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n-Dimensional (S, N)-implications

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ABSTRACT

The n-dimensional fuzzy logic (n-DFL) has been contributed to overcome the insufficiency of traditional fuzzy logic in modeling imperfect and imprecise information, coming from different opinions of many experts by considering the possibility to model not only ordered but also repeated membership degrees. Thus, n-DFL provides a consolidated logical strategy for applied technologies since the ordered evaluations provided by decision makers impact not only by selecting the best solutions for a decision making problem, but also by enabling their comparisons. In such context, this paper studies the n-dimensional fuzzy implications (n-DI) following distinct approaches: (i) analytical studies, presenting the most desirable properties as neutrality, ordering, (contra-)symmetry, exchange and identity principles, discussing their interrelations and exemplifications; (ii) algebraic aspects mainly related to left- and right-continuity of representable n-dimensional fuzzy t-conorms; and (iii) generating n-DI from existing fuzzy implications. As the most relevant contribution, the prospective studies in the class of n-dimensional interval (S,N)-implications include results obtained from t-representable n-dimensional conorms and involutive n-dimensional fuzzy negations. And, these theoretical results are applied to model approximate reasoning of inference schemes, dealing with based-rule in n-dimensional interval fuzzy systems. A synthetic case-study illustrates the solution for a decision-making problem in medical diagnoses.

1. Introduction

Zadeh introduced in 1975 the type-n fuzzy sets [45] (Tn-FSs) whose relevance emerges from the insufficiency of the traditional fuzzy logic (FL) in modeling inherent imperfect information related to distinct opinions of specialists in order to define antecedent and consequent of membership functions in inference systems [18]. Currently, many extensions of fuzzy sets are known, e.g. L-fuzzy sets as proposed by Goguen [23], and the Hesitant Fuzzy Sets introduced by Torra [40,41].

In [38], the notion of an n-dimensional fuzzy set (n-DFS) on L_n-fuzzy set theory was introduced by Shang as a special class of Tn-FSs, generalizing the theories underlying many other multivalued fuzzy logics: the Interval-valued Fuzzy Sets [24, 37], the Intuitionistic Fuzzy Sets [2,3] and the Interval-valued Intuitionistic Fuzzy Sets [4]. In L_n-fuzzy set theory [38], the n-dimensional fuzzy sets membership values are n-tuples of real numbers on $U = [0, 1]$, ordered in increasing order and called n-dimensional intervals.

Lately, in [14], Bedregal et al. notice that in most applications the Typical Hesitant Fuzzy Elements (THFE) are used, i.e., considering finite and non-empty subsets of unitary interval ($U = [0, 1]$) as hesitant fuzzy degrees. In addition, even

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when the repetition of element in n-tuples on the hesitant membership degrees is not considered, they can be defined as a THFE [14]. For a hierarchical and historical analysis of these extensions see [18].

According to [11], the main idea of an n-dimensional fuzzy set is to consider several uncertainty levels in the memberships functions, adding degrees of freedom and making it possible to directly model uncertainties in computational systems based on nDFL. Such uncertainties are frequently associated to systems where time-varying, non-stationary statistical attributes or knowledge of experts using questionnaires, all of them include uncertain words from natural language.

The fuzzy implication class plays an important role in modeling fuzzy conditionals [7,8], covering a wide range of distinct fields, from theoretical to applied research areas. In a broad sense, such class is frequently applied to fuzzy control by techniques of soft-computing and analysis of vagueness in natural language modeled by linguistic fuzzy models [44]. Such class is carrying out all inference processes in any fuzzy rule based system [43]. The analysis of properties of fuzzy implications also contributes to underlying applied research areas such as Approximate Reasoning (AR) [29,28]. In the narrow sense, the study of fuzzy implication properties contributes to a branch of many-valued logic enabling the investigation of deep logical questions [1].

In this paper, both approaches are considered, and for that, various properties of n-dimensional fuzzy implications are investigating, including the study of negations (n-DN) and aggregation operators as t-conorms (n-DS) and their dual constructions on $L_n(U)$. Thus, by making use of the representability of such n-dimensional fuzzy connectives, we are able to extend relevant theoretical results from fuzzy connectives to n-dimensional fuzzy approach.

Focusing on the (S,N)-implication class, representable n-dimensional t-conorms in conjunction with representable n-dimensional strong fuzzy negations are also studied. In particular, several (S,N)-implication properties are also investigated. Finally, we formalize an inference scheme considering the use n-DI, providing an n-dimensional interval fuzzy rule-based expert system. Based on inference schemes, the reasoning method consists on a knowledge base of If-Then rules defined by a binary fuzzy relation on $L_n(U)$, which is stated by representable n-dimensional interval (S,N)-implications.

An application in Approximate Reasoning is also introduced, as methods enabling reasoning with imprecise inputs to obtain meaningful outputs applying n-dimensional interval fuzzy implications.

Our studies contribute with distinct and relevant results:

- **(i)** Consolidating the extension of fuzzy implication on $L_n(U)$, also exploring their representability based on fuzzy implications on U;
- **(ii)** Exploring main properties, as identity, neutrality and exchange principles, the iterative Boolean-like law, the dominance of falsity, among additional ones, showing that they can be preserved from U on $L_n(U)$.
- **(iii)** Discussing constructions and several examples of continuous fuzzy implications on $L_n(U)$, including concepts as left- and right-continuity w.r.t. the Moore-continuity [33,34].
- **(iv)** Exploring properties of n-Dimensional $(\mathcal{S},\mathcal{N})$-implications, as the Law of Excluded Middle and (Right- or Left-) Contraposition w.r.t. an n-dimensional fuzzy negation, which can be performed over dual operators and also considering the action of automorphisms to obtain conjugate operators.
- **(v)** Exploring n-dimensional fuzzy $(\mathcal{S},\mathcal{N})$-implication class in Approximate Reasoning, providing the n-dimensional extension of basic concepts which generalize fuzzy conditional rules;
- **(vi)** Introducing inference schemes as the Generalized Modus Ponens (GMP), when the knowledge base consists of n-dimensional fuzzy IF-THEN rules [28].

Analogously to the fuzzy approach, based on these results, the use of n-dimensional (S,N)-implications can play a similar role in the generalizations of the inference schemes, where reasoning is done with fuzzy statements whose truth-values lie in $L_n(U)$ [7,21].

1.1. Related papers

In [38], the definitions of cut set on an n-dimensional fuzzy set and its corresponding n-dimensional vector level cut set are presented according to Zadeh fuzzy set approach. It also studies the decomposition and representation theorems of the n-dimensional fuzzy sets.

The construction of bounded lattice negations from bounded lattice t-norms is considered in [12], together with a discussion under which these connective conditions are preserved by automorphisms and corresponding conjugate negations and t-norms.

In [11], the authors consider the study of aggregation operators for these new concepts of n-dimensional fuzzy sets, starting from the usual aggregation operator theory and also including a new class of aggregation operators containing an $L_n(U)$-extension of the OWA operator. The results presented in such context allow to extend fuzzy sets to interval-valued Atanassov's intuitionistic fuzzy sets and also preserve their main properties.

The results in [31] provide the class of representable n-dimensional strict fuzzy negations, i.e., an n-dimensional strict fuzzy negation which is determined by strict fuzzy negations.

The authors in [32] and [15] consider the definitions and results obtained for n-dimensional fuzzy negations, applying these studies mainly on natural n-dimensional fuzzy negations for n-dimensional t-norms and n-dimensional t-conorms. And, in [33] Moore Continuous n-dimensional interval fuzzy negations are also discussed.

In [30] the triples formed by a t-norm, t-conorm and standard complement are called De Morgan triples if it fulfills De Morgan laws. Some new important results about t-norm and t-conorm theory are discussed and many of them are not readily found in the literature.

More recently, we can highlight an n-dimensional interval extension of uninorms in [34], a preliminary study in the class of n-dimensional R-implications obtained from representable n-dimensional t-norms is discussed in [47] and the inference schemes making use of n-dimensional fuzzy logic in [48].

Following the results above cited, this paper studies the possibility of dealing with main properties of representable n-dimensional S-implications on $L_n(U)$, exploring their main properties.

1.2. Outline of the paper

The remaining of the paper is set as follows. Section 2 introduces some definitions needed throughout this paper, reporting the main characteristics of fuzzy negations, t-conorms and fuzzy implications.

The concepts structuring the distributive complete lattice $\mathcal{L}_n(U)$ of n-dimensional fuzzy set are reported in Section 3, focusing on the supremum and infimum, both defined w.r.t. the partial natural order, also covering the projection operators and degenerate elements such as the top and bottom elements. In addition, an n-dimensional automorphism on $L(U)$ and their well-known results are both reported.

In Section 4, fuzzy negations on $L_n(U)$ are briefly discussed based on extensions of the main results from [11], including the class of representable and conjugate n-dimensional fuzzy negations.

Section 5 is devoted to the new propositions discussing main properties n-dimensional fuzzy t-conorms, dual and conjugate constructions, projections and examples.

The core of the paper sits in the next three sections. Firstly, in Section 6, the development of the concepts and reasonable properties of n-dimensional fuzzy implications on $L_n(U)$ such as the Moore-continuity, as well as the evidence on properties assuring representability of n-DFI is presented. This section also considers new specific results in the analysis of conjugation operators. In sequence, Section 7 concerns the study of n-dimensional interval fuzzy (S, N)-implications, main characterization of such operators, duality and action of n-dimensional automorphisms. And, Section 7, exploring n-dimensional fuzzy $(\mathcal{S}, \mathcal{N})$-implication in AR, presenting inference schemes, compositional rule-base and exemplification.

The Conclusion highlights main results and briefly comments on further work.

2. Preliminaries

In this section, we will briefly review some basic concepts of FL, concerning the study of n-dimensional intervals, which can be found in [10] and [15].

2.1. Fuzzy negations

A function $N\colon U \to U$ is a fuzzy negation (shortly FN) if

N1 $N(0) = 1$ and $N(1) = 0$;
N2 If $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in U$.

And, a continuous FN is strict [25], when

N3a $x > y$ then $N(x) < N(y)$, $\forall x, y \in U$.

Involutive FNs are called strong FN (shortly SFN):

N3 $N(N(x)) = x$, $\forall x \in U$.

Definition 2.1. Let N be a FN and $f\colon U^n \to U$ be a real function. The N-dual function of f is given by the expression:

$$f_N(\mathbf{x}) = N(f(N(\mathbf{x}))), \forall \mathbf{x} = (x_1, \ldots, x_n) \in U^n, \quad (1)$$

where $N(\mathbf{x}) = (N(x_1), \ldots, N(x_n)) \in U^n$.

Notice that, when N is involutive, $(f_N)_N = f$, that is the N-dual function of f_N coincides with f. In addition, if $f = f_N$ then it is clear that f is a self-dual function.

2.2. Triangular conorm

A function $S : U^2 \to U$ is a triangular-conorm (t-conorm) if and only if it satisfies, for all $x, y, z \in U$, the following properties.

S1 : $S(x, 0) = x$ (neutral element);
S2 : $S(x, y) = S(y, x)$ (commutativity);
S3 : $S(x, S(y, z)) = S(S(x, y), z)$ (associativity);
S4 : if $x \leq x'$, $S(x, y) \leq S(x', y)$ (monotonicity).

The notion of a triangular t-norm $T : U^2 \to U$ can be analogously defined by properties from **T2** to **T4**, with the property **S1** replaced by **T1**: $T(x, 1) = x$, for all $x, y, z \in U$.

Remark 2.1. Let N be a fuzzy negation on U. In the sense of Eq. (1), the N-dual function of a t-conorm S, i.e. S_N, is a t-norm if and only if N is strong. Conversely, the N-dual function of a t-norm T, i.e. T_N, is a t-conorm if and only if N is strong. In this case, the pairs (S, S_N) and (T_N, T) are called of N-mutual duals.

Example 2.1. Let be the SFN $N_S(x) = 1 - x$ and $k \in \mathbb{N}^+$ related to pairs of N_S-mutual dual aggregations:

$$S_M(x, y) = \max(x, y); \qquad T_M(x, y) = \min(x, y);$$
$$S_P(x, y) = x + y - xy; \qquad T_P(x, y) = xy;$$
$$S_{LK}(x, y) = \min(x + y, 1); \qquad T_{LK}(x, y) = \max(x + y - 1, 0);$$
$$S_{nM}(x, y) = \begin{cases} 1, x + y \geq 1, \\ \max(x, y), otherwise; \end{cases} \qquad T_{nM}(x, y) = \begin{cases} 0, x + y \leq 1, \\ \min(x, y), otherwise; \end{cases}$$
$$S^k_Y(x, y) = \min(\sqrt[k]{x^k + y^k}, 1); \qquad T^k_Y(x, y) = \max(\sqrt[k]{1 - (1 - x)^k + (1 - y)^k}, 0).$$

The following comparisons can be requested:

(i) By [26], the following holds: $S_M \leq S_P \leq S_{LK} \leq S_{nM}$ and $S^m_Y \leq S^k_Y$ when $0 < k \leq m$;
(ii) Since $S \leq S'$ implies that $S_N \leq S'_N$ for any t-conorm S and S' and fuzzy negation N, we have that $T_{nM} \leq T_{LK} \leq T_P \leq T_M$ and $T^k_Y \leq T^m_Y$ when $0 < k \leq m$.

2.3. Fuzzy implication

A binary function $I : U^2 \to U$ is a *fuzzy implicator* if I meets the minimal boundary conditions:

I0(a): $I(1, 1) = I(0, 1) = I(0, 0) = 1$; **I0(b)**: $I(1, 0) = 0$;

Definition 2.2. [22, Definition 1.15] An implicator $I : U^2 \to U$ is a *fuzzy implication* if I also satisfies the conditions:

I1: If $x \leq z$ then $I(x, y) \geq I(z, y)$ (first place antitonicity);
I2: If $y \leq z$ then $I(x, y) \leq I(x, z)$ (second place isotonicity).

Let $I(L(U))$ be the family of fuzzy implication on $L(U)$.

Several reasonable properties may be required for fuzzy implications. The properties considered in this paper are listed below and have been extensively studied, see more details in [7,19,39]:

I3: $I(1, y) = y$ (left neutrality principle);
I4: $I(x, 1) = 1$ (dominance of truth of consequent);
I5: $I(x, I(y, z)) = I(y, I(x, z))$ (exchange principle);
I6: $I(x, y) = I(x, I(x, y))$ (iterative boolean-like law);
I7: $I(x, I(y, x)) = 1$ (first axiom of Hilbert system);
I8: $I(x, y) \geq y$ (consequent boundary);
I8: $N_I(x) = I(x, 0)$ is a SFN (natural-negation);
I9: $I(x, y) = I(N(y), N(x))$ (contrapositivity property w.r.t. a FN N and denoted by $CP(N)$);
I9a: $I(x, N(y)) = I(y, N(x))$ (right-contrapositivity w.r.t. a FN N and denoted by $R{-}CP(N)$);
I9b: $I(N(x), y) = I(N(y), x)$ (left-contrapositivity w.r.t. a FN N and denoted by $L{-}CP(N)$);
I10: $I(0, y) = 1$ (dominance of falsity);
I11: $I(x, x) = 1$ (identity principle);
I12: $I(x, y) = 1$ iff $x \leq y$ (ordering property);
I13: $N_I(x) = I(x, 0)$ is a FN;
I13a: $N_I(x) = I(x, 0)$ is a SFN;
I13b: $N_I(x) = I(x, 0)$ is a continuous FN;

I13c: $N_I(x) = I(x, 0)$ is a right invertible[1] FN.

2.4. (S, N)-implication

An (S, N)-implication $I_{S,N} : U^2 \to U$ is defined by the expression:

$$I_{S,N}(x, y) = S(N(x), y), \forall x, y \in U, \tag{2}$$

whenever S is a t-conorm and N is a fuzzy negation. This function is a fuzzy implication which generalizes the following classical logical equivalence: $p \to q \equiv \neg p \vee q$. When N is a strong fuzzy negation, then $I_{S,N}$ is a strong implication referred as S-implication. The name S-implication was firstly introduced in the fuzzy logic framework by [42].

Proposition 2.1. *[7, Theorem 2.4.12] Let $I : U^2 \to U$ be a function. I is an S-implication if and only if the properties **I1**, **I5** and **I13a** are met.*

3. *n*-Dimensional fuzzy sets

In [38], You-guang Shang et al. introduce a new extension of fuzzy sets, namely n-dimensional fuzzy sets in order to generalize in a natural way other two extensions: Interval-valued fuzzy sets [45,37,35] and 3-dimensional fuzzy sets [27]. In sequence, Benjamin Bedregal et al. proposed in [11] the following alternative definition for n-dimensional fuzzy sets:

Let X be a nonempty set, $U = [0, 1]$, $n \in \mathbb{N} - \{0\}$ and $\mathbb{N}_n = \{1, 2, \ldots, n\}$. An n-dimensional fuzzy set A over X is given by

$$A = \{(\mathrm{x}, \mu_{A_1}(\mathrm{x}), \ldots, \mu_{A_n}(\mathrm{x})) : \mathrm{x} \in X\},$$

when, for $i = 1, \ldots, n$, the i-th membership degree of A denoted as $\mu_{A_i} : X \to U$ verifies the condition $\mu_{A_1}(\mathrm{x}) \leq \ldots \leq \mu_{A_n}(\mathrm{x})$, for all $\mathrm{x} \in X$.

In [10], for $n \geq 1$, n-dimensional upper simplex is given as

$$L_n(U) = \{\mathbf{x} = (x_1, \ldots, x_n) \in U^n : x_1 \leq \ldots \leq x_n\}, \tag{3}$$

and its elements are called n-dimensional intervals. For each $i = 1, \ldots, n$, the function $\pi_i : L_n(U) \to U$ defined by $\pi_i(x_1, \ldots, x_n) = x_i$ is called of i-th projection of $L_n(U)$.

An element $\mathbf{x} \in L_n(U)$ is degenerated if

$$\pi_i(\mathbf{x}) = \pi_j(\mathbf{x}), \forall i, j \in \mathbb{N}_n, \tag{4}$$

so, a degenerate element $(x, \ldots, x) \in L_n(U)$ will be denoted by $/x/$.

Remark 3.1. The natural order also called the product order on $L_n(U)$ is defined for each $\mathbf{x}, \mathbf{y} \in L_n(U)$, as follows:

$$\mathbf{x} \leq \mathbf{y} \text{ if and only if } \pi_i(\mathbf{x}) \leq \pi_i(\mathbf{y}), \forall i \in \mathbb{N}_n. \tag{5}$$

In addition, $(L_n(U), \leq)$ is a distributive complete lattice [10]. Additionally, for each $i = 1, \ldots, n$ and for all $\mathbf{x}, \mathbf{y} \in L_n(u)$ the following partial order is also considered

$$\mathbf{x} \preceq \mathbf{y} \Leftrightarrow \mathbf{x} = \mathbf{y} \text{ or } \pi_n(\mathbf{x}) \leq \pi_1(\mathbf{y}). \tag{6}$$

Moreover, one can easily observe that $\preceq$ is more restrictive than $\leq$, meaning that $\mathbf{x} \preceq \mathbf{y} \Rightarrow \mathbf{x} \leq \mathbf{y}$.

According to Bedregal et al. in [11], $\mathcal{L}_n(U) = (L_n(U), \vee, \wedge, /0/, /1/)$ is a distributive complete lattice with $/0/$ and $/1/$ being their bottom and top element, respectively, and $\vee$ and $\wedge$ the supremum and infimum w.r.t. the product order. By [10], for all $\mathbf{x}, \mathbf{y} \in L_n(U)$, the supremum and infimum on $\mathcal{L}_n(U)$ are given as:

$$\mathbf{x} \vee \mathbf{y} = (\max(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, \max(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))) \tag{7}$$

$$\mathbf{x} \wedge \mathbf{y} = (\min(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, \min(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))). \tag{8}$$

[1] A function $f : A \to B$ is right invertible if there exists a function $g : B \to A$ such that $f(g(x)) = x$ for reach $x \in B$.

3.1. Automorphisms and conjugate functions on $L_n(U)$

According to [15] and [36], an n-dimensional automorphism on $L(U)$ and their well-known results are both reported below:

Definition 3.1. A function $\varphi : L_n(U) \to L_n(U)$ is an n-dimensional automorphism, (n-DA) if φ is bijective and the following condition is satisfied

$$\mathbf{x} \le \mathbf{y} \Leftrightarrow \varphi(\mathbf{x}) \le \varphi(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in L_n(U). \tag{9}$$

The family of all automorphism on U and $L_n(U)$ are denoted by $Aut(U)$ and $Aut(L_n(U))$, respectively.

Proposition 3.1. *[11, Theorem 3.4] Given a function $\varphi : L_n(U) \to L_n(U)$, $\varphi \in Aut(L_n(U))$ if and only if there exists $\psi \in Aut(U)$ such that*

$$\varphi(\mathbf{x}) = (\psi(\pi_1(\mathbf{x})), \dots, \psi(\pi_n(\mathbf{x}))), \forall \mathbf{x} \in L_n(U)$$

and, in this case, denote φ by $\widetilde{\psi}$.

Corollary 3.1. *Each n-DA is continuous and strictly increasing.*

Remark 3.2. According to [11, Proposition 3.4], given a $\psi \in Aut(U)$, we have that $\widetilde{\psi}^{-1} = \widetilde{\psi^{-1}}$ and therefore $\widetilde{\psi}^{-1} \in Aut(L_n(U))$, i.e. the inverse of n-dimensional automorphism always exists and it is also an n-dimensional automorphism.

Moreover, when $\varphi \in Aut(L_n(U))$ and $F, F^\varphi : (L_n(U))^k \to L_n(U)$, the function F^φ is called the conjugate of F if for each $\mathbf{x}_1, \dots, \mathbf{x}_k \in L_n(U)$ is verified that

$$F^\varphi(\mathbf{x}_1, \dots, \mathbf{x}_k) = \varphi^{-1}(F(\varphi(\mathbf{x}_1), \dots, \varphi(\mathbf{x}_k))). \tag{10}$$

4. Fuzzy negations on $L_n(U)$

The notion of fuzzy negation on U was extended to $L_n(U)$ in [11], as follows:

Definition 4.1. A function $\mathcal{N} : L_n(U) \to L_n(U)$ is an n-dimensional interval fuzzy negation (n-DN) if it satisfies the following properties:

$\mathcal{N}\mathbf{1}$: $\mathcal{N}(/0/) = /1/$ and $\mathcal{N}(/1/) = /0/$;
$\mathcal{N}\mathbf{2}$: If $\mathbf{x} \le \mathbf{y}$ then $\mathcal{N}(\mathbf{x}) \ge \mathcal{N}(\mathbf{y})$, for all $\mathbf{x}; \mathbf{y} \in U$.

Based on [31], an n-DN $\mathcal{N}$ is strict if it is a continuous function[2] verifying the strict inequality:

$\mathcal{N}\mathbf{3(a)}$: $\mathcal{N}(\mathbf{x}) < \mathcal{N}(\mathbf{y})$ when $\mathbf{y} < \mathbf{x}$.

Additionally, $\mathcal{N}$ is a strong n-DN if $\mathcal{N}$ verifies the involutive property:

$\mathcal{N}\mathbf{3}$: $\mathcal{N}(\mathcal{N}(\mathbf{x})) = \mathbf{x}$.

In [13, Prop. 3.8] was proved that each strong n-DN is also strict.

Example 4.1. The following unary functions on $L_n(U)$ are examples of n-DN.

1. $\mathcal{N}_S(\mathbf{x}) = (1 - \pi_n(\mathbf{x}), \dots, 1 - \pi_1(\mathbf{x}))$;
2. $\mathcal{N}_R(\mathbf{x}) = (\sqrt{1 - \pi_n(\mathbf{x})^2}, \dots, \sqrt{1 - \pi_1(\mathbf{x})^2})$;
3. $\mathcal{N}_{SR}(\mathbf{x}) = (1 - \pi_n(\mathbf{x}), \dots, 1 - \pi_{\lfloor \frac{n}{2} \rfloor + 1}(\mathbf{x}), \sqrt{1 - \pi_{\lfloor \frac{n}{2} \rfloor}(\mathbf{x})^2}, \dots, \sqrt{1 - \pi_1(\mathbf{x})^2})$;
4. $\mathcal{N}_{S^2}(\mathbf{x}) = (1 - \pi_n(\mathbf{x})^2, \dots, 1 - \pi_1(\mathbf{x})^2)$;

[2] For more details on continuity of $L_n(U)$-valued function see [33,34].

5. $\mathcal{N}_{\perp}(\mathbf{x}) = \begin{cases} /1/ & \text{if } \mathbf{x} = /0/, \\ /0/ & \text{otherwise.} \end{cases}$

Notice that $\mathcal{N}_S$ and $\mathcal{N}_R$ are strong n-DN whereas $\mathcal{N}_{SR}$ and $\mathcal{N}_{S^2}$ are strict n-DN. Moreover, $\mathcal{N}_{\perp}$ is a non-continuous n-DN.

Remark 4.1. Let $\mathcal{N}_1, \mathcal{N}_2 : L_n(U) \to L_n(U)$. If $\mathcal{N}_1 \circ \mathcal{N}_2 = Id_{L_n(U)}$ then, in this case, we called $\mathcal{N}_1$ as a left inverse of $\mathcal{N}_2$ and $\mathcal{N}_2$ as a right inverse of $\mathcal{N}_1$.

In addition, one can observe that not all n-DN has a right or left inverse, e.g. $\mathcal{N}_{\perp}$. In addition, a (left) right inverse of an n-DN $\mathcal{N}$, if there exists one, it can not be an n-DN. Indeed, consider the n-DN $\mathcal{N}_1 : L_n(U) \to L_n(U)$ given as follows:

$$\mathcal{N}_1(\mathbf{x}) = \begin{cases} \left(\frac{0.9-\pi_n(\mathbf{x})}{0.8}, \dots, \frac{0.9-\pi_1(\mathbf{x})}{0.8}\right) & \text{if } /0.1/ \leq \mathbf{x} \leq /0.9/ \\ /0/ & \text{if } \mathbf{x} \not\leq /0.9/ \\ /1/ & \text{otherwise.} \end{cases}$$

The function $\mathcal{N}_1$ is the right inverse of the function $\mathcal{N}_2(\mathbf{x}) = \mathcal{N}_S(/0.8/ \cdot \mathbf{x} + /0.1/)$, which is not an n-DN because $\mathcal{N}_2(/0/) = /0.9/$.

So, the results from Remark 4.1 motivate us to the following definition of a (left) right invertible operator:

Definition 4.2. An n-DN $\mathcal{N}$ is (left) right invertible if there exists a (left) right inverse $\mathcal{N}'$ which also is an n-DN.

Proposition 4.1. *[11, Proposition 3.1] If $N_1, \dots, N_n$ are fuzzy negations such that $N_1 \leq \dots \leq N_n$. Then $\widetilde{N_1 \dots N_n} : L_n(U) \to L_n(U)$ given by*

$$\widetilde{N_1 \dots N_n}(\mathbf{x}) = (N_1(\pi_n(\mathbf{x})), \dots, N_n(\pi_1(\mathbf{x}))) \tag{11}$$

is a representable n-DN and $(N_1, \dots, N_n)$ their representants.

Proposition 4.2. *[15, Proposition 3.3] Let $\mathcal{N}$ be an n-DN. The function $\mathcal{N}_{(i)} : U \to U$ is a fuzzy negation given by*

$$\mathcal{N}_{(i)}(x) = \pi_i(\mathcal{N}(/x/)), \forall i = 1, \dots, n. \tag{12}$$

Remark 4.2. In particular, if $\mathcal{N}$ is a representable n-DN then $(\mathcal{N}_{(1)}, \dots, \mathcal{N}_{(n)})$ are their representants [15]. Observe that a representable n-DN $\mathcal{N}$ is strict if and only if their representants are strict fuzzy negations [31, Propositions 4.2 and 4.3].

Proposition 4.3. *Let $\mathcal{N}$ be a representable n-DN. Then $\mathcal{N}$ is right invertible if and only if $\mathcal{N}_{(i)}$ is right invertible for each $i \in \mathbb{N}$.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{N}^{-r}$ be the n-DN which is the right inverse of $\mathcal{N}$. By Proposition 4.2, $(\mathcal{N}^{-r})_{(n-i+1)}$, is a fuzzy negation for each $i \in \mathbb{N}_n$. In addition, for each $x \in [0, 1]$,

$$\begin{aligned} \mathcal{N}_{(i)}((\mathcal{N}^{-r})_{(n-i+1)}(x)) &= \mathcal{N}_{(i)}(\pi_{(n-i+1)}(\mathcal{N}^{-r}(/x/)/)) \text{ by (12)} \\ &= \pi_i(\mathcal{N}(\mathcal{N}^{-r}(/x/))) \text{ by Remark 4.2 and (11)} \\ &= \pi_i(/x/) = x. \end{aligned}$$

Therefore, $\mathcal{N}_{(i)}$ is right invertible for each $i \in \mathbb{N}_n$.

($\Leftarrow$) By Remark 4.2, $\mathcal{N} = \widetilde{\mathcal{N}_{(1)}, \dots, \mathcal{N}_{(n)}}$. So, let N_i^{-r} be the right inverse of $\mathcal{N}_{(i)}$ for each $i \in \mathbb{N}_n$. First observe that, if $i \leq j$ then $N_i^{-r} \geq N_j^{-r}$. So, by (11), the following holds: $\mathcal{N}(\widetilde{N_n^{-r} \dots N_1^{-r}}(\mathbf{x})) = \mathcal{N}(N_n^{-r}(\pi_n(\mathbf{x})), \dots, N_1^{-r}(\pi_1(\mathbf{x}))) = (\mathcal{N}_{(1)}(N_1^{-r}(\pi_1(\mathbf{x}))), \dots, \mathcal{N}_{(n)}(N_n^{-r}(\pi_n(\mathbf{x})))) = \mathbf{x}$, for each $\mathbf{x} \in L_n(U)$. $\square$

Remark 4.3. From the proof of Proposition 4.3 we have that if $\mathcal{N}$ is right invertible and representable then their right inverse $\mathcal{N}^{-r}$ also is representable and $(\mathcal{N}^{-r})_{(n-i+1)}$ is the right inverse of $\mathcal{N}_{(i)}$ for each $i \in \mathbb{N}$.

Proposition 4.4. *If $\mathcal{N}$ is a strict n-DN then, there exists a strict n-DN $\mathcal{N}^{-1}$ such that $\mathcal{N} \circ \mathcal{N}^{-1} = \mathcal{N}^{-1} \circ \mathcal{N} = Id_{L_n(U)}$. In addition, if $\mathcal{N}$ is a representable n-DN then $\mathcal{N}^{-1}$ is also a representable n-DN in $L_n(U)$.*

Proof. Since $\mathcal{N}$ is a strict and representable n-DN, then by Remark 4.2, $\mathcal{N}(\mathbf{x}) = (\mathcal{N}_{(1)}, \dots, \mathcal{N}_{(n)})(\mathbf{x})$ and for each $i \in \mathbb{N}_n$, $\mathcal{N}_{(i)}$ is a strict fuzzy negation and therefore has an inverse $\mathcal{N}_{(i)}^{-1}$. Trivially, if $1 \leq i \leq j \leq n$ then $\mathcal{N}_{(i)}^{-1} \leq \mathcal{N}_{(j)}^{-1}$. So, by Proposition 4.1 and Remark 4.2, $\widetilde{\mathcal{N}_{(1)}^{-1} \dots \mathcal{N}_{(n)}^{-1}}$ is a strict representable n-DN. Then, we obtain that

$$\mathcal{N} \circ \widetilde{\mathcal{N}_{(1)}^{-1} \ldots \mathcal{N}_{(n)}^{-1}}(\mathbf{x}) = \widetilde{\mathcal{N}_{(1)} \ldots \mathcal{N}_{(n)}}(\widetilde{\mathcal{N}_{(1)}^{-1} \ldots \mathcal{N}_{(n)}^{-1}}(\mathbf{x})) = \widetilde{\mathcal{N}_{(1)} \ldots \mathcal{N}_{(n)}}(\mathcal{N}_{(1)}^{-1}(\pi_n(\mathbf{x})), \ldots, \mathcal{N}_{(n)}^{-1}(\pi_1(\mathbf{x})))$$
$$= (\mathcal{N}_{(1)} \circ \mathcal{N}_{(1)}^{-1}(\pi_1(\mathbf{x})), \ldots, (\mathcal{N}_{(n)} \circ \mathcal{N}_{(n)}^{-1}(\pi_n(\mathbf{x})) = (\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x})) = \mathbf{x}.$$

Therefore, $\mathcal{N} \circ \widetilde{\mathcal{N}_{(1)}^{-1} \ldots \mathcal{N}_{(n)}^{-1}} = Id(L_n(U))$ which means that $\mathcal{N}^{-1} = \widetilde{\mathcal{N}_{(1)}^{-1} \ldots \mathcal{N}_{(n)}^{-1}}$. Hence, $\mathcal{N}^{-1}$ is a strict representable n-DN. □

The family of all n-DN will be denoted by $\mathcal{N}(L_n(U))$. Let N be fuzzy negations and $\widetilde{N \ldots N}$ will be denoted just as $\widetilde{N}$.

Theorem 4.1. *[15, Theorem 3.3] A function $\mathcal{N} : L_n(U) \to L_n(U)$ is a strong n-DN if and only if there exists a strong fuzzy negation N such that $\mathcal{N} = \widetilde{N}$.*

Thus, for the strong n-DN in the Example 4.1, we have that $\mathcal{N}_S = \widetilde{N_S}$ where N_S is the standard fuzzy negation $N_S(x) = 1 - x$ and $\mathcal{N}_{S^2} = \widetilde{N_{S^2}}$ where $N_{S^2}(x) = \sqrt{1 - x^2}$.

Proposition 4.5. *[15, Proposition 4.2] Let $\varphi \in Aut(L_n(U))$. $\mathcal{N}$ is (strict, strong) n-DN if and only if $\mathcal{N}^{\varphi}$ is an (strict, strong) n-DN.*

Proposition 4.6. *[15, Proposition 4.3, Theorem 4.2] A function $\mathcal{N} : L_n(U) \to L_n(U)$ is a strong n-DN if and only if there exists an automorphism ψ such that $\mathcal{N} = \widetilde{N_S^{\psi}} = \widetilde{N_S}^{\widetilde{\psi}}$.*

5. Triangular conorms on $L_n(U)$

In [11], the notion of aggregation function was extended for n-dimensional intervals, as follows:

Definition 5.1. [11] Let m and n be positive natural numbers such that $m \geq 2$. A function $P : (L_n(U))^m \to L_n(U)$ is an n-dimensional m-ary aggregation function, if $P(/0/, \ldots, /0/) = /0/$, $P(/1/, \ldots, /1/) = /1/$ and for each $\mathbf{x_1}, \ldots, \mathbf{x_m}, \mathbf{y_1}, \ldots, \mathbf{y_m} \in L_n(U)$ such that $\mathbf{x_i} \leq \mathbf{y_i}$ for all $i \in \mathbb{N}_m$ we have that $P(\mathbf{x_1}, \ldots, \mathbf{x_m}) \leq P(\mathbf{y_1}, \ldots, \mathbf{y_m})$.

Based on the relevance of the t-norm and t-conorm classes as bivariate aggregation operators, their extension on $L_n(U)$ were presented in [32]. Thus, their main concepts and results are reported as follows:

Definition 5.2. A function $\mathcal{S} : L_n(U)^2 \to L_n(U)$ is an n-dimensional t-conorm (n-DS) if it verifies, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$, the following properties:

$\mathcal{S}1$: $\mathcal{S}(\mathbf{x}, /0/) = \mathbf{x}$ (neutral element);
$\mathcal{S}2$: $\mathcal{S}(\mathbf{x}, \mathbf{y}) = \mathcal{S}(\mathbf{y}, \mathbf{x})$ (commutativity);
$\mathcal{S}3$: $\mathcal{S}(\mathbf{x}, \mathcal{S}(\mathbf{y}, \mathbf{z})) = \mathcal{S}(\mathcal{S}(\mathbf{x}, \mathbf{y}), \mathbf{z})$ (associativity);
$\mathcal{S}4$: if $\mathbf{x} \leq \mathbf{x}'$, $\mathcal{S}(\mathbf{x}, \mathbf{y}) \leq \mathcal{S}(\mathbf{x}', \mathbf{y})$ (monotonicity related to the product order in Eq. (5)).

Let $\mathcal{S}$ be an n-DS and $\mathcal{N}$ an n-DN. A pair $(\mathcal{S}, \mathcal{N})$ satisfies the law of excluded middle (LEM) if

$\mathcal{S}5$: $\mathcal{S}(\mathcal{N}(\mathbf{x}), \mathbf{x}) = /1/, \forall \mathbf{x} \in L_n(U)$.

Analogously, an n-dimensional t-norm (n-DT) $\mathcal{T} : L_n(U)^2 \to L_n(U)$ has $/1/$ as the neutral element, is commutative, associative and a monotonic function with respect to the product order.

According to [11], the conditions under which an n-DS can be obtained from a finite subset of t-conorm $S_i : U^2 \to U$, for $i \in \mathbb{N}_{n-1}$, are reported below.

Proposition 5.1. *[32, Theorem 2.1] Let $T_i, S_i : U^2 \to U$ be t-norms and t-conorms with $i \in \mathbb{N}_n$. If $T_i \leq T_{i+1}$ and $S_i \leq S_{i+1}$ for each $i \in \mathbb{N}_{n-1}$ then the functions $\widetilde{T_1 \ldots T_n}, \widetilde{S_1 \ldots S_n} : L_n(U)^2 \to L_n(U)$ defined by*

$$\widetilde{T_1 \ldots T_n}(\mathbf{x}, \mathbf{y}) = (T_1(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, T_n(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))) \tag{13}$$

and

$$\widetilde{S_1 \ldots S_n}(\mathbf{x}, \mathbf{y}) = (S_1(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, S_n(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))) \tag{14}$$

are, respectively, an n-DT and n-DS called as representable operators.

Proposition 5.2. *Let $\mathcal{S}$ be n-DS and $\mathcal{N}$ be a strong n-DN. Then, $\mathcal{S}_{\mathcal{N}} : L_n(U)^2 \to L_n(U)$ defined as*

$$\mathcal{S}_{\mathcal{N}}(\mathbf{x},\mathbf{y}) = \mathcal{N}(\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y})))$$

is n-DT. In addition if $\mathcal{S}$ is representable then $\mathcal{S}_{\mathcal{N}}$ also is.

Proof. Trivially, $\mathcal{S}_{\mathcal{N}}$ is commutative and has $/1/$ as neutral element. If $\mathbf{y} \leq \mathbf{z}$ then $\mathcal{N}(\mathbf{z}) \leq \mathcal{N}(\mathbf{y})$ and therefore, $\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{z})) \leq \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y}))$. Hence, $\mathcal{S}_{\mathcal{N}}(\mathbf{x},\mathbf{y}) = \mathcal{N}(\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y}))) \leq \mathcal{N}(\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{z}))) = \mathcal{S}_{\mathcal{N}}(\mathbf{x},\mathbf{z})$ and therefore $\mathcal{S}_{\mathcal{N}}$ is increasing. Finally, $\mathcal{S}_{\mathcal{N}}(\mathbf{x},\mathcal{S}_{\mathcal{N}}(\mathbf{y},\mathbf{z})) = \mathcal{N}(\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathcal{N}(\mathcal{S}(\mathcal{N}(\mathbf{y}),\mathcal{N}(\mathbf{z})))))) = \mathcal{N}(\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{S}(\mathcal{N}(\mathbf{y}),\mathcal{N}(\mathbf{z}))) = \mathcal{N}(\mathcal{S}(\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y})),\mathcal{N}(\mathbf{z}))) = \mathcal{S}_{\mathcal{N}}(\mathcal{S}_{\mathcal{N}}(\mathbf{x},\mathbf{y}),\mathbf{z}))$. So, $\mathcal{S}_{\mathcal{N}}$ is associative and therefore is n-DT.
In addition, since $\mathcal{N}$ is a strong n-DN by Theorem 4.1, there exists a strong fuzzy negation N such that $\mathcal{N} = \widetilde{N}$. So, if $\mathcal{S}$ is representable, i.e. $\mathcal{S} = \widetilde{S_1 \ldots S_n}$ for some t-conorms $S_1 \leq \ldots \leq S_n$. Then,

$$\begin{aligned}\mathcal{S}_{\mathcal{N}}(\mathbf{x},\mathbf{y}) &= \mathcal{N}(\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y}))) = (N(S_1(N(\pi_1(\mathbf{x})),N(\pi_1(\mathbf{y})))),\ldots,N(S_n(N(\pi_n(\mathbf{x})),N(\pi_n(\mathbf{y})))) \\ &= ((S_1)_N(\pi_1(\mathbf{x}),\pi_1(\mathbf{y})),\ldots,(S_n)_N(\pi_n(\mathbf{x}),\pi_1(\mathbf{y}))) = \widetilde{(S_1)_N \ldots (S_n)_N}(\mathbf{x},\mathbf{y})\end{aligned}$$

So, by Remark 2.1 and Proposition 5.1, $\mathcal{S}_{\mathcal{N}}$ is a representable n-DT. □

Let S be a t-conorm and T be a t-norm. We will denote $\widetilde{S \ldots S}$ and $\widetilde{T \ldots T}$ just as $\widetilde{S}$ and $\widetilde{T}$, respectively.

Example 5.1. Applying the results from Proposition 5.1, some examples are presented below:

(i) Based on Example 2.1, the following operators $\widetilde{S_M, S_P, S_{LK}, S_{nM}}$ and its corresponding $\mathcal{N}_S$-dual construction $\widetilde{T_{nM}, T_{LK}, T_P T_M}$ are representable 4-DS and 4-DT as on $L_4(U)$;
(ii) Analogously, $\widetilde{S_M, S_P, S_{LK}}$ and its corresponding $\mathcal{N}_S$-dual construction $\widetilde{T_{LK}, T_P, T_M}$ are representable 3-DS and 3-DT as on $L_3(U)$;
(iii) $\widetilde{S_M}$, $\widetilde{S_P}$, $\widetilde{S_{LK}}$ and $\widetilde{S_{nM}}$ are representable n-DS $L_n(U)$;
(iv) $\widetilde{T_M}$, $\widetilde{T_P}$, $\widetilde{T_{LK}}$ and $\widetilde{T_{nM}}$ are representable n-DT $L_n(U)$;
(v) For any $n \in \mathbb{N}^+$, $\widetilde{S_Y^{n+1}, \ldots, S_Y^2}$ are other representable n-DS on $L_n(U)$.

Proposition 5.3. *Each representable n-DS $\mathcal{S}$ has an unique representation.*

Proof. Suppose that $\mathcal{S} = \widetilde{S_1 \ldots S_n}$ and $\mathcal{S} = \widetilde{S'_1 \ldots S'_n}$. Then, from Eq. (14), for each $x, y \in U$, $S_i(x,y) = \pi_i(\mathcal{S}(/x/,/y/)) = S'_i(x,y)$. □

Proposition 5.4. *Let $\mathcal{S}$ be a representable n-DS. Then, for $i \in \mathbb{N}_n$, the function $S_{(i)} : U^2 \to U$ given by*

$$S_{(i)}(x,y) = \pi_i(\mathcal{S}(/x/,/y/))$$

is a t conorm.

Proof. Since $\mathcal{S}$ is a representable n-DS then there exist t-conorms $S_1,\ldots,S_n$ such that $\mathcal{S} = \widetilde{S_1 \ldots S_n}$. The proposition follows, once clearly $S_{(i)} = S_i$ for each $i \in \mathbb{N}_n$. In fact, for each $x, y \in U$, $S_{(i)}(x,y) = \pi_i(\mathcal{S}(/x/,/y/)) = S_i(x,y)$. Therefore, Proposition 5.4 is verified. □

Corollary 5.1. *Let $\mathcal{S}$ be a representable n-DS then $\mathcal{S} = \widetilde{S_{(1)} \ldots S_{(n)}}$.*

Proof. Straightforward from Propositions 5.3 and 5.4. □

The next proposition extends results from [46, Proposition 4].

Proposition 5.5. *[11, Theorem 3.6] Let $\mathcal{S}$ be an n-DS and φ be an n-DA. Then $\mathcal{S}^{\varphi}$ is also an n-DS.*

Proposition 5.6. *Let $\mathcal{S}$ be a representable n-DS and $\psi \in Aut(U)$. Then for each $i \in \mathbb{N}_n$, $(\mathcal{S}_{(i)})^{\psi} = (\mathcal{S}^{\widetilde{\psi}})_{(i)}$.*

Proof. Let $i \in \mathbb{N}_n$ and $x, y \in U$. Then

$$\begin{aligned}(\mathcal{S}^{\widetilde{\psi}})_{(i)}(x,y) &= \pi_i(\mathcal{S}^{\widetilde{\psi}}(/x/,/y/)) = \pi_i(\widetilde{\psi}^{-1}(\mathcal{S}(\widetilde{\psi}(/x/),\widetilde{\psi}(/y/)))) = \pi_i(\widetilde{\psi^{-1}}(\mathcal{S}(/\psi(x)/,/\psi(y)/)) \\ &= \psi^{-1}(\pi_i(\mathcal{S}(/\psi(x)/,/\psi(y)/)) = \psi^{-1}(\mathcal{S}_{(i)}(\psi(x),\psi(y))) = (\mathcal{S}_{(i)})^{\psi}(x,y)\end{aligned}$$

Then, Proposition 5.6 is verified. □

Since, each n-dimensional t-norm $\mathcal{T}$ and t-conorm $\mathcal{S}$ are associative operators, then for each natural number $m \geq 2$, they can be naturally extended for an m-ary n-dimensional aggregation function, as follows:

$$\mathcal{T}_{[m]}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \begin{cases} \mathcal{T}(\mathbf{x}_1, \mathbf{x}_2), & \text{if } m = 2, \\ \mathcal{T}(\mathcal{T}_{[m-1]}(\mathbf{x}_1, \ldots, \mathbf{x}_{m-1}), \mathbf{x}_m) & \text{otherwise; } \textit{and} \end{cases} \tag{15}$$

$$\mathcal{S}_{[m]}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \begin{cases} \mathcal{S}(\mathbf{x}_1, \mathbf{x}_2), & \text{if } m = 2, \\ \mathcal{S}(\mathcal{S}_{[m-1]}(\mathbf{x}_1, \ldots, \mathbf{x}_{m-1}), \mathbf{x}_m) & \text{otherwise;} \end{cases} \tag{16}$$

respectively.

6. Fuzzy implications on $L_n(U)$

This section studies n-dimensional fuzzy implications on the lattice $(L_n(U), \leq)$ introduced in [46] extending this work investigating construction methods of n-dimensional fuzzy implications from fuzzy implications preserving their main properties. Additionally, if $n = 2$, the n-dimensional fuzzy implications are the usual interval-valued fuzzy implications as investigated in [1,9,16] and therefore, their corresponding properties are investigated in the more general n-dimensional interval space.

Definition 6.1. A function $\mathcal{I} : L_n(U)^2 \to L_n(U)$ is an n-dimensional fuzzy implicator if $\mathcal{I}$ meets the following minimal boundary conditions:

$\mathcal{I}$**0(a)**: $\mathcal{I}(/1/, /1/) = \mathcal{I}(/0/, /1/) = \mathcal{I}(/0/, /0/) = /1/$;
$\mathcal{I}$**0(b)**: $\mathcal{I}(/1/, /0/) = /0/$.

Definition 6.2. An n-dimensional fuzzy implicator $\mathcal{I}$ is an n-dimensional fuzzy implication (n-DI), if it also satisfies the properties:

$\mathcal{I}$**1**: $\mathbf{x} \leq \mathbf{z} \Rightarrow \mathcal{I}(\mathbf{x}, \mathbf{y}) \geq \mathcal{I}(\mathbf{z}, \mathbf{y})$ (first-place antitonicity);
$\mathcal{I}$**2**: $\mathbf{y} \leq \mathbf{z} \Rightarrow \mathcal{I}(\mathbf{x}, \mathbf{y}) \leq \mathcal{I}(\mathbf{x}, \mathbf{z})$ (right-place isotonicity).

We also consider the following extra properties for n-DIs:

$\mathcal{I}$**3**: $\mathcal{I}(/1/, \mathbf{y}) = \mathbf{y}$ (left neutrality property);
$\mathcal{I}$**4**: $\mathcal{I}(\mathbf{x}, /1/) = /1/$ (right boundary condition);
$\mathcal{I}$**5**: $\mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y}, \mathbf{z})) = \mathcal{I}(\mathbf{y}, \mathcal{I}(\mathbf{x}, \mathbf{z}))$ (exchange principle);
$\mathcal{I}$**6**: $\mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{x}, \mathbf{y}))$ (iterative Boolean law);
$\mathcal{I}$**7**: $\mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y}, \mathbf{x})) = /1/$ (first axiom of Hilbert system);
$\mathcal{I}$**8**: $\mathcal{I}(\mathbf{x}, \mathbf{y}) \geq \mathbf{y}$ (right boundary condition);
$\mathcal{I}$**9**: $\mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{y}), \mathcal{N}(\mathbf{x}))$ (contraposition property w.r.t. an n-DN $\mathcal{N}$ and denoted by $CP(\mathcal{N})$.

And, two other conditions are required in $\mathcal{I}$9, meaning that

$\mathcal{I}$**9(a)**: $\mathcal{I}(\mathbf{x}, \mathcal{N}(\mathbf{y})) = \mathcal{I}(\mathbf{y}, \mathcal{N}(\mathbf{x}))$ (right-contraposition property w.r.t. an n-DN $\mathcal{N}$ and denoted by $R{-}CP(\mathcal{N})$);
$\mathcal{I}$**9(b)**: $\mathcal{I}(\mathcal{N}(\mathbf{x}), \mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{y}), \mathbf{x})$ (left-contraposition property w.r.t. an n-DN $\mathcal{N}$ and denoted by $L{-}CP(\mathcal{N})$);
$\mathcal{I}$**10**: $\mathcal{I}(/0/, \mathbf{y}) = /1/$ (left boundary condition);
$\mathcal{I}$**11**: $\mathcal{I}(\mathbf{x}, \mathbf{x}) = /1/$ (identity principle).
$\mathcal{I}$**12**: $\mathbf{x} \leq \mathbf{y} \Leftrightarrow \mathcal{I}(\mathbf{x}, \mathbf{y}) = /1/$ (ordering principle);
$\mathcal{I}$**13**: $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) = \mathcal{I}(\mathbf{x}, /0/)$ is an n-DN.

Moreover, additional conditions are required in $\mathcal{I}$13, meaning that new properties related to natural negations can be discussed as follows:

$\mathcal{I}$**13(a)**: $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) = \mathcal{I}(\mathbf{x}, /0/)$ is a strong n-DN;
$\mathcal{I}$**13(b)**: $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) = \mathcal{I}(\mathbf{x}, /0/)$ is a continuous n-DN;
$\mathcal{I}$**13(c)**: $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) = \mathcal{I}(\mathbf{x}, /0/)$ is a right invertible n-DN.

Proposition 6.1. *Each n-dimensional fuzzy implication satisfies $\mathcal{I}$4, $\mathcal{I}$10, and $\mathcal{I}$13.*

Proof. Let $\mathbf{x}, \mathbf{y} \in L_n(U)$. Then
$\mathcal{I}4$: By $\mathcal{I}1$, $/1/ = \mathcal{I}(/0/, /1/) \leq \mathcal{I}(\mathbf{x}, /1/)$;
$\mathcal{I}10$: By $\mathcal{I}2$, $/1/ = \mathcal{I}(/0/, /0/) \leq \mathcal{I}(/0/, \mathbf{x})$;
$\mathcal{I}13$: $\mathcal{N}_{\mathcal{I}}(/0/) = \mathcal{I}(/0/, /0/) = /1/$; and $\mathcal{N}_{\mathcal{I}}(/1/) = \mathcal{I}(/1/, /0/) = /0/$;
If $\mathbf{x} \leq \mathbf{y}$ then, by $\mathcal{I}1$, $\mathcal{N}_{\mathcal{I}}(\mathbf{y}) = \mathcal{I}(\mathbf{y}, /0/) \leq \mathcal{I}(\mathbf{x}, /0/) = \mathcal{N}_{\mathcal{I}}(\mathbf{x})$. Therefore, Proposition 6.1 is verified. □

Lemma 6.1. *Let $\mathcal{I} : L_n(U)^2 \to L_n(U)$ be a n-DI and Let $\mathcal{N} : L_n(U) \to L_n(U)$ be a strong n-DN. The following statements are equivalent:*

(i) *$\mathcal{I}$ verify $CP(\mathcal{N})$;*
(ii) *$\mathcal{I}$ verify $L{-}CP(\mathcal{N})$;*
(iii) *$\mathcal{I}$ verify $R{-}CP(\mathcal{N})$.*

Proof. Straightforward. □

Proposition 6.2. *Let $\mathcal{I} : L_n(U)^2 \to L_n(U)$ be a n-dimensional fuzzy implicator which satisfies $\mathcal{I}13(\mathbf{a})$.*

(i) *If $\mathcal{I}$ verifies $CP(\mathcal{N}_{\mathcal{I}})$, then $\mathcal{I}$ verifies $\mathcal{I}3$.*
(ii) *If $\mathcal{I}$ verifies $\mathcal{I}5$, then verifies $\mathcal{I}0(a)$, $\mathcal{I}0(b)$, $\mathcal{I}3$, $CP(\mathcal{N}_{\mathcal{I}})$, $R{-}CP(\mathcal{N}_{\mathcal{I}})$ and $L{-}CP(\mathcal{N}_{\mathcal{I}})$.*

Proof. Since $\mathcal{I}$ satisfies $\mathcal{I}13(\mathbf{a})$, then $\mathcal{N}_{\mathcal{I}}$ is a strong n-DN and therefore satisfies $\mathcal{N}3$. Then,

(i) for each $\mathbf{y} \in L_n(U)$, definition of $\mathcal{N}_{\mathcal{I}}$, we have that:

$$\mathbf{y} = \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{y})) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{y}), \mathcal{N}_{\mathcal{I}}(/1/)) = \mathcal{I}(/1/, \mathbf{y}) \text{ by } \mathcal{N}3\text{ , }\mathcal{N}1 \text{ and } CP(\mathcal{N}_{\mathcal{I}})\text{, respectively.}$$

(ii) By definition of $\mathcal{N}_{\mathcal{I}}$ we have that

$$\mathcal{I}0(a): \ \mathcal{I}(/0/, /0/) = \mathcal{N}_{\mathcal{I}}(/0/) = /1/ \text{ and } \mathcal{I}0(c): \ \mathcal{I}(/1/, /0/) = \mathcal{N}_{\mathcal{I}}(/1/) = /0/$$

$$R{-}CP(\mathcal{N}_{\mathcal{I}}): \ \mathcal{I}(\mathbf{x}, \mathcal{N}_{\mathcal{I}}(\mathbf{y})) = \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y}, /0/)) = \mathcal{I}(\mathbf{y}, \mathcal{I}(\mathbf{x}, /0/)) = \mathcal{I}(\mathbf{y}, \mathcal{N}_{\mathcal{I}}(\mathbf{x})) \text{ by } \mathcal{I}5 \text{ and } \mathcal{I}13.$$

So, by Lemma 6.1, $\mathcal{I}$ also satisfies $CP(\mathcal{N}_{\mathcal{I}})$ and $L{-}CP(\mathcal{N}_{\mathcal{I}})$. Then, by above (a) and (b) items, $\mathcal{I}$ satisfies $\mathcal{I}3$. Finally, from $CP(\mathcal{N}_{\mathcal{I}})$, $\mathcal{I}0(b): \ \mathcal{I}(/1/, /1/) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}(/0/), \mathcal{N}_{\mathcal{I}}(/0/)) = \mathcal{I}(/0/, /0/) = /1/$.

Concluding, Proposition 6.2 is verified. □

Proposition 6.3. *Let $\mathcal{I} : L_n(U)^2 \to L_n(U)$ be an n-DI such that properties $\mathcal{I}5$ and $\mathcal{I}13(c)$ are verified. Then $\mathcal{I}$ verifies $L{-}CP(\mathcal{N}_{\mathcal{I}}^{-r})$ where $\mathcal{N}_{\mathcal{I}}^{-r}$ is the right inverse of $\mathcal{N}_{\mathcal{I}}$.*

Proof. Let $\mathbf{x}, \mathbf{y} \in L_n(U)$. Then by $\mathcal{I}5$, $\mathcal{I}(\mathcal{N}_{\mathcal{I}}^{-r}(\mathbf{x}), \mathbf{y}) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}^{-r}(\mathbf{x}), \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}^{-r}(\mathbf{y}))) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}^{-r}(\mathbf{y}), \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}^{-r}(\mathbf{x}))) = \mathcal{I}(\mathcal{N}_{\mathcal{I}}^{-r}(\mathbf{y}), \mathbf{x})$. Therefore, Proposition 6.3 is verified. □

Proposition 6.4. *If an n-DI $\mathcal{I}$ satisfies $\mathcal{I}5$ and $\mathcal{I}12$ then for each $\mathbf{x} \in L_n(U)$ we have that*

1. *$\mathbf{x} \leq \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}))$;*
2. *$\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}))) \leq \mathcal{N}_{\mathcal{I}}(\mathbf{x})$.*

Proof. Let $\mathbf{x} \in L_n(U)$, the following holds:

$$\begin{aligned}\mathcal{I}(\mathbf{x}, \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}))) &= \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathcal{I}(\mathbf{x}, /0/), /0/)) = \mathcal{I}(\mathcal{I}(\mathbf{x}, /0/), \mathcal{I}(\mathbf{x}, /0/)) \text{ by } \mathcal{I}5 \\ &= \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}), \mathcal{N}_{\mathcal{I}}(\mathbf{x})) = /1/ \text{ by } \mathcal{I}12.\end{aligned}$$

So, $\mathcal{I}(\mathbf{x}, \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}))) = /1/$ and then, by $\mathcal{I}12$, $\mathbf{x} \leq \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}))$. In addition, since N_I is decreasing, it implies that $\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}))) \leq \mathcal{N}_{\mathcal{I}}(\mathbf{x})$ for each $\mathbf{x} \in L_n(U)$. Therefore, Proposition 6.4 is verified. □

6.1. Representable n-DI on $L_n(U)$

Proposition 6.5. *[46, Prop. 6] Let $I_1, \ldots, I_n : U^2 \to U$ be functions such that $I_1 \leq \ldots \leq I_n$. Then, for all $\mathbf{x}, \mathbf{y} \in L_n(U)$, the function $\widetilde{I_1 \ldots I_n} : L_n(U)^2 \to L_n(U)$ given by*

$$\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) = (I_1(\pi_n(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, I_n(\pi_1(\mathbf{x}), \pi_n(\mathbf{y}))), \tag{17}$$

is an n-DI (n-dimensional fuzzy implicator) if and only if $I_1, \ldots, I_n$ are also fuzzy implications (implicators).

Based on Proposition 6.5, $\mathcal{I}$ is called representable n-DI (n-dimensional fuzzy implicator) if there exist fuzzy implications (implicators) $I_1 \leq \ldots \leq I_n$ such that $\mathcal{I} = \widetilde{I_1 \ldots I_n}$. In addition, the n-tuple of implications $(I_1, \ldots, I_n)$ is called a representant of $\mathcal{I}$. Moreover, when $I_1 = \ldots = I_n = I$, expression $\widetilde{I_1 \ldots I_n}$ in (17) is denoted by $\widetilde{I}$.

Remark 6.1. Let $\mathcal{I}(L_n(U))$ be the set of all n-DI. For all $\mathbf{x}, \mathbf{y} \in L_n(U)$ when $\widetilde{I_1 \ldots I_n}, \widetilde{I} \in \mathcal{I}(L_n(U))$, we have that

(i) $\pi_i(\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y})) = I_i(\pi_{n+1-i}(\mathbf{x}), \mathbf{y}_i)$, for $i = 1, \ldots, n$;
(ii) $\pi_i(\widetilde{I_1 \ldots I_n}(/x/, /y/)) = I_i(x, y)$;
(iii) $\pi_i(\widetilde{I}(/x/, /y/)) = I(x, y)$.

The next proposition shows that a conjugate operation w.r.t. an n-DI also is an n-DI.

Proposition 6.6. *Let $\mathcal{I}$ be an n-DI and $\varphi \in Aut(L_n(U))$. Then $\mathcal{I}^\varphi$ also is an n-DI.*

Proof. Trivial, once $\varphi(/0/) = /0/ = \varphi^{-1}(/0/)$, $\varphi(/1/) = /1/ = \varphi^{-1}(/1/)$ and both, φ and φ^{-1}, are increasing functions. □

Lemma 6.2. *Let $\mathcal{I}$ be an n-DI (n-dimensional fuzzy implicator) and $i \in \mathbb{N}_n$. Then the function $\mathcal{I}_{(i)} : [0, 1]^2 \to [0, 1]$ defined by*

$$\mathcal{I}_{(i)} = \pi_i(\mathcal{I}(/x/, /y/)$$

is a fuzzy implication (implicator).

Proof. We have that $\mathcal{I}_{(i)}(0, x) = \pi_i(\mathcal{I}(/0/, /x/)) = \pi(/1/) = 1$, $\mathcal{I}_{(i)}(x, 1) = \pi_i(\mathcal{I}(/x/, /1/)) = \pi(/1/) = 1$ and $\mathcal{I}_{(i)}(1, 0) = \pi_i(\mathcal{I}(/1/, /0/)) = \pi(/0/) = 0$. So, $\mathcal{I}_{(i)}$ satisfies the boundary conditions of fuzzy implications, i.e. it is a fuzzy implicator when $\mathcal{I}$ is an n-dimensional fuzzy implicator. Now, if $x \leq z$ then $/x/ \leq /z/$ and therefore by $\mathcal{I}1$, it holds that $\mathcal{I}_{(i)}(x, y) = \pi_i(\mathcal{I}(/x/, /y/)) \geq \pi_i(\mathcal{I}(/z/, /y/)) = \mathcal{I}_{(i)}(z, y)$. Analogously, it is possible to prove that $\mathcal{I}_{(i)}(x, y) \leq \mathcal{I}_{(i)}(x, z)$ whenever $y \leq z$. And, Lemma 6.2 holds. □

Proposition 6.7. *Let $\mathcal{I}$ be an n-DI (n-dimensional fuzzy implicator), $\psi \in Aut(U)$ and $\varphi = \widetilde{\psi} \in Aut(L_n(U))$. Then the following statements are equivalent:*

1. *$\mathcal{I}$ is representable;*
2. *$\mathcal{I} = \widetilde{\mathcal{I}_{(1)}, \ldots, \mathcal{I}_{(n)}}$;*
3. *$(\mathcal{I}^\varphi)_{(i)} = (\mathcal{I}_{(i)})^\psi$ for each $i \in \mathbb{N}_n$ and $\mathcal{I}^\varphi$ is a representable n-DI (n-dimensional fuzzy implicator).*

Proof. $(1 \Rightarrow 2)$ If $\mathcal{I}$ is representable then there exists fuzzy implications I_i, with $i = 1, \ldots, n$, such that $I_i \leq I_{i+1}$ and $\mathcal{I} = \widetilde{I_1 \ldots I_n}$. Let $x, y \in U$ then $\mathcal{I}_{(i)}(x, y) = \pi_i(\mathcal{I}(/x/, /y/)) = \pi_i(\widetilde{I_1 \ldots I_n}(/x/, /y/)) = I_i(x, y)$. Therefore, $\mathcal{I} = \widetilde{\mathcal{I}_{(1)}, \ldots, \mathcal{I}_{(n)}}$.

$(2 \Rightarrow 3)$ Since, by Proposition 6.6, $\mathcal{I}^\varphi$ is an n-DI then, by Lemma 6.2, $(\mathcal{I}^\varphi)_{(i)}$ for each $i \in \mathbb{N}_n$ is a fuzzy implication (implicator) and $\varphi^{-1} = \widetilde{\psi}^{-1} = \widetilde{\psi^{-1}}$ then

$$\begin{aligned}(\mathcal{I}^\varphi)_{(i)}(x, y) &= \pi_i(\varphi^{-1}(\mathcal{I}(\varphi(/x/), \varphi(/y/)))) = \pi_i(\widetilde{\psi^{-1}}(\mathcal{I}(/\psi(x)/, /\psi(y)/))) \\ &= \psi^{-1}(\pi_i(\mathcal{I}(/\psi(x)/, /\psi(y)/))) = \psi^{-1}(\mathcal{I}_{(i)}(\psi(x), \psi(y)))) = (\mathcal{I}_{(i)})^\psi(x, y).\end{aligned}$$

On the other hand, for each $\mathbf{x}, \mathbf{y} \in L_n(U)$, it holds that

$$\begin{aligned}\mathcal{I}^\varphi(\mathbf{x}, \mathbf{y}) &= \varphi^{-1}(\mathcal{I}(\varphi(\mathbf{x}), \varphi(\mathbf{y}))) = \widetilde{\psi^{-1}}(\widetilde{\mathcal{I}_{(1)} \ldots \mathcal{I}_{(n)}}(\widetilde{\psi}(\mathbf{x}), \widetilde{\psi}(\mathbf{y}))) \\ &= \widetilde{\psi^{-1}}(\mathcal{I}_{(1)}(\pi_n(\widetilde{\psi}(\mathbf{x}), \pi_1(\widetilde{\psi}(\mathbf{y}))), \ldots, \mathcal{I}_{(n)}(\pi_1(\widetilde{\psi}(\mathbf{x}), \pi_n(\widetilde{\psi}(\mathbf{y})))) \\ &= \widetilde{\psi^{-1}}(\mathcal{I}_{(1)}(\psi(\pi_n(\mathbf{x}), \psi(\pi_1(\mathbf{y}))), \ldots, \mathcal{I}_{(n)}(\psi(\pi_1(\mathbf{x}), \psi(\pi_n(\mathbf{y})))) \\ &= ((\mathcal{I}_{(1)})^\psi(\pi_n(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, (\mathcal{I}_{(n)})^\psi(\pi_1(\mathbf{x}), \pi_n(\mathbf{y})) \\ &= \widetilde{(\mathcal{I}_{(1)})^\psi \ldots (\mathcal{I}_{(n)})^\psi}(\mathbf{x}, \mathbf{y}) = \widetilde{(\mathcal{I}^\varphi)_{(1)} \ldots (\mathcal{I}^\varphi)_{(n)}}(\mathbf{x}, \mathbf{y}).\end{aligned}$$

Therefore, $\mathcal{I}^\varphi$ is representable.

$(3 \Rightarrow 1)$ Since $\varphi^{-1} \in Aut(L_n(U))$, then $\mathcal{I} = (\mathcal{I}^\varphi)^{\varphi^{-1}}$ and therefore, since $\mathcal{I}^\varphi$ is representable, then there exist fuzzy implications (implicators) $I_1 \leq \ldots \leq I_n$ such that $\mathcal{I}^\varphi = \widetilde{I_1 \ldots I_n}$ and by 3.1 there exists an automorphism ψ such that $\varphi = \widetilde{\psi}$. So, for each $\mathbf{x}, \mathbf{y} \in L_n(U)$ we have that

$$\begin{aligned}
\mathcal{I}(\mathbf{x},\mathbf{y}) &= (\mathcal{I}^{\varphi})^{\varphi^{-1}}(\mathbf{x},\mathbf{y})\\
&= \varphi(\mathcal{I}^{\varphi}(\varphi^{-1}(\mathbf{x}),\varphi^{-1}(\mathbf{y})))\\
&= \widetilde{\psi}(\widetilde{I_1\ldots I_n}(\widetilde{\psi}^{-1}(\mathbf{x}),\widetilde{\psi}^{-1}(\mathbf{y})))\\
&= \widetilde{\psi}(\widetilde{I_1\ldots I_n}(\widetilde{\psi^{-1}}(\mathbf{x}),\widetilde{\psi^{-1}}(\mathbf{y}))) \text{ by Remark 3.2}\\
&= (I_1^{\psi}(\pi_n(\mathbf{x}),\pi_1(\mathbf{y})),\ldots,I_n^{\psi}(\pi_1(\mathbf{x}),\pi_n(\mathbf{y})))
\end{aligned}$$

and since each I_i^{ψ} is a fuzzy implication (implicator) and $I_1^{\psi} \leq \ldots \leq I_n^{\psi}$ then $\mathcal{I}$ is representable. □

Corollary 6.1. *Each representable n-DI has exactly a unique representant n-tuple of fuzzy implications.*

Corollary 6.2. *For all representable n-DI $\mathcal{I}$ we have that $\mathcal{N}_{\mathcal{I}_{(1)}} \leq \ldots \leq \mathcal{N}_{\mathcal{I}_{(n)}}$.*

Proof. From Proposition 6.7, we have that $\mathcal{I}_{(1)} \leq \ldots \leq \mathcal{I}_{(n)}$. So, for each $x \in [0,1]$ we have that $\mathcal{I}_{(1)}(x,0) \leq \ldots \leq \mathcal{I}_{(n)}(x,0)$ or equivalently that $\mathcal{N}_{\mathcal{I}_{(1)}}(x) \leq \ldots \leq \mathcal{N}_{\mathcal{I}_{(n)}}(x)$. □

6.2. Continuity of n-dimensional fuzzy implications

The condition under which an n-dimensional interval fuzzy implication verifies the continuity on $\mathcal{I}(L_n(U))$ based on the continuity of family $\mathcal{I}(U^n)$ of fuzzy implications on U^n is considered in the following.

Definition 6.3. Let $\phi : U^n \to L_n(U)$ be the $(U^n, L_n(U))$-permutation expressed by the increase ordering, meaning that $\phi(x_1,\ldots,x_n) = (x_{(1)},\ldots,x_{(n)})$ such that $\{(1),\ldots,(n)\} = \{1,\ldots,n\}$ and $x_{(i)} \leq x_{(i+1)}, \forall i = 1,\ldots,n-1$. A function $\mathcal{F} : L_n(U)^2 \to L_n(U)$ is continuous if the related function $\mathcal{F}^{\phi} : U^n \times U^n \to U^n$ given by

$$\mathcal{F}^{\phi}(\mathbf{x},\mathbf{y}) = \mathcal{F}(\phi(\mathbf{x}),\phi(\mathbf{y})) \tag{18}$$

is continuous in the usual sense.

Observe that, since $L_n(U) \subset U^n$, then $\mathcal{F}^{\phi}$ is well defined.

Proposition 6.8. *Let $\mathcal{I}$ be a representable n-DI. Then $\mathcal{I}$ is continuous if and only if $\mathcal{I}_{(i)}$ is continuous for each $i = 1,\ldots,n$.*

Proof. (⇒) Let $\phi : U^n \to L_n(U)$ be the $(U^n, L_n(U))$-permutation and $\delta : U^n \to U^n$ the function $\delta(x_1,\ldots,x_n) = (x_n,\ldots,x_1)$. Since, $\widetilde{I_1\ldots I_n}^{\phi}$ is continuous and $\widetilde{I_1\ldots I_n}^{\phi} = (I_1 \times \cdots \times I_n) \circ ((\delta \circ \phi) \times \phi)$ then each I_i is continuous. So, from Proposition 5.6, $\mathcal{I}_{(i)}$ is continuous for each $i \in \mathbb{N}_n$.

(⇐) If $\mathcal{I}_{(i)}$ is continuous for each $i \in \mathbb{N}_n$, then $\mathcal{I}_{(1)} \times \ldots \times \mathcal{I}_{(n)}$ also is continuous. Therefore, since $\widetilde{\mathcal{I}_{(1)}\ldots\mathcal{I}_{(n)}}^{\phi} = (\mathcal{I}_{(1)} \times \ldots \times \mathcal{I}_{(n)}) \circ ((\delta \circ \phi) \times \phi)$ and δ as well as ψ are continuous, $\widetilde{\mathcal{I}_1(1)\ldots\mathcal{I}_{(n)}}^{\phi}$ is continuous. Hence, by Definition 6.3, $\widetilde{\mathcal{I}_{(1)}\ldots\mathcal{I}_{(n)}}$ is continuous. □

6.3. Other main properties of n-dimensional fuzzy implications

In the following, main properties of fuzzy implications on $L(U)$ are preserved by the representable n-dimensional fuzzy implications on $L_n(U)$.

Proposition 6.9. *[46, Propositions 6 and 11] A representable n-DI $\mathcal{I}$ verifies the property $\mathcal{I}k$, for $k = 1,\ldots,6,8,10$ if and only if each $\mathcal{I}_{(i)}$, with $i \in \mathbb{N}_n$, verifies the corresponding property* **Ik**.

Proposition 6.10. *No representable n-DI satisfies $\mathcal{I}7$.*

Proof. Let $\mathcal{I}$ be a representable n-DI and $\mathbf{x} = (0, \underbrace{1,\ldots,1}_{(n-1)-times})$ we have that

$$\mathcal{I}(\mathbf{x}, \mathcal{I}(/1/,\mathbf{x})) = \mathcal{I}(\mathbf{x}, (\mathcal{I}_{(1)}(1,0), \mathcal{I}_{(2)}(1,1),\ldots,\mathcal{I}_{(n)}(1,1)) = \mathcal{I}(\mathbf{x},\mathbf{x}) = \mathbf{x} \neq /1/.$$

Therefore, Proposition 6.10 is verified. □

Proposition 6.11. *Let $\mathcal{N}$ be a representable n-DN and $\mathcal{I}$ a representable n-DI. The pair $(\mathcal{N},\mathcal{I})$ verifies $\mathcal{I}9$ ($\mathcal{I}9(a)$, $\mathcal{I}9(b)$) and $\mathcal{N}_{(1)}=\ldots=\mathcal{N}_{(n)}=N$) if and only if for each $i=1,\ldots,n$,*

1. *the pair $(\mathcal{N}_{(i)},\mathcal{I}_{(i)})$ verifies corresponding property* **I9(a)**;
2. *the pair $(\mathcal{N}_{(n-i+1)},\mathcal{I}_{(i)})$ verifies corresponding property* **I9(b)**;
3. *$\mathcal{N}_{(1)}=\ldots=\mathcal{N}_{(n)}=N$ and the pair $(N,\mathcal{I}_{(i)})$ verifies corresponding property* **I9**

respectively.

Proof. By Remark 4.2 and Proposition 6.7 we have that $\mathcal{N}=\widetilde{\mathcal{N}_{(1)}\ldots\mathcal{N}_{(n)}}$ and $\mathcal{I}=\widetilde{\mathcal{I}_{(1)}\ldots\mathcal{I}_{(n)}}$.

($\Leftarrow$) Since the pair $(\mathcal{N}_{(i)},\mathcal{I}_{(i)})$ verifies **I9(a)** for each $i=1,\ldots,n$, then the following holds:

$$\begin{aligned}\mathcal{I}9(a):\mathcal{I}(\mathbf{x},\mathcal{N}(\mathbf{y}))&=(\mathcal{I}_{(1)}(\pi_n(\mathbf{x}),\mathcal{N}_{(1)}(\pi_n(\mathbf{y}))),\ldots,\mathcal{I}_{(n)}(\pi_1(\mathbf{x}),\mathcal{N}_{(n)}(\pi_1(\mathbf{y})))) \text{ by (11) and (17)}\\&=(\mathcal{I}_{(1)}(\pi_n(\mathbf{y}),\mathcal{N}_{(1)}(\pi_n(\mathbf{x}))),\ldots,\mathcal{I}_{(n)}(\pi_1(\mathbf{y}),\mathcal{N}_{(n)}(\pi_1(\mathbf{x})))) \text{ by } \mathbf{I9(a)}\\&=\mathcal{I}(\mathbf{y},\mathcal{N}(\mathbf{x})) \text{ by (17) and (11).}\end{aligned}$$

Since, the pair $(\mathcal{N}_{(n-i+1)},\mathcal{I}_{(i)})$ verifies **I9**(b), for each $i=1,\ldots,n$, then we have that:

$$\begin{aligned}\mathcal{I}9(b):\mathcal{I}(\mathcal{N}(\mathbf{y}),\mathbf{x})&=(\mathcal{I}_{(1)}(\mathcal{N}_{(n)}(\pi_1(\mathbf{y})),\pi_1(\mathbf{x})),\ldots,\mathcal{I}_{(n)}(\mathcal{N}_{(1)}(\pi_n(\mathbf{y})),\pi_n(\mathbf{x}))) \text{ by (11) and (17)}\\&=(\mathcal{I}_{(1)}(\mathcal{N}_{(n)}(\pi_1(\mathbf{x})),\pi_1(\mathbf{y})),\ldots,\mathcal{I}_{(n)}(\mathcal{N}_{(1)}(\pi_n(\mathbf{x})),\pi_n(\mathbf{y}))) \text{ by } \mathbf{I9(b)}\\&=\mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y}) \text{ by (11) and (17).}\end{aligned}$$

In addition, since the pair $(N,\mathcal{I}_{(i)})$ verifies **I9**, for each $i=1,\ldots,n$, it holds that:

$$\begin{aligned}\mathcal{I}9:\mathcal{I}(\widetilde{N}(\mathbf{y}),\widetilde{N}(\mathbf{x}))&=(\mathcal{I}_{(1)}(N(\pi_1(\mathbf{y})),N(\pi_n(\mathbf{x}))),\ldots,\mathcal{I}_{(n)}(N(\pi_n(\mathbf{y})),N(\pi_1(\mathbf{x})))) \text{ by (11) and (17)}\\&=(\mathcal{I}_{(1)}(\pi_n(\mathbf{x}),\pi_1(\mathbf{y})),\ldots,\mathcal{I}_{(n)}(\pi_1(\mathbf{x}),\pi_n(\mathbf{y}))) \text{ by } \mathbf{I9}\\&=\mathcal{I}(\mathbf{x},\mathbf{y}) \text{ by (17).}\end{aligned}$$

($\Rightarrow$) Conversely, since $(\mathcal{N},\mathcal{I})$ verifies $\mathcal{I}9(a)$, ($\mathcal{I}9(b)$, $\mathcal{I}9$), we have the following results:

$$\begin{aligned}\mathbf{I9(a)}:\mathcal{I}_{(i)}(y,\mathcal{N}_{(i)}(x))&=\pi_i(\mathcal{I}_{(1)}(y,\mathcal{N}_{(1)}(x)),\ldots,\mathcal{I}_{(n)}(y,\mathcal{N}_{(n)}(x)))\\&=\pi_i(\mathcal{I}(/y/,\mathcal{N}(/x/))) \text{ by (17) and (11)}\\&=\pi_i(\mathcal{I}(/x/,\mathcal{N}(/y/))) \text{ by } \mathcal{I}9\\&=\pi_i(\mathcal{I}_{(1)}(x,\mathcal{N}_{(1)}(y)),\ldots,\mathcal{I}_{(n)}(x,\mathcal{N}_{(n)}(y)))\mathcal{I}_{(i)}(y,\mathcal{N}_{(i)}(x)) \text{ by (11) and (17)}.\\\mathbf{I9(b)}:\mathcal{I}_{(i)}(\mathcal{N}_{(n-i+1)}(x),y)&=\pi_i(\mathcal{I}_{(1)}(\mathcal{N}_{(n)}(x),y),\ldots,\mathcal{I}_{(n)}(\mathcal{N}_{(1)}(x),y))\\&=\pi_i(\mathcal{I}((\mathcal{N}_{(1)}(x),\ldots,\mathcal{N}_{(n)}(x)),/y/))) \text{ by (17)}\\&=\pi_i(\mathcal{I}(\mathcal{N}(/x/),/y/)) \text{ by (11)}\\&=\pi_i(\mathcal{I}(\mathcal{N}(/y/),/x/)) \text{ by } \mathcal{I}9(b)\\&=\pi_i(\mathcal{I}_{(1)}(\mathcal{N}_{(n)}(y),x)\ldots,\mathcal{I}_{(n)}(\mathcal{N}_{(1)}(y),x))=\mathcal{I}_{(i)}(\mathcal{N}_{(n-i+1)}(y),x) \text{ by (11) and (17)}.\\\mathbf{I9}:\mathcal{I}_{(i)}(N(y),N(x))&=\pi_i(\mathcal{I}_{(1)}(N(y),N(x)),\ldots,\mathcal{I}_{(n)}(N(y),N(x)))\\&=\pi_i(\mathcal{I}(\widetilde{N}(/y/),\widetilde{N}(/x/)))\text{by (17) and (11)}\\&=\pi_i(\mathcal{I}(/x/,/y/))=\mathcal{I}_{(i)}(x,y) \text{ by } \mathcal{I}9 \text{ and (17).}\end{aligned}$$

Therefore, Proposition 6.11 is verified. □

Proposition 6.12. *No representable n-DI satisfies $\mathcal{I}11$.*

Proof. Let $\mathcal{I}$ be a representable n-DI. If $n\geq 2$ taking $\mathbf{x}=(\underbrace{0,\ldots,0}_{(n-1)-times},1)$ we have that

$$\mathcal{I}(\mathbf{x},\mathbf{x})=(\mathcal{I}_{(1)}(1,0),\mathcal{I}_{(2)}(0,0),\ldots,\mathcal{I}_{(n-1)}(0,0),\mathcal{I}_{(n)}(0,1))=(0,\underbrace{1,\ldots,1}_{(n-1)-times})\neq /1/. \quad \square$$

Corollary 6.3. *No representable n-DI satisfies $\mathcal{I}12$.*

Proof. Straightforward. □

Lemma 6.3. *Let $\mathcal{I}$ be a representable n-DI. Then $\mathcal{N}_{\mathcal{I}}$ is a representable n-DN and $(\mathcal{N}_{\mathcal{I}})_{(i)} = N_{\mathcal{I}_{(i)}}, \forall i \in \mathbb{N}_n$.*

Proof. By Proposition 6.1 we have that $\mathcal{N}_{\mathcal{I}}$ is an n-DN. So, in Proposition 4.2 and for each $i \in \mathbb{N}_n$, $(\mathcal{N}_{\mathcal{I}})_{(i)}$ is a fuzzy negation and, by Remark 4.2, they are the representant of $\mathcal{N}_{\mathcal{I}}$. In addition, $(\mathcal{N}_{\mathcal{I}})_{(i)}(x) = \pi_i(\mathcal{N}_{\mathcal{I}}(/x/)) = \pi_i(\mathcal{I}(/x/,/0/)) = \mathcal{I}_{(i)}(x,0) = \mathcal{N}_{\mathcal{I}_{(i)}}(x)$ for each $x \in U$. And, Lemma 6.3 holds. □

From Proposition 6.1, each n-DI $\mathcal{I}$ satisfies $\mathcal{I}13$ and $\mathcal{N}_{\mathcal{I}}$ is a representable n-DN.

Proposition 6.13. *Let $\mathcal{I}$ be a representable n-dimensional fuzzy implication. If $\mathcal{I}$ satisfies $\mathcal{I}13(a)$ ($\mathcal{I}13(b)$, $\mathcal{I}13(c)$) then for each $i \in \mathbb{N}_n$, $\mathcal{I}_{(i)}$ satisfies* **I13**(**a**) (**I13**(**b**), **I13**(**c**)).

Proof. By Proposition 6.1 and Lemma 6.3 we have that $\mathcal{N}_{\mathcal{I}}$ is a representable n-DN and $(\mathcal{N}_{\mathcal{I}})_{(i)} = N_{\mathcal{I}_{(i)}}$ for each $i \in \mathbb{N}_n$. In addition, the following holds:
($\mathcal{I}13(a) \Rightarrow$ **I13**(**a**)) Since $\mathcal{I}$ satisfies $\mathcal{I}13(a)$ then by Lemma 6.3 and Theorem 4.1, $(\mathcal{N}_{\mathcal{I}})_{(i)} = (\mathcal{N}_{\mathcal{I}})_{(j)}$ (or equivalently $N_{\mathcal{I}_{(i)}} = N_{\mathcal{I}_{(j)}}$) for each $i, j \in \mathbb{N}_n$. Hence, $\mathcal{N}_{\mathcal{I}} = \widetilde{N_{\mathcal{I}_{(1)}}}$ and therefore for each $x \in U$ and $i \in \mathbb{N}_n$ we have that $/\pi_i(\mathcal{N}_{\mathcal{I}}(/x/))/ = \mathcal{N}_{\mathcal{I}}(/x/)$. Consequently, $N_{\mathcal{I}_{(i)}}(N_{\mathcal{I}_{(i)}}(x)) = \pi_i(\mathcal{N}_{\mathcal{I}}(/\pi_i(\mathcal{N}_{\mathcal{I}}(/x/))/)) = \pi_i(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(/x/))) = \pi_i(/x/) = x$.
($\mathcal{I}13(b) \Rightarrow$ **I13**(**b**)) Since $\mathcal{N}_{\mathcal{I}}$ is a continuous n-DN then $(\mathcal{N}_{\mathcal{I}})_{(i)}$ is a continuous function, for each $i \in \mathbb{N}_n$. By Proposition 6.1 and Lemma 6.3, $N_{\mathcal{I}_{(i)}}$ is a continuous fuzzy negation. Consequently, $\mathcal{I}_{(i)}$ satisfies **I13**(**b**).
($\mathcal{I}13(c) \Rightarrow$ **I13**(**c**)) By Proposition 6.1 and Lemma 6.3 we have that $\mathcal{N}_{\mathcal{I}}$ is a representable n-DN. From Corollary 6.2, $N_{\mathcal{I}_{(i)}} \leq N_{\mathcal{I}_{(j)}}$ or equivalently, $(\mathcal{N}_{\mathcal{I}})_{(i)} \leq (\mathcal{N}_{\mathcal{I}})_{(j)}$, whenever $i \neq j$. Since $\mathcal{I}$ satisfies $\mathcal{I}13(c)$ then $\mathcal{N}_{\mathcal{I}}$ is right invertible, and therefore by Proposition 4.3, each $N_{\mathcal{I}_{(i)}}$ is also right invertible. Concluding the proof, the Corollary 6.13 holds. □

Remark 6.2. Observe that the converse construction of Proposition 6.13 does not always hold. In fact, let $\mathcal{I} = \widetilde{I_{LK^1} \ldots I_{LK^n}}$ be an n-DI where

$$I_{LK^i}(x, y) = \min(1, \sqrt[m]{1 - x^i + y^i}), \forall i \in \mathbb{N}_n.$$

It is an example of representable n-DI which neither satisfies $\mathcal{I}13(a)$ nor satisfies $\mathcal{I}13(c)$. However, each I_{LK^i} satisfies **I13**(**a**) and therefore **I13**(**c**), since $N_{I_{LK^i}}(x) = \sqrt[i]{1 - x^i}$ is a strong fuzzy negation. Finally, one can also easily verify that the converse of the other item holds, meaning that **I13**(**b**) $\Rightarrow \mathcal{I}13(b)$.

7. $(\mathcal{S}, \mathcal{N})$-Implications on $L_n(U)$

Properties in the class of (S, N)-Implication on $L_n(U)$ are analyzed in the following propositions.

7.1. Definition of n-dimensional $(\mathcal{S}, \mathcal{N})$-implication

Proposition 7.1. *Let $\mathcal{S}$ be a n-DS and $\mathcal{N}$ be a n-DN. The function $\mathcal{I}_{\mathcal{S},\mathcal{N}} : L_n(U)^2 \to L_n(U)$ given by*

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x}, \mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathbf{x}), \mathbf{y}) \tag{19}$$

is an n-dimensional fuzzy implication called as n-dimensional $(\mathcal{S}, \mathcal{N})$-implication.

Proof. Let $\mathcal{S}$ be an n-DITS and $\mathcal{N}$ be an n-DIFN. The following holds:

$\mathcal{I}$**0**: The boundary conditions $\mathcal{I}0(a)$ and $\mathcal{I}0(b)$ are verified:
$\mathcal{I}_{\mathcal{S},\mathcal{N}}(/1/,/1/) = \mathcal{S}(/0/,/1/) = /1/$; $\mathcal{I}_{\mathcal{S},\mathcal{N}}(/0/,/1/) = \mathcal{S}(/1/,/0/) = /1/$;
$\mathcal{I}_{\mathcal{S},\mathcal{N}}(/0/,/0/) = \mathcal{S}(/1/,/0/) = /1/$; $\mathcal{I}_{\mathcal{S},\mathcal{N}}(/1/,/0/) = \mathcal{S}(/0/,/0/) = /0/$;
$\mathcal{I}$**1**: $\mathbf{x} \leq \mathbf{z} \Rightarrow \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x}, \mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathbf{x}), \mathbf{y}) \geq \mathcal{S}(\mathcal{N}(\mathbf{z}), \mathbf{y}) = \mathcal{I}(\mathbf{z}, \mathbf{y})$, based on both properties, the monotonicity of $\mathcal{S}$ and the monotonicity of $\mathcal{N}$;
$\mathcal{I}$**2**: Analogously, $\mathbf{y} \leq \mathbf{z} \Rightarrow \mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathbf{x}), \mathbf{y}) \leq \mathcal{S}(\mathcal{N}(\mathbf{x}), \mathbf{z}) = \mathcal{I}(\mathbf{x}, \mathbf{z})$, based on the monotonicity of $\mathcal{S}$.

Therefore, $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ satisfies the conditions of Definition 6.2 and Proposition 7.1 is verified. □

Remark 7.1. The underlying n-DS and n-DN of an n-dimensional $(\mathcal{S}, \mathcal{N})$-implication $\mathcal{I}$ are called the pair of generators. Let $\mathcal{N}$ be a strong n-DN. $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is denoted by $\mathcal{I}_{\mathcal{S}}$ and it is called as $\mathcal{S}$-implication.

7.2. Characterizing n-dimensional $(\mathcal{S},\mathcal{N})$-implication

Proposition 7.2. *Let $\mathcal{I}$ be an n-dimensional $(\mathcal{S},\mathcal{N})$-implication and $(\mathcal{S},\mathcal{N})$ be the generator pair of $\mathcal{I}$. Then, the following properties hold:*

(i) *$\mathcal{I}$ verifies $\mathcal{I}3$ and $\mathcal{I}5$;*
(ii) *$\mathcal{N}_{\mathcal{I}} = \mathcal{N}$;*
(iii) *$\mathcal{I}$ verifies $R{-}CP(\mathcal{N})$;*
(iv) *If $\mathcal{N}$ is right invertible with right inverse $\mathcal{N}^{-r}$ then $\mathcal{I}$ verifies $L{-}CP(\mathcal{N}^{-r})$;*
(v) *$\mathcal{N}$ is strong if and only if $\mathcal{I}$ verifies $CP(\mathcal{N})$.*

Proof. For all $\mathbf{x},\mathbf{y},\mathbf{z}$, the following holds:

(i) For each $\mathbf{y} \in Ł_n(U)$ we have that $\mathcal{I}(/1/,\mathbf{y}) = \mathcal{S}(/0/,\mathbf{y}) = \mathbf{y}$ and therefore $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ verify $\mathcal{I}3$. Since $\mathcal{S}$ verifies the $\mathcal{S}2$ and $\mathcal{S}3$ properties, the following holds for each $\mathbf{x},\mathbf{y} \in L_n(U)$:
$\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{S}(\mathcal{N}(\mathbf{y}),\mathbf{z}))) = \mathcal{S}(\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y})),\mathbf{z}))) = \mathcal{S}(\mathcal{S}(\mathcal{N}(\mathbf{y}),\mathcal{N}(\mathbf{x})),\mathbf{z}))) = \mathcal{S}(\mathcal{N}(\mathbf{y}),\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{z})))$.
Therefore, $\mathcal{I}(\mathbf{x},\mathcal{I}(\mathbf{y},\mathbf{z})) = \mathcal{I}(\mathbf{y},\mathcal{I}(\mathbf{x},\mathbf{z}))$. So $\mathcal{I}$ verifies $\mathcal{I}5$.
(ii) For each $\mathbf{x} \in Ł_n(U)$ we have that $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) = \mathcal{I}(\mathbf{x},/0/) = \mathcal{S}(\mathcal{N}(\mathbf{x}),/0/) = \mathcal{N}(\mathbf{x})$.
(iii) Straightforward from the previous item and Proposition 6.2(ii).
(iv) For each $\mathbf{x},\mathbf{y} \in Ł_n(U)$ we have that $\mathcal{I}(\mathcal{N}^{-r}(\mathbf{x}),\mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathcal{N}^{-r}(\mathbf{x})),\mathbf{y}) = \mathcal{S}(\mathbf{x},\mathbf{y}) = \mathcal{S}(\mathbf{y},\mathbf{x}) = \mathcal{S}(\mathcal{N}(\mathcal{N}^{-r}(\mathbf{y})),\mathbf{x}) = \mathcal{I}(\mathcal{N}^{-r}(\mathbf{y}),\mathbf{x})$.
(v) $(\Rightarrow)$ Since, $\mathcal{N}$ is strong then for each $\mathbf{x},\mathbf{y} \in Ł_n(U)$ we have that $\mathcal{I}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y})) = \mathcal{S}(\mathcal{N}(\mathcal{N}(\mathbf{x})),\mathcal{N}(\mathbf{y})) = \mathcal{S}(\mathcal{N}(\mathbf{y}),\mathbf{x}) = \mathcal{I}(\mathbf{y},\mathbf{x})$, i.e. $\mathcal{I}$ satisfies $CP(\mathcal{N})$.
$(\Leftarrow)$ When $\mathbf{x} \in L_n(U)$, then the following holds:

$$\begin{aligned}\mathcal{N}(\mathcal{N}(\mathbf{x})) &= \mathcal{S}(\mathcal{N}(\mathcal{N}(\mathbf{x})),\mathcal{N}(/1/)) \text{ by } \mathcal{N}1 \text{ and } \mathcal{S}1\\ &= \mathcal{I}(\mathcal{N}(\mathbf{x}),\mathcal{N}(/1/)) \text{ by } (19)\\ &= \mathcal{I}(/1/,\mathbf{x}) \text{ by } CP(\mathcal{N}) = \mathbf{x} \text{ by item (i).}\end{aligned}$$

Therefore, Proposition 7.2 is verified. □

Proposition 7.3. *Let $\mathcal{I} : L_n(U)^2 \to L_n(U)$ be an n-DI and $\mathcal{N}$ be a n-DN. If $\mathcal{S}_{\mathcal{I},\mathcal{N}} : (L_n(U))^2 \to L_n(U)$ is the given function as follows:*

$$\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in L_n(U).$$

Then the following holds:

(i) *$\mathcal{S}_{\mathcal{I},\mathcal{N}}(/1/,\mathbf{x}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},/1/) = /1/$;*
(ii) *If $\mathcal{I}$ verifies $\mathcal{I}3$ then $\mathcal{S}_{\mathcal{I},\mathcal{N}}(/0/,\mathbf{x}) = \mathbf{x}$;*
(iii) *$\mathcal{S}_{\mathcal{I},\mathcal{N}}$ is increasing in both variables;*
(iv) *$\mathcal{S}_{\mathcal{I},\mathcal{N}}$ is commutative if and only if $\mathcal{I}$ satisfies $L{-}CP(\mathcal{N})$;*
(v) *if $\mathcal{I}$ verifies $\mathcal{I}_5$ and $L{-}CP(\mathcal{N})$ then $\mathcal{S}_{\mathcal{I},\mathcal{N}}$ is associative.*

Proof. By Proposition 6.1, $\mathcal{I}$ satisfies $\mathcal{I}4$ and $\mathcal{I}10$. So, for $\mathbf{x},\mathbf{y},\mathbf{z} \in (L_n(U))$, we obtain the following results:

$$\begin{aligned}
&(i)\ \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},/1/) = \mathcal{I}(\mathcal{N}(\mathbf{x}),/1/) = /1/ \text{ by } \mathcal{I}4;\\
&\quad \mathcal{S}_{\mathcal{I},\mathcal{N}}(/1/,\mathbf{x}) = \mathcal{I}(\mathcal{N}(/1/),\mathbf{x}) = /1/ \text{ by } \mathcal{I}10.\\
&(ii)\ \mathcal{S}_{\mathcal{I},\mathcal{N}}(/0/,\mathbf{x}) = \mathcal{I}(\mathcal{N}(/0/),\mathbf{x}) = \mathbf{x} \text{ by } \mathcal{I}3.\\
&(iii)\ \text{if } \mathbf{x}_1 \le \mathbf{x}_2 \text{ then } \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}_1,\mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{x}_1),\mathbf{y}) \le \mathcal{I}(\mathcal{N}(\mathbf{x}_2),\mathbf{y}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}_2,\mathbf{y});\\
&\quad \text{if } \mathbf{y}_1 \le \mathbf{y}_2 \text{ then } \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathbf{y}_1) = \mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y}_1) \le \mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y}_2) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathbf{y}_2).\\
&(iv)\ (\Leftarrow)\ \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y}) = \mathcal{I}(N(\mathbf{y}),\mathbf{x}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{y},\mathbf{x}) \text{ by } L{-}CP(\mathcal{N});\\
&\quad (\Rightarrow)\ \mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathbf{y}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{y},\mathbf{x}) = \mathcal{I}(\mathcal{N}(\mathbf{y}),\mathbf{x}) \text{ by } \mathbf{S2};\\
&(v)\ \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathbf{y}),\mathbf{z}) = \mathcal{I}(\mathcal{N}(\mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y})),\mathbf{z}) = \mathcal{I}(\mathcal{N}(\mathbf{z}),\mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y})) = \mathcal{I}(\mathcal{N}(\mathbf{x}),\mathcal{I}(\mathcal{N}(\mathbf{z}),\mathbf{y}))\\
&\qquad = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{z},\mathbf{y})) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{y},\mathbf{z})) \text{ by } \mathcal{I}5,\ L{-}CP(\mathcal{N}) \text{ and (iv).}
\end{aligned}$$

Therefore, Proposition 7.3 is verified. □

Corollary 7.1. *Let $\mathcal{I} : L_n(U)^2 \to L_n(U)$ be an n-dimensional $(\mathcal{S}, \mathcal{N})$-implication. If $\mathcal{I}$ satisfies $\mathcal{I}13(c)$ then $\mathcal{S}_{\mathcal{I},\mathcal{N}}$ is an n-DS.*

Proof. Straightforward from Propositions 6.3, 7.2 and 7.3. □

Proposition 7.4. *Let $\mathcal{I} : L_n(U)^2 \to L_n(U)$ be an n-dimensional $(\mathcal{S}, \mathcal{N})$-implication such that $\mathcal{N}_\mathcal{I}$ is right invertible n-DN. Then $(\mathcal{S}_{\mathcal{I},\mathcal{N}_\mathcal{I}^{-r}}, \mathcal{N}_\mathcal{I})$ is the generator pair of $\mathcal{I}$ whenever $\mathcal{N}_\mathcal{I}^{-r}$ is a right inverse of $\mathcal{N}_\mathcal{I}$.*

Proof. Since $\mathcal{I}$ is an n-dimensional $(\mathcal{S}, \mathcal{N})$-implication then there exists an n-DS $\mathcal{S}$ and n-DN $\mathcal{N}$ such that $\mathcal{I} = \mathcal{I}_{\mathcal{S},\mathcal{N}}$. By Proposition 7.2(ii), $\mathcal{N} = \mathcal{N}_\mathcal{I}$ and once $\mathcal{N}_\mathcal{I}$ is right invertible then there is an n-DN $\mathcal{N}_\mathcal{I}^{-r}$ such that $\mathcal{N}_\mathcal{I} \circ \mathcal{N}_\mathcal{I}^{-r} = Id_{L_n(U)}$. So, $\mathcal{S}_{\mathcal{I},\mathcal{N}_\mathcal{I}^{-r}}(\mathbf{x}, \mathbf{y}) = \mathcal{I}(\mathcal{N}_\mathcal{I}^{-r}(\mathbf{x}), \mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathcal{N}_\mathcal{I}^{-r}(\mathbf{x})), \mathbf{y}) = \mathcal{S}(\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}^{-r}(\mathbf{x})), \mathbf{y}) = \mathcal{S}(\mathbf{x}, \mathbf{y})$ and therefore $\mathcal{S}_{\mathcal{I},\mathcal{N}_\mathcal{I}^{-r}} = \mathcal{S}$. Hence, $(\mathcal{S}_{\mathcal{I},\mathcal{N}_\mathcal{I}^{-r}}, \mathcal{N}_\mathcal{I}) = (\mathcal{S}, \mathcal{N})$, i.e. is a generator pair of $\mathcal{I}$. □

Theorem 7.1. *Let $\mathcal{I} : L_n(U)^2 \to L_n(U)$ be an n-DI such that $\mathcal{N}_\mathcal{I}$ is right invertible. Then the following statements are equivalent:*

(i) *$\mathcal{I}$ is an n-dimensional $(\mathcal{S}, \mathcal{N})$-implication;*
(ii) *$\mathcal{I}$ satisfies $\mathcal{I}3$ and $\mathcal{I}5$.*

Proof. $(i) \Rightarrow (ii)$: By Proposition 7.4, $\mathcal{I} = \mathcal{I}_{\mathcal{S}_{\mathcal{I},\mathcal{N}_\mathcal{I}^{-r}},\mathcal{N}_\mathcal{I}}$ and then, by Proposition 7.2(i), $\mathcal{I}$ satisfies $\mathcal{I}3$ and $\mathcal{I}5$.

$(ii) \Rightarrow (i)$: Since $\mathcal{I}$ satisfies $\mathcal{I}3$, $\mathcal{I}5$ and $\mathcal{I}13(c)$ then by Proposition 6.3, $\mathcal{I}$ also satisfies $L{-}CP(\mathcal{N}_\mathcal{I}^{-r})$, where $\mathcal{N}_\mathcal{I}^{-r}$ is the right inverse of $\mathcal{N}_\mathcal{I}$. So, by Proposition 7.3, $\mathcal{S}_{\mathcal{I},\mathcal{N}_\mathcal{I}}$ is an n-DS. On the other hand, for each $\mathbf{x}, \mathbf{y} \in L_n(U)$ we have that $\mathcal{I}_{\mathcal{S}_{\mathcal{I},\mathcal{N}_\mathcal{I}},\mathcal{N}_\mathcal{I}^{-r}}(\mathbf{x}, \mathbf{y}) = \mathcal{S}_{\mathcal{I},\mathcal{N}_\mathcal{I}}(\mathcal{N}_\mathcal{I}^{-r}(\mathbf{x}), \mathbf{y}) = \mathcal{I}(\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}^{-r}(\mathbf{x})), \mathbf{y}) = \mathcal{I}(\mathbf{x}, \mathbf{y})$. So, $\mathcal{I}$ is an n-dimensional $(\mathcal{S}, \mathcal{N})$-implication with $(\mathcal{S}_{\mathcal{I},\mathcal{N}_\mathcal{I}}, \mathcal{N}_\mathcal{I}^{-r})$ as the generator pair. □

7.3. $(\mathcal{S}, \mathcal{N})$-Implications – the first axiom of Hilbert system and the identity principle

The Proposition 6.10 and Proposition 6.12 claim that each representable n-dimensional $(\mathcal{S}, \mathcal{N})$-implication satisfies neither the first axiom of Hilbert system nor the identity principle. Nevertheless, there are n-dimensional $(\mathcal{S}, \mathcal{N})$-implication satisfying $\mathcal{I}7$ and $\mathcal{I}11$, for instance, the n-dimensional version of the Weber-implication:

$$\mathcal{I}_{WB}(X, Y) = \begin{cases} Y & \text{if } X = /1/; \\ /1/ & \text{otherwise.} \end{cases}$$

Proposition 7.5. *Let $\mathcal{I}$ be n-dimensional $(\mathcal{S}, \mathcal{N})$-implication. $\mathcal{I}$ satisfy $\mathcal{I}7$ if and only if $\mathcal{I}$ satisfy $\mathcal{I}11$.*

Proof. $(\Rightarrow)$ Since by Proposition 7.2, $\mathcal{I}$ satisfies $\mathcal{I}3$ then, by $\mathcal{I}3$ and $\mathcal{I}7$, for each $\mathbf{x} \in L_n(U)$, $\mathcal{I}(\mathbf{x}, \mathbf{x}) = \mathcal{I}(\mathbf{x}, \mathcal{I}(/1/, \mathbf{x})) = /1/$. Therefore, $\mathcal{I}$ satisfies $\mathcal{I}11$.
$(\Leftarrow)$ Let $\mathbf{x}, \mathbf{y} \in L_n(U)$. By $\mathcal{I}11$ we have that, $\mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y}, \mathbf{x})) = \mathcal{S}(\mathcal{N}(\mathbf{x}), \mathcal{S}(\mathcal{N}(\mathbf{y}), \mathbf{x})) = \mathcal{S}(\mathcal{S}(\mathcal{N}(\mathbf{x}), \mathbf{x}), \mathcal{N}(\mathbf{y})) = \mathcal{S}(\mathcal{I}(\mathbf{x}, \mathbf{x}), \mathcal{N}(\mathbf{y})) = \mathcal{S}(/1/, \mathcal{N}(\mathbf{y})) = /1/$. Therefore, $\mathcal{I}$ satisfies $\mathcal{I}7$. □

Since not all $(\mathcal{S}, \mathcal{N})$-implication, even $\mathcal{S}$-implications, satisfy the identity principle, we analyze this property for this family in the following propositions.

Proposition 7.6. *For an n-DS $\mathcal{S}$ and an n-DN $\mathcal{N}$ the following statements are equivalent:*

(i) *The $(\mathcal{S}, \mathcal{N})$-implication $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ satisfies $\mathcal{I}11$;*
(ii) *The pair $(\mathcal{S}, \mathcal{N})$ satisfies LEM expressed as $\mathcal{S}5$.*

Proof. If $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ satisfies $\mathcal{I}11$, then $\mathcal{S}(\mathcal{N}(\mathbf{x}), \mathbf{x}) = \mathcal{I}(\mathbf{x}, \mathbf{x}) = /1/$, for all $\mathbf{x} \in L_n(U)$.
Conversely, if the pair $(\mathcal{S}, \mathcal{N})$ satisfies $\mathcal{S}5$, then $\mathcal{I}(\mathbf{x}, \mathbf{x}) = \mathcal{S}(\mathcal{N}(\mathbf{x}), \mathbf{x}) = /1/$ for all $\mathbf{x} \in L_n(U)$. Therefore, Proposition 7.6 is verified. □

Proposition 7.7. *Let $\mathcal{I}$ be n-dimensional $(\mathcal{S}, \mathcal{N})$-implication. If $\mathcal{I}$ satisfies $\mathcal{I}11$ then $\inf\{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathbf{x}, \mathbf{y}) = /1/\} \leq \mathbf{x}$ and $\mathcal{N}(\mathbf{x}) \geq \inf\{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathcal{N}(\mathbf{x}), \mathbf{y}) = /1/$.*

Proof. For each $\mathbf{x} \in L_n(U)$, by $\mathcal{I}11$, $\mathcal{I}(\mathbf{x}, \mathbf{x}) = /1/$ and therefore $\inf\{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathbf{x}, \mathbf{y}) = /1/\} \leq \mathbf{x}$. On the other hand, by $\mathcal{I}11$, $\mathcal{I}(\mathcal{N}(\mathbf{x}), \mathcal{N}(\mathbf{x})) = /1/$ and therefore $\inf\{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathcal{N}(\mathbf{x}), \mathbf{y}) = /1/\} \leq \mathcal{N}(\mathbf{x})$. Then, Proposition 7.7 holds. □

From the results achieved in Proposition 7.5, the Propositions 7.6 and 7.7 also hold when $\mathcal{I}11$ is substituted by $\mathcal{I}7$.

7.4. $(\mathcal{S},\mathcal{N})$-Implications and the ordering property

As noted earlier, not all natural generalizations of the classical implication to multi-valued logic satisfy ordering property $\mathcal{I}12$. In the following section we discuss results on $(\mathcal{S},\mathcal{N})$-implications with respect to their ordering property.

Lemma 7.1. *Let $\mathcal{I}$ be n-dimensional $(\mathcal{S},\mathcal{N})$-implication satisfying $\mathcal{I}12$. Then $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) = \inf\{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}),\mathbf{y}) = /1/\}$ for each $\mathbf{x} \in L_n(U)$.*

Proof. By $\mathcal{I}12$, $\mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}),\mathcal{N}_{\mathcal{I}}(\mathbf{x})) = /1/$ and therefore $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) \geq \inf\{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}),\mathbf{y}) = /1/\}$. Suppose that $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) > \inf\{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}),\mathbf{y}) = /1/\}$ and take $\mathbf{z} \in L_n(U)$ such that $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) > \mathbf{z} > \inf\{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}),\mathbf{y}) = /1/\}$. Then, there exist $\mathbf{y}' \in \{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}),\mathbf{y}) = /1/\}$ such that $\mathbf{z} \geq \mathbf{y}'$. So, by $\mathcal{I}2$, $\mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}),\mathbf{z}) \geq \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}),\mathbf{y}') = /1/$ and therefore, by $\mathcal{I}12$, $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) \leq \mathbf{z}$ which is a contradiction. Hence, $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) = \inf\{\mathbf{y} \in L_n(U) : \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}),\mathbf{y}) = /1/\}$. Then, Proposition 7.1 holds. □

Theorem 7.2. *Let $\mathcal{I}$ be n-dimensional $(\mathcal{S},\mathcal{N})$-implication. Then the following statements are equivalent:*

(i) *$\mathcal{I}$ satisfies $\mathcal{I}12$ and $\mathcal{I}13(c)$;*
(ii) *$\mathcal{N}_{\mathcal{I}}$ is a strong negation and $\mathcal{I}$ satisfies $\mathcal{I}11$.*

Proof. $(i) \Rightarrow (ii)$ Since $\mathcal{I}12$ implies $\mathcal{I}11$ and, by Proposition 7.2, $\mathcal{I}$ satisfies $\mathcal{I}3$ and $\mathcal{I}5$, then, by Proposition 6.4(2), we have that $\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}))) \leq \mathcal{N}_{\mathcal{I}}(\mathbf{x})$ for each $\mathbf{x} \in L_n(U)$. But N_I is right invertible and so has a right inverse, denoted by $\mathcal{N}^{-r}$. Then for each $\mathbf{x} \in L_n(U)$, $\mathbf{x} = \mathcal{N}_{\mathcal{I}}(\mathcal{N}^{-r}(\mathbf{x})) \geq \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}^{-r}(\mathbf{x})))) = \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}))$. Therefore, by Proposition 6.4(1), we conclude that $\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x})) = \mathbf{x}$.

$(ii) \Rightarrow (i)$ Since $\mathcal{N}_{\mathcal{I}}$ is strong, trivially is right invertible. Let $\mathbf{x},\mathbf{y} \in L_n(U)$, $\mathcal{I}(\mathbf{x},\mathbf{y}) = /1/$. So, since $\mathcal{N}_{\mathcal{I}}$ is strong, it holds that

$$\mathbf{y} \geq \inf\{\mathbf{z} \in L_n(U) : \mathcal{I}(\mathbf{x},\mathbf{z}) = /1/\} = \inf\{\mathbf{z} \in L_n(U) : \mathcal{I}(\mathcal{N}(\mathcal{N}(\mathbf{x})),\mathbf{z}) = /1/\}.$$

Then, by Lemma 7.1, $\mathbf{y} \geq \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x})) = \mathbf{x}$. On the other hand, if $\mathbf{x} \leq \mathbf{y}$ then, because $\mathcal{N}_{\mathcal{I}}$ is strong, $\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x})) \leq \mathbf{y}$ and therefore, by Lemma 7.1, $\inf\{\mathbf{z} \in L_n(U) : \mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{z}) = /1/\} = \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathbf{x})) \leq \mathbf{y}$. So, $\mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y}) = /1/$. Therefore, Theorem 7.2 is verified. □

7.5. Representing of n-dimensional $(\mathcal{S},\mathcal{N})$-implication

As aforementioned, there is no representable n-dimensional $(\mathcal{S},\mathcal{N})$-implication satisfying the identity principle. However, this does not imply in the no existence of representable n-dimensional $(\mathcal{S},\mathcal{N})$-implication and the study of classes of representable n-dimensional $(\mathcal{S},\mathcal{N})$-implications deserves to be studied.

Proposition 7.8. *Let $\mathcal{S}$ and $\mathcal{N}$ be representable n-DS and n-DN, respectively. Then, $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is representable.*

Proof. Since $\mathcal{S}$ and $\mathcal{N}$ are representable then there exists t-conorms S_i and fuzzy negations N_i, with $i = 1,\ldots,n$, such that $\mathcal{S} = \widetilde{S_1,\ldots,S_n}$ and $\mathcal{N} = \widetilde{N_1,\ldots,N_n}$ and so obey the conditions stated in Propositions 5.1 and 4.1. Then, we obtained the results below:

$$\begin{aligned}\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{y})) &= \widetilde{S_1,\ldots,S_n}\,((N_1(\pi_n(\mathbf{x})),\ldots,N_n(\pi_1(\mathbf{x}))),(\pi_1(\mathbf{y}),\ldots,\pi_n(\mathbf{y})))\\ &= (S_1(N_1(\pi_n(\mathbf{x})),\pi_1(\mathbf{y})),\ldots,S_n(N_n(\pi_1(\mathbf{x})),\pi_n(\mathbf{y})))\\ &= (I_{S_1,N_1}(\pi_n(\mathbf{x}),\pi_1(\mathbf{y})),\ldots,I_{S_n,N_n}(\pi_1(\mathbf{x}),\pi_n(\mathbf{y}))) = \widetilde{I_{S_1,N_1},\ldots,I_{S_n,N_n}}(\mathbf{x},\mathbf{y}).\end{aligned}$$

Therefore, $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is also a representable function on $L_n(U)$ and so, Proposition 7.8 holds. □

Proposition 7.9. *Let $\mathcal{S}$ be a n-DS and $\mathcal{N}$ be a n-DN. If $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is representable then $\mathcal{N}$ is representable. In addition, if $\mathcal{N}$ is right invertible then $\mathcal{S}$ is representable.*

Proof. Since $\mathcal{I} = \mathcal{I}_{\mathcal{S},\mathcal{N}}$ is a representable n-dimensional $(\mathcal{S},\mathcal{N})$-implication then, by Proposition 6.7, each $\mathcal{I}_{(i)}$ with $i \in \mathbb{N}_n$ is an n-DI and $\mathcal{I} = \widetilde{\mathcal{I}_{(1)}\ldots\mathcal{I}_{(n)}}$. Then for each $\mathbf{x} \in L_n(U)$, we have that $\mathcal{N}(\mathbf{x}) = \mathcal{S}(\mathcal{N}(\mathbf{x}),/0/) = \mathcal{I}(\mathbf{x},/0/) = (\mathcal{I}_{(1)}(\pi_n(\mathbf{x}),0),\ldots\mathcal{I}_{(n)}(\pi_1(\mathbf{x}),0)) = (N_{\mathcal{I}_{(1)}}(\pi_n(\mathbf{x})),\ldots,N_{\mathcal{I}_{(n)}}(\pi_1(\mathbf{x}))) = \widetilde{N_{\mathcal{I}_{(1)}}\ldots N_{\mathcal{I}_{(n)}}}(\mathbf{x})$, that is, $\mathcal{N}$ is representable. In addition, let $\mathcal{N}^{-r}$ the right inverse of $\mathcal{N}$. By Proposition 4.3, $\mathcal{N}_{(i)}$ is right invertible and, Remark 4.3, holds that $\mathcal{N}^{-r}$ is representable. Then, for $\mathbf{x},\mathbf{y} \in L_n(U)$, $\mathcal{N}(\mathcal{N}^{-r}(\mathbf{x})) = \mathbf{x}$, and we obtained the results below:

$$\begin{aligned}\mathcal{S}(\mathbf{x},\mathbf{y}) &= \mathcal{S}(\mathcal{N}(\mathcal{N}^{-r}(\mathbf{x})),\mathbf{y}) = \mathcal{I}(\mathcal{N}^{-r}(\mathbf{x}),\mathbf{y})\\ &= \widetilde{\mathcal{I}_{(1)},\ldots,\mathcal{I}_{(n)}}\left(((\mathcal{N}^{-r})_{(1)}(\pi_n(\mathbf{x})),\ldots,(\mathcal{N}^{-r})_{(n)}(\pi_1(\mathbf{x}))),(\pi_1(\mathbf{y}),\ldots,\pi_n(\mathbf{y}))\right)\\ &= (\mathcal{I}_{(1)}((\mathcal{N}^{-r})_{(n)}(\pi_1(\mathbf{x})),\pi_1(\mathbf{y})),\ldots,\mathcal{I}_{(n)}((\mathcal{N}^{-r})_{(1)}(\pi_n(\mathbf{x})),\pi_n(\mathbf{y})))\\ &= (S_1(\pi_1(\mathbf{x}),\pi_1(\mathbf{y})),\ldots,S_n(\pi_n(\mathbf{x}),\pi_n(\mathbf{y})))\end{aligned}$$

with $S_i(x,y) = \mathcal{I}_{(i)}((\mathcal{N}^{-r})_{(n-i+1)}(x),y)$ for each $i \in \mathbb{N}_n$. Moreover, for each $x,y,z \in [0,1]$ and $i \in \mathbb{N}_n$, we have that: $S_i(x,0) = \pi_i(\mathcal{S}(/x/,/0/)) = \pi_i(/x/) = x$, $S_i(x,y) = \pi_i(\mathcal{S}(/x/,/y/)) = \pi_i(\mathcal{S}(/y/,/x/)) = S_i(y,x)$, if $y \leq z$ then $/y/ \leq /z/$ and so $S_i(x,y) = \pi_i(\mathcal{S}(/x/,/y/)) \leq \pi_i(\mathcal{S}(/x/,/z/)) = S_i(x,z)$. And, finally, since $\mathcal{I}_{(i)}$ is an $(S.N)$-implication and $\mathcal{I}$ satisfies $\mathcal{I}5$. Consequently, by Proposition 6.9, $\mathcal{I}_{(i)}$ satisfies **I5**. So, we obtained the following results:

$$\begin{aligned}S_i(x,S_i(y,z)) &= \mathcal{I}_{(i)}((\mathcal{N}^{-r})_{(n-i+1)}(x),\mathcal{I}_{(i)}((\mathcal{N}^{-r})_{(n-i+1)}(y),z))\\ &= \mathcal{I}_{(i)}((\mathcal{N}^{-r})_{(n-i+1)}(y),\mathcal{I}_{(i)}((\mathcal{N}^{-r})_{(n-i+1)}(x),z)) = S_i(y,S_i(x,z)).\end{aligned}$$

Therefore, S_i is associative and therefore, it is a t-conorm. And, Proposition 7.9 holds. □

Example 7.1. In the following, an example of a representable n-dimensional fuzzy implication is presented:
Let $I_{KD}, I_{RC}, I_{LK}, I_{FD} : U^2 \to U$ be operators given by the expressions below:

- I_{KD} be the Kleene-Dienes (S_M, N_S)-implication: $I_{KD}(x,y) = S_M(N_S(x),y) = \max(1-x,y)$;
- I_{RC} be the Reichenbach (S_P, N_S)-implication: $I_{RC}(x,y) = S_P(N_S(x),y) = 1-x+xy$;
- I_{LK} be the Lukasiewicz (S_{LK}, N_S)-implication: $I_{LK}(x,y) = S_{LK}(N_S(x),y) = \min(1,1-x+y)$;
- I_{FD} be the Fodor (S_{nM}, N_S)-implication: $I_{FD}(x,y) = S_{nM}(N_S(x),y) = \begin{cases}\max(1-x,y), & y \leq x,\\ 1, & \text{otherwise}.\end{cases}$

(i) When $\mathbf{x} = (0.0, 0.1, 0.5, 0.8)$ and $\mathbf{y} = (0.2, 0.6, 0.9, 1.0)$, it holds that:

$$\begin{aligned}\widetilde{I_{KD}}(\mathbf{x},\mathbf{y}) &= (I_{KD}(0.8,0.2), I_{KD}(0.5,0.6), I_{KD}(0.1,0.9), I_{KD}(0,1)) = (0.2, 0.6, 0.9, 1.0);\\ \widetilde{I_{RC}}(\mathbf{x},\mathbf{y}) &= (I_{RC}(0.8,0.2), I_{RC}(0.5,0.6), I_{RC}(0.1,0.9), I_{RC}(0,1)) = (0.36, 0.8, 0.99, 1.0);\\ \widetilde{I_{LK}}(\mathbf{x},\mathbf{y}) &= (I_{LK}(0.8,0.2), I_{LK}(0.5,0.6), I_{LK}(0.1,0.9), I_{LK}(0,1)) = /1/;\\ \widetilde{I_{FD}}(\mathbf{x},\mathbf{y}) &= (I_{FD}(0.8,0.2), I_{FD}(0.5,0.6), I_{FD}(0.1,0.9), I_{FD}(0,1)) = (0.2, 1.0, 1.0, 1.0).\end{aligned}$$

(ii) Based on results in [6] and [5] we have that $I_{KD} \leq I_{RC} \leq I_{LK}$. So, see two examples of representable n-DI obtained from Eq. (17):

$$\widetilde{I_{KD}, I_{RC}, I_{LK}}(\mathbf{x},\mathbf{y}) = (\max(1-x_3,y_1), 1-x_2+x_2\cdot y_2, \min(1,1-x_1+y_3)). \tag{20}$$

Therefore, the following results are obtained:

$$\widetilde{I_{KD}, I_{RC}, I_{LK}}((0.1,0.5,0.8),(0.2,0.6,0.9)) = (0.2, 0.2, 1.0).$$

(iii) Analogously, by results from [6] and [5], $I_{KD} \leq I_{RC} \leq I_{LK} \leq I_{FD}$ and the following holds:

$$\widetilde{I_{KD}, I_{RC}, I_{LK}, I_{FD}}((0.0,0.1,0.5,0.8),(0.2,0.6,0.9,1.0)) = (0.2, 0.8, 1.0, 1.0).$$

(iv) And, if $\mathcal{I} \equiv \widetilde{I_{KD}, I_{RC}, I_{LK}, I_{FD}}$, $\mathcal{S} = \widetilde{S_{KD}, S_{RC}, S_{LK}, S_{FD}}$ and $\mathcal{N} = \widetilde{N_S}$, it is immediate the following $\mathcal{I}(\mathbf{x},\mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{y})$.

7.6. Conjugation of n-dimensional $(\mathcal{S},\mathcal{N})$-implication

Concluding this section, the next proposition extends the results in [7, Theorem 2.4.5.] and discusses the action of automorphisms on the class of n-dimensional fuzzy $(\mathcal{S},\mathcal{N})$-implication,

Proposition 7.10. *If $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is an n-dimensional $(\mathcal{S},\mathcal{N})$-implication, then the φ-conjugate of $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is also an n-dimensional $(\mathcal{S},\mathcal{N})$-implication generated from the φ-conjugate of $\mathcal{S}$ and $\mathcal{N}$, that is,*

$$(\mathcal{I}_{\mathcal{S},\mathcal{N}})^{\varphi}(\mathbf{x},\mathbf{y}) = \mathcal{I}_{\mathcal{S}^{\varphi},\mathcal{N}^{\varphi}}(\mathbf{x},\mathbf{y}). \tag{21}$$

In addition, given $\varphi \in Aut(L_n(U))$, we have that $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is representable if and only if $(\mathcal{I}_{\mathcal{S},\mathcal{N}})^{\varphi}$ is representable.

Proof. Let $\varphi \in Aut(L_n(U))$ and let $\mathcal{S}, \mathcal{N}$ be an n-DS and an n-DN, respectively. So, by Propositions 5.5 and 4.5, the functions $\mathcal{S}^\varphi, \mathcal{N}^\varphi$ are also an n-DS and an n-DN, thus

$$\begin{aligned}(\mathcal{I}_{\mathcal{S},\mathcal{N}})^\varphi(\mathbf{x},\mathbf{y}) &= \varphi^{-1}(\mathcal{I}_{\mathcal{S},\mathcal{N}}(\varphi(\mathbf{x}),\varphi(\mathbf{y}))) = \varphi^{-1}(\mathcal{S}(\mathcal{N}(\varphi(\mathbf{x})),\varphi(\mathbf{y}))) \text{ by Eqs. (10) and (19)}\\ &= \varphi^{-1}(\mathcal{S}(\varphi(\varphi^{-1}(\mathcal{N}(\varphi(\mathbf{x})))),\varphi(\mathbf{y}))) = \mathcal{S}^\varphi(\mathcal{N}^\varphi(\mathbf{x}),\mathbf{y}) \text{ by Eq. (10)}\\ &= \mathcal{I}_{\mathcal{S}^\varphi,\mathcal{N}^\varphi}(\mathbf{x},\mathbf{y}) \text{ by Eq. (19)}.\end{aligned}$$

Therefore, $(\mathcal{I}_{\mathcal{S},\mathcal{N}})^\varphi$ is an n-dimensional $(\mathcal{S},\mathcal{N})$-implication. In addition, by Proposition 6.7, $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is representable if and only if $(\mathcal{I}_{\mathcal{S},\mathcal{N}})^\varphi$ is also representable. Therefore, Proposition 7.10 is verified. □

8. Exploring n-dimensional fuzzy $(\mathcal{S},\mathcal{N})$-implication in approximate reasoning

Owing to the effective and reasonable description to the uncertainty information, the expression ability related to the concepts in the n-dimensional simplex $L_n(U)$ is stronger than Zadeh's fuzzy sets. So, in this section, first results in the extension of the basic concepts of AR are considered, by using n-dimensional intervals. In particular, the class of n-dimensional fuzzy (S, N)-implication can be employed to relate fuzzy propositional formulae in n-dimensional fuzzy logic inference schemes. For example, if A, B are any n-dimensional fuzzy logic propositional formulae, then $A \to B$ is called an n-dimensional fuzzy conditional statement or more commonly, as an n-dimensional fuzzy IF-THEN rule and it is again interpreted as "A implies B". This construction can be carried out considering both aspects:

(i) n-dimensional intervals and fuzzy statements
An expression of the form "$\mathbf{x}$ is A" is termed as a fuzzy statement, where A is an n-dimensional fuzzy set on the n-dimensional simplex $L_n(U)$, with reference to the context. Thus, we can say that the above statement can be interpreted as follows:
– Let "$\mathbf{x}$ is A" and also that $\mathbf{x}$ assumes the precise value, let us say, $\mu_A(\mathbf{x}) = u \in L_n(U)$, the domain of A. Then the truth value of the above fuzzy statement is obtained as $t(\mathbf{x}$ is $A) = A(u)$. Thus, the greater the membership degree of $\mathbf{x}$ in the concept A is, the higher the truth value of the fuzzy statement.
While in the above case a fuzzy statement was looked upon as a fuzzy proposition to be evaluated based on some precise information, it can also be used to express something precise when the only information regarding the variable $\mathbf{x}$ is imprecise.

(ii) n-dimensional intervals compounding n-dimensional IF-THEN rules
We can also interpret an n-dimensional fuzzy statement as a linguistic statement on the suitable domain $L_n(U)$. Then A represents a concept and hence can be thought of as a linguistic value. Then a symbol $\mathbf{x}$ can assume or be assigned to a linguistic value. Then a linguistic statement "$\mathbf{x}$ is A" is interpreted as the linguistic variable $\mathbf{x}$ taking the linguistic value A.

8.1. *n-Dimensional intervals and inference schemes in approximate reasoning*

This section describes a structure in the fuzzy rules of deduction for inference schemes in AR on the n-dimensional simplex domain, which is analogous to the fuzzy logic approach.

In the GMP methodology, a fuzzy logic rule of deduction considers an inequality explicit by a conjunction, defined as an n-dimensional t-norm $\mathcal{T}$ together with an n-dimensional fuzzy $(\mathcal{S},\mathcal{N})$-implication.

The inference schemes are performed based on the combination-projection principle, providing the Compositional Rules of Inference (CRI) [44], which has the structure fuzzy rules based on the GMP inference patterns as follows:

(i) the fuzzy rule has the form "IF $\mathbf{x}$ is A THEN $\mathbf{y}$ is B", and the fact "$\mathbf{x}$ is A'";
(ii) a conclusion to be drawn has the form "$\mathbf{y}$ is B'" when $A, A' \in \mathcal{F}$ and $B, B' \in \mathcal{F}$.

In fuzzy approach, neither A' is necessarily identical to A nor B' is also necessarily identical to B.

8.2. *Compositional rule of inference on $L_n(U)$*

This section describes the application of compositional rules of inference (CRI) systems on $L_n(U)$. For that, let FS_χ be the set of all n-dimensional fuzzy sets w.r.t. a universe χ.

The Cartesian Product among n-DFS is given in the next definition.

Definition 8.1. Let $\chi_1, \ldots, \chi_m$ be non-empty and finite universe-sets and, for each $i \in \mathcal{N}_m$, $A_i \in FS_{\chi_i}$. Then, the Cartesian Product $A_1 \times \ldots \times A_m \equiv \Pi_{i\in\mathbb{N}_m}(A_i)$ w.r.t. the universe-set $\chi_1 \times \ldots \times \chi_m \equiv \Pi_{i\in\mathbb{N}_m}(\chi_i)$ is the function $\Pi_{i\in\mathbb{N}_m}(A_i) : \Pi_{i\in\mathbb{N}_m}(\chi_i) \to (L_n(U))^m$ defined as follows

$$\Pi_{i\in\mathbb{N}_m}(A_i)(\mathrm{x}_1,\ldots,\mathrm{x}_m) = (A_1(\mathrm{x}_1),\ldots,A_m(\mathrm{x}_m)), \forall \mathrm{x}_1 \in \chi_1,\ldots,\forall \mathrm{x}_m \in \chi_m.$$

In particular, let $A_1, \ldots, A_m \in FS(\chi)$ w.r.t. the same universe $\chi = \{x_j : j \in \mathbb{N}_{\#\chi}\}$. So, when $i \in \mathbb{N}_m$, for each $A_i \in FS(\chi)$ the related membership function $\mu_{A_i} : \chi \to L_n(U)$ is given as $\mu_{A_i}(x_j) \equiv A_i(x_j) = \mathbf{x}_{ij}$, $\forall j \in \mathbb{N}_{\#\chi}$. The Cartesian Product of these n-DFS is given in the next definition.

Definition 8.2. Let χ be a non-empty and finite universe-set and $\mathcal{A} = \{A_1, \ldots, A_{\#\mathcal{A}}\} \subseteq FS_\chi$. Taking $\#\mathcal{A} = m$, then the Cartesian Product of the n-dimensional fuzzy sets $A_1, \ldots, A_m$, which is denoted as $A_1 \times \ldots \times A_m \equiv \Pi_{j\in\mathbb{N}_m}(A_j)$, is the function $\mu_{\Pi_{j\in\mathbb{N}_m}(A_j)} \equiv \Pi_{j\in\mathbb{N}_m}(A_j) : \chi^m \to (L_n(U))^m$ defined as follows

$$\Pi_{i\in\mathbb{N}_m}(A_i)(x_1, \ldots, x_m) = (A_1(x_1), \ldots, A_m(x_m)), \forall (x_1, \ldots, x_m) \in \chi^m.$$

Clearly, $\Pi_{j\in\mathbb{N}_m}(A_j)(x_1, \ldots, x_m)$ is well defined. Moreover, for each $(x_1, \ldots, x_m) \in \chi^m$, we have that $\Pi_{j\in\mathbb{N}_m}(A_j)(x_1, \ldots x_m) = (\mathbf{x}_1, \ldots, \mathbf{x}_m) \in (L_n(U))^m$. Thus, the relation $\Pi_{j\in\mathbb{N}_m}(A_j)$ can be expressed as a matrix $\mathbf{X}$ on $(L_n(U))^{m\times l}$ given as

$$\mathbf{X} = (\mathbf{x}_{ij})_{m\times l} = \left(\Pi_{i\in\mathbb{N}_m}(A_i)(x_j)\right)_{j\in\mathcal{N}_l},$$

where $l = \#\chi$ and whose elements $\mathbf{x}_{ij} = A_i(x_j) \in L_n(U)$, for all $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_l$.

Definition 8.3. Let $\mathcal{P} = \{P_1, \ldots, P_l\}$ be a family of n-dimensional m-ary aggregation functions, χ be a non-empty and finite universe-set, $l = \#\chi$ and $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq FS_\chi$. An operator $\mathcal{P} : (L_n(U))^{m\times l} \to (L_n(U))^l$ is defined as follows

$$\mathcal{P}(\mathbf{X}) = \mathcal{P}\left(\left(\Pi_{i\in\mathbb{N}_m}(A_i)(x_j)\right)_{j\in\mathcal{N}_l}\right) = (P_j(A_1(x_j), \ldots, A_m(x_j)))_{j\in\mathbb{N}_l}. \tag{22}$$

The expression given by Eq. (22) provides a method to generate new members on FS_χ, based on the action of a family of aggregation operators.

Example 8.1. Let $\chi = \{1, 2\}$, meaning that $i \in \mathbb{N}_2$. For each $j \in \mathbb{N}_4$, consider the 4-dimensional fuzzy sets $\mathcal{A} = \{A_1, A_2, A_3\}$ over χ defined as follows

$$\pi_j(A_1(x_i)) = \frac{x_i}{n-j+2} \Rightarrow A_1 = \{(0.2, 0.25, 0.\overline{3}, 0.5), (0.4, 0.5, 0.\overline{6}, 1.0)\};$$
$$\pi_j(A_2(x_i)) = \frac{x_i}{n-j+3} \Rightarrow A_2 = \{(0.1\overline{6}, 0.2, 0.25, 0.5), (0.\overline{3}, 0.4, 0.5, 0.\overline{6})\};$$
$$\pi_j(A_3(x_i)) = \frac{x_i}{n-j+4} \Rightarrow A_3 = \{(0.14, 0.1\overline{6}, 0.2, 0.25), (0.28, 0.\overline{3}, 0.4, 0.5)\}.$$

Now, taking the associative operators $\bigwedge, \bigvee : (L_4(U))^{3\times 2} \to (L_4(U))^2$ we have that

$$\begin{aligned}
\bigwedge\left(\left(\Pi_{i\in\mathbb{N}_3}(A_i)(x_j)\right)_{j\in\mathcal{N}_2}\right) &= (\wedge(A_1(x_1), A_2(x_1), A_3(x_1)), \wedge(A_1(x_2), A_2(x_2), A_3(x_2)))\,;\\
&= ((0.14, 0.1\overline{6}, 0.2, 0.25), (0.28, 0.\overline{3}, 0.4, 0.5))\\
\bigvee\left(\left(\Pi_{i\in\mathbb{N}_3}(A_i)(x_j)\right)_{j\in\mathcal{N}_2}\right) &= (\vee(A_1(x_1), A_2(x_1), A_3(x_1), \vee(A_1(x_2), A_2(x_2), A_3(x_2)))\\
&= ((0.2, 0.25, 0.\overline{3}, 0.5), (0.4, 0.5, 0.\overline{6}, 1.0)).
\end{aligned}$$

Definition 8.4. [48] Let χ_1, χ_2 be finite, nonempty sets and $\mathcal{A} = \{A_1, A_2\} \subseteq FS_{\chi_i}$. The cartesian product of the n-DFS A_1 and A_2 related to an n-dimensional t-norm $\mathcal{T}$ is an n-dimensional fuzzy set on $FS_{\chi_1\times\chi_2}$ defined as follows:

$$\mathcal{T}(A_1, A_2)(x_1, x_2) = \mathcal{T}(A_1(x_1), A_2(x_2)), \quad \forall x_1 \in \chi_1, x_2 \in \chi_2.$$

Analogously, an IF-THEN rule is represented by a binary n-dimensional fuzzy relation $\mathcal{R}_\mathcal{I}(A_1, A_2) : (L_n(U))^2 \to L_n(U)$ given as:

$$\mathcal{R}_\mathcal{I}(A_1, A_2)(x_1, x_2) = \mathcal{I}(A_1(x_1), A_2(x_2)), \quad \forall x_1 \in \chi_1, x_2 \in \chi_2 \tag{23}$$

when $\mathcal{I}$ is usually an n-dimensional fuzzy $(\mathcal{S}, \mathcal{N})$-implication and A_1, A_2 are n-dimensional fuzzy sets on their respective universe domains χ_1, χ_2.

Therefore, given a fact “x_1 is A_1”, the inferred output “x_2 is A'_2” is obtained as sup-$\mathcal{T}$ composition of $A'_1(x_1)$ and $\mathcal{R}_\mathcal{I}(A_1, A_2)(x_1, x_2)$, as follows:

$$A'_2(x_2) = (A'_1 \circ^{\mathcal{T}} \mathcal{R}_{\mathcal{I}}(A_1, A_2))(x_2) = \bigvee_{x_1 \in \chi_1} \mathcal{T}(A'_1(x_1), \mathcal{R}_{\mathcal{I}}(A_1, A_2)(x_1, x_2)) = \bigvee_{x_1 \in \chi_1} \mathcal{T}(A'_1(x_1), \mathcal{I}(A_1(x_1), A_2(x_2))). \tag{24}$$

Let A_1, A_2, A_3 be n-DFS on their respective universe domains χ_1, χ_2, χ_3. So, considering the two following cases:

1. Firstly, considering a SISO system given by Eq. (24) attaining normality at an $x'_1 \in \chi_1$, then the related output constructing when the input A'_1 is the singleton n-dimensional fuzzy set $A'_1(x) = /1/$ for each $x \in \chi_1$, is obtained as follows:

$$A'_2(x_2) = A'_1(x_1) \circ^{\mathcal{T}} \mathcal{R}_{\mathcal{I}}(A_1, A_2)(x_1, x_2) = \bigvee_{x_1 \in \chi_1} \mathcal{T}(A'_1(x_1), \mathcal{R}_{\mathcal{I}}(A_1, A_2)(x_1, x_2)) = \mathcal{T}(/1/, \mathcal{R}_{\mathcal{I}}(A_1, A_2)({x'}_1, x_2)) = \mathcal{R}_{\mathcal{I}}(A_1, A_2)({x'}_1, x_2).$$

2. And, in the another case, considering the rule-base in a Multi-Input Single-Output (MISO) system, the relation R is given by

$$R(A_1(x_1), \dots, A_m(x_m), A_{m+1}(x_{m+1})) = \mathcal{I}\left(\bigodot_{i=1}^{m} A_i(x_i), A_{m+1}(x_{m+1})\right), \tag{25}$$

where operator $\bigodot$, called the n-dimensional antecedent combiner, is usually given as an m-ary n-dimensional t-norm as in Eq. (15). Thus, we have $\bigodot_{i=1}^{m} A_i(x_i$ defines the Cartesian Product between A_i's with respect to an m-ary n-dimensional t-norm $\bigodot$. So, given a multiple-input (A'_1, A'_2) and taking the sup-$\mathcal{T}_M$ composition, the inferred output A'_{m+1} is given by the following expression:

$$A'_{m+1} = \bigodot(A'_1, \dots, A'_m) \circ^{\mathcal{T}} \left(\bigodot(A_1, \dots, A_m) \to A_{m+1}\right). \tag{26}$$

Then, by applying results of Eq. (26), for all $x_{m+1} \in \chi_{m=1}$, we obtain the following expression for an output in the IF-THEN base-rule in a MISO system:

$$A'_{m+1}(x_{m+1}) = \bigvee_{(x_1,\dots,x_m) \in \chi_1 \times \dots \times \chi_m} \mathcal{T}\left(\bigodot_{i=1}^{m} A'_i(x_i), \mathcal{I}\left(\bigodot_{i=1}^{m} A_i(x_i), A_{m+1}(x_{m+1})\right)\right). \tag{27}$$

So, when $m = 2$, $\bigodot$ is in fact an n-dimensional t-norm, just denoted by $\odot$. In the following, an example exploring the structure presented in Eq. (27) for $m = 2$ and considering the n-DFS A'_1 and A'_2 as singleton inputs, is presented.

8.3. Exemplification of IF-THEN base-rule in MISO n-dimensional fuzzy system

Consider a virtual application in developing method to medical diagnosis for a patient-analysis with the given five symptoms: fever (a_1), sore throat (a_2), (head)ache (a_3), (dry)cough (a_4), anosmia (a_5), which are described in terms of $L_3(U)$-fuzzy set theory by $\chi_A = \{a_1, \dots, a_5\}$ in order to contemplate the opinions of three experts from distinct researches areas (infectology, epidemiology, and pneumology).

In addition, consider the medical knowledge base components: *Influenzavirus Subtype A-H1N1* (b_1), *COVID-19* (b_2) and *Atopic Bacterial Pneumonia* (b_3), which can enable a proper diagnosis from the set $\chi_B = \{b_1, b_2, b_3\}$. The resulting data provide the worst, moderate and best estimates to each one of diagnoses, modeled by $\chi_C = \{c_1, c_2, c_3\}$ in $L_3(U)$.

The proposed computational evaluation process is conceived to add degrees of freedom and to directly model uncertainty levels of experts knowledge, also including uncertain words from natural language and possible repetition of parameters related to the collected data.

So, let $\chi_A = \{a_1, \dots, a_5\}$, $\chi_B = \{b_1, b_2, b_3\}$ and $\chi_C = \{c_1, c_2, c_3\}$ be universe-sets related to the membership function $A : \chi_1 \to (L_3(U))^5$, $B : \chi_2 \to (L_3(U))^3$ and $A : \chi_3 \to (L_3(U))^3$, defining the corresponding 3-DFS in the following:

$$A = (\mathbf{x}_1, \dots, \mathbf{x}_5) \in (L_3(U))^5,$$
$$B = (\mathbf{y}_1, \dots, \mathbf{y}_3) \in (L_3(U))^3,$$
$$C = (\mathbf{z}_1, \dots, \mathbf{z}_3) \in (L_3(U))^3.$$

where $\mathbf{x}_i = A(a_i)$, $\mathbf{y}_i = B(b_i)$ and $\mathbf{z}_i = C(c_i)$ for each $i \in \mathbb{N}_5$ and $j, k \in \mathbb{N}_3$. In this application, instances of such 3-DFS are, respectively, given as follows:

$$A = ((0.55, 0.6, 0.65), (0.45, 0.50, 0.55), (0.35, 0.40, 0.45), (0.35, 0.40, 0.45), (0.60, 0.65, 0.70)),$$
$$B = ((0.75, 0.80, 0.85), (0.35, 0.40, 0.45), (0.55, 0.60, 0.65)),$$
$$C = ((0.10, 0.15, 0.20), (0.15, 0.20, 0.25), (0.20, 0.25, 0.30)).$$

Let $A' = (/0/, /0/, /1/, /0/, /0/) \in (L_3(U))^5$, $B' = (/0/, /1/, /0/) \in (L_3(U))^3$ as the given singleton inputs. Moreover, we consider the following operators:

(i) $\odot \equiv \widetilde{T_{LK}, T_P, T_M}$, the representable n-DT modeling the Cartesian Product operator (see, Example 5.1);
(ii) $\rightarrow \equiv \widetilde{I_{KD}, I_{RC}, I_{LK}}$, the representable n-dimensional fuzzy implication (see, Example 7.1, Eq. (20));
(iii) $(\bigvee, \widetilde{T_M})$, the operators providing the sup $-\mathcal{T}_M$ composition, (see, Example 5.1).

By Eq. (27), the expression of IF-THEN base-rules in MISO n-DFL is given as follows:

$$\begin{aligned} C'(z) &= \bigvee_{(x,y)\in\chi_1\times\chi_2} \widetilde{T_M}\left(\odot(A'(x), B'(y)), \mathcal{I}\left(\odot(A(x), B(y)), C(z)\right)\right), \\ &= \bigvee_{(x,y)\in\chi_1\times\chi_2} \left(\widetilde{T_M}\left(\widetilde{T_{LK}, T_P, T_M}\left(A'(x), B'(y)\right), \widetilde{I_{KD}, I_{RC}, I_{LK}}\left(\odot(A(x), B(y)), C(z)\right)\right)\right), \forall z \in \chi_3. \end{aligned} \tag{28}$$

The steps to consolidate Eq. (28) are described in the following.

(I) Firstly, the Cartesian Product $A \times B$ considering $\widetilde{T_{LK}, T_P, T_M}$ is defined as follows:

$$\mathbf{w}_{ij} = \widetilde{T_{LK}, T_P, T_M}(\mathbf{x}_i, \mathbf{y}_j) = \left(T_{LK}(\pi_1(\mathbf{x}_i), \pi_1(\mathbf{y}_j)), T_P(\pi_2(\mathbf{x}_i), \pi_2(\mathbf{y}_j)), T_M((\pi_3(\mathbf{x}_i), \pi_3(\mathbf{y}_j))\right), \forall i \in \mathbb{N}_5, j \in \mathbb{N}_3.$$

where $\mathbf{x}_i = A(a_i)$ and $\mathbf{y}_j = B(b_j)$ for each $i \in \mathbb{N}_5$ and $j \in \mathbb{N}_3$. Thus, for example, the first component, taking $i = j = 1$ is given as

$$\begin{aligned} \mathbf{w}_{11} &= \widetilde{T_{LK}, T_P, T_M}(\mathbf{x}_1, \mathbf{y}_1) = (T_{LK}(0.55, 0.75), T_P(0.60, 0.80), T_M(0.65, 0.85)) \\ &= (\max(0.55 + 0.75 - 1, 0), 0.60 \cdot 0.80, \min(0.65, 0.85)) = (0.30, 0.48, 0.65). \end{aligned}$$

Analogously, the other components can be obtained. They are described as a matrix structure below:

$$(\mathbf{w}_{ij})_{i\in\mathbb{N}_5, j\in\mathbb{N}_3} = \begin{pmatrix} (0,30, 0,48, 0,65) & (0,00, 0,24, 0,45) & (0,10, 0,36, 0,65) \\ (0,20, 0,40, 0,55) & (0,00, 0,20, 0,45) & (0.00, 0,30, 0,55) \\ (0,10, 0,32, 0,45) & (0,00, 0,16, 0,45) & (0.00, 0,24, 0,45) \\ (0,10, 0,32, 0,45) & (0,00, 0,16, 0,45) & (0.00, 0,24, 0,45) \\ (0,35, 0,52, 0,70) & (0,00, 0,26, 0,45) & (0,15, 0,39, 0,65) \end{pmatrix}$$

(II) Let $I_{KD}, I_{RC}, I_{LK} : U^2 \to U$ be the (S,N)-implications given in Example 7.1 related to representable n-DI $\widetilde{I_{KD}, I_{RC}, I_{LK}} : (L_3(U))^2 \to L_3(U)$ reported in Example 7.1, Eq. (20).
In the following, see the results from operator

$$(\mathbf{v}_{ij})_k = \widetilde{I_{KD}, I_{RC}, I_{LK}}\left(\mathbf{w}_{ij}, \mathbf{z}_k\right), \forall i \in \mathbb{N}_5, \forall j, k \in \mathbb{N}_3. \tag{29}$$

where $\mathbf{z}_k = C(c_k)$ for each $k \in \mathbb{N}_3$. For $k = i = 1$ and $j \in \mathbb{N}_3$:

$$(\mathbf{v}_{1j})_1 = \widetilde{I_{KD}, I_{RC}, I_{LK}}(\mathbf{w}_{1j}, \mathbf{c}_1) = \left(I_{KD}(\pi_3(\mathbf{w}_{1j}), \pi_1(\mathbf{c}_1)), I_{RC}(\pi_2(\mathbf{w}_{1j}), \pi_2(\mathbf{c}_1)), I_{LK}((\pi_3(\mathbf{w}_{1j}), \pi_3(\mathbf{c}_1))\right)$$

$$\begin{aligned} (\mathbf{v}_{11})_1 &= \widetilde{I_{KD}, I_{RC}, I_{LK}}(\mathbf{w}_{11}, \mathbf{c}_1) = \widetilde{I_{KD}, I_{RC}, I_{LK}}((0.30, 0.48, 0.65), (0.10, 0.15, 0.20)) \\ &= (I_{KD}(0.65, 0.10), I_{RC}(0.48, 0.15), I_{LK}(0.30, 0.20)) \\ &= (\max(1 - 0.65; 0.10), 1 - 0.48 + 0.48 \cdot 0.15, \min(1, 1 - 0.30 + 0.20)) \\ &= ((0.350, 0.592, 0.9000) \end{aligned}$$

$$\begin{aligned} (\mathbf{v}_{12})_1 &= \widetilde{I_{KD}, I_{RC}, I_{LK}}(\mathbf{w}_{12}, \mathbf{c}_1) = \widetilde{I_{KD}, I_{RC}, I_{LK}}((0.00, 0.24, 0.45), (0.10, 0.15, 0.20)) \\ &= (I_{KD}(0.45, 0.10), I_{RC}(0.24, 0.15), I_{LK}(0.00, 0.20)) \\ &= (\max(1 - 0.45; 0.10), 1 - 0.24 + 0.24 \cdot 0.15, \min(1, 1 - 0.00 + 0.20)) \\ &= ((0.550, 0.796, 1.0000) \end{aligned}$$

$$\begin{aligned}(\mathbf{v}_{13})_1 = \widetilde{I_{KD}, I_{RC}, I_{LK}}(\mathbf{w}_{13}, \mathbf{c}_1) &= \widetilde{I_{KD}, I_{RC}, I_{LK}}((0.10, 0.36, 0.65), (0.10, 0.15, 0.20)) \\ &= (I_{KD}(0.65, 0.10), I_{RC}(0.36, 0.15), I_{LK}(0.10, 0.20)) \\ &= (\max(1-0.65; 0.10), 1-0.36+0.36\cdot 0.15, \min(1, 1-0.10+0.20)) \\ &= ((0.350, 0.694, 1.0000)\end{aligned}$$

Then, the above results constitute the first line in the $[(\mathbf{v}_{ij})_1]$-matrix. The other coefficients can be analogous obtained. See the final results concluding this step in the three matrices $[(\mathbf{v}_{ij})_1]$, $[(\mathbf{v}_{ij})_2]$ and $[(\mathbf{v}_{ij})_3]$ in the following:

$$[(\mathbf{v}_{ij})_1] = \begin{pmatrix} (0.3500, 0.5920, 0.9000), & (0.5500, 0.7960, 1.0000), & (0.3500, 0.6940, 1.0000) \\ (0.4500, 0.5400, 1.0000), & (0.5500, 0.8300, 1.0000), & (0.4500, 0.7450, 1.0000) \\ (0.5500, 0.7280, 1.0000), & (0.5500, 0.8640, 1.0000), & (0.5500, 0.79601.0000) \\ (0.5500, 0.7280, 1.0000), & (0.5500, 0.8640, 1.0000), & (0.5500, 0.79601.0000) \\ (0.3000, 0.5580, 0.8500), & (0.5500, 0.7790, 1.0000), & (0.35000.6685, 1.0000) \end{pmatrix}$$

$$[(\mathbf{v}_{ij})_2] = \begin{pmatrix} (0.3500, 0.6160, 0, 9500) & (0, 5500, 0.8080, 1.0000) & (1, 0000, 0.7120, 1.0000) \\ (0.4500, 0.6800, 1.0000) & (0.5500, 0.8400, 1.0000) & (1, 0000, 0.7600, 1.0000) \\ (0.5500, 0.7440, 1.0000) & (0.5500, 0.8720, 1.0000) & (1, 0000, 0.8080, 1, 0000) \\ (0.5500, 0.7440, 1.0000) & (0.5500, 0.8720, 1.0000) & (1, 0000, 0.8080, 1, 0000) \\ (0.3000, 0.5840, 0.9000) & (0.7323, 0.7920, 0.8425) & (1, 0000, 0.6880, 1, 0000) \end{pmatrix}$$

$$[(\mathbf{v}_{ij})_3] = \begin{pmatrix} (0.3500, 0.6400, 0.9500) & (0.2000, 0.8200, 1.0000) & (0.3500, 0.7300, 1.0000) \\ (0.4500, 0.7000, 1.0000) & (0.2000, 0.8500, 1.0000) & (0.4500, 0.7750, 1.0000) \\ (0.5500, 0.7600, 1.0000) & (0.2000, 0.8800, 1.0000) & (0.5500, 0.8200, 1.0000) \\ (0.5500, 0.7600, 1.0000) & (0.2000, 0.8800, 1.0000) & (0.5500, 0.8200, 1.0000) \\ (0.3000, 0.6100, 0.9000) & (0.2000, 0.8050, 1.0000) & (0.3500, 0.7075, 1.0000) \end{pmatrix}$$

(III) The Cartesian Product $A' \times B'$ also considers the n-DT $\widetilde{T_{LK}, T_P, T_M}$ and, $\forall i \in \mathbb{N}_5,\ j \in \mathbb{N}_3$, such operator is defined by as follows:

$$\mathbf{u}_{ij} = \widetilde{T_{LK}, T_P, T_M}(\mathbf{x}'_i, \mathbf{y}'_j) = \left(T_{LK}(\pi_3(\mathbf{x}'_i), \pi_1(\mathbf{y}'_i)), T_P(\pi_2(\mathbf{x}'_i), \pi_2(\mathbf{y}'_i)), T_M((\pi_1(\mathbf{x}'_i), \pi_3(\mathbf{y}'_i))\right),$$

and graphically represented by the matrix below:

$$[\mathbf{u}_{ij}]_{i\in\mathbb{N}_5, j\in\mathbb{N}_3} = \widetilde{T_{LK}, T_P, T_M}(\mathbf{x}', \mathbf{y}') = \begin{pmatrix} /0/ & /0/ & /0/ \\ /0/ & /0/ & /0/ \\ /0/ & /1/ & /0/ \\ /0/ & /0/ & /0/ \\ /0/ & /0/ & /0/ \end{pmatrix}.$$

(IV) For each $i \in \mathbb{N}_5,\ j \in \mathbb{N}_3$. Consider $\left(\mathbf{t}_{(ij)}\right)_{k\in\mathbb{N}_3} = \left(\widetilde{T_M}\left(\mathbf{u}_{ij}, \mathbf{v}_{ij}\right)\right)_{k\in\mathbb{N}_3}$ resulting on the matrices below:

$$[\mathbf{t}_1] = \begin{pmatrix} /0/ & /0/ & /0/ \\ /0/ & /0/ & /0/ \\ /0/ & (0.550, 0.864, 1.000) & /0/ \\ /0/ & /0/ & /0/ \\ /0/ & /0/ & /0/ \end{pmatrix} \tag{30}$$

$$[\mathbf{t}_2] = \begin{pmatrix} /0/ & /0/ & /0/ \\ /0/ & /0/ & /0/ \\ /0/ & (0.550, 0.872, 1.000) & /0/ \\ /0/ & /0/ & /0/ \\ /0/ & /0/ & /0/ \end{pmatrix} \tag{31}$$

$$[\mathbf{t}_3] = \begin{pmatrix} /0/ & /0/ & /0/ \\ /0/ & /0/ & /0/ \\ /0/ & (0.200, 0.880, 1.000) & /0/ \\ /0/ & /0/ & /0/ \\ /0/ & /0/ & /0/ \end{pmatrix} \tag{32}$$

Table 1
The constructor of IF-THEN base-rules in MISO n-DFL.

$(\mathcal{T},\mathcal{I})$-operator	Results from CRI Execution on $(L_3(U))^3$
$(\widetilde{\mathcal{T}_M,\mathcal{T}_P},\mathcal{T}_{LK};\widetilde{\mathcal{I}_{KD},\mathcal{I}_{RC}},\mathcal{I}_{LK})$	$((\mathbf{0.5500},\mathbf{0.8640},\mathbf{1.0000}),(\mathbf{0.5500},\mathbf{0.8720},\mathbf{1.0000}),(\mathbf{0.2000},\mathbf{0.8800},\mathbf{1.0000}))$
$(\widetilde{\mathcal{T}_M};\widetilde{\mathcal{I}_{KD}})$	$((\mathbf{0.5500},0.6000,0.6500),\ (\mathbf{0.5500},0.6000,0.6500),\ (\mathbf{0.2000},0.6000,0.6500))$
$(\widetilde{\mathcal{T}_P};\widetilde{\mathcal{I}_{RC}})$	$((0.8178,\mathbf{0.8640},0.9020),\ (0.8279,\mathbf{0.8720},0.9881),\ (0.8380,\mathbf{0.8800},0.9143))$
$(\widetilde{\mathcal{T}_{LK}};\widetilde{\mathcal{I}_{LK}})$	$((1.0000,1.0000,\mathbf{1.0000}),\ (1.0000,1.0000,\mathbf{1.0000}),\ (1.0000,1.0000,\mathbf{1.0000}))$

(V) Concluding, in the $\bigvee - \widetilde{\mathcal{T}_M}$ composition, we apply the operator $\bigvee : (L_3(U))^5 \to (L_3(U))$ considering the five lines of each matrix $[\mathbf{t}_1]$, $[\mathbf{t}_2]$ and $[\mathbf{t}_3]$. It results on the following n-DFS:

$$C' = \bigvee_{k\in\mathbb{N}_3} \left(\mathbf{t}_{(ij)}\right)_k$$

$$C' = ((0.550, 0.864, 1.000), (0.550, 0.872, 1.000), (0.200, 0.880, 1.000))$$

The constructor of IF-THEN base-rules in MISO n-DFL can be obtained considering other three operators, as reported in Table 1. In these cases, the first pair-operators defined as $(\widetilde{\mathcal{T}_M,\mathcal{T}_P},\mathcal{T}_{LK};\widetilde{\mathcal{I}_{KD},\mathcal{I}_{RC}},\mathcal{I}_{LK})$ presents in the same execution inference of based-rules, the worst, moderate and best estimates. In addition, it also partially includes results from other pairs $(\widetilde{\mathcal{T}_M};\widetilde{\mathcal{I}_{KD}})$, $(\widetilde{\mathcal{T}_P};\widetilde{\mathcal{I}_{RC}})$ and $(\widetilde{\mathcal{T}_{LK}};\widetilde{\mathcal{I}_{LK}})$, as emphasized by bold numbers.

9. Conclusion

This work discussed the n-dimensional interval fuzzy implications, considering the study of continuity, duality, conjugation and their representability based on fuzzy implications from U to $L_n(U)$.

As the main contribution, relevant properties characterizing the class of n-dimensional interval (S,N)-implications on $L_n(U)$ are studied. In sequence, this study contemplated the discussion of such extension of fuzzy connectives on $L_n(U)$. It is worth mentioning that we considered the case and provided a characterization in Theorem 7.1, for n-dimensional interval (S,N)-implications on $L_n(U)$ when the n-DN is right reversible. Moreover, since n-DFS generalize fuzzy sets and interval-valued fuzzy set and such class of (interval-valued) (S,N)-implications were not studied, then Theorem 7.1 can also contribute in the study of (interval-valued) (S,N)-implications. In particular, once right invertible fuzzy negation generalizes continuous fuzzy negations, then this result generalizes the characterization of continuous (interval-valued) (S,N)-implications, as presented in [7, Theorem 2.4.10]. In addition, n-dimensional intervals and inference schemes in approximate reasoning were presented and an example was also developed.

Since inherent ordering related to n-dimensional intervals, admissible linear orders contributing with research areas as making decisions based on multi-attributes. Ongoing work overcomes the restriction of selected representable n-DFI verifying the increasing sequence of fuzzy implications, by considering the use of admissible linear $\preceq$-orders on $L_n(U)$ as studied in [20]. Thus, we intend to analyze properties as anti/iso monotonicity, continuity, reversibility w.r.t. admissible $\preceq$-orders on $L_n(U)$.

Further work also considers studying other special classes of fuzzy implications as D-, QL- and R-implications and others as power-implications, Yager-implications, (T,N)-implications and H-implications.

Concluding, the studied properties in the class of n-dimensional interval (S,N)-implications on $L_n(U)$ unable us to apply the obtained results in definition of new consensus measures in the sense as proposed by Beliakov [17].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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