



Various aggregation operators of the generalized hesitant fuzzy numbers based on Archimedean t -norm and t -conorm functions

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Abstract

This paper intends to introduce mathematical tools for aggregation of the generalized hesitant fuzzy numbers in order to increase the use of them in the real world. The proposed operators, are based on general form of t -norm and t -conorm functions, enable us to do some mathematical computations and aggregate the given generalized hesitant fuzzy numbers. At first, some famous Archimedean t -norms and t -conorms, i.e., Algebraic, Einstein, Hamacher, and Frank t -norms and t -conorms, and their properties, have been developed to be employed with generalized hesitant fuzzy numbers. Then, several averaging and geometric-based aggregation operators for generalized hesitant fuzzy numbers have been proposed. Later on, a decision-making algorithm has been defined based on such operators to address the problems. The necessity and application of the proposed concepts have been explained by some numerical examples.

Keywords Generalized hesitant fuzzy numbers · Hesitant fuzzy numbers · Hesitant fuzzy sets · t -norm and t -conorm functions · Hybrid assessment

1 Introduction

Today, uncertainty is an accepted scientific factor in modeling of real-world problems. Probability theory, belief function theory, possibility theory, fuzzy sets theory, etc., are some of the proposed tools to deal with the uncertainty (Weaver 1948; Smithson 1989; Selvachandran et al. 2019; Yilin et al. 2021; Giang et al. 2019; Bashir et al. 2018). As human societies grew, so did the complexity of their problems, and it was necessary to develop appropriate solution tools to the triple classification of problems (organized simplicity, organized complexity, and disorganized complexity problems) (Weaver 1948). For example, fuzzy sets (FSs) theory (Zadeh 1965) and some of its generalizations, i.e., type-2 fuzzy sets

(Karnik and Mendel 2001), intuitionistic fuzzy sets (IFSs) (Atanassov 1983, 2022), Pythagorean fuzzy sets (Wang and Garg 2020), Hesitant fuzzy sets (HFSs) (Torra 2010), intuitionistic Hesitant fuzzy sets (Mahmood et al. 2021), Hesitant fuzzy numbers (HFNs) (Keikha 2021), and generalized Hesitant fuzzy numbers (GHFNs) (Keikha 2021) are justifiable from this perspective.

Most of the practical problems are composed of many but a finite number of factors, and fall into the organized complexity problems category (Weaver 1948). Recently, the HFSs which deployed hesitant fuzzy elements (HFEs), i.e., a confined collection of several amounts between 0 and 1 as unsureness grades, have received a lot of attention to be used in different areas alone or in combination with some other methods (Jin et al. 2022; Qin et al. 2022; Shen et al. 2021). Due to various practical problems, the conventional model of HFSs did not meet the needs of researchers, and as a result they have been developed to interval-valued HFSs (Yahya et al. 2021), generalized trapezoidal hesitant fuzzy numbers (Deli 2020), HFNs with two completely different and separate definitions (Ranjbar et al. 2020; Keikha 2021), type-2 HFSs (Liu et al. 2018), interval type-2 HFSs (Hu et al. 2015), generalized HFNs (Keikha 2021) to solve problems more accurately. Considering that these are in fact kinds of quan-

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tification of non-quantitative/uncertain values, it is therefore urgently necessary to invent methods which can be used to carry out arithmetical operations, and aggregation operators (Keikha 2021; Liao and Xu 2014b; Lobillo et al. July 2021; Tan et al. 2015; Zhang 2016), etc. Therefore, mathematical formalization and calculus of HFSs, like operation laws (Torra 2010; Liao and Xu 2014a), distance and similarity measures (Xu and Xia 2011a, b; Tong and Yu 2016), correlation coefficient (Xu and Xia 2011b; Tong and Yu 2016), entropy measure (Xu and Xia 2012), and aggregation operators (Xia and Xu 2011; Wei 2012; Zhang 2013; Liao and Xu 2014b), have been investigated simultaneously with their usage in solving practical problems, very soon.

The calculus development of these uncertainty theories is based on functions $g : [0, 1] \times [0, 1] \rightarrow [0, 1]$, with associativity, commutativity, boundary condition, and monotonicity properties, which are called t -norm and t -conorm (Klir and Yuan 1995; Lobillo et al. July 2021). They are defined and known in different forms as Frank t -norm and t -conorm, Hamacher t -norm and t -conorm, Einstein t -norm and t -conorm, Algebraic t -norm and t -conorm, etc., and extended to be used with uncertain data. For instance, some of these famous t -norms and t -conorms, such as Frank, Hamacher, Einstein, Dombi, and Algebraic, have been applied as aggregation operators of HFSs (Tang et al. 2018; Zhou et al. 2014; Zhang 2016; Tan et al. 2015; Yahya et al. 2021; Mahmood et al. 2021; Liu et al. 2020).

Although quantitative modeling of the uncertainty of some problems is possible with the help of some existing methods, sometimes researchers have no choice but to generalize them or invent a new method to increase the accuracy of modeling. It should be noted that this process is unstoppable, because despite the aforementioned three classifications of real problems remaining constant, their variety and the demand for increasing accuracy modeling are unstoppable. From this point of view, GHFNs can be rational, if they are used in the right place.

It may be asked that, what are the GHFNs? and what is the need for them? In answer must be said: the GHFNs have been introduced to model discrete vague information, where incorrectly transferred to continuous spaces, and modeled via other types of fuzzy numbers, i.e., an infinite set of values along with an infinite set of membership degrees. The following examples show that this type of fuzzy numbers have already been existed, but are incorrectly modeled by other types of them.

- (1) Suppose an astronomer estimates the distance of a celestial body from Earth to be 18, 19, 20, 21, and at most 22 million light-years, and expresses the result with the linguistic phrase “approximately 20 million light-years.” Also, assume that the researcher avails a finite set of amounts from $[0,1]$ to express her/his degrees of skepticism

about the evaluation values. We can model it by interval-number [18, 22], trapezoidal fuzzy number [18,19,20,22], which are containing all values between 18 and 22, i.e., an infinite set of possible values.

- (2) Suppose a farmer has a certain amount of farm, and is willing to plan for the next crop year based on the relevant information recorded in the last years. It is clear that for the forward crop year, the climate changes, the price and the amount of harvest are unknown, but they may be close to the previous recorded values. Applying the averaging-based methods, type-1/type-2/intuitionistic fuzzy numbers, and probability-based methods although common, cannot cover all requirements of such problems, or may add their complexity. Because some past results may have been achieved under ideal conditions, some in very bad conditions, and some in normal conditions, which may lead to different degrees of satisfaction for the farmer. In other words, the harvest in the past may be unfavorable in ideal weather conditions and the farmer change it, albeit with some degree of doubt. In other scenarios, such changes (incremental or decreasing) may occur with a confined set of grades of indecision/satisfaction. Generalized hesitant fuzzy numbers, in addition to being able to directly use the available data without changing, or apply the farmer’s opinions, for each of these conditions, also take into account the farmer’s degree of skepticism/satisfaction about those conditions and achievements.

In such cases, the use of interval numbers, trapezoidal/triangular/intuitionistic/type-2 fuzzy numbers cause the quantification space to be transformed from a discrete space to a continuous space. That is, two finite sets of values along with membership degrees are replaced by relevant infinite sets. Although this transfer provides many analytical methods available to the researcher by unintentionally changing the nature of the problem from the category of organized complexity to the category of disorganized complexity (Klir 2006), but by solving a problem other than the main problem, the main problem remains practically unresolved. Therefore, improper use of modeling tools in some situations will add to the inaccuracy of the problem. On the other hand, the use of an infinite number of values, as is common in other types of fuzzy numbers, conflicts with human mental structures.

A GHFN $\tilde{P}^H = \langle \{p_1, p_2, \dots, p_m\}; \{\lambda_1, \lambda_2, \dots, \lambda_n\} \rangle$ is a method for modeling situations where the decision maker (DM) is faced with a finite set of non-rejectable/acceptable real numbers $\{p_1, p_2, \dots, p_m\}$, and expresses her/his doubtness with a finite set of values between 0 and 1, i.e., $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. One of the important applications of these numbers is in stock exchanges and future markets. Where the trader, based on the knowledge she/he has gained from the market, doubtfully reaches a finite set of amounts for

the selling/buying each unit of an asset, and the doubtness degrees are expressed as a HFE. Futurists, managers, and decision makers will be another important users of GHFNs. Economists, agricultural planners, development planners, politicians, social scientists and archaeologists, deep learning and machine learning researchers, etc., will be able to solve many of their problems with the help of GHFNs. Also, facing new situations, such as group decision making, e-learning, especially in COVID-19 pandemic, the best stock portfolio selection, etc., Yilin et al. (2021), showed that existing tools need to be developed, or new tools must be invented, to increase modeling accuracy.

The establishment of mathematical computational instruments is a vital step for any newly proposed modeling concept to extend its applications (Klir 2006). So, after any new concept we encounter with many paper, where discussed and researched mathematical computational methods (Deli and Karaaslan 2021; Deli 2021). In order to expand the applications of GHFNs and gain the trust of researchers in other scientific fields, they must have a strong methodology and mathematical support. It will be discussed for GHFNs in this article, to lead to the practical development of GHFNs, and finally finding more accurate responses for real problems in other research fields such as medical diagnosis, fair evaluation, cosmology, artificial intelligence, robotics, decision making, computer science, deep learning, machine learning, robotics, image processing, etc. Rodriguez et al. (2012), Zhang et al. (2017, 2020).

Aggregation of partial given values is an important step to solve a multi-criteria decision making problem. As we know, Archimedean t-norm and t-conorm (ATT) are the most useful mathematical tools in aggregation process. They are the extensions of some other t-norms and t-conorms such as Hamacher, Einstein, Frank, and algebraic t-norms and t-conorms. They are defined based on an additive function $g(\tau)$, called generator function, and its dual $f(\tau) = g(1 - \tau)$, in which we can obtain the specific t-norm and t-conorm by choosing the special form of additive function. The study of ATT-based aggregation operators of GHFNs and their usage to decision making problems under real hesitant fuzzy environments is important in that they make it easier to state/model the DMs' opinions and unsureness in the decision process, and the ATT supply more flexibility than others.

In this paper, given the novelty of the GHFNs, based on ATT functions, some mathematical and aggregation operators of GHFNs have been proposed. The objectives of the study are listed as below:

- (1) To explore the idea of the generalized hesitant fuzzy numbers to demonstrate the uncertainties in the data.
- (2) To define some mathematical operations between the pairs of the GHFNs and studied their properties.
- (3) To propose some weighted averaging and geometric operators based on the ATT functions of GHFNs.
- (4) To set up a decision-making algorithm based on the proposed operators.
- (5) To demonstrate the working of the algorithm with two numerical examples and compare their results with the existing algorithms.

It should be noted that the conditions of the problem and the diagnosis of the user determine the need to use each of the above operators. Therefore, none of them have priority in nature, but their existence is mandatory.

In the following, the paper is structured as follows. Section 2 describes the basic preliminaries. In Sect. 3, we defined the concepts of arithmetic operations of GHFNs, and some ATT-based aggregation operators along with their properties. In Sect. 4, we illustrate the proposed approach with numerical example. Finally, some conclusion is given in Sect. 5.

2 Basic concepts and definitions

This section contains the concepts based on hesitant fuzzy sets, required by the other Sections. Mathematical representation of a HFS is $D = \{ \langle x, d(x) \rangle \mid x \in X \}$, where $d(x) = \{ \lambda_1, \lambda_2, \dots, \lambda_n \}$, $\lambda_i \in [0, 1]$ (Torra 2010; Xia and Xu 2011). Operational laws and calculus of HFSs, which have been discussed simultaneously with their practical applications, have been defined based on bounded, commutated, associated, and monotone functions $h : [0, 1] \times [0, 1] \rightarrow [0, 1]$, named triangular norm (*t*-norm) and triangular conorm (*t*-conorm) (Klir and Yuan 1995; Nguyen and Walker 1997).

Definition 1 Klir and Yuan (1995) The *t*-norm $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ and *t*-conorm $S : [0, 1] \times [0, 1] \rightarrow [0, 1]$ are two arbitrary functions, that $\forall a, b, c \in [0, 1]$:

- | | |
|---|---|
| <p>(I) $T(1, a) = a,$</p> <p>(II) $T(a, b) = T(b, a),$</p> <p>(III) $T(a, T(b, c)) = T(T(a, b), c),$</p> <p>(IV) $a \leq a' \& b \leq b' \Rightarrow T(a, b) \leq T(a', b'),$</p> | <p>(I') $S(0, a) = a$</p> <p>(II') $S(a, b) = S(b, a),$</p> <p>(III') $S(a, S(b, c)) = S(S(a, b), c),$</p> <p>(IV') $a \leq a' \& b \leq b' \Rightarrow S(a, b) \leq S(a', b')$</p> |
|---|---|

If for all $a \in [0, 1]$: $T(a, a) < a$, and $S(a, a) > a$, then T and S are called Archimedean. Consider an additive generator $g : [0, 1] \rightarrow [0, +\infty)$, and $f(t) = g(1 - t)$. Then, Archimedean t -norm $T(x, y) = g^{-1}(g(x) + g(y))$, and Archimedean t -conorm $S(x, y) = f^{-1}(f(x) + f(y))$, which are strictly increasing, are called strictly Archimedean (Klement and Mesiar 2005). The t -norm T and t -conorm S are called: Algebraic if $g(t) = -\log t$, Einstein if $g(t) = \log \frac{2-t}{t}$, and Hamacher if $g(t) = \log \frac{v+(1-v)t}{t}$, $v > 0$ (Klir and Yuan 1995). Using these concepts, arithmetic operations of any arbitrary HFEs d, d_1 , and d_2 , would be defined as follows Zhang (2016):

$$\begin{aligned}
 (1) d_1 \oplus d_2 &= \bigcup_{\lambda_1 \in d_1, \lambda_2 \in d_2} \{S(\lambda_1, \lambda_2)\} \\
 &= \bigcup_{\lambda_1 \in d_1, \lambda_2 \in d_2} \left\{ f^{-1}(f(\lambda_1) + f(\lambda_2)) \right\}; \\
 (2) d_1 \otimes d_2 &= \bigcup_{\lambda_1 \in d_1, \lambda_2 \in d_2} \{T(\lambda_1, \lambda_2)\} \\
 &= \bigcup_{\lambda_1 \in d_1, \lambda_2 \in d_2} \left\{ g^{-1}(g(\lambda_1) + g(\lambda_2)) \right\}; \\
 (3) \lambda d &= \bigcup_{\lambda \in d} \left\{ f^{-1}(\lambda f(\lambda)) \right\}, \quad \lambda > 0; \\
 (4) d^\lambda &= \bigcup_{\lambda \in d} \left\{ g^{-1}(\lambda g(\lambda)) \right\}, \quad \lambda > 0;
 \end{aligned}$$

where, λ is a positive real value.

Using these, and based on t -norm and t -conorm we can define some Archimedean hesitant fuzzy (A-HF) operators as follows.

Definition 2 Zhang (2016) Let $d_i (i = 1, 2, \dots, n)$ be given HFEs, that are weighted by $0 \leq w_i \leq 1$ with $\sum_{i=1}^n w_i = 1$. Then

(1) A-HF weighted averaging (A-HFWA) operator,

$$\begin{aligned}
 \text{A-HFWA}(d_1, d_2, \dots, d_n) &= \bigoplus_{i=1}^n (w_i d_i) \\
 &= \bigcup_{\lambda_i \in d_i} \left\{ f^{-1} \left(\sum_{i=1}^n w_i f(\lambda_i) \right) \right\},
 \end{aligned}$$

(2) A-HF weighted geometric (A-HFWG) operator,

$$\begin{aligned}
 \text{A-HFWG}(d_1, d_2, \dots, d_n) &= \bigotimes_{i=1}^n (d_i^{w_i}) \\
 &= \bigcup_{\lambda_i \in d_i} \left\{ g^{-1} \left(\sum_{i=1}^n w_i g(\lambda_i) \right) \right\},
 \end{aligned}$$

(3) A-HF ordered weighted averaging (A-HFOWA) operator

$$\begin{aligned}
 \text{A-HFOWA}(d_1, d_2, \dots, d_n) &= \bigoplus_{i=1}^n (w_i d_{\sigma(i)}) \\
 &= \bigcup_{\lambda_{\sigma(i)} \in d_{\sigma(i)}} \left\{ f^{-1} \left(\sum_{i=1}^n w_i f(\lambda_{\sigma(i)}) \right) \right\},
 \end{aligned}$$

(4) A-HF ordered weighted geometric (A-HFOWG) operator

$$\begin{aligned}
 \text{A-HFOWG}(d_1, d_2, \dots, d_n) &= \bigotimes_{i=1}^n (d_{\sigma(i)}^{w_i}) \\
 &= \bigcup_{\lambda_{\sigma(i)} \in d_{\sigma(i)}} \left\{ g^{-1} \left(\sum_{i=1}^n w_i g(\lambda_{\sigma(i)}) \right) \right\},
 \end{aligned}$$

in which $d_{\sigma(i)} i = 1, 2, \dots, n$ is a permutation of $d_i, i = 1, 2, \dots, n$ such that $d_{\sigma(1)} \leq d_{\sigma(2)} \leq \dots \leq d_{\sigma(n)}$.

In some tangible decision making problems, there may be exist some pre-determined amounts that are not fully avowed by DM, and should be considered in decision making process. HFNs, as the outreach of HFSs, are apt modeling such cases (Keikha 2021).

Definition 3 Keikha (2021) Suppose x be a predetermined positive real value about an element z of the reference set Z , which is hesitated by HFE

$$h(x) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}, \lambda_i \in [0, 1]$$

A HFN \tilde{X}^H with mathematical representation $\langle x; d(x) \rangle$ merges these two separate information on element $z \in Z$, i.e., $\tilde{X}^H = \langle x; \{\lambda_1, \lambda_2, \dots, \lambda_n\} \rangle$.

It means that each HFN is an ordered pair that its first component is a predetermined real value, and the second one is a confined set of doubtness grades, called real part and membership part, respectively. HFNs are called adjusted, if their membership parts are adjusted HFEs.

Definition 4 Keikha (2021) Suppose $\tilde{X}^H = \langle x; d(x) \rangle, \tilde{Y}^H = \langle y; d(y) \rangle$ be two adjusted HFNs. Then

$$\begin{aligned}
 (1) \tilde{X}^H \oplus \tilde{Y}^H &= \left\langle x + y; \bigcup_{\substack{\lambda_{1(i)} \in d(x), \\ \lambda_{2(i)} \in d(y)}} \{S(\lambda_{1(i)}, \lambda_{2(i)})\} \right\rangle \\
 &= \left\langle x + y; \bigcup_{\substack{\lambda_{1(i)} \in d(x), \\ \lambda_{2(i)} \in d(y)}} \left\{ f^{-1}(f(\lambda_{1(i)}) + f(\lambda_{2(i)})) \right\} \right\rangle, \\
 (2) \tilde{X}^H \otimes \tilde{Y}^H &
 \end{aligned}$$

$$= \left\langle x.y; \bigcup_{\substack{\lambda_{1(i)} \in d(x), \\ \lambda_{2(i)} \in d(y)}} \{T(\lambda_{1(i)}, \lambda_{2(i)})\} \right\rangle$$

$$= \left\langle x.y; \bigcup_{\substack{\lambda_{1(i)} \in h(x), \\ \lambda_{2(i)} \in h(y)}} \{g^{-1}(g(\lambda_{1(i)}) + g(\lambda_{2(i)}))\} \right\rangle,$$

$$(3) \lambda \tilde{X}^H = \left\langle \lambda p; \bigcup_{\lambda \in d(x)} \{f^{-1}(\lambda f(\lambda))\} \right\rangle,$$

$$(4) (\tilde{X}^H)^\lambda = \left\langle x^\lambda; \bigcup_{\lambda \in d(x)} \{g^{-1}(\lambda g(\lambda))\} \right\rangle,$$

where, $\lambda \in \mathbb{R}^{\geq 0}$, $\{\lambda_{l(1)}, \lambda_{l(2)}, \dots\}$ is a permutation of $\{\lambda_{l1}, \lambda_{l2}, \dots\}$ such that $\lambda_{l(1)} \leq \lambda_{l(2)} \leq \dots$.

In some cases, there may be different amounts reported on a single subject, such as the inhabitants of an ancient site, the victims of the pandemic in a particular state, the income of individuals/companies, etc., which cannot be ruled out and the degrees doubts about them can be expressed with a hesitant fuzzy set. Generalized hesitant fuzzy numbers, which are defined as follows, are the most appropriate modeling tools in such situations.

Definition 5 Keikha (2021) For the universal set \mathbb{R} , positive real numbers $x_l, l = 1, 2, \dots, m$, and doubtness/membership /satisfaction degrees $\lambda_j \in [0, 1], j = 1, 2, \dots, n$, the generalized hesitant fuzzy number is defined as

$$\tilde{X}^H = \langle \{x_1, x_2, \dots, x_m\}; \{\lambda_1, \lambda_2, \dots, \lambda_n\} \rangle$$

As it is seen, a GHFN similar to a HFN contain two parts: a real values part, and a hesitation degrees part. For any two GHFNs, if cardinalities of their real values parts are equal, and simultaneously the cardinalities of their hesitation degrees parts are also the same, they are called adjusted GHFNs (AGHFNs).

Definition 6 Keikha (2021) For each GHFNs as $\tilde{X}^H = \langle \{x_1, x_2, \dots, x_m\}; \{\lambda_1, \lambda_2, \dots, \lambda_n\} \rangle$, its mean value, score, and variance, displayed by $M(\tilde{X}^H)$, $Score(\tilde{X}^H)$, and $Var(\tilde{X}^H)$, respectively, can be obtained as follows:

$$M(\tilde{X}^H) = (\bar{x}, \bar{\lambda}) = \left(\frac{\sum_{l=1}^m x_l}{m}, \frac{\sum_{j=1}^n \lambda_j}{n} \right), \tag{1}$$

$$Score(\tilde{X}^H) = \bar{x} \times \bar{\lambda} = \frac{\sum_{l=1}^m x_l}{m} \times \frac{\sum_{j=1}^n \lambda_j}{n}, \tag{2}$$

$$Var(\tilde{X}^H) = \sqrt{\frac{\sum_{l=1}^m (x_l - \bar{x})^2}{m} + \sum_{i \neq j} (\lambda_i - \lambda_j)^2}. \tag{3}$$

Definition 7 Keikha (2021) Let $\tilde{X}^H = \langle \{x_1, x_2, \dots, x_m\}; \{\lambda_1, \lambda_2, \dots, \lambda_n\} \rangle$ and $\tilde{Y}^H = \langle \{y_1, y_2, \dots, y_k\}; \{\gamma_1, \gamma_2, \dots, \gamma_l\} \rangle$ be two arbitrary GHFNs, with mean values $M(\tilde{X}^H) = (\bar{x}, \bar{\lambda})$ and $M(\tilde{Y}^H) = (\bar{y}, \bar{\gamma})$, respectively. Then, strongly superior (s.s.), superior (s.), weakly superior (w.s.), and almost equal (a.eq.) relations can be defined as follows:

- (i) \tilde{X}^H is s.s. to \tilde{Y}^H ($\tilde{X}^H \succ_{s.s.} \tilde{Y}^H$) if $M(\tilde{X}^H) > M(\tilde{Y}^H)$, i.e., $\bar{x} > \bar{y}$ & $\bar{\lambda} > \bar{\gamma}$.
- (ii) \tilde{X}^H is s. to \tilde{Y}^H ($\tilde{X}^H \succ_s \tilde{Y}^H$), if $Score(\tilde{X}^H) > Score(\tilde{Y}^H)$, and

$$\bar{x} = \bar{y} \quad \& \quad \bar{\lambda} > \bar{\gamma},$$

or

$$\bar{x} > \bar{y} \quad \& \quad \bar{\lambda} = \bar{\gamma}.$$

- (iii) \tilde{X}^H is w.s. to \tilde{Y}^H ($\tilde{X}^H \succ_{w.s.} \tilde{Y}^H$), if

$$Score(\tilde{X}^H) \geq Score(\tilde{Y}^H), \quad \& \quad \bar{s} > \bar{t} \quad \& \quad \bar{\lambda} < \bar{\gamma} \\ (\bar{x} < \bar{y} \quad \& \quad \bar{\lambda} > \bar{\gamma})$$

or

$$M(\tilde{X}^H) = M(\tilde{Y}^H) \quad \& \quad Var(\tilde{X}^H) < Var(\tilde{Y}^H)$$

- (iv) \tilde{X}^H is a.eq. to \tilde{Y}^H ($\tilde{X}^H \equiv_{a.eq.} \tilde{Y}^H$), if

$$M(\tilde{X}^H) = M(\tilde{Y}^H) \quad \& \quad Var(\tilde{X}^H) = Var(\tilde{Y}^H)$$

Definition 8 Keikha (2021) Let $u > 0$, $\tilde{X}^H = \langle \{x_1, x_2, \dots, x_m\}; \{\lambda_1, \lambda_2, \dots, \lambda_n\} \rangle$ and $\tilde{Y}^H = \langle \{y_1, y_2, \dots, y_k\}; \{\gamma_1, \gamma_2, \dots, \gamma_l\} \rangle$ be two AGHFNs. Then

- (i) $u\tilde{X}^H = \langle \{ux_1, ux_2, \dots, ux_m\}; \{\lambda_1, \lambda_2, \dots, \lambda_n\} \rangle$
- (ii) $(\tilde{X}^H)^u = \langle \{(x_1)^u, (x_2)^u, \dots, (x_m)^u\}; \{\lambda_1, \lambda_2, \dots, \lambda_n\} \rangle$
- (iii) $\tilde{X}^H \oplus \tilde{Y}^H = \left\langle \bigcup_i \{x_{(i)} + y_{(i)}\}; \bigcup_i \{\lambda_i, \gamma_i\} \right\rangle$
- (iv) $\tilde{X}^H \otimes \tilde{Y}^H = \left\langle \bigcup_i \{x_{(i)}y_{(i)}\}; \bigcup_i \min\{\lambda_i, \gamma_i\} \right\rangle$

where $x_{(1)}, x_{(2)}, \dots, x_{(m)}$ is a permutation of x_1, x_2, \dots, x_m such that $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(m)}$, and $y_{(1)}, y_{(2)}, \dots, y_{(m)}$ is a permutation of y_1, y_2, \dots, y_m such that $y_{(1)} \leq y_{(2)} \leq \dots \leq y_{(m)}$.

Definition 9 Keikha (2021) Let $U = (u_1, u_2, \dots, u_k)$ with $u_l \in [0, 1]$ and $\sum_{l=1}^k u_l = 1$ be the weight vector of GHFNs $\tilde{X}_l^H = \langle h(X_l), mh(X_l) \rangle$, $l = 1, 2, \dots, k$, where $h(X_l) = \{x_{l1}, x_{l2}, \dots, x_{lm}\}$, $mh(X_l) = \{\lambda_{l1}, \lambda_{l2}, \dots, \lambda_{ln}\}$. Then

$$GHWA A_u \left(\tilde{X}_1^H, \tilde{X}_2^H, \dots, \tilde{X}_k^H \right) = \left\langle \bigcup_{r=1}^m \left\{ \sum_{l=1}^k u_l x_{l(r)} \right\}; \bigcup_{l=1}^k mh(X_l) \right\rangle,$$

is called GHWAA operator.

The GHWAA operator will be named GHAA operator if $U = \left(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k} \right)$.

Definition 10 Keikha (2021) For GHFNs $\tilde{X}_l^H = \langle h(X_l), mh(X_l) \rangle$, $l = 1, 2, \dots, k$ which are weighted with $u_l \in [0, 1]$ and $\sum_{l=1}^k u_l = 1$, GHWGA operator has been defined as follows.

$$GHWGA_u \left(\tilde{X}_1^H, \tilde{X}_2^H, \dots, \tilde{X}_r^H \right) = \left\langle \bigcup_{r=1}^m \left\{ \prod_{l=1}^k x_{l(r)}^{u_l} \right\}; \bigcap_{l=1}^k mh(X_l) \right\rangle,$$

where,

$\bigcap_{l=1}^k mh(X_l) = \bigcup_{\lambda_l \in h(X_l)} \min\{\lambda_1, \lambda_2, \dots, \lambda_k\}$. For $U = \left(\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k} \right)$, the GHWGA operator is called GHGA operator.

Definition 11 Keikha (2021) Consider the weight vector $U = (u_1, u_2, \dots, u_k)$ with $u_l \in [0, 1]$ and $\sum_{l=1}^k u_l = 1$. Then GHOWA and GHOWG operators of the given GHFNs $\tilde{X}_l^H = \langle h(X_l), mh(X_l) \rangle$, $l = 1, 2, \dots, k$, which are ordered as $\tilde{X}_{(1)}^H < \tilde{X}_{(2)}^H < \dots < \tilde{X}_{(k)}^H$, i.e., $\tilde{X}_{(l)}^H = \langle h(X_{(l)}), mh(X_{(l)}) \rangle$, $l = 1, 2, \dots, k$ is the l th largest value of them, can be defined as

$$GHOWA_u \left(\tilde{X}_1^H, \tilde{X}_2^H, \dots, \tilde{X}_k^H \right) = \left\langle \bigcup_{r=1}^m \left\{ \sum_{l=1}^k u_l x_{(l)(r)} \right\}; \bigcup_{l=1}^k mh(X_{(l)}) \right\rangle,$$

and

$$GHOWG_u \left(\tilde{X}_1^H, \tilde{X}_2^H, \dots, \tilde{X}_k^H \right) = \left\langle \bigcup_{r=1}^m \left\{ \prod_{l=1}^k x_{(l)(r)}^{u_l} \right\}; \bigcap_{l=1}^k mh(X_{(l)}) \right\rangle,$$

where, $\bigcap_{l=1}^k mh(X_{(l)}) = \bigcup_{\lambda_l \in h(p_{(l)})} \min\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

Because GHFNs have just been introduced, like any new theory, the development of computational tools can provide strong scientific support for their applications in solving real-world problems. Therefore, in the next section, we will introduce several aggregation operators (AOs) to be used with GHFNs.

3 Mathematical development of GHFNs

In this section, we proposed some ATT -based operators which enable us to employing GHFNs in solving practical problems, or extending some other techniques by them. Also, as an application of GHFNs, an algorithm has been proposed to solve multi criteria group decision making problems.

3.1 Aggregation operators of GHFNs

The aim of this sub-section is to update some existing Archimedean t -norms and t -conorms with GHFNs, and based on them proposed several AOs for GHFNs.

Definition 12 Consider a positive real value η , and two adjusted GHFNs as $\tilde{X}^H = \langle \{x_1, x_2, \dots, x_m\}; \{\lambda_1, \lambda_2, \dots, \lambda_n\} \rangle$, $\tilde{Y}^H = \langle \{y_1, y_2, \dots, y_m\}; \{\gamma_1, \gamma_2, \dots, \gamma_n\} \rangle$. Then

- (1) $\tilde{X}^H \oplus \tilde{Y}^H = \left\langle \bigcup_{l=1}^m \{x_{(l)} + y_{(l)}\}; \bigcup_{\substack{\lambda_{(r)} \in h(X), \\ \gamma_{(r)} \in h(Y)}} \{S(\lambda_{(r)}, \gamma_{(r)})\} \right\rangle = \left\langle \bigcup_{l=1}^m \{x_{(l)} + y_{(l)}\}; \bigcup_{\substack{\lambda_{(r)} \in h(X), \\ \gamma_{(r)} \in h(Y)}} \{f^{-1}(f(\lambda_{(r)}) + f(\gamma_{(r)}))\} \right\rangle,$
- (2) $\tilde{X}^H \otimes \tilde{Y}^H = \left\langle \bigcup_{l=1}^m \{x_{(l)} \cdot y_{(l)}\}; \bigcup_{\substack{\lambda_{(r)} \in h(X), \\ \gamma_{(r)} \in h(Y)}} \{T(\lambda_{(r)}, \gamma_{(r)})\} \right\rangle = \left\langle \bigcup_{l=1}^m \{x_{(l)} \cdot y_{(l)}\}; \bigcup_{\substack{\lambda_{(r)} \in h(X), \\ \gamma_{(r)} \in h(Y)}} \{g^{-1}(g(\lambda_{(r)}) + g(\gamma_{(r)}))\} \right\rangle,$
- (3) $\eta \tilde{X}^H = \left\langle \bigcup_{l=1}^m \{\eta x_l\}; \bigcup_{\lambda \in h(X)} \{f^{-1}(\eta f(\lambda))\} \right\rangle,$
- (4) $(\tilde{X}^H)^\eta = \left\langle \bigcup_{l=1}^m \{x_l^\eta\}; \bigcup_{\lambda \in h(X)} \{g^{-1}(\eta g(\lambda))\} \right\rangle,$

where, $\{(1), (2), \dots\}$ are permutations of $\{1, 2, \dots\}$ such that $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(m)}$; $\lambda_{(1)} \leq \lambda_{(2)} \leq \dots \leq \lambda_{(n)}$; $y_{(1)} \leq y_{(2)} \leq \dots \leq y_{(m)}$; $\gamma_{(1)} \leq \gamma_{(2)} \leq \dots \leq \gamma_{(n)}$.

Theorem 1 For any GHFNs $\tilde{X}^H = \langle x_1, x_2, \dots, x_m \rangle; \{\lambda_1, \lambda_2, \dots, \lambda_n\}, \tilde{Y}^H = \langle \{y_1, y_2, \dots, y_m\}; \{\gamma_1, \gamma_2, \dots, \gamma_n\},$ and positive real numbers $\eta, \eta_1, \eta_2,$ we have

- (1) $\tilde{X}^H \oplus \tilde{Y}^H = \tilde{Y}^H \oplus \tilde{X}^H$
- (2) $\tilde{X}^H \otimes \tilde{Y}^H = \tilde{Y}^H \otimes \tilde{X}^H$
- (3) $\eta(\tilde{X}^H \oplus \tilde{Y}^H) = \eta\tilde{X}^H \oplus \eta\tilde{Y}^H$
- (4) $(\tilde{X}^H \otimes \tilde{Y}^H)^\eta = (\tilde{X}^H)^\eta \otimes (\tilde{Y}^H)^\eta$
- (5) $(\eta_1 + \eta_2)\tilde{X}^H = \eta_1\tilde{X}^H + \eta_2\tilde{X}^H$
- (6) $(\tilde{X}^H)^{\eta_1 + \eta_2} = (\tilde{X}^H)^{\eta_1} \otimes (\tilde{X}^H)^{\eta_2}$

Proof Due to the commutative property of t -norms and t -conorms, and also multiplication and addition of real numbers, the proof of the theorem is done simply. \square

Some special Archimedean t -norms and t -conorms have been defined based on special cases of additive generator g as follows:

(i) For $g(t) = -\log t,$ we have Algebraic t -norm and t -conorm:

- (i1) $\tilde{X}^H \oplus \tilde{Y}^H = \left\langle \bigcup_l \{x_{(l)} + y_{(l)}\}; \bigcup_{\substack{\lambda_{1(r)} \in h(X), \\ \lambda_{2(r)} \in h(Y)}} \{\lambda_{1(r)} + \lambda_{2(r)} - \lambda_{1(r)} \cdot \lambda_{2(r)}\} \right\rangle,$
- (i2) $\tilde{X}^H \otimes \tilde{Y}^H = \left\langle \bigcup_l \{x_{(l)} \cdot y_{(l)}\}; \bigcup_{\substack{\lambda_{1(r)} \in h(X), \\ \lambda_{2(r)} \in h(Y)}} \{\lambda_{1(r)} \cdot \lambda_{2(r)}\} \right\rangle,$
- (i3) $\eta\tilde{X}^H = \left\langle \bigcup_l \{\eta x_l\}; \bigcup_{\lambda \in h(X)} \{1 - (1 - \lambda)^\eta\} \right\rangle,$
- (i4) $(\tilde{X}^H)^\eta = \left\langle \bigcup_l \{x_l^\eta\}; \bigcup_{\lambda \in h(X)} \{\lambda^\eta\} \right\rangle.$

(ii) With $g(t) = \log \frac{2-t}{t},$ we have Einstein t -norm and t -conorm, i.e.,

- (ii1) $\tilde{X}^H \oplus \tilde{Y}^H = \left\langle \bigcup_l \{x_{(l)} + y_{(l)}\}; \bigcup_{\substack{\lambda_{1(r)} \in h(X), \\ \lambda_{2(r)} \in h(Y)}} \left\{ \frac{\lambda_{1(r)} + \lambda_{2(r)}}{1 + \lambda_{1(r)} \cdot \lambda_{2(r)}} \right\} \right\rangle,$
- (ii2) $\tilde{X}^H \otimes \tilde{Y}^H = \left\langle \bigcup_l \{x_{(l)} \cdot y_{(l)}\}; \bigcup_{\substack{\lambda_{1(r)} \in h(X), \\ \lambda_{2(r)} \in h(Y)}} \left\{ \frac{\lambda_{1(r)} \cdot \lambda_{2(r)}}{1 - (1 - \lambda_{1(r)})(1 - \lambda_{2(r)})} \right\} \right\rangle,$
- (ii3) $\eta\tilde{X}^H = \left\langle \bigcup_l \{\eta x_l\}; \bigcup_{\lambda \in h(X)} \left\{ \frac{(1 + \lambda)^\eta - (1 - \lambda)^\eta}{(1 + \lambda)^\eta + (1 - \lambda)^\eta} \right\} \right\rangle,$

$$(ii4) (\tilde{X}^H)^\eta = \left\langle \bigcup_l \{x_l^\eta\}; \bigcup_{\lambda \in h(X)} \left\{ \frac{2\lambda^\eta}{(2 - \lambda)^\eta + \lambda^\eta} \right\} \right\rangle.$$

(iii) Hamacher t -norm and t -conorm is the result of putting

$$g(t) = \begin{cases} \frac{1-t}{t} & v = 0 \\ \log \frac{v+(1-v)t}{t} & 0 < v \leq +\infty \end{cases}, \text{ i.e.,}$$

- (iii1) $\tilde{X}^H \oplus \tilde{Y}^H = \left\langle \bigcup_l \{x_{(l)} + y_{(l)}\}; \bigcup_{\substack{\lambda_{1(r)} \in h(X), \\ \lambda_{2(r)} \in h(Y)}} \left\{ \frac{\lambda_{1(r)} + \lambda_{2(r)} - \lambda_{1(r)} \cdot \lambda_{2(r)} - (1 - v)\lambda_{1(r)} \cdot \lambda_{2(r)}}{1 - (1 - v)\lambda_{1(r)} \cdot \lambda_{2(r)}} \right\} \right\rangle,$
- (iii2) $\tilde{X}^H \otimes \tilde{Y}^H = \left\langle \bigcup_l \{x_{(l)} \cdot y_{(l)}\}; \bigcup_{\substack{\lambda_{1(r)} \in h(X), \\ \lambda_{2(r)} \in h(Y)}} \left\{ \frac{\lambda_{1(r)} \cdot \lambda_{2(r)}}{v + (1 - v)(\lambda_{1(r)} + \lambda_{2(r)} - \lambda_{1(r)} \cdot \lambda_{2(r)})} \right\} \right\rangle,$
- (iii3) $\eta\tilde{X}^H = \left\langle \bigcup_l \{\eta x_l\}; \bigcup_{\lambda \in h(X)} \left\{ \frac{(1 + (v - 1)\lambda)^\eta - (1 - \lambda)^\eta}{(1 + (v - 1)\lambda)^\eta + (v - 1)(1 - \lambda)^\eta} \right\} \right\rangle,$
- (iii4) $(\tilde{X}^H)^\eta = \left\langle \bigcup_l \{x_l^\eta\}; \bigcup_{\lambda \in h(X)} \left\{ \frac{v\lambda^\eta}{(1 + (v - 1)(1 - \lambda)^\eta) + (v - 1)\lambda^\eta} \right\} \right\rangle.$

(iv) Frank t -norm and t -conorm is the result of putting

$$g(t) = \begin{cases} -\log t & v = 1 \\ 1 - t & v = +\infty \\ \log \frac{v-1}{v^t-1} & \text{otherwise} \end{cases}, \text{ i.e.,}$$

- (iv1) $\tilde{X}^H \oplus \tilde{Y}^H = \left\langle \bigcup_l \{x_{(l)} + y_{(l)}\}; \bigcup_{\substack{\lambda_{1(r)} \in h(X), \\ \lambda_{2(r)} \in h(Y)}} \left\{ 1 - \log_v \left(1 + \frac{(v^{1-\lambda_{1(r)}} - 1)(v^{1-\lambda_{2(r)}} - 1)}{v - 1} \right) \right\} \right\rangle,$
- (iv2) $\tilde{X}^H \otimes \tilde{Y}^H = \left\langle \bigcup_l \{x_{(l)} \cdot y_{(l)}\}; \bigcup_{\substack{\lambda_{1(r)} \in h(X), \\ \lambda_{2(r)} \in h(Y)}} \left\{ \log_v \left(1 + \frac{(v^{\lambda_{1(r)}} - 1)(v^{\lambda_{2(r)}} - 1)}{v - 1} \right) \right\} \right\rangle,$
- (iv3) $\eta\tilde{X}^H = \left\langle \bigcup_l \{\eta x_l\}; \bigcup_{\lambda \in h(X)} \left\{ 1 - \log_v \left(1 + \frac{(v^{1-\lambda} - 1)^\eta}{(v - 1)^{\eta-1}} \right) \right\} \right\rangle,$

$$(iv4) \quad (\tilde{X}^H)^\eta = \left\langle \bigcup_l \{x_l^\eta\}; \right. \\ \left. \bigcup_{\lambda \in h(X)} \left\{ \log_\nu \left(1 + \frac{(\nu^\lambda - 1)^\eta}{(\nu - 1)^{\eta-1}} \right) \right\} \right\rangle \\ = \left\langle \bigcup_{r=1}^m \left\{ \sum_{l=1}^k u_l x_{l(r)} \right\}; \right. \\ \left. \bigcup_{j=1}^n \left\{ f^{-1} \left(\sum_{l=1}^k u_l f(\lambda_{\sigma(l)(j)}) \right) \mid \lambda_{\sigma(l)(j)} \in mh(X_{\sigma(l)}) \right\} \right\rangle, \tag{6}$$

Notice that the Hamacher t -norm and t -conorm is the generalized form of Algebraic t -norm and t -conorm, and also Einstein t -norm and t -conorm. In other words, (iii1) – (iii4) reducing to (ii1) – (ii4) by replacing $\nu = 1$, and (iii1) – (iii4) reducing to (i1) – (i4) by replacing $\nu = 2$. In the Frank t -norm and t -conorm, if $\nu \rightarrow 1$, then (iv1) – (iv4) reducing to (i1) – (i4), respectively. The given operators in Definition 2, i.e., $A - HFWA$, $A - HFWG$, $A - HFOWA$ and $A - HFLOWG$ would be developed to be applied with GHFNs, which are called $A - GHFNWA$, $A - GHFNWG$, $A - GHFNOWA$ and $A - GHFNOWG$ operators, respectively.

Definition 13 Let $\tilde{X}_l^H = \langle h(X_l), mh(X_l) \rangle$, $l = 1, 2, \dots, k$, where $h(X_l) = \{x_{l1}, x_{l2}, \dots, x_{lm}\}$, $mh(X_l) = \{\lambda_{l1}, \lambda_{l2}, \dots, \lambda_{ln}\}$ be a collection of GHFNs, and $0 \leq u_l \leq 1$ with $\sum_{l=1}^k u_l = 1$ be the weight vector of given GHFNs. Then

(1) ATT-based GHFN weighted averaging (A-GHFNWA) operator

$$A - GHFNWA(\tilde{X}_1^H, \tilde{X}_2^H, \dots, \tilde{X}_k^H) = \bigoplus_{l=1}^k (u_l \tilde{X}_l^H) \\ = \left\langle \bigcup_{r=1}^m \left\{ \sum_{l=1}^k u_l x_{l(r)} \right\}; \right. \\ \left. \bigcup_{j=1}^n \left\{ f^{-1} \left(\sum_{l=1}^k u_l f(\lambda_{l(j)}) \right) \mid \lambda_{l(j)} \in mh(X_l) \right\} \right\rangle, \tag{4}$$

(2) ATT-based GHFN weighted geometric (A-GHFNWG) operator

$$A - GHFNWG(\tilde{X}_1^H, \tilde{X}_2^H, \dots, \tilde{X}_k^H) = \bigotimes_{l=1}^k ((\tilde{X}_l^H)^{u_l}) \\ = \left\langle \bigcup_{r=1}^m \left\{ \prod_{l=1}^k p_{l(r)}^{u_l} \right\}; \right. \\ \left. \bigcup_{j=1}^n \left\{ g^{-1} \left(\sum_{l=1}^k u_l g(\lambda_{l(j)}) \right) \mid \lambda_{l(j)} \in mh(X_l) \right\} \right\rangle, \tag{5}$$

(3) ATT-based GHFN ordered weighted averaging (A-GHFNOWA) operator

$$A - GHFNOWA(\tilde{X}_1^H, \tilde{X}_2^H, \dots, \tilde{X}_k^H) = \bigoplus_{l=1}^k (u_l \tilde{X}_{(l)}^H)$$

(4) ATT-based GHFN ordered weighted geometric (A-GHFNOWG) operator

$$A - GHFNOWG(\tilde{X}_1^H, \tilde{X}_2^H, \dots, \tilde{X}_k^H) = \bigotimes_{l=1}^k ((\tilde{X}_{(l)}^H)^{u_l}) \\ = \left\langle \bigcup_{r=1}^m \left\{ \prod_{l=1}^k x_{(l)(r)}^{u_l} \right\}; \right. \\ \left. \bigcup_{j=1}^n \left\{ g^{-1} \left(\sum_{l=1}^k u_l g(\lambda_{\sigma(l)(j)}) \right) \mid \lambda_{\sigma(l)(j)} \in mh(X_{\sigma(l)}) \right\} \right\rangle, \tag{7}$$

where $\tilde{X}_{\sigma(l)}^H$ ($l = 1, 2, \dots, k$) is a permutation of \tilde{X}_l^H ($l = 1, 2, \dots, k$), such that $\tilde{X}_{\sigma(1)}^H \leq \tilde{X}_{\sigma(2)}^H \leq \dots \leq \tilde{X}_{\sigma(k)}^H$.

Theorem 2 Consider weighted GHFNs $\tilde{X}_l^H = \langle h(X_l), mh(X_l) \rangle$, $l = 1, 2, \dots, k$, with weight vector $U = (u_1, u_2, \dots, u_k)$, where $h(X_l) = \{x_{l1}, x_{l2}, \dots, x_{lm}\}$, $mh(X_l) = \{\lambda_{l1}, \lambda_{l2}, \dots, \lambda_{ln}\}$, and $0 \leq u_l \leq 1$ that $\sum_{l=1}^k u_l = 1$. Then, the aggregated values using any of the operators $A - GHFNWA$, $A - GHFNWG$, $A - GHFNOWA$ and $A - GHFNOWG$ are GHFNs.

Proof The Theorem will be proved only for $A - GHFNWA$ operator. The others can be proved in a similar manner. As in Definition 13, let

$$A - GHFNWA(\tilde{X}_1^H, \tilde{X}_2^H, \dots, \tilde{X}_k^H) \\ = \left\langle \bigcup_{r=1}^m \left\{ \sum_{l=1}^k u_l x_{l(r)} \right\}; \right. \\ \left. \bigcup_{j=1}^n \left\{ f^{-1} \left(\sum_{l=1}^k u_l f(\lambda_{l(j)}) \right) \mid \lambda_{l(j)} \in mh(X_l) \right\} \right\rangle = \tilde{X}^H.$$

We know that a GHFN has two parts: a real value part, and a membership part. Thus, it is enough to see that \tilde{X}^H is a generalized hesitant fuzzy number, i.e., $\bigcup_{r=1}^m \{ \sum_{l=1}^k u_l x_{l(r)} \}$ is a finite set of real values, and also $\bigcup_{j=1}^n \left\{ f^{-1} \left(\sum_{l=1}^k u_l f(\lambda_{l(j)}) \right) \mid \lambda_{l(j)} \in mh(X_l) \right\}$ is a finite set of membership degrees in $[0, 1]$. Based on properties of real values, it is obvious

that $\sum_{l=1}^k u_l x_{l(r)} \in \mathbb{R}^+$, and $\bigcup_{r=1}^m \{\sum_{l=1}^k u_l x_{l(r)}\} \subset \mathbb{R}^+$. Furthermore, $f : [0, 1] \rightarrow [0, +\infty]$ is a strictly increasing function, then it is reversible, and there exist $f^{-1} : [0, +\infty] \rightarrow [0, 1]$. That is $f^{-1} \left(\sum_{l=1}^k u_l f(\lambda_{l(j)}) \right) \in [0, 1]$, and $\bigcup_{j=1}^n \left\{ f^{-1} \left(\sum_{l=1}^k u_l f(\lambda_{l(j)}) \right) \mid \lambda_{l(j)} \in mh(X_l) \right\} \subset [0, 1]$ is a finite set of membership degrees. \square

3.2 A group decision-making algorithm

We summarize the steps of the proposed decision-making algorithm as follows.

- **Step 1** Determine the sets of criteria/attributes C , decision makers M , and options/alternatives O .
- **Step 2** Determine the weight vectors of DMs (W), and criteria (v), if it is necessary.
- **Step 3** Make a GHFNs matrix (called decision matrix $\tilde{\tilde{D}} = [\tilde{\tilde{d}}_{ij}]_{|O| \times |C|}$) whose number of rows is equal to the number of options and its columns are equal to the number of criteria.
- **Step 4** Muster the evaluation values of all DMs on i th option against j th criterion as ij th element of GHFNS matrix $\tilde{\tilde{D}}$, in which the real part of $\tilde{\tilde{d}}_{ij}$ shows the direct scores of the evaluators and the membership part shows their levels of satisfaction/ambiguity with the evaluation amounts/conditions.
- **Step 5** Pick the r th row of the GHFNs matrix $\tilde{\tilde{D}}$, $r = 1, 2, \dots, |O|$, and aggregate its elements by choosing one of the appropriate proposed operators in this article.
- **Step 6** Set the i th obtained values in Step 5 as the score of i th option, $i = 1, 2, \dots, |O|$. Then, rank the options similar to the ranking of their scores.

The flowchart of the algorithm is shown in Fig. 1.

4 Numerical examples

Example 1 Consider three students A, B and C with courses in Mathematics, Chemistry, Physics, and Literature, to be ranked via hybrid assessment at the end of the semester. Suppose in addition to the monthly exams (4 exams), the professors' qualitative assessments based on their academic readiness and classroom activities during the semester are also included in final assessment. Let the results of the evaluations have been merged to construct generalized hesitant fuzzy numbers as in Table 1, in which the real part contains the results of the monthly exams, and the degree of doubt includes the qualitative evaluations.

Assuming that the courses have the same weight, we will calculate the average of each student using one of the

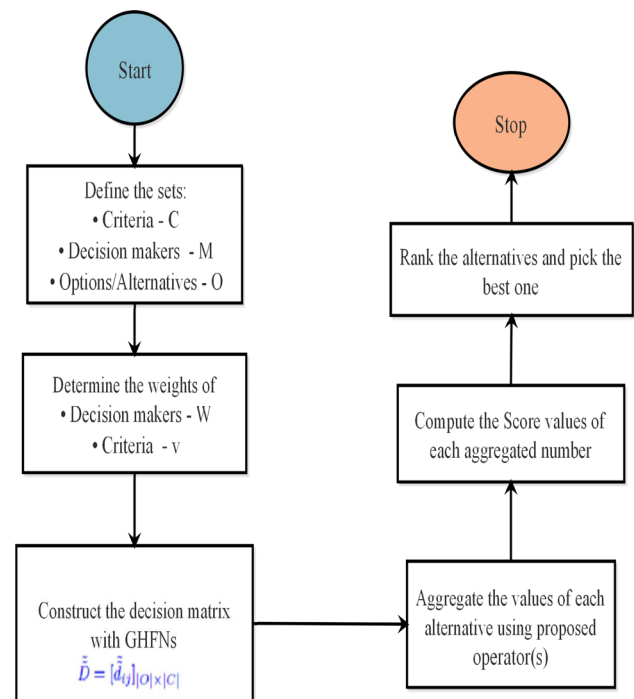


Fig. 1 Flowchart of the proposed algorithm

proposed average-based operators in this article. Using the Hamacher t -norm and t -conorm with A-GHFNWA operator (Eq. 4), and $v = 2$, we get Table 2, in which the average score of each student is a GHFN. The students can be ranked based on their total corresponding GHFNs marks, are given in Table 2:

$$A \succ_s C \succ_{w.s} B.$$

Example 2 Usually in the planting season, farmers try to have the highest income by estimating the harvest per unit area of each agricultural item(Ton/Hectare), as well as estimating the price (Dollar/Ton) of those products in the future, by diversifying the cultivated items. We classify the estimates into three general categories: optimistic, normal, and pessimistic, according to the influence of many factors on crop yields and their prices. Let the estimated values for each product in any of the above three cases are determined using the recorded values in recent years. In fact, the historical values have been ranked in ascending order, firstly. Then, as in Table 3, the lower ones have been classified in pessimistic class, the best ones in optimistic class, and the intermediate values in normal class. It is natural for the farmer or consulting firm to be skeptical of achieving any of the above values, and be free to use them directly or change. Secondly, suppose the given values in Table 3 are evaluated, and the degrees of doubt about them are expressed with HFEs, as in Table 4. Finally, with the merging of these two data sets, GHFNs will be obtained (see Tables 5 and 6).

Table 1 The students' marks in hybrid assessments as GHFNs

Students	Mathematics	Chemistry	Physics	Literature
A	$\langle\{88, 95, 92, 97\}; \{0.6, 0.2, 0.7, 0.9, 0.4, 0.9\}\rangle$	$\langle\{78, 92, 97, 98\}; \{0.1, 0.7, 0.8, 0.5, 0.9, 0.9\}\rangle$	$\langle\{83, 92, 96, 91\}; \{0.8, 0.3, 0.7, 0.8, 0.9, 0.3\}\rangle$	$\langle\{97, 99, 98, 97\}; \{0.1, 0.2, 1, 0.9, 0.9, 1\}\rangle$
B	$\langle\{87, 96, 94, 96\}; \{0.3, 0.8, 0.6, 0.5, 0.4, 0.5\}\rangle$	$\langle\{78, 94, 95, 98\}; \{0.2, 0.1, 0.7, 0.5, 0.4, 0.4\}\rangle$	$\langle\{87, 95, 92, 93\}; \{0.3, 0.5, 0.6, 0.2, 0.7, 0.6\}\rangle$	$\langle\{98, 99, 100, 97\}; \{0.3, 0.1, 0.6, 0.4, 0.2, 0.5\}\rangle$
C	$\langle\{89, 96, 94, 95\}; \{0.6, 0.5, 0.9, 0.7, 0.4, 0.4\}\rangle$	$\langle\{80, 94, 92, 96\}; \{0.3, 0.3, 0.6, 0.6, 0.4, 0.4\}\rangle$	$\langle\{84, 94, 92, 90\}; \{0.3, 0.7, 0.5, 0.4, 0.9, 0.6\}\rangle$	$\langle\{96, 100, 100, 98\}; \{0.1, 0.8, 0.2, 0.6, 0.3, 0.5\}\rangle$

Table 2 The average marks of students

Students	GHFNs	Final score	Rank
A	$\langle\{86.5, 93, 95.5, 97.5\}; \{0.176, 0.5, 0.909, 0.990, 1, 1\}\rangle$	(93.125, 0.762)	1
B	$\langle\{87.25, 94.5, 95.75, 97.5\}; \{0.176, 0.432, 0.726, 0.894, 0.968, 0.994\}\rangle$	(93.75, 0.693)	3
C	$\langle\{87.25, 93.5, 95.25, 96.5\}; \{0.278, 0.555, 0.794, 0.932, 0.985, 0.999\}\rangle$	(93.125, 0.757)	2

Table 3 The predicted harvest values in three scenarios

	Item	Optimistic	Normal	Pessimistic
Harvest	A	{10, 11, 13}	{7, 8, 9}	{4, 5, 6}
	B	{14, 15, 16}	{10, 12, 13}	{8.5, 9, 9.5}
	C	{25, 27, 30}	{15, 17, 18}	{10, 12, 13}
Price	A	{100, 120, 115}	{85, 88, 90}	{65, 70, 77}
	B	{110, 100, 95}	{80, 85, 88}	{75, 80, 82}
	C	{80, 90, 95}	{68, 70, 75}	{48, 55, 60}

Now, we are able to determine the total income of the farmer in three levels, by aggregating of given GHFNs in each column in Tables 5 and 6. To do it, we can choose one of the proposed aggregation operators in this article. For example, Algebraic *t*-norm and *t*-conorm gives us:

– Optimistic mode:

$$\langle\{10, 11, 13\}; \{0.6, 0.7, 0.9\}\rangle \otimes \langle\{100, 120, 115\}; \{0.5, 0.7, 0.9\}\rangle \oplus \langle\{14, 15, 16\}; \{0.6, 0.8, 1\}\rangle \otimes \langle\{110, 100, 95\}; \{0.4, 0.6, 0.9\}\rangle \oplus \langle\{25, 27, 30\}; \{0.7, 0.8, 0.9\}\rangle \otimes \langle\{80, 90, 95\}; \{0.4, 0.6, 0.9\}\rangle = \langle\{4330, 5195, 6170\}; \{0.505, 0.883, 0.996\}\rangle;$$

– Normal mode:

$$\langle\{2415, 2914, 3304\}; \{0.569, 0.909, 0.989\}\rangle;$$

– Pessimistic mode:

$$\langle\{1377.5, 1730, 2021\}; \{0.187, 0.545, 0.765\}\rangle.$$

4.1 Numerical analysis

In this subsection, the above two examples will be analyzed.

- Consider Example 1 to be analyzed. In the current methods, first, the average score of each course is calculated, and the result is considered as the final score of the student in the relevant course. Then, the average of the final scores of the student's semester courses expresses its grade point average in the semester. With these explanations, and using the simple additive weighted (SAW) method, the results of quantitative evaluation of students are summarized in Table 7.

Now, if we consider only the class evaluations of each course as a HFE. After aligning these HFES with the help of the Hamacher operator, we will have a single HFE corresponding to each student, which will be compared and the students' ranking will be determined as in Table 8.

Comparing the results of these two methods shows a completely different ranking, which of course is natural because the natures of the usage data are different. We can compute the similarity coefficients (Salabun and Urbaniak 2020) of the ranking orders with what obtained in the case of GHFNs. Then, this parameter for monthly evaluations and hybrid evaluation is 0.25, and for HFES assessment values and hybrid evaluation is 1. It means that continuous assessments are much more similar to hybrid assessments than monthly exams.

- Let us aggregate the given GHFNs in Table 1 by some other ATT-based operators and compare the resulting orders.
 - Using Algebraic operator and based on Table 9 we get $A \succ_s C \succ_{w,s} B$.

Table 4 The hesitant degrees of achieving predicted harvest

	Item	Optimistic	Normal	Pessimistic
Harvest	A	{0.6, 0.7, 0.9}	{0.7, 0.8, 0.9}	{0.3, 0.2, 0.5}
	B	{0.6, 0.8, 1}	{0.5, 0.7, 0.8}	{0.4, 0.5, 0.7}
	C	{0.7, 0.8, 0.9}	{0.8, 0.9, 1}	{0.1, 0.2, 0.3}
Price	A	{0.5, 0.7, 0.9}	{0.3, 0.6, 0.9}	{0.2, 0.3, 0.25}
	B	{0.4, 0.6, 0.9}	{0.7, 0.75, 0.9}	{0.9, 0.8, 0.2}
	C	{0.1, 0.8, 0.9}	{0.7, 0.8, 0.2}	{0.8, 0.9, 1}

Table 5 The predicted harvest values as GHFNs

	Item	Optimistic	Normal	Pessimistic
Harvest	A	$\langle\{10, 11, 13\}; \{0.6, 0.7, 0.9\}\rangle$	$\langle\{7, 8, 9\}; \{0.7, 0.8, 0.9\}\rangle$	$\langle\{4, 5, 6\}; \{0.3, 0.2, 0.5\}\rangle$
	B	$\langle\{14, 15, 16\}; \{0.6, 0.8, 1\}\rangle$	$\langle\{10, 12, 13\}; \{0.5, 0.7, 0.8\}\rangle$	$\langle\{8.5, 9, 9.5\}; \{0.4, 0.5, 0.7\}\rangle$
	C	$\langle\{25, 27, 30\}; \{0.7, 0.8, 0.9\}\rangle$	$\langle\{15, 17, 18\}; \{0.8, 0.9, 1\}\rangle$	$\langle\{10, 12, 13\}; \{0.1, 0.2, 0.3\}\rangle$

Table 6 The predicted price values as GHFNs

	Item	Optimistic	Normal	Pessimistic
Price	A	$\langle\{100, 120, 115\}; \{0.5, 0.7, 0.9\}\rangle$	$\langle\{85, 88, 90\}; \{0.3, 0.6, 0.9\}\rangle$	$\langle\{65, 70, 77\}; \{0.2, 0.3, 0.25\}\rangle$
	B	$\langle\{110, 100, 95\}; \{0.4, 0.6, 0.9\}\rangle$	$\langle\{80, 85, 88\}; \{0.7, 0.75, 0.9\}\rangle$	$\langle\{75, 80, 82\}; \{0.9, 0.8, 0.2\}\rangle$
	C	$\langle\{80, 90, 95\}; \{0.1, 0.8, 0.9\}\rangle$	$\langle\{68, 70, 75\}; \{0.7, 0.8, 0.2\}\rangle$	$\langle\{48, 55, 60\}; \{0.8, 0.9, 1\}\rangle$

– By Einstein operator we have Table 10, and then $A \succ_s C \succ_{w.s} B$.

Although using different aggregation operators resulted the same ranking orders in this special problem, but this not means it may be happen always. Because, the membership part of obtained GHFN from aggregation of given GHFNs will change due to the use of different operators (with their own properties).

– To analyze Example 2, suppose we use type-1 triangular fuzzy numbers, as in Table 11, for modeling. The calculated farmer’ income will then be a triangular fuzzy number which are given in Table 12, in each of the three scenarios. That is, infinite amounts and, consequently, infinite degrees of membership are the introduced candidates as the final incomes, which increased the ambiguity of the problem.

Generalized hesitant fuzzy numbers, by combining both types of data, have the advantages of both types of evaluation and the results will be closer to the reality. For example, in the case of education, hybrid assessments, i.e., qualitative evaluation along with the formal quantitative tests, would help improving its real quality. Because, the student is required to

Table 7 The average of students’ marks by SAW method

Students	Math.	Chem.	Phy.	Lit.	Average	Rank
A	93	91.25	90.5	97.75	93.125	2
B	93.25	91.25	91.75	98.5	93.687	1
C	93.5	90.5	90	98.5	93.125	2

participate in the teaching process in addition to the cross-sectional activities of the exams season, while maintaining scientific readiness. In the field of agriculture, in addition to the recorded data, attention has also been paid to the individual experiences of the exploiter. In reviewing the results of a medical test, not only the recorded values are relied on, but also the physician’s experiences and his personal analysis will be effective in the diagnosis and treatment of diseases.

5 Conclusion

Generalized hesitant fuzzy numbers have the capability of modeling positions in which the DMs are hesitant between a finite set of real values about a special thing, and expressed

Table 8 The final students' marks as HFE

Students	Aggregated HFE	Score	Rank
A	{0.176, 0.5, 0.909, 0.990, 1, 1}	0.762	1
B	{0.176, 0.432, 0.726, 0.894, 0.968, 0.994}	0.698	3
C	{0.278, 0.555, 0.794, 0.932, 0.985, 0.999}	0.757	2

Table 9 The Algebraic ATT-based operator average marks

Students	GHFNs	Final score	Rank
A	{{86.5, 93, 95.5, 97.5}; {0.179, 0.36, 0.748, 0.822, 1, 1}}	(93.125, 0.685)	1
B	{{87.25, 94.5, 95.75, 97.5}; {0.179, 0.280, 0.431, 0.482, 0.553, 0.712}}	(93.75, 0.439)	3
C	{{87.25, 93.5, 95.25, 96.5}; {0.282, 0.395, 0.431, 0.532, 0.653, 0.832}}	(93.125, 0.521)	2

Table 10 The Einstein ATT-based operator average marks

Students	GHFNs	Final score	Rank
A	{{86.5, 93, 95.5, 97.5}; {0.176, 0.355, 0.751, 0.818, 1, 1}}	(93.125, 0.683)	1
B	{{87.25, 94.5, 95.75, 97.5}; {0.176, 0.277, 0.428, 0.480, 0.552, 0.703}}	(93.75, 0.436)	3
C	{{87.25, 93.5, 95.25, 96.5}; {0.278, 0.327, 0.428, 0.530, 0.653, 0.968}}	(93.125, 0.531)	2

Table 11 The predicted harvest and price values in three scenarios

	Item	Optimistic	Normal	Pessimistic
Harvest	A	(10, 11, 13)	(7, 8, 9)	(4, 5, 6)
	B	(14, 15, 16)	(10, 12, 13)	(8.5, 9, 9.5)
	C	(25, 27, 30)	(15, 17, 18)	(10, 12, 13)
Price	A	(100, 115, 120)	(85, 88, 90)	(65, 70, 77)
	B	(95, 100, 110)	(80, 85, 88)	(75, 80, 82)
	C	(80, 90, 95)	(68, 70, 75)	(48, 55, 60)

Table 12 The farmer's incomes as triangular fuzzy number

Scenario	Income
Optimistic	(4330, 5195, 6170)
Normal	(2415, 2914, 3304)
Pessimistic	(1377.5, 1730, 2021)

their hesitation/satisfaction degrees as a HFE. As they are at the beginning of a hard way including application and theoretical extension, development of mathematical computation tools will make navigation easier. In this paper, we focused on Archimedean t -norm and t -conorm functions, and then we proposed the general form of arithmetic operations of GHFNs. In the following, the arithmetic operations of GHFNs have been defined based on the Algebraic, Einstein, Hamacher, Frank t -norms and t -conorms, as the special cases of ATT. Finally, some averaging-based aggregation operators using the ATT functions have been proposed to aggregate a finite set of given GHFNs to a single GHFNs. These operators are called briefly A-GHFNWA operator, A-GHFNWG operator, A-GHFNOWA operator, and A-GHFNOWG operator.

A lot of research should be done in the future, both in the field of computational development and in the field of application, because the newly proposed generalized hesitant fuzzy numbers are at the starting point of an important scien-

tific way. For example, we need to define several measures, as similarity, entropy, distance, etc., for GHFNs, to solve multi-attribute decision making problems. It will be required to update some existing methods such as Best-Worst Method (BWM), TOPSIS, VIKOR, ELECTREE, ORESTE, aggregation operators (Maleki et al. 2022; Xu and Zhang 2012; Wu and Liao 2018), etc., in the future, too.

Also, GHFNs can be used in solving linear programming problems, data envelopment analysis, medical image processing, medical decision making, cognitive science, future studies, spatial planning, social networking, graph theory, etc. Liu et al. (2022), De Souza et al. (2021).

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