta} : 0.04780903$ [0.005900272]	168.8337
GE	$\hat{\alpha} : 0.67803191; \hat{\lambda} : 0.01880112$ [0.14502855] [0.004808501]	153.6516
Weibull	$\hat{\alpha} : 0.7764249; \hat{\beta} : 35.3612950$ [0.1075087] [8.378008]	153.5868
PF	$\hat{\alpha} : 3.6; \hat{\beta} : 179.4; \hat{\gamma} : 7.4$ [1.560389] [9.086103] [0.7138179]	174.7961
IKum	$\hat{a} : 0.7198268; \hat{b} : 4.3555596$ [0.1185141] [1.3004900]	156.0465

Table 12. Goodness-of-fit of the fitted models for the Myelogenous Leukemia data.

Models	Information criteria			K-S	
	AIC	BIC	AICc	statistic	$p - value$
TPF	335.5109	337.0074	335.6399	0.2824595	0.01033151
α PTTPF	315.9169	318.9099	316.3169	0.1325820	0.60763410
Exponential	312.9003	314.3969	313.0294	0.2182042	0.08634559
Rayleigh	379.2713	380.7678	379.4003	0.4254536	1.296101×10^{-5}
Lindley	339.6674	341.1639	339.7964	0.3251196	0.00186735
GE	311.3033	314.2963	311.7033	0.1383340	0.55284150
Weibull	311.1737	314.1667	311.5737	0.1366803	0.56845140
PF	355.5922	360.6588	356.2589	0.2863200	0.00893900
IKum	316.0929	319.1457	316.4800	0.1510000	0.43930000

Table 13. Parameter estimates, standard errors (in bracket), and the log-likelihood of the fitted models for Vinyl data.

Models	MLEs Parameters (SEs)	$-\widehat{\mathcal{L}}(\Theta)$
TPF	$\hat{\gamma} : 1.13527$ [0.1947167]	59.63520
α PTTPF	$\hat{\gamma} : 2.021596; \hat{\alpha} : 21.074041$ [0.3564867] [23.067017]	55.85258
Exponential	$\hat{\lambda} : 0.5320809$ [0.09126827]	55.45260
Rayleigh	$\hat{\sigma} : 1.901662$ [0.1631069]	74.59166
Lindley	$\hat{\theta} : 0.8238207$ [0.1054387]	56.30364
GE	$\hat{\alpha} : 1.076410; \hat{\lambda} : 0.558048$ [0.2474184] [0.1241997]	55.40187
Weibull	$\hat{\alpha} : 1.010223; \hat{\beta} : 1.887931$ [0.1326663] [0.3391093]	55.44962
PF	$\hat{\alpha} : 0.100000; \hat{\beta} : 8.066825; \hat{\gamma} : 14.000000$ [NaN] [0.01926371] [0.63898061]	161.6394
IKum	$\hat{a} : 1.740881; \hat{b} : 2.105867$ [0.3238838] [0.5374866]	55.77069

Table 14. Goodness-of-fit of the fitted models for the Vinyl data.

Models	Information criteria			K-S	
	AIC	BIC	AICc	statistic	$p - value$
TPF	121.2704	122.7968	121.3954	0.199850	0.1323
α PTTPF	115.7052	118.7579	116.0923	0.086723	0.9602
Exponential	112.9052	114.4316	113.0302	0.088957	0.9507
Rayleigh	151.1833	152.7097	151.3083	0.379860	1.096×10^{-4}
Lindley	114.6073	116.1336	114.7323	0.132620	0.5881
GE	114.8037	117.8565	115.1908	0.097777	0.9012
Weibull	114.8992	117.9520	115.2863	0.091840	0.9366
PF	329.2788	333.8578	330.0788	0.390360	6.324×10^{-5}
IKum	115.5414	118.5941	115.9285	0.096576	0.9090

α PTTPF distribution gave the smallest BIC and K-S statistic values and the largest K-S $p - value$ on one hand while on another hand the IKum distribution gave the smallest AIC and AICc values, therefore, while we say that the α PTTPF distribution offers the best fit than the rest of the distributions under comparison it is also important to mention that the IKum distribution is a major competitor with the α PTTPF distribution for this data.

For the Myelogenous Leukemia data we observe that only the α PTTPF distribution, exponential distribution, GE distribution, Weibull distribution, and the IKum distribution indicated a reasonably good fit but, the Weibull distribution gave the smallest AIC, BIC, AICc followed by the GE distribution and then the α PTTPF distribution while the α PTTPF distribution gave the smallest K-S statistic value and the largest K-S $p - value$ followed by the Weibull distribution, the GE distribution, and the IKum distribution hence, because of this; the α PTTPF distribution whilst a strong competitor with the Weibull and the GE distribution could be regarded as the best fitting distribution for the Myelogenous Leukemia data since it gave the largest K-S $p - value$.

For the Vinyl data, we note that the α PTTPF distribution gave the smallest K-S statistic values and the largest K-S $p - value$ followed by the exponential distribution, Weibull distribution, IKum distribution, and the GE distribution hence, suggesting that the α PTTPF distribution provides the best fit for the Vinyl data than the other distributions under examination. But comparing the α PTTPF distribution and the IKum distribution in terms of the Information type criteria we found that the IKum distribution gave smaller AIC, BIC, and AICc values thus, suggesting that the IKum distribution provide a better fit to the data than the α PTTPF distribution. That being said, the exponential distribution and the IKum distribution could be viewed as strong competitors with the α PTTPF distribution, however, the case of the exponential distribution is not surprising because, the scaled TTT plot in Fig. 5 although depicting a convex curve, still it appears to be apparently linear suggesting that a constant hazard rate could be suspected.

The summary of the goodness-of-fit of the fitted models in this section makes sense, especially for the K-S based test because, the K-S test essentially measures the total distance between the empirical and the estimated distribution. The performance of the α PTTPF distribution is further illustrated in the plots of the PDF, survival function and P-P plots in Figs. 3, 4 and 5.

We can also observe from Tables 9, 10, 11, 12, 13, and 14 that, by generalizing the TPF distribution through the α -power transformation framework we were able to improve its fitting performance by a considerable amount.

Also, we note that the α PTTPF distribution gives a better fit for the breakdown data than the two-parameter New Pareto-type (NP) distribution (Bourguignon et al., 2016) because, Nik et al. (2019) reported that the NP distribution gave a K-S statistic of 0.25, and K-S $p - value$ of 0.22 for the same data and the associated K-S test values for the α PTTPF distribution are better than those in Nik et al. (2019) by far. For the Myelogenous Leukemia data, the two-parameter generalized Ramos-Louzada (GRL) distribution by Al-Mofleh et al. (2020) gave a K-S statistic of 0.13637 which is only marginally higher than the K-S statistic of the α PTTPF distribution in Table 12 thus, both distributions are considered strong competitors for the Myelogenous Leukemia data.

5. Concluding remarks

This paper introduces the one-parameter Transformed Power function (TPF) distribution and further extended it to the two-parameter α -Power Transformed Transformed Power Function (α PTTPF) distribution. Some of the basic properties including the asymptotic behaviors of the α PTTPF distribution were studied. The hazard rate function of the α PTTPF distribution could either be decreasing or upside-down bathtub shaped depending on the value of the shape parameter. The α PTTPF distribution contains the TPF distribution as a special case. We presented and discussed the MLE and MCMC methods of parameter estimation

for the TPF distribution and we recommend the MLE method of parameter estimation for the α PTTPF distribution because it appears to be better than the LSE and the WLSE methods. Numerical examples based on three real data-sets were used to illustrate the usefulness of the new model. Future work includes comparing the MLE method with the MCMC method of parameter estimation for the α PTTPF distribution in a rigorous numerical experiment.

Declarations

Author contribution statement

I.E. Okorie: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper. J. Ohakwe, B.O. Osu, C.U. Onyemachi: Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability statement

Data included in article/supplementary material/referenced in article.

Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Acknowledgements

The authors wish to thank the Editor and the two reviewers for their careful reading and comments which greatly improved the paper.

Appendix A

The MAPLE code for calculating the first four ordinary moments in Corollary 2.2.1 is given below.

```
k := k; r := r; a := a # choose specific parameter values.
evalf(ln(a) * (sum(sum(ln(a)^s * (-1)^t * binomial(k, t) *
Beta((t+r-k)/r, s+1)/factorial(s),
t = 0 .. k), s = 0 .. infinity)) / (a-1))
```

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