Acute threshold dynamics of an epidemic system with quarantine strategy driven by correlated white noises and Lévy jumps associated with infinite measure

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Abstract

Several studies have previously been conducted on the dynamics of probabilistic epidemic models driven by Lévy disorder. All of these works have used the Poisson counting process with finite Lévy measures. However, this scope disregards a considerable category of correlated Lévy jump processes governed by an infinite Lévy measure. In this research, we take into consideration this general framework applied to an epidemic model with a quarantine strategy. Under an appropriate hypothetical setting, we infer the exact threshold value between the ergodicity and the disease disappearance. Our analysis completes the work presented by Privault and Wang (J Nonlinear Sci 31(1):1–28, 2021) and puts forward a novel analytical aspect to deal with other stochastic models in several areas. As a numerical application, we implement the algorithm of Rosinski (Stoch Process Appl 117:677–707, 2007) for tempered stable Lévy processes with an infinite Lévy measure.

1 Study background and problematic

Transmissible illness surveillance relies on analytical modeling and future forecasting as a key decision-making tool [1]. However, each illness is modeled and described by its own mode of transmission, so in each specific case, the selection of an appropriate method to adequately characterize disease dynamics is highly demanded [2]. In this regard, mathematical biology, especially through compartmental systems, is the most famous approach for obtaining an understandable view of the disease spread. Many of the mathematical models adopted in the study of epidemics are derived from the basic SIR system suggested by Kermack and McKendrick in 1927 [3]. From then on, diverse formulations of this model

have been investigated by many researchers due to their theoretical and functional importance [4,5]. However, the mentioned epidemiological model is not sufficient to describe the mechanism of the spread of highly prevalent viruses such as COVID-19, and some hypotheses or strategies must be included. In fact, many individual public health measures have been practiced during the spread of the COVID-19 pandemic such as staying at home and maximizing physical distancing from others for better protection. By considering the application of the quarantine strategy and the impact of immune deterioration, in this study, we focus on an epidemic model with the following fourth classes:

1. susceptible class C_1, 2. infected class C_2, 3. quarantined class C_3, 4. recovered class C_4.

In this epidemic system, infected individuals may be isolated and evolve transitory resistance after infection, and recovered persons, with diminished immunity, come back to the susceptible class. The transfer rates between the above classes are characterized by this dynamical system:

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Table 1 Definition of the positive parameters appearing in system (1.1)

Parameter	Epidemiological meaning
$\mathfrak{a}$	The flow into the host population $\mathbf{C}_1$
$\mathfrak{b}$	The prevalence rate between $\mathbf{C}_1$ *and* $\mathbf{C}_2$
$\mathfrak{d}$	The normal mortality rate of $\mathbf{C}_k, k = 1, 2, 3, 4$
$\mathfrak{d}_2$	The disease-related mortality rate of $\mathbf{C}_2$
$\mathfrak{d}_3$	The disease-related mortality rate of $\mathbf{C}_3$
$\mathfrak{q}$	The quarantined rate of $\mathbf{C}_2$
$\mathfrak{c}_2$	The cure ratio of of $\mathbf{C}_2$
$\mathfrak{c}_3$	The cure ratio of of $\mathbf{C}_3$
$\mathfrak{h}$	The immune deterioration rate of $\mathbf{C}_4$

$$\begin{cases} d\mathbf{C}_1(t) = \big(\mathfrak{a} - \mathfrak{d}\mathbf{C}_1(t) - \mathfrak{b}\mathbf{C}_1(t)\mathbf{C}_2(t) + \mathfrak{h}\mathbf{C}_4(t)\big)dt, \\ d\mathbf{C}_2(t) = \big(\mathfrak{b}\mathbf{C}_1(t)\mathbf{C}_2(t) - (\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2)\mathbf{C}_2(t)\big)dt, \\ d\mathbf{C}_3(t) = \big(\mathfrak{q}\mathbf{C}_2(t) - (\mathfrak{d} + \mathfrak{d}_3 + \mathfrak{c}_3)\mathbf{C}_3(t)\big)dt, \\ d\mathbf{C}_4(t) = \big(\mathfrak{c}_2\mathbf{C}_2(t) + \mathfrak{c}_3\mathbf{C}_3(t) - (\mathfrak{d} + \mathfrak{h})\mathbf{C}_4(t)\big)dt, \\ \mathbf{C}_k(0) > 0, \ k = 1, 2, 3, 4, \end{cases}$$

(1.1)

where the positive constants $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{d}_2$, $\mathfrak{d}_3$, $\mathfrak{q}$, $\mathfrak{c}_2$, $\mathfrak{c}_3$ and $\mathfrak{h}$ are defined in Table 1.

In dealing with transmissible disease systems, one of the main goals is to determine the long-run behavior of the model. Analytically, it is shown that the asymptotic behavior of the deterministic model (1.1) depends on the sign of the expression $\lambda_0 = \frac{\mathfrak{a}\mathfrak{b}}{\mathfrak{d}} - (\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2)$. Minutely,

* ⋆ If $\lambda_0 > 0$, then the illness is continued in the population.
* ⋆ If $\lambda_0 \leq 0$, then the illness dies out.

Predominantly, λ_0 can be rewritten as the basic reproduction number $\mathcal{R}_\circ = \frac{\mathfrak{a}\mathfrak{b}}{\mathfrak{d}(\mathfrak{d}+\mathfrak{d}_2+\mathfrak{q}+\mathfrak{c}_2)}$ and we can compare this ratio with the number 1 to assort the large time behavior of the deterministic system (1.1) [6].

As far as we know, environmental disturbances affect the spread of an epidemic and make it more difficult to predict its behavior [7–18]. In such situations, deterministic systems, while able to make very instructive predictions and forecasts [19], are not actually enough. Hence, we need a developed and sophisticated mathematical model that takes into consideration the randomness effect, especially when studying the prevalence of a highly harmful infectious disease like COVID-19 [20–26]. In this vein, a large number of authors have suggested and evolved many stochastic models that describe the dynamic of many illnesses from various angles and prospects [27–37]. In all these works, the passage from the deterministic formulation to the probabilistic one is done by assuming that the solution of this first wiggles normally around its value, which is often expressed

by perturbing some system parameters with white noises. The addition of these variations is considered to be one of the most logical and prominent ways of describing any real phenomenon under small and continuous fluctuations [38–40]. Unfortunately, this approach is insufficient to model the spread of disease under massive and sudden environmental disturbances, during some economic crises, or through the application of some human interventions (isolation and vaccination in the case of COVID-19 [41,42]). For this reason, we resort to the Lévy processes which are renowned for their ability to correctly formulate this type of randomness [43–51]. Inspired by the above facts and motivations, this study puts forward a stochastic formulation of the illness model (1.1) driven by Lévy jumps of the form:

$$\begin{cases} d\mathbf{C}_1(t) = \overbrace{\big(\mathfrak{a} - \mathfrak{d}\mathbf{C}_1(t) - \mathfrak{b}\mathbf{C}_1(t)\mathbf{C}_2(t) + \mathfrak{h}\mathbf{C}_4(t)\big)dt}^{\text{Deterministic part}} + \overbrace{\mathbf{C}_1(t_-)d\mathbb{A}_1(t)}^{\text{Jumps-diffusion part,}} \\ d\mathbf{C}_2(t) = \big(\mathfrak{b}\mathbf{C}_1(t)\mathbf{C}_2(t) - (\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2)\mathbf{C}_2(t)\big)dt \\ \quad + \mathbf{C}_2(t_-)d\mathbb{A}_2(t), \\ d\mathbf{C}_3(t) = \big(\mathfrak{q}\mathbf{C}_2(t) - (\mathfrak{d} + \mathfrak{d}_3 + \mathfrak{c}_3)\mathbf{C}_3(t)\big)dt + \mathbf{C}_3(t_-)d\mathbb{A}_3(t), \\ d\mathbf{C}_4(t) = \big(\mathfrak{c}_2\mathbf{C}_2(t) + \mathfrak{c}_3\mathbf{C}_3(t) - (\mathfrak{d} + \mathfrak{h})\mathbf{C}_4(t)\big)dt \\ \quad + \mathbf{C}_4(t_-)d\mathbb{A}_4(t), \\ \mathbf{C}_k(0) > 0, \ k = 1, 2, 3, 4, \end{cases}$$

(1.2)

where $\mathbb{A} = (\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4)$ indicates the vector of the random process that describes the intensity of sudden events shocks. Here and elsewhere, $\mathbf{C}_k(t_-)$ ($k = 1, 2, 3, 4$) are respectively the left limits of the Markov processes $\mathbf{C}_k(t)$ ($k = 1, 2, 3, 4$). For the convenience of the reader and for a finer overview of the formulation of system (1.2), we introduce two categories of the process $\mathbb{A}$.

* Jumps-diffusion with independent Brownian motions and finite Lévy measure:

In [52], the authors considered an SIQS model (a particular case of (1.1)) with the following stochastic process:

$$\mathbb{A}_k(t) = \overbrace{\mathfrak{m}_k\mathbb{B}_k(t)}^{\text{B.m. part}} + \overbrace{\int_0^t \int_{\mathcal{H}} \vartheta_k(u)\mathbb{J}(ds, du)}^{\text{Jumps part}}, \ (k = 1, 2, 3, 4),$$

(1.3)

where the positive constants $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3$ and $\mathfrak{m}_4$ indicate the intensities of the independent Brownian motions (B.m.s) $\mathbb{B}_1(t), \mathbb{B}_2(t), \mathbb{B}_3(t)$ and $\mathbb{B}_4(t)$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq0}, \mathbb{P})$ such that $\{\mathcal{F}_t\}_{t\geq0}$ is an increasing, right continuous filtration and $\mathcal{F}_0$ includes all $\mathbb{P}$-null sets. $\mathcal{N}_\chi$ is a Poisson measure which is independent of $\mathbb{B}_k$ with a finite specific measure χ defined on a measurable subdomain $\mathcal{H} \subset \mathbb{R}_+ \setminus \{0\}$. $\mathbb{J}$ is the compensator process with its

associated Lévy measure (L.m.) χ, where

$$\mathbb{J}(t, \mathrm{d}u) = \mathcal{N}_\chi(t, \mathrm{d}u) - t\chi(\mathrm{d}u).$$

The jumps magnitude functions $\vartheta_k : \mathcal{H} \to (-1, \infty)$ ($k = 1, 2, 3, 4$) are assumed to be continuous on $\mathcal{H}$.

Remark 1.1 The above-mentioned work offers the long-run characteristics of an SIQS epidemic system driven by jumps with independent B.M.s and a finite L.m. $\chi(\cdot) < \infty$. Nevertheless, this scope eliminates a special class of Lévy jump processes with two characteristics: the infinitude of Lévy measure χ and the interdependence between the random noise items of model (1.2).

Thoroughly, Lévy process increments driven by finite measures have partially-weighty tails, and they have limited potential to simulate radical and brutal phenomena which usually lead to unexpected variations in the total number of individuals [53]. In the next category, we present an alternative frame that considers a general L.m. and the relationship between B.m.s components.

- *Jumps-diffusion with general L.m. and correlated B.m.s*

In [54], Privault and Wang proposed a novel class of Lévy-jumps perturbation by considering a process $\mathbb{A}$ with the associated Lévy–Khintchine formula $\mathbb{E}[e^{\Phi(t)}] = e^{\phi(t,u)}$, where

$$\Phi(t) = ia_1\mathbb{A}_1(t) + ia_2\mathbb{A}_2(t) + ia_3\mathbb{A}_3(t) + ia_4\mathbb{A}_4(t),$$
$$\phi(t, u) = -0.5t\langle a, \mathfrak{M}a\rangle$$
$$+ t\int_{\mathcal{H}} \left(e^{i\langle a, \vartheta(u)\rangle} - i\langle a, \vartheta(u)\rangle - 1\right)\chi(\mathrm{d}u).$$

Here and elsewhere, we use the flowing notations and definitions:

- $\star$ $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$.
- $\star$ $\mathfrak{M} = (\mathfrak{m}_{k,j})_{1 \leq k,j \leq 4}$ is a positive definite matrix.
- $\star$ The Lévy intensities $\vartheta_k : \mathcal{H} \subset \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}$ ($k = 1, 2, 3, 4$) are continuous functions.
- $\star$ χ verifies that $\int_{\mathcal{H}} \min\left(|\vartheta_k(u)|^2, 1\right)\chi(\mathrm{d}u) < \infty$, ($k = 1, 2, 3, 4$).

Motivated by the theory presented in [53] and [49], the authors in [54] expressed $\mathbb{A}$ by

$$\mathbb{A}_k(t) = \overbrace{\mathbb{B}_k^{\mathfrak{m}}(t)}^{\text{Gaussian process}}$$
$$+ \overbrace{\int_0^t \int_{\mathcal{H}} \vartheta_k(u)\mathbb{J}(\mathrm{d}s, \mathrm{d}u)}^{\text{General Lévy tails}}, \quad (k = 1, 2, 3, 4). \quad (1.4)$$

Here, $\mathbb{B}^{\mathfrak{m}} = \left(\mathbb{B}_1^{\mathfrak{m}}, \mathbb{B}_2^{\mathfrak{m}}, \mathbb{B}_3^{\mathfrak{m}}, \mathbb{B}_4^{\mathfrak{m}}\right)$ is referring to a Gaussian process with the following hypotheses:

- $\star$ $\mathbb{B}^{\mathfrak{m}}$ has independent and stationary increments.
- $\star$ The associated co-variance matrix of $\mathbb{B}^{\mathfrak{m}}$ is donated by $\mathfrak{M}$.
- $\star$ $\mathcal{N}_\chi$ is independent of $\mathbb{B}^{\mathfrak{m}}$.

Furthermore, it is supposed that χ can be infinite or finite and the conveniences of $\mathbb{A}$ are expressed by

$$\mathbb{E}\left[\mathbb{A}_k(t)\mathbb{A}_j(t)\right] = \mathfrak{m}_{k,j}t$$
$$+ t\int_{\mathcal{H}} \vartheta_k(u)\vartheta_j(u)\chi(\mathrm{d}u), k, j = 1, 2, 3, 4.$$

Remark 1.2 In [54], Privault and Wang obtained sufficient criteria for the disease vanishing and its insistence in the case of SIR model with the second representation of $\mathbb{A}$. However, the ergodicity property has not been investigated due to some technical difficulties. It must be mentioned that the ergodicity is an important statistical property for random systems. In this survey, we properly deal with this question.

Specifically, this study exhibits a novel approach to treat the long-run of the perturbed model (1.2) with the representation (1.4). Under an appropriate hypothetical framework, we present the sufficient and necessary condition for ergodicity and extinction of the model. Based on some nice characteristics of an auxiliary equation with linear jump-diffusion, we establish the exact expression of the threshold $\overline{\mathcal{R}}_\circ$. In other words, if $\overline{\mathcal{R}}_\circ > 1$, then system (1.2) has a single ergodic stable distribution, and if $\overline{\mathcal{R}}_\circ < 1$, then the illness will tend to disappearance exponentially. We mention that our proof to demonstrate the disease disappearance differs from that presented in [54].

As an instance where the proposed methodology is appropriate, we present and study numerically a robust class of tempered stable distributions. The discrete increments of tempered stable processes have power tails that are strongly applied in infinite Lévy measure cases [54]. In line with the survey presented in [53], the tempered α-stable Lévy measure is expressed as follows:

$$\chi(\mathcal{Y}) = \int_{\mathbb{R}_+} \int_{\mathcal{H}} e^{-\tau}\tau^{-\alpha-1}\mathbb{1}_{\mathcal{Y}}(\tau y)\mathfrak{Z}_\alpha(\mathrm{d}y)\mathrm{d}\tau,$$
$$\alpha \in (0, 2). \quad (1.5)$$

Here, $\mathfrak{Z}_\alpha(\cdot)$ denotes a measure on $\mathcal{H}$ such that $\int_{\mathcal{H}} \min\left(\|y\|^2, '\|y\|^\alpha\right)\mathfrak{Z}_\alpha(\mathrm{d}y) < \infty$. We take $\mathfrak{Z}_\alpha(\mathrm{d}y) = \phi_-(\mathrm{d}y) + \phi_+(\mathrm{d}y)$, where $\phi_- = \zeta_-\theta_-^\alpha\delta_{(-1/\theta_-,-1/\theta_-,-1/\theta_-,-1/\theta_-)}$, $\phi_+ = \zeta_+\theta_+^\alpha\delta_{(1/\theta_+,1/\theta_+,1/\theta_+,1/\theta_+)}$, for all $\zeta_-, \zeta_+, \theta_-, \theta_+ > 0$ and

δ_z is the Dirac mass measure at point z in $\mathbb{R}^4$. From (1.5), we infer that the infinite measure χ is rewritten as follows:

$$\chi(\mathcal{Y}) = \int_{\mathbb{R}_+} \psi_- e^{-\tau}\tau^{-\alpha-1}\mathrm{d}\tau + \int_{\mathbb{R}_+} \psi_+ e^{-\tau}\tau^{-\alpha-1}\mathrm{d}\tau,$$
$$\alpha \in (0,2) \tag{1.6}$$

where $\psi_- = \zeta_- \mathbb{1}_{\mathcal{Y}}(-\tau/\theta_-, -\tau/\theta_-, -\tau/\theta_-, -\tau/\theta_-)$ and $\psi_+ = \zeta_+ \mathbb{1}_{\mathcal{Y}}(\tau/\theta_+, \tau/\theta_+, \tau/\theta_+, \tau/\theta_+)$.

The remnant of this study is organized into three following parts:

- ⋆ Second Sect. 2 presents the hypothetical framework as well as some required lemmas.
- ⋆ Third Sect. 3 shows that $\overline{\mathcal{R}}_\circ$ is the global sill of the system (1.2) with the representation (1.4).
- ⋆ Final Sect. 4 introduces some numerical examples to clarify and emphasize our outcomes.

2 Hypothetical framework and some required lemmas

To properly study our model (1.2) with the representation (1.4), we impose the following assumptions:

- H_1: $\int_{\mathcal{H}} \vartheta_k^2(u)\chi(\mathrm{d}u) < \infty$, $k = 1,2,3,4$,
- H_2: $\vartheta_k(u) \in (-1,\infty)$ and $\int_{\mathcal{H}}\left(\vartheta_k(u) - \ln\left(1 + \vartheta_k(u)\right)\right)\chi(\mathrm{d}u) < \infty$, $k = 1,2,3,4$,
- H_3: $\Theta = \mathfrak{d} - \frac{2p-1}{2}\|\mathfrak{m}\|_\infty - \frac{\mathcal{I}(2p)}{2p} > 0$, $p > 1$,
- H_4: $\int_{\mathcal{H}} |(1 + \vartheta_\star(u))^{2p} - 1|\chi(\mathrm{d}u) < \infty, p > 1$,

where

$$\|\mathfrak{m}\|_\infty = \max_{k=1,2,3,4}\sum_{j=1}^4 |\mathfrak{m}_{k,j}|,$$
$$\mathcal{I}(2p) = p(2p-1)\max\left(2^{2p-3},1\right)$$
$$\int_{\mathcal{H}}\left(\vartheta_\star^2(u) + \vartheta_\star^{2p}(u)\right)\chi(\mathrm{d}u), p > 1,$$
$$\vartheta_\star(u) = \max\left\{\vartheta_1(u),\vartheta_2(u),\vartheta_3(u),\vartheta_4(u)\right\}.$$

Remark 2.1 Biologically, if $\vartheta_k(u) > 0$ the Lévy jumps increase the quantity of the host population. Otherwise, if $-1 < \vartheta_k(u) < 0$, the number of individuals is minimized gradually.

Remark 2.2 The assumptions H_2 and H_4 mean biologically that the intensity of Lévy jumps cannot exceed the environmental carrying capacity.

The next consequence guarantees the well-posedness of the model (1.2) with the representation (1.4).

Lemma 2.1 *Under the hypotheses H_1 and H_2, the probabilistic system (1.2) is well posed.*

By the approach used in [43], we can easily prove that for any positive initial data $\mathbf{C}_0 = (\mathbf{C}_1(0), \mathbf{C}_2(0), \mathbf{C}_3(0), \mathbf{C}_4(0))$, there corresponds one and only one global solution $\mathbf{C} = (\mathbf{C}_1(t), \mathbf{C}_2(t), \mathbf{C}_3(t), \mathbf{C}_4(t)) \in \mathbb{R}_+^4$ of the model (1.2) on $t \geq 0$.

Now and based on the positivity of the solution $\mathbf{C}$, we give an estimate of the total class $T_{\mathbf{C}}(t) = \sum_{k=1}^4 \mathbf{C}_k(t)$.

Lemma 2.2 *Let hypotheses H_1, H_2, H_3 hold and let $\mathbf{C}$ be the solution of (1.2) with initial value $\mathbf{C}_0 \in \mathbb{R}_+^4$, then for any $p > 1$ such that $\Theta > 0$, it holds that*

$$\limsup_{t\to\infty} t^{-1}\int_0^t \mathbb{E}\left[\left(1 + T_{\mathbf{C}}(s)\right)^{2p}\right]\mathrm{d}s \leq \frac{2p\Gamma}{\gamma},$$

where $\gamma \in (0, 2p\Theta)$ and $\Gamma = 1 + \sup_{x>0}\left\{x^{2p-2}\left(-\left(\Theta - \frac{\gamma}{2p}\right)x^2 + \left(\mathfrak{a} - \mathfrak{d} + \frac{\gamma}{p}\right)x + \mathfrak{a} + \frac{\gamma}{2p}\right)\right\}$.

The above result can be proved using an analysis analogous to that of the proof of Lemma 2.2. in [2].

Lemma 2.3 *Let hypotheses H_1, H_2, H_3, H_4 hold and let $\mathbf{C}_0$ be a given positive value. If $\mathbf{C}$ indicates the unique solution of model (1.2) that begins from $\mathbf{C}_0$, then*

1. $\lim_{t\to\infty} \frac{\mathbf{C}_k(t)}{t} = 0$ a.s. $k = 1,2,3,4$.
2. $\lim_{t\to\infty} t^{-1}\int_0^t \mathbf{C}_k(s)\mathrm{d}\mathbb{B}_k^\flat(s) = 0$ a.s. $k = 1,2,3,4$.
3. $\lim_{t\to\infty} t^{-1}\int_0^t \int_{\mathcal{H}} \vartheta_k(u)\mathbf{C}_k(s_-)\mathbb{J}(\mathrm{d}s,\mathrm{d}u) = 0$ a.s. $k = 1,2,3,4$.

By employing the method presented in [44] and Kunita's inequality [55], we can readily demonstrate the above Lemma. A detailed proof is presented in [54] (see Lemmas 2.2, 2.3 and 2.5).

To deal with the new stochastic system (1.2), we propose an alternative method based on a second system very close to the equation of the total population $T_{\mathbf{C}}(t)$. This new auxiliary system characterizes the epidemic dynamics in limit conditions when the infection is absent. Keeping the same probabilistic part of $T_{\mathbf{C}}(t)$, the auxiliary system is expressed

by the following boundary equation:

$$\begin{cases} d\mathbf{D}(t) = \big(\mathfrak{a} - \mathfrak{d}\mathbf{D}(t)\big)dt + \sum_{k=1}^{4} \mathbf{C}_k(t_-)d\mathbb{A}_k(t), \\ \mathbf{D}(0) = T_{\mathbf{C}}(0) \in \mathbb{R}_+. \end{cases} \quad (2.1)$$

The stochastic system (2.1) is biologically well-posed and admits a unique positive solution $\mathbf{D}(t)$. Moreover, $\mathbf{D}(t)$ is a Markov process which satisfies nice analytical properties. As an example, we present the next lemma.

Lemma 2.4 *Let* H_1, H_2, H_3, H_4 *hold. Then,* $\lim_{t\to\infty} t^{-1} \int_0^t \mathbf{D}(s)ds = \frac{\mathfrak{a}}{\mathfrak{d}}$ *a.s.*

Proof By integrating (2.1) and using Lemma 2.3, we can effortlessly and directly prove this result. $\square$

Remark 2.3 Via the probabilistic comparison theorem [56], we conclude that $T_{\mathbf{C}}(t) \le \mathbf{D}(t)$ for all $t \in [0, \infty)$ a.s.

Different from the Hasminskii's method, in this paper, we use the alternately limited possibilities lemma of Feller processes to get the sufficient and almost necessary criterion for the ergodicity of our system.

Lemma 2.5 (Alternately limited possibilities lemma, [57]) *We consider a stochastic process* $\mathcal{G} \in \mathbb{R}^n$ *that verifies the Feller property. Then, two possibilities are available:*

1. *A single ergodic stationary distribution exists, or*
2. *The following result is satisfied*

$$\limsup_{t\to\infty} t^{-1} \int_0^t \int_{\mathbb{R}^n} \mathbb{P}(x; s, \mathbb{U}) \hat{\varrho}(dx) ds = 0, \quad (2.2)$$

for a given compact domain $\mathbb{U} \subset \mathbb{R}^n$, *where* $\hat{\varrho}$ *is the initial distribution on* $\mathbb{R}^n$ *and* $\mathbb{P}(x; s, \mathbb{U})$ *stands for the probability of* $\mathcal{G}$ *belongs to* $\mathbb{U}$ *with* $\mathcal{G}(0) = x \in \mathbb{R}^n$.

3 Threshold analysis: stationary distribution and extinction

As stated in the introduction, when analyzing a mathematical model that describes the spread of a particular illness, our main preoccupation is to know if it will end or will last. For this reason, we will prove that

$$\overline{\mathcal{R}}_\circ = \mathcal{R}_\circ - \frac{\left(0.5\mathfrak{m}_{2,2} + \int_{\mathcal{H}} \big(\vartheta_2(u) - \ln(1 + \vartheta_2(u))\big)\chi(du)\right)}{\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2},$$

is the threshold among stationarity and extinction of the stochastic model (1.2) with the representation (1.4). But

before doing so, let us first introduce the following assumption:

- $\mathbf{H}_5$: $\int_{\mathcal{H}} \Big(\ln\big(1 + \vartheta_k(u)\big)\Big)^2 \chi(du) < \infty$, $k = 1, 2, 3, 4$.

Theorem 3.1 *Assume that* H_1, H_2, H_3, H_4 *and* H_5 *hold. The parameter* $\overline{\mathcal{R}}_\circ$ *is the sill of the stochastic model* (1.2) *with the representation* (1.4). *That is to say that,*

1. *If* $\overline{\mathcal{R}}_\circ > 1$, *then the stationarity and ergodicity of our model are verified.*
2. *If* $\overline{\mathcal{R}}_\circ < 1$, *then the illness dies out exponentially (rapidly) with probability one.*

    Biological interpretation 3.1. *By Theorem 3.1, we show that:*

1. *The stationarity and ergodicity reveal that the stochastic model* (1.2) *has a limiting stable distribution that prophesies the continuation of the illness. That implies that the infected population persevers for a long time.*
2. *The quantity* $\overline{\mathcal{R}}_\circ$ *contains linear random intensities, which are related to the infected class* $\mathbf{C}_2$. *This designates that if* $\overline{\mathcal{R}}_\circ$ *is strictly less than one, the stochastic fluctuations help to the inhibition of the illness.*

Proof Analogous to the demonstration of (Lemma 3.2. in [58]), we can confirm the Feller property of the Markov process $\mathbf{C}$. In the next step, we prove that (2.2) is not verified for the system (1.2). Let $\mathbf{G}(t) = \mathbf{C}_1(t)\mathbf{C}_2(t)$ and apply Itô's formula to function $\mathcal{V}(t) = \ln \mathbf{C}_2(t) - \frac{\mathfrak{b}}{\mathfrak{d}}\big(\mathbf{D}(t) - \mathbf{C}_1(t)\big)$, then

$$\begin{aligned} d\mathcal{V}(t) \ge & \bigg(\mathfrak{b}\mathbf{D}(t) - (\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2) - 0.5\mathfrak{m}_{2,2} \\ & - \int_{\mathcal{H}} \Big(\vartheta_2(u) - \ln\big(1 + \vartheta_2(u)\big)\Big)\chi(du) \bigg)dt \\ & - \frac{\mathfrak{b}^2}{\mathfrak{d}}\mathbf{G}(t)dt \\ & + d\overline{\mathbb{B}}_2^{\mathfrak{m}}(t) + \int_{\mathcal{H}} \ln\big(1 + \vartheta_2(u)\big)\mathbb{J}(dt, du) \\ & - \frac{\mathfrak{b}}{\mathfrak{d}}\sum_{k=2}^{4} \mathbf{C}_k(t)d\mathbb{B}_k^{\mathfrak{m}}(t) - \frac{\mathfrak{b}}{\mathfrak{d}}\sum_{k=2}^{4} \\ & \int_{\mathcal{H}} \vartheta_k(u)\mathbf{C}_k(t_-)\mathbb{J}(dt, du). \end{aligned} \quad (3.1)$$

We integrate from 0 to t on both sides of (3.1), then we get

$$\mathcal{V}(t) - \mathcal{V}(0) \ge \int_0^t \mathfrak{b}\mathbf{D}(s)ds - (\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2)t$$

$$- 0.5\mathfrak{m}_{2,2}t - t$$
$$\int_{\mathcal{H}} \Big(\vartheta_2(u) - \ln\big(1 + \vartheta_2(u)\big)\Big)\chi(\mathrm{d}u)$$
$$- \frac{\mathfrak{b}^2}{\mathfrak{d}} \int_0^t \mathbf{G}(s)\mathrm{d}s$$
$$+ \mathbb{B}_2^{\mathfrak{m}}(t) + \int_0^t \int_{\mathcal{H}} \ln\big(1 + \vartheta_2(u)\big)\mathbb{J}(\mathrm{d}s, \mathrm{d}u)$$
$$- \frac{\mathfrak{b}}{\mathfrak{d}} \sum_{k=2}^4 \int_0^t \mathbf{C}_k(s)\mathrm{d}\mathbb{B}_k^{\mathfrak{m}}(s)$$
$$- \frac{\mathfrak{b}}{\mathfrak{d}} \sum_{k=2}^4 \int_0^t \int_{\mathcal{H}} \vartheta_k(u)\mathbf{C}_k(s_-)\mathbb{J}(\mathrm{d}s, \mathrm{d}u).$$

Thus, we have

$$\int_0^t \mathfrak{b}\mathbf{G}(s)\mathrm{d}s \geq \frac{\mathfrak{d}}{\mathfrak{b}} \int_0^t \mathfrak{b}\mathbf{D}(s)\mathrm{d}s$$
$$- \frac{\mathfrak{d}}{\mathfrak{b}}\bigg((\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2) + 0.5\mathfrak{m}_{2,2}$$
$$+ \int_{\mathcal{H}} \Big(\vartheta_2(u) - \ln\big(1 + \vartheta_2(u)\big)\Big)\chi(\mathrm{d}u)\bigg)t$$
$$+ \frac{\mathfrak{d}}{\mathfrak{b}}\mathbb{B}_2^{\mathfrak{m}}(t) + \frac{\mathfrak{d}}{\mathfrak{b}} \int_0^t \int_{\mathcal{H}} \ln\big(1 + \vartheta_2(u)\big)\mathbb{J}(\mathrm{d}s, \mathrm{d}u)$$
$$- \sum_{k=2}^4 \int_0^t \mathbf{C}_k(s)\mathrm{d}\mathbb{B}_k^{\mathfrak{m}}(s)$$
$$- \sum_{k=2}^4 \int_0^t \int_{\mathcal{H}} \vartheta_k(u)\mathbf{C}_k(s_-)\mathbb{J}(\mathrm{d}s, \mathrm{d}u)$$
$$+ \big(\mathbf{D}(t) - \mathbf{C}_1(t)\big) - \big(\mathbf{D}(0) - \mathbf{C}_1(0)\big)$$
$$- \frac{\mathfrak{d}}{\mathfrak{b}} \ln\frac{\mathbf{C}_2(t)}{\mathbf{C}_2(0)}. \tag{3.2}$$

Let

$$f(t) = \frac{\mathfrak{d}}{\mathfrak{b}}\mathbb{B}_2^{\mathfrak{m}}(t)$$
$$\overbrace{+ \frac{\mathfrak{d}}{\mathfrak{b}} \int_0^t \int_{\mathcal{H}} \ln\big(1 + \vartheta_2(u)\big)\mathbb{J}(\mathrm{d}s, \mathrm{d}u)}^{=g(t)}$$
$$- \sum_{k=2}^4 \int_0^t \mathbf{C}_k(s)\mathrm{d}\mathbb{B}_k^{\mathfrak{m}}(s)$$
$$- \sum_{k=2}^4 \int_0^t \int_{\mathcal{H}} \vartheta_k(u)\mathbf{C}_k(s_-)\mathbb{J}(\mathrm{d}s, \mathrm{d}u).$$

The quadratic variation associated with the local martingale $g(t)$ is given by

$$\langle g(t), g(t)\rangle = \bigg(\int_{\mathcal{H}} \big(\ln(1 + \vartheta_2(u))\big)^2\chi(\mathrm{d}u)\bigg)t.$$

By employing the strong law of large numbers for martingale [55], Lemma 2.3 and hypothesis $\mathbf{H}_5$, we obtain

$$\lim_{t\to\infty} t^{-1}f(t) = 0, \quad \text{a.s.}$$

From Lemma 3.1 of [45], we conclude that $\limsup\limits_{t\to\infty} t^{-1}\ln\frac{\mathbf{C}_2(t)}{\mathbf{C}_2(0)} \leq \limsup\limits_{t\to\infty} t^{-1}\ln\frac{T_{\mathbf{C}}(t)}{\mathbf{C}_2(0)} \leq 0$ a.s. So,

$$\liminf_{t\to\infty} t^{-1} \int_0^t \mathfrak{b}\mathbf{G}(s)\mathrm{d}s$$
$$\geq \frac{\mathfrak{d}}{\mathfrak{b}}\bigg(\liminf_{t\to\infty} t^{-1} \int_0^t \mathfrak{b}\mathbf{D}(s)\mathrm{d}s$$
$$- \bigg((\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2) + 0.5\mathfrak{m}_{2,2}$$
$$+ \int_{\mathcal{H}} \Big(\vartheta_2(u) - \ln\big(1 + \vartheta_2(u)\big)\Big)\chi(\mathrm{d}u)\bigg)\bigg)$$
$$= \frac{\mathfrak{d}}{\mathfrak{b}}(\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1) > 0.$$

To carry on with our proof, we need to consider the following subsets:

$$\mathbb{S}_1 = \Big\{(t, \omega) \in \mathbb{R}_+ \times \Omega | \mathbf{C}_1(t, \omega) \geq \theta, \text{ and, } \mathbf{C}_2(t, \omega) \geq \theta\Big\},$$
$$\mathbb{S}_2 = \Big\{(t, \omega) \in \mathbb{R}_+ \times \Omega | \mathbf{C}_1(t, \omega) \leq \theta\Big\},$$
$$\mathbb{S}_3 = \Big\{(t, \omega) \in \mathbb{R}_+ \times \Omega | \mathbf{C}_2(t, \omega) \leq \theta\Big\},$$

where $\theta > 0$ is a constant to be specified in the following. Therefore, we obtain

$$\liminf_{t\to\infty} t^{-1} \int_0^t \mathbb{E}\Big[\mathfrak{b}\mathbf{G}(s)\mathbb{1}_{\mathbb{S}_1}\Big]\mathrm{d}s$$
$$\geq -\limsup_{t\to\infty} t^{-1} \int_0^t \mathbb{E}\Big[\mathfrak{b}\mathbf{G}(s)\mathbb{1}_{\mathbb{S}_2}\Big]\mathrm{d}s$$
$$- \limsup_{t\to\infty} t^{-1} \int_0^t \mathbb{E}\Big[\mathfrak{b}\mathbf{G}(s)\mathbb{1}_{\mathbb{S}_3}\Big]\mathrm{d}s$$
$$+ \liminf_{t\to\infty} t^{-1} \int_0^t \mathbb{E}\Big[\mathfrak{b}\mathbf{G}(s)\Big]\mathrm{d}s$$
$$\geq -\mathfrak{b}\theta\limsup_{t\to\infty} t^{-1} \int_0^t \mathbb{E}\big[\mathbf{C}_1(s)\big]\mathrm{d}s$$
$$- \mathfrak{b}\theta\limsup_{t\to\infty} t^{-1} \int_0^t \mathbb{E}\big[\mathbf{C}_2(s)\big]\mathrm{d}s$$
$$+ \frac{\mathfrak{d}}{\mathfrak{b}}(\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1).$$

Consequently

$$\liminf_{t\to\infty} t^{-1} \int_0^t \mathbb{E}\Big[\mathfrak{b}\mathbf{G}(s)\mathbb{1}_{\mathbb{S}_1}\Big]\mathrm{d}s$$
$$\geq -\frac{2\mathfrak{a}\mathfrak{b}\theta}{\mathfrak{d}} + \frac{\mathfrak{d}}{\mathfrak{b}}(\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1).$$

We can choose $\theta \leq \frac{\eth^2}{4\mathfrak{b}^2\mathfrak{a}}(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1)$, and then we have

$$\liminf_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\Big[\mathfrak{b}\mathbf{G}(s)\mathbb{1}_{\mathbb{S}_1}\Big]ds$$
$$\geq \frac{\eth}{2\mathfrak{b}}(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1) > 0. \tag{3.3}$$

Let $p > 1$ such that $\Theta = \eth - \frac{(2p-1)}{2}\|\mathfrak{m}\|_\infty - \frac{\mathcal{I}(2p)}{2p} > 0$ and q is given by $\frac{1}{q} + \frac{1}{p} = 1$. By employing the Young inequality [55], we get

$$\liminf_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\Big[\mathfrak{b}\mathbf{G}(s)\mathbb{1}_{\mathbb{S}_1}\Big]ds$$
$$\leq \liminf_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\Big[\big(p^{-1}(\varrho\mathfrak{b}\mathbf{G}(s))^p + q^{-1}\varrho^{-q}\mathbb{1}_{\mathbb{S}_1}\big)\Big]ds$$
$$\leq p^{-1}(\varrho\mathfrak{b})^p \limsup_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\Big[(1 + T_{\mathbf{C}}(s))^{2p}\Big]ds$$
$$+ \liminf_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\Big[q^{-1}\varrho^{-q}\mathbb{1}_{\mathbb{S}_1}\Big]ds,$$

where $\varrho > 0$ is a constant verifying $\varrho^p \leq \frac{\eth\gamma\mathfrak{b}^{-(p+1)}}{8\Gamma}(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1)$. By Lemma 2.2 and (3.3), we conclude that

$$\liminf_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\Big[\mathbb{1}_{\mathbb{S}_1}\Big]ds$$
$$\geq q\varrho^q\Big(\frac{\eth}{2\mathfrak{b}}(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1) - \frac{2\Gamma\varrho^p\mathfrak{b}^p}{\gamma}\Big)$$
$$\geq \frac{\eth q\varrho^q}{4\mathfrak{b}}(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1) > 0. \tag{3.4}$$

Setting

$$\mathbb{S}_4 = \{(t, \omega) \in \mathbb{R}_+ \times \Omega | \ \mathbf{C}_1(t, \omega) \geq \zeta, \ \text{or}, \ \mathbf{C}_2(t, \omega) \geq \zeta\},$$
$$\mathbb{S}^\star = \{(t, \omega) \in \mathbb{R}_+ \times \Omega | \ \theta \leq \mathbf{C}_1(t, \omega) \leq \zeta, \ \text{and},$$
$$\theta \leq \mathbf{C}_2(t, \omega) \leq \zeta\},$$

where $\zeta > \theta > 0$ is a constant value to be described in the next. By using the Markov's inequality [55], we can find that

$$\int_{\mathbb{S}} \mathbb{1}_{\mathbb{S}_4}(t, \omega) d\mathbb{P}(\omega) \leq \mathbb{P}(\mathbf{C}_1(t) \geq \zeta) + \mathbb{P}(\mathbf{C}_2(t) \geq \zeta)$$
$$\leq \frac{1}{\zeta}\mathbb{E}\big[\mathbf{C}_1(t) + \mathbf{C}_2(t)\big].$$

We choose $\frac{1}{\zeta} \leq \frac{\eth^2 q\varrho^q}{8\mathfrak{b}\mathfrak{a}}(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1)$, then we obtain

$$\limsup_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\Big[\mathbb{1}_{\mathbb{S}_4}\Big]ds$$

$$\leq \frac{\eth q\varrho^q}{8\mathfrak{b}}(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1).$$

By (3.4), we have

$$\liminf_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\big[\mathbb{1}_{\mathbb{S}^\star}\big]ds$$
$$\geq -\limsup_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\big[\mathbb{1}_{\mathbb{S}_4}\big]ds$$
$$+ \liminf_{t \to \infty} t^{-1} \int_0^t \mathbb{E}\big[\mathbb{1}_{\mathbb{S}_1}\big]ds$$
$$\geq \frac{\eth q\varrho^q}{8\mathfrak{b}}(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1) > 0.$$

Ultimately and according to the above treatment, we have specified a compact domain $\mathbb{S}^\star$ such that

$$\liminf_{t \to \infty} t^{-1} \int_0^t \mathbb{P}(\mathbf{C}_0; s, \mathbb{S}^\star)ds$$
$$\geq \frac{\eth q\varrho^q}{8\mathfrak{b}}(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1) > 0. \tag{3.5}$$

In contrary, we check easily that if $\overline{\mathcal{R}}_\circ < 1$, the illness will extinct. In accordance with Lemma 2.3, we obtain

$$\limsup_{t \to \infty} t^{-1} \ln \frac{\mathbf{C}_2(t)}{\mathbf{C}_2(0)}$$
$$= \mathfrak{b}\limsup_{t \to \infty} t^{-1} \int_0^t \mathbf{C}_1(s)ds$$
$$- \big((\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2) + 0.5\mathfrak{m}_{2,2}$$
$$+ \int_{\mathcal{H}} \big(\vartheta_2(u) - \ln(1 + \vartheta_2(u))\big)\chi(du)\big)$$
$$\leq \mathfrak{b}\lim_{t \to \infty} t^{-1} \int_0^t \mathbf{D}(s)ds$$
$$- \big((\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2) + 0.5\mathfrak{m}_{2,2}$$
$$+ \int_{\mathcal{H}} \big(\vartheta_2(u) - \ln(1 + \vartheta_2(u))\big)\chi(du)\big)$$
$$= (\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1) < 0 \quad \text{a.s.}$$

So, $\lim_{t \to \infty} \mathbf{C}_2(t) = 0$ a.s. To put it another way, the epidemic of the system (1.2) will quickly be removed, and its deterioration rate is at least $(\eth + \eth_2 + \mathfrak{q} + \mathfrak{c}_2)(\overline{\mathcal{R}}_\circ - 1)$. This ends the demonstration. $\quad\square$

4 Application: epidemic model (1.2) driven by tempered stable Poisson process

This part is devoted to introducing the numerical examples and checking the correctness of Theorem 3.1. Through computer simulations, we acquire the trajectories plot, and corresponding histograms, which can more obviously reflect

the complex dynamical attitude of the perturbed system (1.2). Moreover, we choose some reasonable parameter values to verify our hypothetical framework. According to the work presented in [54], we use the following compensated tempered Poisson process:

$$\mathbb{Y}(t) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} u \, \mathbb{J}(\mathrm{d}s, \mathrm{d}u), \qquad (4.1)$$

with the associated infinite L.m. (1.6). To numerically apply the method proposed in [53] on the process (4.1) with related measure (1.6), we use the following setup.

4.1 Algorithm configuration and inputs tuning

- We suppose that $\mathbb{B}^{\mathfrak{m}}$ is generated as follows:

$$\mathbb{B}_1^{\mathfrak{m}} = \mathfrak{m}_{1,1}\mathcal{B}_a,$$
$$\mathbb{B}_2^{\mathfrak{m}} = \mathfrak{m}_{2,1}\mathcal{B}_a + \mathfrak{m}_{2,2}\mathcal{B}_b,$$
$$\mathbb{B}_3^{\mathfrak{m}} = \mathfrak{m}_{3,1}\mathcal{B}_a + \mathfrak{m}_{3,2}\mathcal{B}_b + \mathfrak{m}_{3,3}\mathcal{B}_c,$$
$$\mathbb{B}_4^{\mathfrak{m}} = \mathfrak{m}_{4,1}\mathcal{B}_a + \mathfrak{m}_{4,2}\mathcal{B}_b + \mathfrak{m}_{4,3}\mathcal{B}_c + \mathfrak{m}_{4,4}\mathcal{B}_d,$$

  where $\mathcal{B}_a, \mathcal{B}_b, \mathcal{B}_c$ and $\mathcal{B}_d$ stand for independent Brownian motions.
- $(a_j)_{j\geq 1}$ is an i.i.d. Bernoulli random sequence with the associated distribution $\left(\zeta_-/(\zeta_- + \zeta_+), \zeta_+/(\zeta_- + \zeta_+)\right)$.
- $(b_j)_{j\geq 1}$ and $(b_j')_{j\geq 1}$ are i.i.d. exponential random variables with the parameter 1, where $B_j = b_1' + \cdots + b_j'$.
- $(c_j)_{j\geq 1}$ are i.i.d. uniform random variables.
- $(d_j)_{j\geq 1}$ is an i.i.d. uniform $U(0,1)$ random sequence.

According to Theorem 5.3 in [53], all above sequences are supposed to be mutually independent. Furthermore, the process $\mathbb{Y}$ with (1.6) can be presented as follows:

- When $0 < \alpha < 1$, then $\mathbb{Y}(t) = \sum_{j=1}^{\infty} \mathbb{1}_{(0,t]}(c_j)\frac{a_j}{|a_j|}\tilde{S}_1$, for all $0 \leq t \leq T$, where $\tilde{S}_1 = \min\left\{\left(\frac{(\zeta_- + \zeta_+)T}{\alpha B_j}\right)^{\frac{1}{\alpha}}, \frac{b_j}{|a_j|}d_j^{\frac{1}{\alpha}}\right\}$.

- When $1 \leq \alpha < 2$, then
$$\mathbb{Y}(t) = \sum_{j=1}^{\infty}\left(\mathbb{1}_{(0,t]}(c_j)\frac{a_j}{|a_j|}\tilde{S}_2 - z_0\frac{t}{T}\left(\frac{\zeta_- + \zeta_+}{\alpha j/T}\right)^{\frac{1}{\alpha}}\right) + t\Phi_T, \text{ for all } 0 \leq t \leq T \text{ where } \tilde{S}_2 = \min\left\{\left(\frac{\zeta_- + \zeta_+}{\alpha B_j/T}\right)^{\frac{1}{\alpha}}, \frac{b_j}{|a_j|}d_j^{\frac{1}{\alpha}}\right\}, z_0 = (\zeta_- - \zeta_+)/(\zeta_- + \zeta_+), z_1 = \zeta_+\theta_+^{-1-\alpha} - \zeta_-\theta_-^{-1-\alpha}$ and

$$\Phi_T = \begin{cases} \frac{z_0}{T}\zeta\left(\frac{1}{\alpha}\right)\left(\frac{T(\zeta_- + \zeta_+)}{\alpha}\right)^{\frac{1}{\alpha}} - z_1\Gamma(\alpha - 1), & 1 < \alpha < 2, \\ (2\gamma_e + \ln(\zeta_- + \zeta_+))z_1 - \int_{\mathcal{H}} x\ln(|x|)\mathfrak{Z}_\alpha(\mathrm{d}x) & \alpha = 1, \end{cases}$$

where $\zeta(\cdot)$ denotes the Riemann zeta function, $\Gamma(\cdot)$ is the Gamma function, and γ_e is the Euler constant.

Now, we choose $\vartheta_k(u) = e_k u$, where $e_k > 0$, $(k = 1, 2, 3, 4)$, and we introduce the following perturbed model

$$\begin{cases} \mathrm{d}\mathbf{C}_1(t) = \big(\mathfrak{a} - \mathfrak{d}\mathbf{C}_1(t) - \mathfrak{b}\mathbf{C}_1(t)\mathbf{C}_2(t) + \mathfrak{h}\mathbf{C}_4(t)\big)\mathrm{d}t \\ \qquad + \mathbf{C}_1(t)\mathrm{d}\mathbb{B}_1^{\mathfrak{m}}(t) + e_1\mathbf{C}_1(t_-)\mathrm{d}\mathbb{Y}(t), \\ \mathrm{d}\mathbf{C}_2(t) = \big(\mathfrak{b}\mathbf{C}_1(t)\mathbf{C}_2(t) - (\mathfrak{d} + \mathfrak{d}_2 + \mathfrak{q} + \mathfrak{c}_2)\mathbf{C}_2(t)\big)\mathrm{d}t \\ \qquad + \mathbf{C}_2(t)\mathrm{d}\mathbb{B}_2^{\mathfrak{m}}(t) + e_2\mathbf{C}_2(t_-)\mathrm{d}\mathbb{Y}(t), \\ \mathrm{d}\mathbf{C}_3(t) = \big(\mathfrak{q}\mathbf{C}_2(t) - (\mathfrak{d} + \mathfrak{d}_3 + \mathfrak{c}_3)\mathbf{C}_3(t)\big)\mathrm{d}t \\ \qquad + \mathbf{C}_3(t)\mathrm{d}\mathbb{B}_3^{\mathfrak{m}}(t) + e_3\mathbf{C}_3(t_-)\mathrm{d}\mathbb{Y}(t), \\ \mathrm{d}\mathbf{C}_4(t) = \big(\mathfrak{c}_2\mathbf{C}_2(t) + \mathfrak{c}_3\mathbf{C}_3(t) - (\mathfrak{d} + \mathfrak{h})\mathbf{C}_4(t)\big)\mathrm{d}t \\ \qquad + \mathbf{C}_4(t)\mathrm{d}\mathbb{B}_4^{\mathfrak{m}}(t) + e_4\mathbf{C}_4(t_-)\mathrm{d}\mathbb{Y}(t), \\ \mathbf{C}_1(0) = 1.6, \mathbf{C}_2(0) = 0.4, \mathbf{C}_3(0) = 0.3, \mathbf{C}_4(0) = 0.1. \end{cases}$$
$$(4.2)$$

Remark 4.1 In fact, we noticed that:

1. the assumptions on the jump-diffusion intensities $\mathbf{H}_1$, $\mathbf{H}_2$ and $\mathbf{H}_5$ are naturally verified in our case.
2. the condition $\mathbf{H}_4$ holds just for $\alpha \in (0,1)$, $p > 1$.
3. $\mathcal{I}(2p)$ is finite when $p > \alpha$.
4. the condition $\mathbf{H}_3$ will be checked according to the choice of other parameters.

In view of the last remark, we will give some numerical simulation results in the case of the one-sided tempered stable process $\mathbb{Y}(t)$ with $\zeta_- = 0$ and $\alpha \in (0,1)$. So, we choose

- $\alpha = 0.7$, $\zeta_+ = 2.8$ and $\theta_- = \theta_+ = 1.2$.
- $\mathfrak{m}_{1,1} = 0.11$, $\mathfrak{m}_{2,1} = 0.16$, $\mathfrak{m}_{3,1} = 0.15$, $\mathfrak{m}_{4,1} = 0.101$, $\mathfrak{m}_{2,2} = 0.22$, $\mathfrak{m}_{3,2} = 0.12$, $\mathfrak{m}_{4,2} = 0.17$, $\mathfrak{m}_{3,3} = 0.1$, $\mathfrak{m}_{4,3} = 0.135$, $\mathfrak{m}_{4,4} = 0.115$.
- $e_1 = 0.1$, $e_2 = 0.11$, $e_3 = 0.101$ and $e_4 = 0.1025$.

4.2 Theoretical results check

For the probabilistic model (4.2), the deterministic parameters are taken as follows: $\mathfrak{a} = 0.2$, $\mathfrak{b} = 0.35$, $\mathfrak{d} = 0.11$, $\mathfrak{d}_2 = 0.03$, $\mathfrak{d}_3 = 0.025$, $\mathfrak{h} = 0.1$ $\mathfrak{q} = 0.13$, $\mathfrak{c}_2 = 0.2$, $\mathfrak{c}_3 = 0.1$. Then, $\mathbf{H}_3$ holds and $\overline{\mathcal{R}}_\circ = 1.0098 > 1$. By using Theorem 3.1, we conclude that there is only one stable distribution. In Figs. 1 and 2, we plot the two-dimensional empirical distribution in order to offer a comprehensive overview of the marginal densities of the solution. In Fig. 3, we show the permanence of all trajectories. Now, we decrease the value of $\mathfrak{b}$ to 0.285 which indicates the reduction of the disease prevalence between $\mathbf{C}_1$ and $\mathbf{C}_2$. Then, $\mathbf{H}_3$ holds and $\overline{\mathcal{R}}_\circ = 0.9803 < 1$. From Theorem 3.1, we establish that
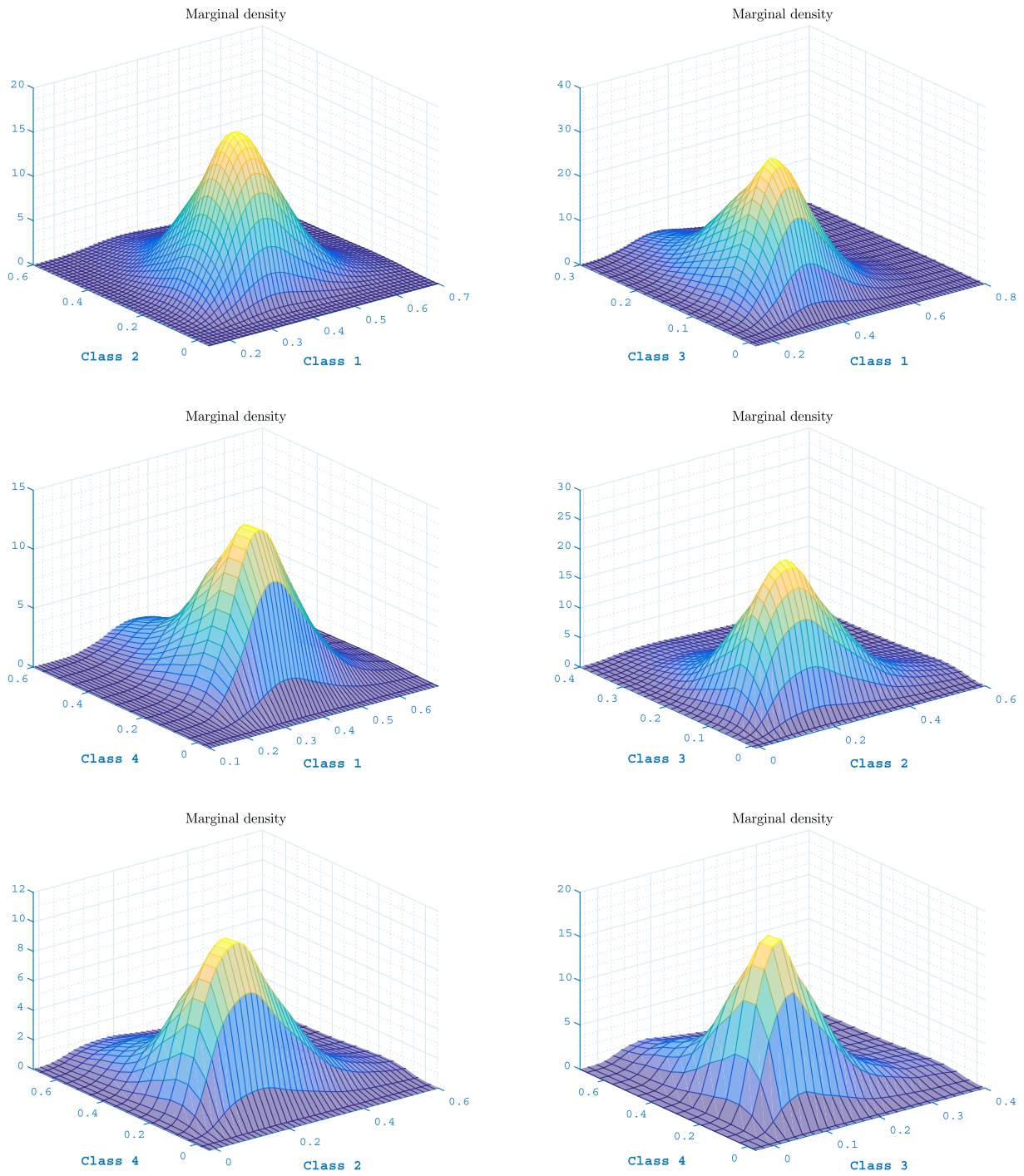
Fig. 1 The 3D graph of the joint two dimensional density at time $t = 5000$ of the classes $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_4$
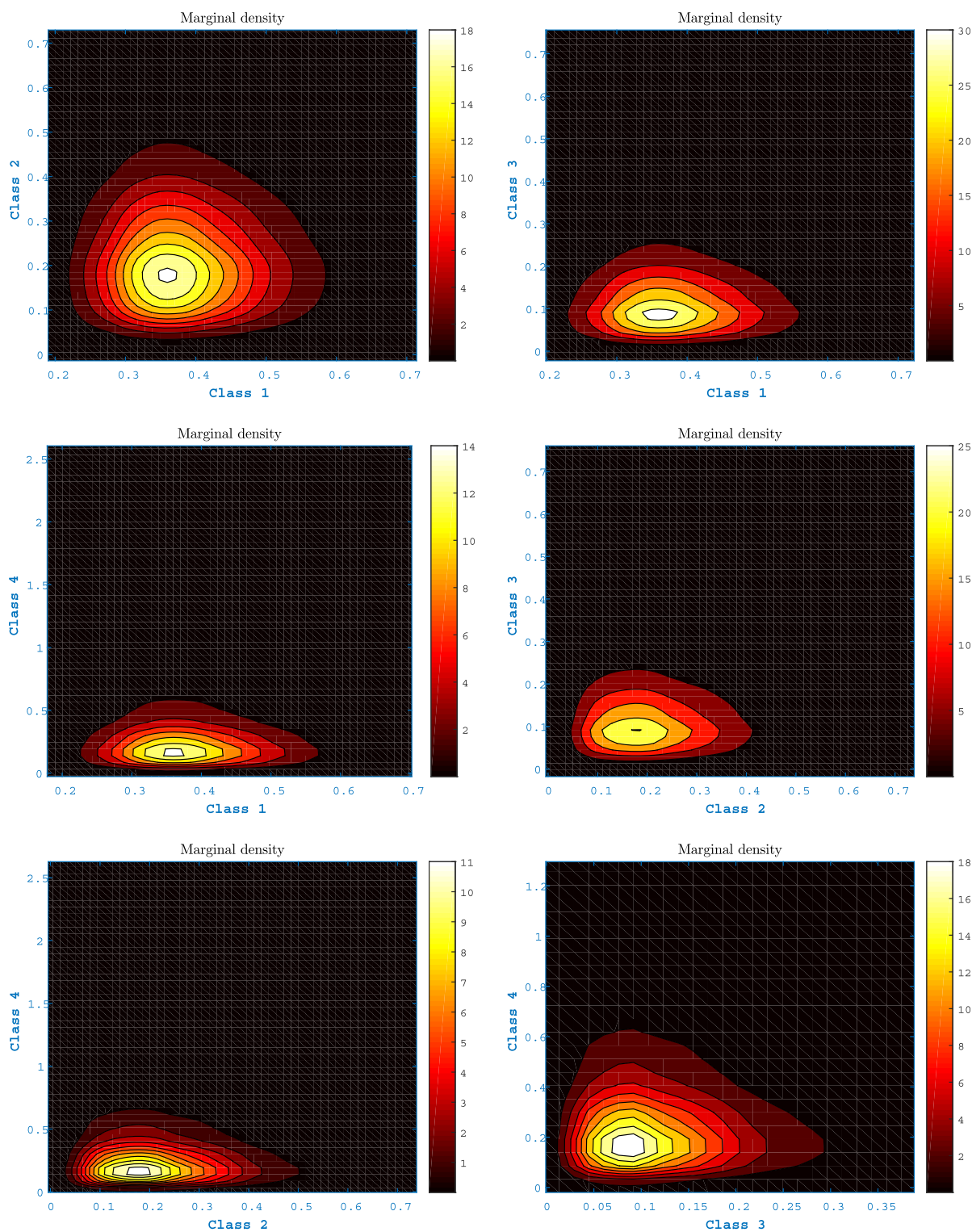
Fig. 2 The upper view of the joint two dimensional densities at time $t = 5000$ of the classes $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ and $\mathbf{C}_4$
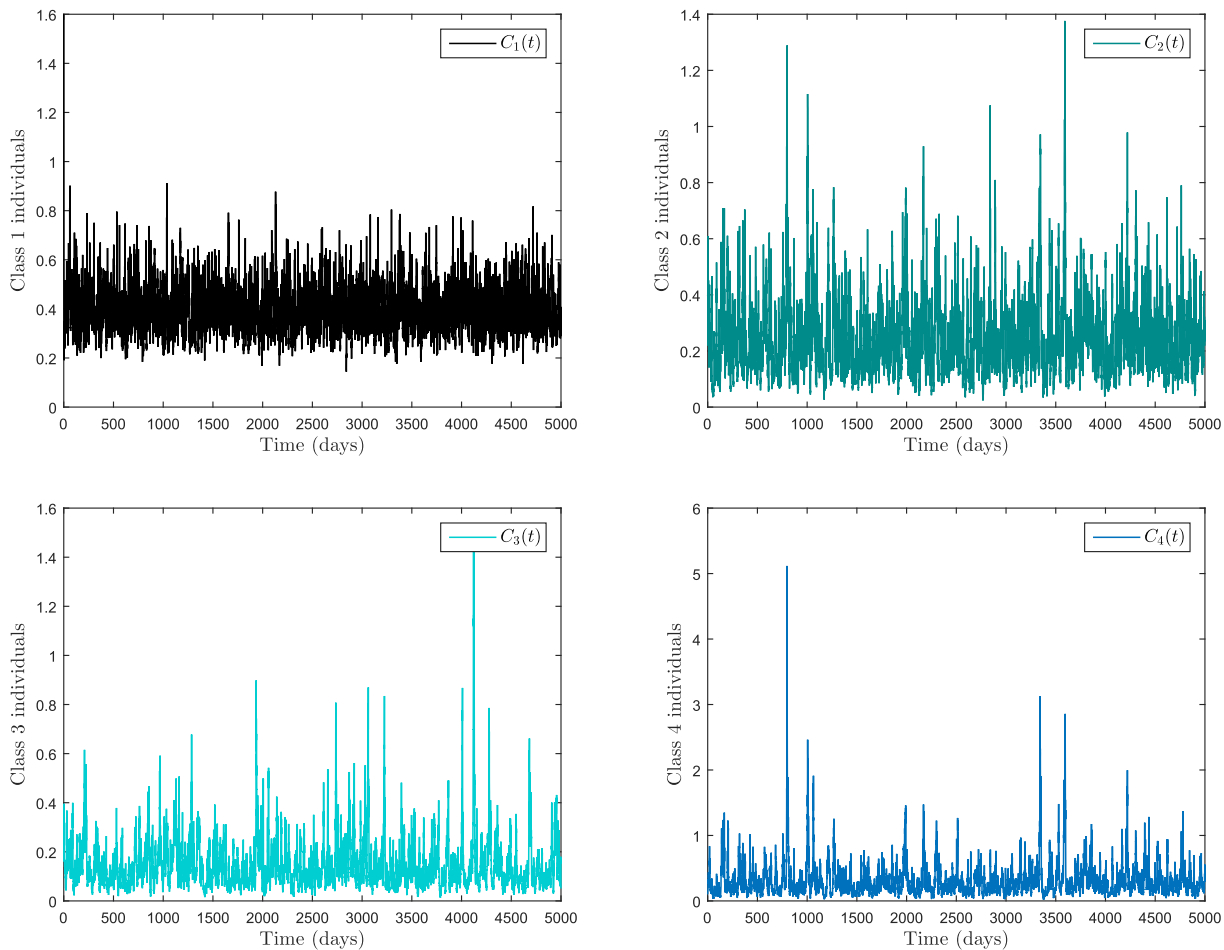
Fig. 3 Computer simulation of the solution of the probabilistic model (4.2) *with tempered process*

the illness will almost certainly extinct. To explore the Lévy jumps effect in this case, we compare the trajectories of (4.2) with the deterministic solution. A simple calculation shows that $\mathcal{R}_0 = 1.1025 > 1$. From the Fig. 4, we notice that the Lévy jumps conduct to the cancellation of the disease while the deterministic path persists. Thus, discontinuous jumps have a passive impact on the continuation of the disease and this means that Lévy jumps with infinite measure can change the propagation pattern remarkably in the long term.

Remark 4.2 We have theoretically chosen the parameters used in the simulations according to two criteria:

1. To verify and check appropriately the obtained analytical results in both cases: permanence and extinction of the diseases.
2. To show numerically the sharpness of the obtained thresholds.

It should be pointed out that our theoretical findings are general and can be applied to study many transmissible diseases, for example, COVID 19 epidemic (please see [59]).

Conclusion

In this study, we have analyzed a classical illness model with quarantine strategy and Lévy fluctuations. By considering a general Lévy measure and correlated noise items, we have proposed an analytical framework to deal with our constructed model. Explicitly, we have investigated the properties of stationarity and extinction by using the stochastic comparison theorem, exponential inequalities for martingales, Feller's property, the mutually limited possibilities lemma, and other mathematical tools. Our method differs from the well-known Khasminskii approach by providing the sufficient and necessary condition for ergodicity and disease suppression, and this is the strong point of our work. It only remains to verify what happens in the situation of $\overline{\mathcal{R}}_\circ = 1$. We will process this open question in the future.
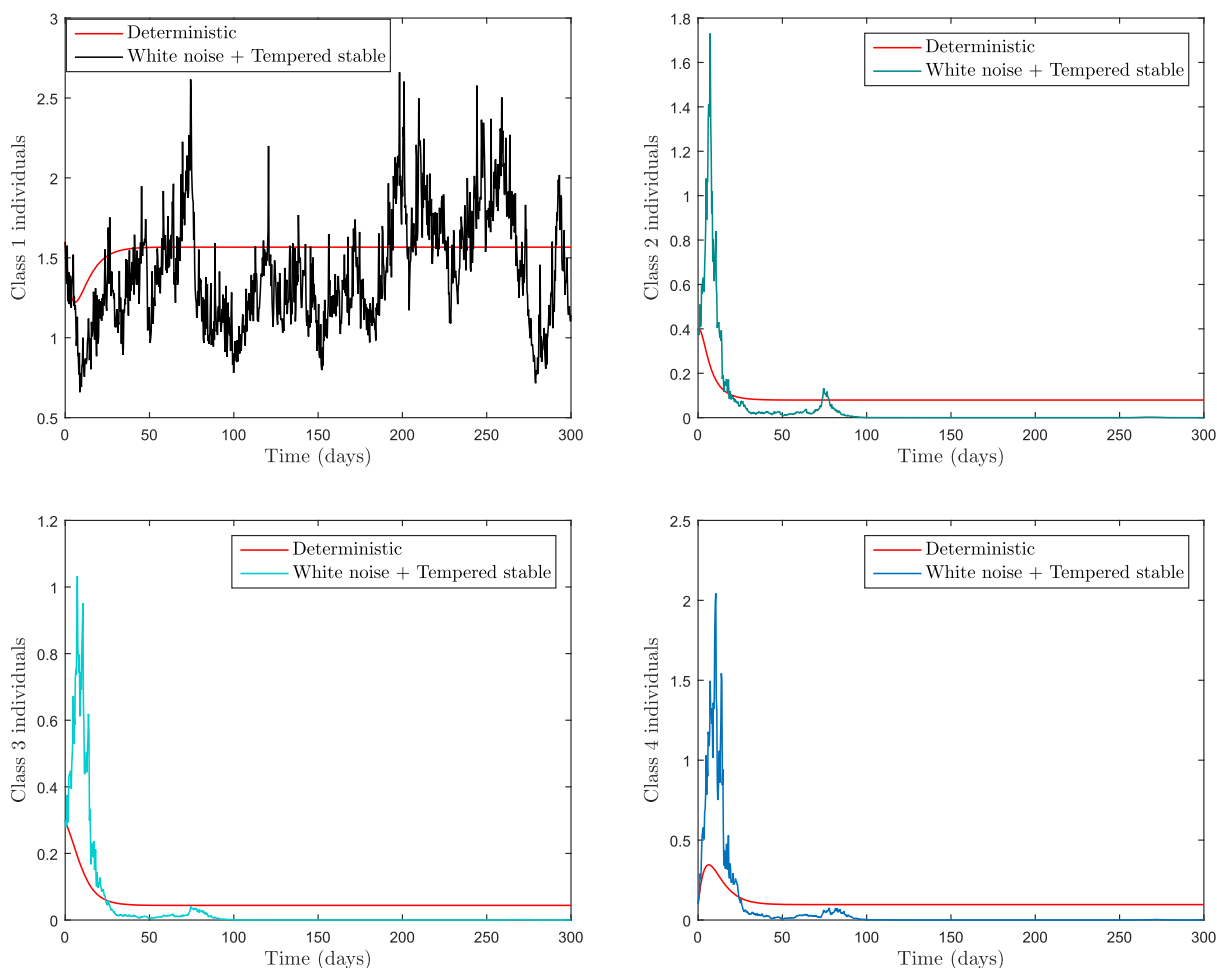
Fig. 4 Computer simulation of the solution of the probabilistic model (4.2) *with tempered process*

Declarations

Author Contributions Yassine Sabbar: Writing original draft, Formal analysis, Investiga- tion, Conceptualization, Methodology, Software. Driss Kiouach: Methodology, Investigation, Writing review and editing. S.P. Rajasekar: Conceptualization, Investigation, Writing - reviewing and editing.

Data availability The theoretical data used to support the findings of this study are already included in the article.

Code Availability The Matlab code of the numerical simulation can be requested from the corresponding author (Dr. Yassine Sabbar).

Conflict of interest The corresponding author states that there is no conflict of interest.

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