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Research article

Dynamic characteristics of expectations of short-term interest rate and a generalized Vasicek model

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A R T I C L E I N F O

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A B S T R A C T

This study's principal mathematical deduction exploits the importance of the specification of long-term equilibrium level in the mean-reversion short-term interest rate model—such as the CKLS (Chan, Karolyi, Longstaff, and Sanders) model—to describe the dynamic characteristics of future short-term interest rate expectations, especially long-horizon expectations. Therefore, we present a preferred model by introducing a stochastic long-run equilibrium level factor to extend the specifications of the Vasicek model's short-term interest rate dynamics. Using this new type of short-term interest rate as a driver we develop a two-factor affine arbitrage-free model of term structure, the generalized Vasicek mode. The empirical results show that the model not only has a good sample fitting ability to the Chinese government bond yield curve, but also has better performance in capturing the dynamic characteristics of short-term interest rates and short-term interest rate expectations. This study provides not only a promising avenue for future research on improving interest rate modeling techniques and their practical applications in the financial industry but also a new literature base for accurately identifying public expectations and explaining the underlying mechanisms of expectation changes.

1. Introduction

The term structure of interest rates describes the relationship between yields to maturity and the maturity of default-free securities and plays a very important role in economic theory. Several theoretical models have been proposed to explain this relationship and its dynamic changes over time in various ways. Vasicek [1] pioneered the method of modeling the term structure of interest rates in continuous time, assuming that short-term interest rates follow the Ornstein–Uhlenbeck process [2] and that the bond yield obtained by imposing no-arbitrage restrictions is an affine function of the short-term interest rate. Researchers have since explored the formulations of multifactor models to overcome the single-factor limitations of the Vasicek model, where the representation of the short-term rate is extended by introducing latent factors following more general diffusion processes, and the short-term rate is a linear combination of these latent factors [e.g., Refs. [3–5]].

Dai and Singleton [6] provide a thorough specification analysis of affine term structure models (ATSMs), including the Vasicek model. For computational ease, various improved classes of ATSMs with closed-form bond pricing solutions, such as by Ang and

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Piazzesi [7] and Christensen and Rudebusch [8], have gained popularity. These ATSMs can provide a better fit to the Treasury yield curve data and macroeconomic theoretical explanations of the Treasury yield curve dynamics, but the long-run equilibria of the short-term interest rate or the state factors determining the short-term interest rate implied by these ATSMs remain constant, consistent with the Vasicek model. Our mathematical deduction exploits the fact that this specification causes the long-horizon expectations of future short-term interest rates implicit in the model to lack dynamism, which is obviously inconsistent with the actual situation.

Even special research on the short-term interest rate model does not pay sufficient attention to this relationship between the long-term equilibrium level and different horizon expectations of short-term interest rates in terms of dynamic characteristics. Chan, Karolyi, Longstaff, and Sanders [9], hereafter CKLS, generalized a conclusion that the dynamics for the short-term interest rate can be nested within the following stochastic differential equation:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma r(t)^{\gamma} dW(t), \tag{1}$$

where $r(t)$ denotes the instantaneous spot rate, κ, σ, θ, and γ are nonnegative parameters, and $W(t)$ is a standard Brownian motion. κ and θ represent the speed of mean reversion and the long-run equilibrium level of the short-term interest rate, respectively. The $\sigma r(t)^{\gamma}$ the evolution of the short-term intermediate. By simply placing the appropriate restrictions on the parameter γ, different specifications of the short-term interest rate dynamics in many well-known term structure models can be obtained. For example, setting $\gamma = 0$ corresponds to Vasicek [1], setting $\gamma = 1/2$ corresponds to Cox et al. [10], and setting $\gamma = 1$ corresponds to Brennan and Schwartz [11]. CKLS's empirical comparisons indicate that the models in which there is a higher level of parameter γ were more successful in capturing the dynamics of the short-term interest rate.

However, the CKLS model only emphasizes the sensitivity of interest rate volatility to the level of the interest rate by the $\sigma r(t)^{\gamma}$ term and ignores the high persistence of interest rate volatility. To better describe the dynamics of interest rate volatility, Brenner et al. [12] and Koedijk et al. [13] remodel the volatility of short-term interest rates by considering both the level effect of CKLS and the conditional heteroskedasticity effect of the GARCH class of models. However, Andersen and Lund [14] extend the CKLS model to a multi-factor diffusion process by incorporating a stochastic volatility factor. Most researchers regard volatility as a critical component for improving the applicability of short-term interest rate models. Durham [[15] shows that allowing for additional flexibility beyond a constant drift term provides minimal benefit, but this conclusion was completely derived from the nonlinear time-varying rather than the stochastic long-run equilibrium level in the short-term interest rate model. He also overlooked the importance of the specification of long-term equilibrium level θ to describe the dynamic characteristics of short-term interest rate expectations $\mathrm{E}_t[r(s)](0 \le t < s)$. Scholars such as Yu and Phillips [16], Ahangarani [17], and Phillips and Yu [18] also discussed estimation methods for short-term interest rate models using methods such as maximum likelihood and Gaussian for continuous-time models in finance [19,20].

In this study, we present a preferred model of short-term interest rates by introducing a stochastic long-run equilibrium level factor to extend the specifications of short-term interest rate dynamics in the Vasicek model. In contrast to Chen [21] and Balduzzi et al. [22], we derive the stochastic properties of the long-run equilibrium level in our model entirely from the short-term interest rate r(t) without any additional uncertainties, consistent with the actual situation. Generally, the short-term interest rate model with a stochastic long-run equilibrium level factor can adequately describe the dynamics of short-term interest rate expectations, especially long-horizon expectations (2–5 years). With this new way of driving short-term interest rates, we further develop a two-factor affine arbitrage-free model of the term structure, the generalized Vasicek model, whose factors are the short-term interest rate and the long-run equilibrium level of the short-term interest rate. The estimation is performed by implementing the Kalman filter in the state-space model, where the above factors are considered as the two unobserved state variables. A comparison with existing models, which are empirically represented by the Vasicek model, shows that the model can provide better performance in capturing the dynamics of short-term interest rates and short-term interest rate expectations.

2. Model construction

2.1. Short-term interest rate and short-term interest rate expectation

Based on equation [1], which describes the dynamics of the short-term interest rate, we derive an analytical expression of short-term interest rate expectations and obtain the relationship between short-term interest rate expectations and the long-run equilibrium level of the short-term interest rate. As equation [1] is an It ô process, we use It ô's lemma to obtain equation [2]

$$de^{\kappa t} r(t) = \kappa\theta e^{\kappa t} dt + \sigma e^{\kappa t} r(t)^{\gamma} dW(t). \tag{2}$$

By integrating both sides of the above equation [2] for time t to $(s \ge t)$, we can get equation [3]

$$r(s) = e^{-\kappa(s-t)} r(t) + \theta\left(1 - e^{-\kappa(s-t)}\right) + \sigma \int_t^s e^{-\kappa(t-u)} r(u)^{\gamma} dW(u). \tag{3}$$

Equation [4] presents the conditional expectation of $r(s)$ at time t is

$$\mathrm{E}_t[r(s)] = e^{-\kappa(s-t)}r(t) + \theta\left(1 - e^{-\kappa(s-t)}\right) + \mathrm{E}_t[\sigma\int_t^s e^{-\kappa(t-u)}r(u)^\gamma dW(u)]. \tag{4}$$

According to the martingale property of It $\widehat{\text{o}}$ integral, as shown in equation [5], the expected value of the integral is zero

$$\mathrm{E}_t\left[\int_t^s e^{-\kappa(t-u)}r(u)^\gamma dW(u)\right] = \int_t^t e^{-\kappa(t-u)}r(u)^\gamma dW(u) = 0. \tag{5}$$

Therefore, the expression for the market's rational expectation of the short-term interest rate $r(s)$ at time t is then given by equation [6]

$$\mathrm{E}_t[r(s)] = e^{-\kappa(s-t)}r(t) + \theta\left(1 - e^{-\kappa(s-t)}\right). \tag{6}$$

The above expression demonstrates that the expectation of future short-term interest rate $\mathrm{E}_t[r(s)]$ is the weighted average of the current values of short-term interest rate $r(t)$ and long-run equilibrium level θ, and their influence weights are the functions of forecast horizon s: $g_1(s) = e^{-\kappa(s-t)}$ and $g_2(s) = 1 - e^{-\kappa(s-t)}$. The monotonicity of $g_1(s)$ and $g_2(s)$ with respect to s describes the backward- and forward-looking characteristics of public expectations; the smaller the s, the greater the influence weight of $r(t)$, and the greater the s, the greater the influence weight of θ.[1] Unfortunately, in the limit $s\to\infty$, we have $e^{-\kappa(s-t)}\to 0$ and therefore, the behavior of the infinite horizon expectation $\mathrm{E}_t[r(\infty)]$ is trivially a constant θ. Besides, the model has a smaller value of κ because of the highly persistent nature of interest rates, and it may be difficult to accommodate a considerable time variation in long-horizon expectations (2–5 years).

Moreover, ATSMs [7,8] based on Dai and Singleton's [6] theory assume that the short-term interest rate is determined by a linear combination of several observable or unobservable state factors. However, these state factors still follow a mean-reversion process in which the long-run equilibrium level of all state factors is constant. Thus, the instantaneous spot rate is defined as stated in equation [7], can be expressed as

$$r(t) = \delta_0 + \delta_1^{'}X(t), \tag{7}$$

where $\delta_0 \in R$, $\delta_1 \in R^n$. $X(t)$ is a column vector consisting of n state factors $X_1(t), X_2(t), \cdots, X_n(t)$. $X(t)$ follows an affine Gaussian process with constant volatility under the actual probability measure P as shown in equation [8], with dynamics in continuous time given by the solution to the following stochastic differential equations (SDEs):

$$dX(t) = \mathrm{K}(\theta - X(t))dt + \Sigma dW(t), \tag{8}$$

where $\mathrm{K} \in R^{n\times n}$ is a mean-reversion matrix, $\theta \in R^n$ is a vector of mean levels, $\Sigma \in R^{n\times n}$ a volatility matrix and $W(t)$ is a n-dimensional vector of independent standard Brownian motions under P. According to the It $\widehat{\text{o}}$–Deblin formula and the property of the matrix exponential function,[2] get equation [9].

$$d(e^{\mathrm{K}t}X(t)) = e^{\mathrm{K}t}\mathrm{K}X(t)dt + e^{\mathrm{K}t}dX(t) = e^{\mathrm{K}t}\mathrm{K}\theta dt + e^{\mathrm{K}t}\Sigma dW(t). \tag{9}$$

Then, the following expression [10] may be obtained by integrating both sides of the above equation [9] for time t to ($s \geq t$):

$$X(s) = e^{-\mathrm{K}(s-t)}X(t) + \left(I - e^{-\mathrm{K}(s-t)}\right)\theta + \int_t^s e^{-\mathrm{K}(s-u)}\Sigma dW(u), \tag{10}$$

According to the martingale property of the It $\widehat{\text{o}}$ integral, we can obtain the conditional expectation of $X(s)$ at time t in the same manner., presented in equation [11]

$$\mathrm{E}_t[X(s)] = e^{-\mathrm{K}(s-t)}X(t) + \left(I - e^{-\mathrm{K}(s-t)}\right)\theta. \tag{11}$$

Finally, the expectations of short-term interest rates can be expressed as equation [12]:

$$\mathrm{E}_t[r(s)] = \delta_0 + \delta_1^{'}\left(e^{-\mathrm{K}(s-t)}X(t) + \left(I - e^{-\mathrm{K}(s-t)}\right)\theta\right). \tag{12}$$

That is, in the limit $s\to\infty$, there are $e^{-K(s-t)}\to 0$, so the infinite horizon expectation $\mathrm{E}_t[r(\infty)]$ is still a constant $\delta_0 + \delta_1^{'}\theta$.

To solve these problems, we draw inspiration from Cox et al. [10,23] and describe the dynamic process of short-term interest rates as follows.

[1] *As the short-term interest rate r(t) can only fluctuate in θ around its long-term equilibrium level, the long-term equilibrium level can reflect the future direction of short-term interest rates to some extent.*

[2] *According to the definition and properties of the exponential function of the matrix, if A is a square matrix of constants and t is an independent variable, then* $e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$, $\frac{de^{At}}{dt} = Ae^{At} = e^{At}A$.

$$dr(t) = \kappa(\theta(t) - r(t))dt + \sigma dW_r(t), \tag{13}$$

$$\theta(t) = \int_0^\infty \beta e^{-\beta u} r(t-u)du, \tag{14}$$

where equation [13], $\kappa > 0$, $\sigma > 0$ is a constant and $W_r(t)$ is a standard Brownian motion. As $\int_0^\infty \beta e^{-\beta u} du = 1$, we consider that $\theta(t)$ in equation [14] is the exponentially weighted average of the past values of short-term interest rate $r(t)$. Let $\tau = t - u$, and then equation [14] can be changed to

$$\theta(t) = \beta e^{-\beta t} \int_{-\infty}^{t} e^{\beta \tau} r(\tau) d\tau. \tag{15}$$

Differentiating both sides of the equation [15], we get

$$d\theta(t) = \beta(r(t) - \theta(t))dt. \tag{16}$$

Our specifications for the long-run equilibrium level $\theta(t)$ are different from those in Chen [21] and Balduzzi et al. [22]. There is no diffusion term in equation [16]; thus, $\theta(t)$ has zero quadratic variation, which means that as long as $r(t)$ and $\theta(t)$ are known at time t, $\theta(t)$ at the next moment is determined, and there is no additional uncertainty in the model. Nevertheless, since $r(t)$ is random, $\theta(t)$ is also random over longer periods. It is in line with the reality that $\theta(t)$ is the long-run equilibrium level of the short-term interest rate, whose stochastic properties entirely derive from the short-term interest rate $r(t)$ itself, without any additional uncertainty.

We derive an expression for the expectation of short-term interest rates within the framework of this new short-term interest rate model. First, reconcile the relations defined by Equations [13,16] into a unified matrix form. This formulation culminates in equation [17]:

$$dX(t) = -\mathrm{K}_0 X(t)dt + \Sigma_0 dW(t), \tag{17}$$

where

$$X(t) = \begin{pmatrix} r(t) \\ \theta(t) \end{pmatrix}, \mathrm{K}_0 = \begin{pmatrix} \kappa & -\kappa \\ -\beta & \beta \end{pmatrix}, \Sigma_0 = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, W(t) = \begin{pmatrix} W_r(t) \\ W_\theta(t) \end{pmatrix},$$

and $W_\theta(t)$ is also a standard Brownian motion. Using a similar approach as before, we obtain equation [18]

$$X(s) = e^{-\mathrm{K}_0(s-t)} X(t) + \int_t^s e^{-\mathrm{K}_0(s-u)} \Sigma_0 dW(u), s \geq t, \tag{18}$$

and the conditional expectation of $X(s)$ at time t in equation [19]:

$$\mathrm{E}_t(X(s)) = e^{-\mathrm{K}_0(s-t)} X(t). \tag{19}$$

See Appendix 2 for the specific form of $e^{-\mathrm{K}_0(s-t)}$. The expectations of the short-term interest rate have been

$$\mathrm{E}_t[r(s)] = \left[\frac{\beta}{\beta+\kappa} + \frac{\kappa}{\beta+\kappa} e^{-(\beta+\kappa)(s-t)}\right] r(t) + \frac{\kappa}{\beta+\kappa}\left(1 - e^{-(\beta+\kappa)(s-t)}\right)\theta(t). \tag{20}$$

Obviously, in our short-term interest rate model with random long-run equilibrium level, the expectation of future short-term interest rate is determined by two factors: the current values of short-term interest rate $r(t)$ and long-term equilibrium level $\theta(t)$. In addition to $f_1(s) + f_2(s) = 1$, the factor-loading functions $f_1(s) = \frac{\beta}{\beta+\kappa} + \frac{\kappa}{\beta+\kappa} e^{-(\beta+\kappa)(s-t)}$ and $f_2(s) = \frac{\kappa}{\beta+\kappa}(1 - e^{-(\beta+\kappa)(s-t)})$ are a monotone decreasing function and a monotone increasing function of s, respectively. These characteristics are consistent with those of the old models. Whereas, in the limit $s \to \infty$, we have $e^{-\kappa(s-t)} \to 0$, and hence, the behavior of the infinite horizon expectation $\mathrm{E}_t[r(\infty)] = \frac{\beta}{\beta+\kappa} r(t) + \frac{\kappa}{\beta+\kappa}\theta(t)$. As a result, the long-horizon expectation has considerable time-varying and stochastic property yet, and may be more in line with the reality that the long-horizon expectation has backward-looking characteristics to a certain extent.

2.2. Modeling the term structure

Let $R(t, s)$ denote the yield at time t of a discount bond with maturity date s, where $t \leq s$. The yield, as expressed in equation [21], The expectation, market segmentation, and liquidity preference hypotheses conform to the following assumption.

$$R(t,s) = \mathrm{E}_t\left(\frac{1}{s-t}\int_t^s r(\tau)d\tau) + \pi(t,s,r(t)\right), \tag{21}$$

with various specifications for the function π that denote the risk premium. The dynamics of the short-term interest rate $r(t)$ can be characterized by the following SDEs.

$$dr(t) = \kappa(\theta(t) - r(t))dt + \sigma dW_r(t), \tag{22}$$

$$d\theta(t) = \beta(r(t) - \theta(t))dt. \tag{23}$$

Equation [23] establishes that both $r(t)$ and $\theta(t)$ are Markov processes as the solutions of SDEs. There must be a Borel-measurable function $f(t,r,\theta)$ of the dummy variable t, r and θ such that shown in equation [24]

$$\mathrm{E}_t\left(\frac{1}{s-t}\int_t^s r(\tau)d\tau) = f(t,r(t),\theta(t)\right), \tag{24}$$

and $R(t,s)$ is a function of t, $r(t)$ and $\theta(t)$. Once the yield-to-maturity has been computed, we can define the discount bond price as shown in equation [25]

$$P(t,s) = e^{-(s-t)R(t,s)}. \tag{25}$$

Thus, $P(t,s)$ is also a function of t, $r(t)$ and $\theta(t)$: $P(t,s) = P(t,s,r,\theta)$. From the It ô-Deblin formula, we derive the partial differential equation in equation [26]

$$dP(t,s,r,\theta) = P_t dt + P_r dr + P_\theta d\theta + \frac{1}{2}P_{rr}dr^2 + \frac{1}{2}P_{\theta\theta}d\theta^2 + P_{r\theta}drd\theta, \tag{26}$$

where $P_t = \frac{\partial P}{\partial t}$, $P_r = \frac{\partial P}{\partial r}$, $P_\theta = \frac{\partial P}{\partial \theta}$, $P_{rr} = \frac{\partial^2 P}{\partial r^2}$, $P_{\theta\theta} = \frac{\partial^2 P}{\partial \theta^2}$ and $P_{r\theta} = \frac{\partial^2 P}{\partial r \partial \theta}$ are some partial derivatives of the function $P(t,s,r,\theta)$, and $dr^2 = \sigma^2 dt$, $d\theta^2 = 0$, $drd\theta = 0$. Substituting equations [22,23] yields

$$dP(t,s,r,\theta) = P\mu(t,s,r,\theta)dt - P\rho(t,s,r,\theta)dW_r(t), \tag{27}$$

which

$$\mu(t,s,r,\theta) = \frac{1}{P}\left[P_t + \kappa(\theta(t) - r(t))P_r + \beta(r(t) - \theta(t))P_\theta + \frac{1}{2}\sigma^2 P_{rr}\right],$$

$$\rho(t,s,r,\theta) = -\frac{1}{P}\sigma P_r.$$

Further, the differential form of the discounting process $D(t)$ is:

$$dD(t) = -r(t)D(t)dt, \tag{28}$$

Using Ito's product rule, the discounted price of the bond $D(t)P(t,s,r,\theta)$ satisfies the following equation (29).

$$dD(t)P(t,s,r,\theta) = D(t)dP(t,s,r,\theta) + P(t,s,r,\theta)dD(t) + dP(t,s,r,\theta)dD(t). \tag{29}$$

The substitution of equation [27] and equation [28] leads to equation (30)

$$dD(t)P(t,s,r,\theta) = -D(t)P(t,s,r,\theta)\rho(t,s,r,\theta)\left[-\frac{\mu(t,s,r,\theta) - r(t)}{\rho(t,s,r,\theta)}dt + dW_r(t)\right]. \tag{30}$$

If there is a market price of risk $\lambda(t,r,\theta)$ independent of the bond's maturity, then

$$\lambda(t,r,\theta) = \frac{\mu(t,s,r,\theta) - r(t)}{\rho(t,s,r,\theta)}. \tag{31}$$

According to Gosanov's theorem, the model has a risk-neutral probability measure $\widetilde{P}$ under which the discounted price of any maturity bond $D(t)P(t,s,r,\theta)$ is a martingale, and it does not admit arbitrage. We assume that $\lambda(t,r,\theta)$ exists and is constant [1], that is $\lambda(t,r,\theta) = \lambda$. Substituting the expressions for $\mu(t,s,r,\theta)$, $\rho(t,s,r,\theta)$ in equation [27] into equation (31), we collate get equation (32)

$$P_t + [\kappa(\theta(t) - r(t)) + \lambda\sigma]P_r + \beta(r(t) - \theta(t))P_\theta + \frac{1}{2}\sigma^2 P_{rr} - rP = 0. \tag{32}$$

Solving this partial differential equation (see Appendix 1 for the detailed solution process) yields to seen in equation (33)

$$P(t,s,r,\theta) = exp\{-A(t,s) - B(t,s)r(t) - C(t,s)\theta(t)\}, \tag{33}$$

where

$$A(t,s) = \left[\frac{\lambda\sigma\kappa}{(\beta+\kappa)^2} - \frac{\sigma^2\kappa^2}{2(\beta+\kappa)^4}\right](s-t) + \left[\frac{\lambda\sigma\beta}{2(\beta+\kappa)} - \frac{\sigma^2\beta\kappa}{2(\beta+\kappa)^3}\right](s-t)^2 - \frac{\sigma^2\beta^2}{6(\beta+\kappa)^2} - t)^3 + \left[\frac{\lambda\sigma\kappa}{(\beta+\kappa)^3} - \frac{\sigma^2\beta\kappa}{(\beta+\kappa)^5} - \frac{\sigma^2\kappa^2}{(\beta+\kappa)^5}\right]e^{-(\beta+\kappa)(s-t)} + \frac{\sigma^2\kappa^2}{4(\beta+\kappa)^5}e^{-2(\beta+\kappa)(s-t)} - \frac{\sigma^2\beta\kappa}{(\beta+\kappa)^4}(s-t)e^{-(\beta+\kappa)(s-t)} - \frac{\lambda\sigma\kappa}{(\beta+\kappa)^3} + \frac{\sigma^2\beta\kappa}{(\beta+\kappa)^5} + \frac{3\sigma^2\kappa^2}{4(\beta+\kappa)^5},$$

$$B(t,s) = \frac{\beta}{\beta+\kappa}(s-t) + \frac{\kappa}{(\beta+\kappa)^2}\left(1 - e^{-(\beta+\kappa)(s-t)}\right),$$

$$C(t,s) = \frac{\kappa}{\beta+\kappa}(s-t) - \frac{\kappa}{(\beta+\kappa)^2}\left(1 - e^{-(\beta+\kappa)(s-t)}\right).$$

Finally, the discount bond yields as shown in equation (34)

$$R(t,s) = \frac{A(t,s)}{s-t} + \frac{B(t,s)}{s-t}r(t) + \frac{C(t,s)}{s-t}\theta(t), \tag{34}$$

which satisfies the specifications of arbitrage-free affine term structure models. We call this term the generalized Vasicek model, constructed following technical studies such as Marin et al. [24,25].

3. Empirical test

3.1. Data

This study uses daily closing price data from Wind's Treasury Bond Benchmark Yield (CGBB) (for specific algorithms, refer to Wind's Description of the Treasury Bond Benchmark Yield Algorithm) for empirical research. From August 9, 2010 to August 8, 2016, 1498 sets of observations and 1492 sets were retained, after six outliers were excluded. Each set of observation data includes the Treasury yield for four periods: 3-month, 6-month, 1-year, and 5-year, and all rates are compounded continuously on an annualized basis. Missing data were not treated because of the high frequency and small percentage of missing data in the overall dataset.

3.2. Estimation method

To estimate the model parameters, this study transforms the interest rate term structure model constructed in continuous time, as described above, into a state-space form, using the short-term interest rate and short-term interest rate regression mean as unobservable state variables and the discounted bond yield as an observable state variable.

Many scholars have attempted to transform the diffusion process similar to that in equation [22] into a discrete time form. Brennan et al. [12], Dietrich-Campbell et al. [26], and Sanders et al. [28] use an approximate discretization method. Following this approximate discretization, the discrete-time form of equation [13] is:

$$r_t - r_{t-1} = \kappa(\theta_{t-1} - r_{t-1}) + v_t, v_t \sim N(0, \sigma^2), \tag{35}$$

$t = 1, 2\cdots, n$ denotes the point in time at which the sample is located. If we treat equation [22] using the Euler discretization technique, then we get equation (36)

$$r_t - r_{t-1} = \kappa(\theta_{t-1} - r_{t-1}) \cdot \Delta t + v_t, v_t \sim N(0, \sigma^2 h), \tag{36}$$

where Δt denotes the time interval between observations. If the time unit chosen by the model is year and the sample data are quarterly data, then Δt should be equal to 1/4, which shows that equation (35) holds only if it is ensured that the time unit selected by the model is the same as that of the sample. The Eulerian method is also an approximate discretization method, and the approximation error increases as the sample time interval increases. Therefore, according to the calculation results in equation [18], we can transform equations [22,23] into a discrete-time form as follows.

$$X_t = e^{-K_0\Delta t}X_{t-1} + v_t, \tag{37}$$

where

$$X_t = \begin{pmatrix} r_t \\ \theta_t \end{pmatrix} = X(t), X_{t-1} = \begin{pmatrix} r_{t-1} \\ \theta_{t-1} \end{pmatrix} = X(t-\Delta t),$$

$$v_t = \int_{t-\Delta t}^{t} e^{-K_0(t-u)} \Sigma dW(u) \sim i.i.d.N_2\left(0, \int_0^{\Delta t} e^{-K_0 u} \Sigma_0 \Sigma_0' e^{-K_0' u} du\right)$$

Equation (A3) is the transfer equation for the state-space model. See Appendix 2 for the explicit expressions of $e^{-K_0\Delta t}$ and $\int_0^{\Delta t} e^{-K_0 u}\Sigma_0\Sigma_0' e^{-K_0' u} du$.

Suppose that at each time t,t $=1,2\cdots$,n, there are M discount bonds with maturitie $T_1, T_2, \cdots, T_M$. If the discount bond yield $R(t, T_i)$ is taken as the observable variable and there are measurement errors $\varepsilon_{it} \sim$ i.i.d.$N(0, \sigma_i^2)$, i $=1,\cdots$,Mi $=1,2,\cdots$,M, which are independent of each other, then according to equation (34), the following measurement equation is obtained:

$$R_{it} = \frac{A(t, t+T_i)}{T_i} + \frac{B(t, t+T_i)}{T_i} r_t + \frac{C(t, t+T_i)}{T_i} \theta_t + e_{it}, i = 1, 2, \cdots, M, \tag{38}$$

where $R_{it} = R(t, T_i)$, $r_t = r(t)$, $\theta_t = \theta(t)$. Assuming that $\varepsilon_t = (\varepsilon_{1t}, \cdots, \varepsilon_{Mt})'$ is independent o v_t, the covariance matrix between ε_t and v_t can be written as follows:

$$\begin{pmatrix} v_t \\ \varepsilon_t \end{pmatrix} \sim N\left(\begin{matrix} \int_0^{\Delta t} e^{-K_0 u}\Sigma_0\Sigma_0' e^{-K_0' u} du & 0 \\ 0 & \Phi \end{matrix} \right), \Phi = \begin{pmatrix} \sigma^2(T_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma^2(T_N) \end{pmatrix}. \tag{39}$$

The state-space model consists of equation (39)* and equation (38), which can be estimated by implementing the Kalman filter to compute the estimators of the unobserved variables and the log-likelihood function via prediction error decomposition; this process has been discussed by Kim and Nelson [27]. We use year as the unit of time; thus, the bond maturity T_i is expressed in years, for example, six months is expressed as 1/2 year. The estimates of all fixed and unobserved variables can be interpreted as annualized values. As the sample data used in the present paper are daily data, the observation time interval Δt is equal to 1/365 year.

3.3. Estimated results and analysis

We completed the estimation using the Gaussian programming language. Table 1 reports the estimates of the fixed parameters in the generalized Vasicek model. The results show that all parameters are significantly nonzero. The continuous-time mean-reversion parameters $\kappa = 1.82$ and $\beta = 1.61$ are used to compute the one-day conditional mean-reversion matrix $e^{-K_0\Delta t}$ in the VAR structure of equation (35):

$$e^{-K_0\Delta t} = \begin{pmatrix} 0.9950 & 0.0050 \\ 0.0044 & 0.9956 \end{pmatrix},$$

reflect the high persistence of short-term interest rates. Besides, the market price of risk $\lambda = 1.33$ and the volatility of short-term interest rate $\sigma = 0.7580$. In equation (31), $\mu(t, s, r, \theta)$ is the mean rate of the discount bond yield, $r(t)$ is the short-term riskless rate, and $\lambda\rho(t, s, r, \theta) = \mu(t, s, r, \theta) - r(t)$ indicates the instantaneous risk premium. Therefore, $\lambda\rho(t, s, r, \theta) = -\lambda\sigma P_r/P > 0$ indicates that the Chinese Treasury bond market exists in a positive risk premium during the sample period. The standard deviations of the measurement errors of the 3-month, 6-month, 1-year, and 5-year treasury bond rates are $\sigma(1/4) = 0.3832$, $\sigma(1/2) = 0.1538$, $\sigma(1) = 0.1306$, $\sigma(5) = 0.2006$, respectively. This shows that the non-determinacy of the observed Chinese Treasury bond yields gradually decreased with maturity.

Fig. 1 illustrates the interaction between short-term interest rates and long-run equilibrium levels over time. First, short-term interest rates always move up and down around the long-run equilibrium level and tend to converge towards the long-run

Table 1
Estimates of the generalized Vasicek model parameters.

Parameters	Estimated Value	Standard Deviation	t-value
κ	1.82	0.1986	9.18
β	1.61	0.0882	18.26
λ	1.33	0.0103	129.36
σ	0.7580	0.0344	22.03
$\sigma(1/4)$	0.3832	0.0078	49.36
$\sigma(1/2)$	0.1538	0.0037	41.66
$\sigma(1)$	0.1306	0.0033	39.61
$\sigma(5)$	0.2006	0.0045	44.62
log-likelihood	704.80		

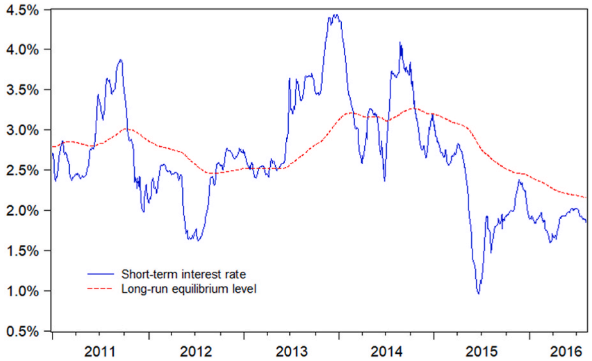

Fig. 1. Estimates of the short-term interest rate and the long-run equilibrium level of short-term interest rate.

equilibrium level. This cyclical pattern underscores the underlying economic theory that short-term interest rates converge to a long-run equilibrium, reflecting the self-correcting mechanism of the market to adjust to underlying economic fundamentals over time.

Second, the volatility of short-term interest rates is significantly higher than that of long-run equilibrium levels. When short-term interest rates start to fall, the long-run equilibrium level follows. When short-term interest rates start to rise, the long-run equilibrium level follows. Meanwhile, the average yield of short-term interest rates is important in guiding short-term interest rate expectations. This difference in volatility indicates how short-term interest rates are susceptible to direct economic forces such as policy changes, investor sentiment, and global events, which can lead to rapid fluctuations. By contrast, the long-term equilibrium level is more inert and adjusts to changes more gradually, suggesting that it acts as a proxy for the underlying economic trends. This lagged movement

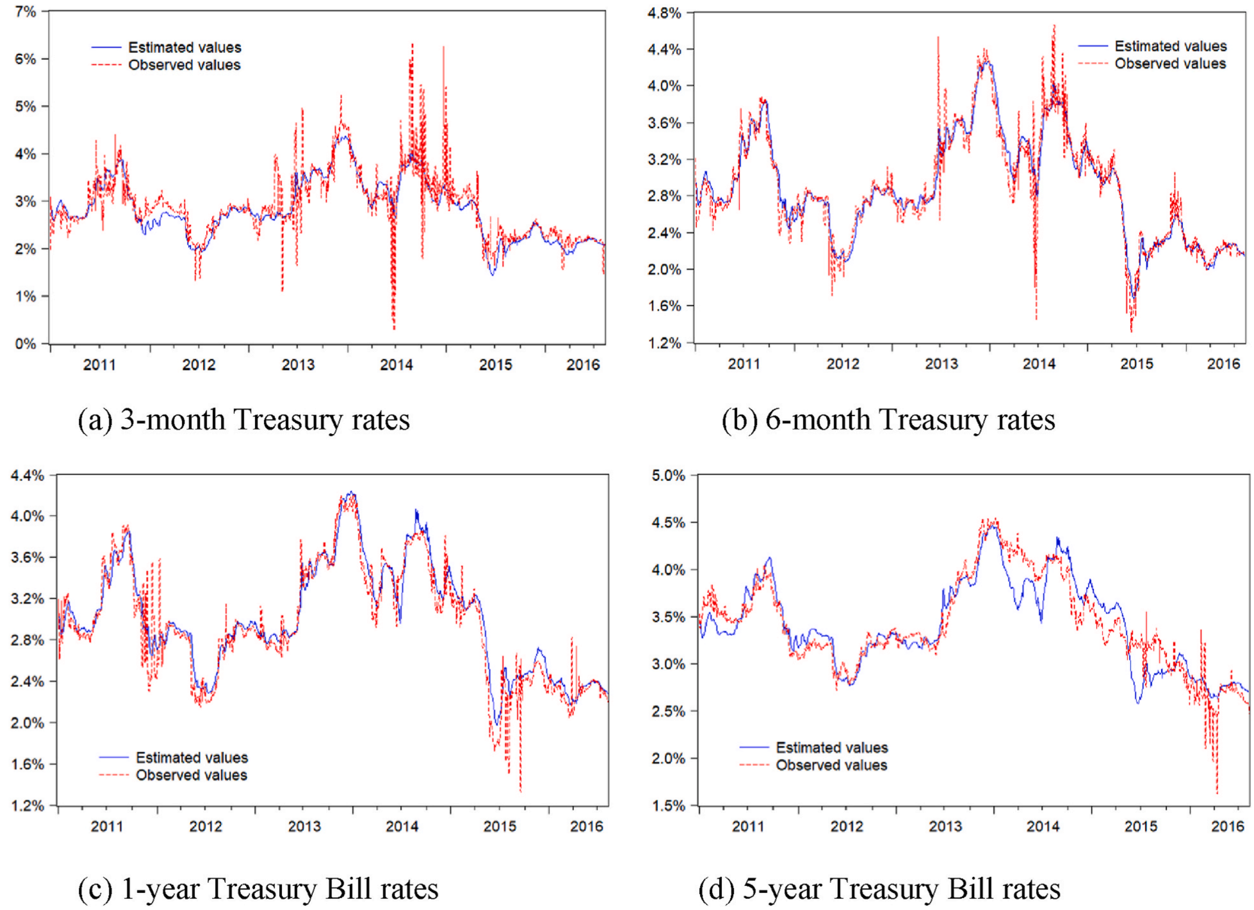

Fig. 2. Comparison of estimated and actual observed Treasury rates. (a) 3-month Treasury rates (b) 6-month Treasury rates (c) 1-year Treasury Bill rates (d) 5-year Treasury Bill rates.

suggests that the equilibrium level is not only smoother but also emerges as a follower rather than a leader in the relationship, adjusting its path in response to movements in short-term interest rates.

The dynamic process in Fig. 1 confirms our proposed theoretical framework, in which the short-term interest rate is not only influenced by the long-run equilibrium level but also acts as a weather vane for the long-run equilibrium level. The behavior of these two state variables highlights the intrinsic link between current economic conditions and long-term expectations. When the short-term interest rates begin to fall or rise, the long-term equilibrium level responds with a time lag, representing a form of economic inertia or momentum.

However, the upward and downward ranges of the long-term equilibrium level are smaller than those of short-term interest rates. The empirical results confirm the specifications of the two-factor dynamic process used in this study.

The following facts also exist. When the short-term interest rate is above the long-run equilibrium level, regardless of whether it is on an upward or downward trend, it eventually declines and converges toward the long-run equilibrium level. Conversely, when the short-term interest rate is below the long-term equilibrium, it eventually converges. Again, when the time interval Δt is sufficiently small, then equation [16] can be completely rewritten as

$$\theta(t+\Delta t)=\theta(t)+\beta\Delta t(r(t)-\theta(t)). \tag{40}$$

If $\theta(t)$ is taken as $\mathrm{E}_{t-\Delta t}[r(t)]$, then the above equation (40) is precisely consistent with the theoretical model of adaptive expectations, which leads us to the formulation of equation (41)

$$\mathrm{E}_t[r(t+\Delta t)]=\mathrm{E}_{t-\Delta t}[r(t)]+\beta\Delta t(r(t)-\mathrm{E}_{t-\Delta t}[r(t)]). \tag{41}$$

When a forecast error is observed, the current expectation can be adjusted by the previous forecast error, and a new expectation emerges. Thus, the long-run equilibrium level of short-term interest rates contains information on short-term interest rate expectations, which can be used as a basis for predicting the future paths of short-term interest rates.

By substituting the estimates of the fixed and unobserved variables into equation (34), we can calculate the model-estimated term structure of the interest rate. As shown in Fig. 2, we estimate the 3-month (Fig. 2(a)), 6-month (Fig. 2(b)), 1-year (Fig. 2(c)), and 5-year Treasury bill rates (Fig. 2(d)) separately for comparison with the actual observed Treasury bill rates. The estimated (solid line) and observed (dashed line) values generally follow the same trend over time. This finding suggests that the model effectively captures the overall movement in treasury yields. The estimates follow the same trend as the actual observations, indicating an excellent in-sample fit.

Fig. 2(a) shows a strong correlation between the estimated and observed rates, and the model captures the short-term fluctuations well. This suggests that in a sample, the model can accurately track the most immediate responses to changes in economic indicators and policies that affect short-term interest rates. Fig. 2(b) on 6-month rates also shows a good fit between the estimates and observations but with a slight bias, suggesting that the model may not be able to fully capture all market dynamics at this slightly longer maturity. Nevertheless, the model captures the general trend well, indicating its validity at the short end of the yield curve.

Fig. 2(c) on the 1-year rate shows good overall agreement between the model's estimates and the market-observed rate, with the two lines tracking each other closely throughout the period. Some differences are evident, particularly during periods of market stress or economic uncertainty, but the model maintains a good fit, although the fit declines relative to shorter-term Treasury bills. Fig. 2(d) shows that the difference between estimated and observed rates is larger for 5-year Treasury bills than for shorter maturities. This reflects the challenges of forecasting long-term interest rates, which are influenced by economic and financial conditions.

Nevertheless, the model follows a general downward trend in the observed rates, suggesting a reasonable medium-term fit, albeit with less precision than the short-term model, which fits short-term Treasury yields better than medium- and long-term Treasury yields, a finding supported by the results for the root mean square error.

3.4. Dynamic features of short-term interest rate expectations

As with the original Vasicek model, numerous other classical ATSMs share the characteristics of time-invariant long-run equilibrium levels. We theoretically show that this specification is a major cause of underestimating the volatility of short-term interest rate expectations in the model. Thus, an empirical comparison with the Vasicek model is sufficient to illustrate that our model is preferable in capturing the dynamics of long-horizon expectations of the short-term interest rate. As mentioned, the original Vasicek model

Table 2
Estimates of the original Vasicek mode parameters.

parameters	Estimated value	Standard deviation	*t*-value
κ	0.1347	0.0082	16.47
θ	1.11	2.06	0.5407
λ	1.74	0.5352	3.25
σ	0.5170	0.0114	45.53
$\sigma(1/4)$	0.3786	0.0075	50.47
$\sigma(1/2)$	0.1614	0.0037	43.40
$\sigma(1)$	0.1194	0.0029	41.03
$\sigma(5)$	0.2245	0.0048	46.72
log-likelihood	1898.00		

assumes that the short-term interest rate $r(t)$ follows the Ornstein–Uhlenbeck process which is expressed in Equation (42) as:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma r dW(t). \tag{42}$$

With arbitrage-free constraints, the yield $R(t,s)$ at time t of a discount bond with maturity s, $t \leq s$, shown in equation (43), as:

$$R(t,s) = \left(\theta + \frac{\lambda\sigma}{\kappa} - \frac{\sigma^2}{2\kappa^2}\right) + \frac{\sigma^2}{4\kappa^3(s-t)}\left(1 - e^{-\kappa(s-t)}\right)^2 - \frac{1}{\kappa(s-t)}\left(1 - e^{-\kappa(s-t)}\right)\left(\theta + \frac{\lambda\sigma}{\kappa} - \frac{\sigma^2}{2\kappa^2} - r(t)\right). \tag{43}$$

This is the same as our preferred model with $\beta = 0$, so the original one-factor Vasicek model is our generalized model's subclass. State-space models and Kalman filtering were also applied to estimate this model. Table 2 reports the estimation results.

The estimation results show that all parameters are significantly non-zero except for the long-run equilibrium level θ. The mean reversion coefficient $\kappa = 0.1347$ is much smaller than the generalized model estimate of 1.82, and the volatility $\sigma = 0.5170$ is also smaller than the generalized model estimate of 0.7580. Thus, the original Vasicek model overestimates the persistence of short-term interest rates and underestimates volatility. As the original Vasicek model sets the long-run equilibrium level θ to a constant, it subjectively rejects the dynamic property of θ It has the effect of imposing the persistence of the short-term interest rate, which should have been captured by the dynamic process of θ, on that of r.

Fig. 3 also compares the short-term interest rates estimated using the original Vasicek model with those estimated using the generalized Vasicek model. The short-term interest rate trend estimated by the two models is consistent, but the distance between the neighboring peaks and troughs of the oscillation estimated by the original Vasicek model is shorter, indicating less volatility and persistence. This empirical result confirms the validity of the analysis.

Using equations [6,20], the original and generalized Vasicek models can be used to estimate the expectations of short-term interest rates at different horizons. Fig. 4 plots (a), (b), (c), and (d), comparing the expectations of short-term interest rates for different time horizons estimated by the original and generalized Vasicek models. Fig. 4(a) 1-month horizon expectation, Fig. 4 (b) 6-month horizon expectation Fig. 4 (c) 1-year horizon expectation Fig. 4 (d) 2-year horizon expectation. First, Fig. 4(a) shows the one-month horizon expectations for which the estimates of both the Vasicek models are very close to each other, following the same trend and being close from 2011 to 2016. This finding suggests that the two models perform similarly in terms of short-term expectations. For 6-month expectations, some differences between the two models began to emerge, but owing to interruptions, the figures have not been fully checked. Typically, as the time horizon increases, the differences in model assumptions begin to have a more pronounced effect on the estimates.

However, expectations at the 1-year horizon are shown in Fig. 4(c), where clear differences between the models begin to emerge. The original Vasicek model produces much lower and less volatile estimates than the generalized model, suggesting that it may not fully capture the dynamics that affect long-term interest rates. The estimates for the 2-year horizon show that the original Vasicek model produces an almost flat line, suggesting little change in short-term interest rates predictable over this period. By contrast, the estimates from the generalized model are much more volatile, reflecting the dynamics of the economy over the 2-year horizon.

Overall, the estimates from the original Vasicek model become progressively less volatile and more linear as the expected maturity increases from one month to two years, suggesting a lack of responsiveness to factors that typically cause changes in long-term interest rates. By contrast, the generalized Vasicek model estimates retain their volatility and do not show the same degree of flattening, suggesting that the model is more sensitive to the dynamics affecting interest rates at such maturities. This comparison highlights the potential limitations of the original Vasicek model in capturing long-term expectations, and suggests that the generalized model may reflect expectations of future short-term interest rates more accurately and dynamically.

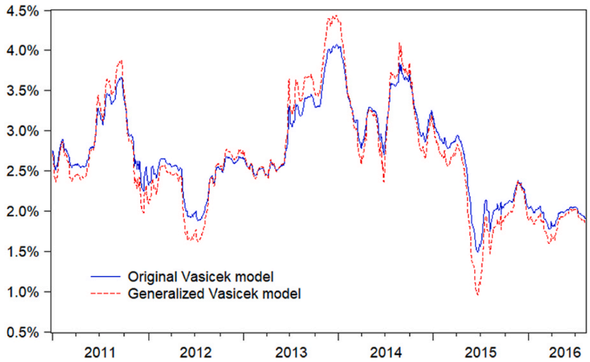

Fig. 3. Comparison of short-term interest rates estimated by the original Vasicek model and by the generalized Vasicek model.

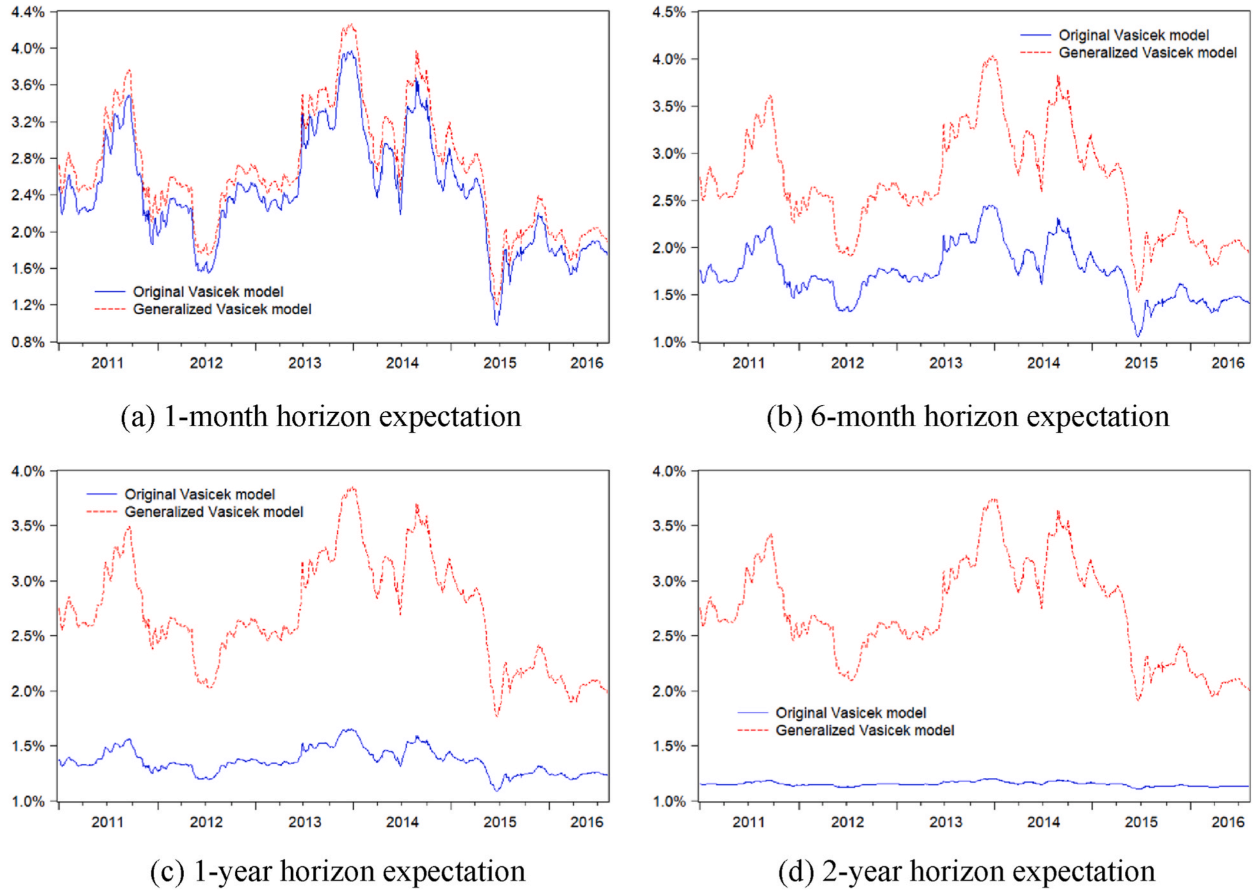

Fig. 4. Comparison of the expectations of short-term interest estimated by the original Vasicek model and by the generalized Vasicek. (a) 1-month horizon expectation (b) 6-month horizon expectation (c) 1-year horizon expectation (d) 2-year horizon expectation.

4. Conclusion

Arbitrage-free affine term structure models are currently the most effective in explaining the relationship between yields to maturity, the maturities of non-default risk bonds, and their dynamic changes over time. In this model class, short-term interest rate dynamics are expressed as mean reversion diffusion processes, and the long-run equilibrium level of the short-term interest rate is constrained to be constant. Through mathematical deduction, we find that the invariant long-run equilibrium level causes expectations of short-term interest rates to lack dynamism. Thus, inspired by Cox et al. [10,23], we ascribe the stochastic property to the long-run equilibrium of the short-term interest rate in the Vasicek model.

In contrast to Chen [21] and Balduzzi et al. [22], the dynamics of the long-run equilibrium level of the short-term interest rate are derived entirely from the short-term interest rate, without introducing additional uncertainties, in accordance with reality. With this new way of driving the short-term interest rate, we develop a two-factor affine arbitrage-free model of the term structure—the generalized Vasicek model, whose state factors are the short-term interest rate and the long-run equilibrium level of the short-term interest rate—and the original one-factor Vasicek model, a subclass of our generalized model.

The empirical results confirm that the model can better reflect the dynamic properties of the short-term interest rate and long-run equilibrium level, as well as the mean reversion of the short-term interest rate. Additionally, the long-run equilibrium level contains information on short-term interest rate expectations, which may provide a basis for predicting the future paths of short-term interest rates. As the original Vasicek model has the same specifications for the long-run equilibrium level of short-term interest rates as numerous other classical ATSMs, comparing the empirical performance of this representative model with our generalized Vasicek model, we find that the invariant long-run equilibrium level of short-term interest rates causes exaggerated persistence of short-term interest rates and underestimation of short-term interest rate volatility. Moreover, our generalized Vasicek model can provide better performance in capturing the dynamics of short-term interest rate and short-term interest rate expectations, especially long-horizon (2–5 years) expectations.

However, this study does not establish a link between short-term interest rates, especially the long-run equilibrium level of short-term interest rates, and other macroeconomic variables. Therefore, in future research, we propose to build on the idea of Ang and Piazzesi [7] to introduce macro factors into the generalized Vasicek model, which can not only better represent the formation mechanism of short-term interest rate expectations and their influencing factors, but also further improve the model's forecasting performance.

Data availability statement

Data will be made available on request.

CRediT authorship contribution statement

Yu Guan: Writing – original draft, Methodology, Data curation, Conceptualization. **Zhongzheng Fang:** Writing – review & editing, Writing – original draft, Methodology, Formal analysis, Conceptualization. **Xingshuai Wang:** Writing – original draft, Software, Methodology, Formal analysis. **Xi Wang:** Software, Methodology, Conceptualization. **Ting Yu:** Project administration, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix 1

We initially assumed and subsequently verified that the solution of partial differential equation (32) has the following form:

$$P(t,s,r,\theta) = exp\{-A(t,s) - B(t,s)r(t) - C(t,s)\theta(t)\}, \tag{A1}$$

In equation (A1), for the nonrandom functions $A(t,s)$, $B(t,s)$ and $C(t,s)$ to be determined. These are functions of the current time $t \geq 0$, and the maturity T is fixed. In this case,

$$P_t = (-A'(t,s) - B'(t,s)r(t) - C'(t,s)\theta(t))P(t,s,r,\theta),$$

$$P_r = -B(t,s)P(t,s,r,\theta),$$

$$P_\theta = -C(t,s)P(t,s,r,\theta),$$

$$P_{rr} = B^2(t,s)P(t,s,r,\theta),$$

where $A'(t,s) = \frac{dA(t,s)}{dt}$, $B'(t,s) = \frac{dB(t,s)}{dt}$ and $C'(t,s) = \frac{dC(t,s)}{dt}$. Substitution into the partial differential equation (32) gives

$$\Big[(-B'(t,s) + \kappa B(t,s) - \beta C(t,s) - 1)r(t) + (-C'(t,s) - \kappa B(t,s) + \beta C(t,s))\theta(t) - A'(t,s) - \lambda\sigma B(t,s) + \frac{1}{2}\sigma^2 B^2(t,s)\Big]P(t,s,r,\theta) = 0. \tag{A2}$$

As the equation must hold for all $r(t)$ and $\theta(t)$, both the term that multiplies $r(t)$ and the term that multiplies $\theta(t)$ must be zero. Otherwise, changing the values of $r(t)$ and $\theta(t)$ would change the value of the left-hand side of the equation, and hence it could not always be equal to zero. This gives us two ordinary differential equations in t:

$$-B'(t,s) + \kappa B(t,s) - \beta C(t,s) - 1 = 0, \tag{A3}$$

$$-C'(t,s) - \kappa B(t,s) + \beta C(t,s) = 0. \tag{A4}$$

Setting these two terms equal to zero in equation (A2), we obtain

$$A'(t,s) = -\lambda\sigma B(t,s) + \frac{1}{2}\sigma^2 B^2(t,s). \tag{A5}$$

The terminal condition $P(s,s) = e^{-(s-s)R(s,s)} = 1$ must hold for the values of $r(s)$ and $\theta(s)$ in arbitrary $s > 0$, and this implies that $A(s,$

$s) = B(s,s) = C(s,s) = 0$. Equations (A3), (A4) and (A5) and these terminal condition provide enough information to determine the functions $A(t,s)$, $B(t,s)$ and $C(t,s)$ for $0 \le t \le s$. They are given by equation (A6) (A7) (A8)

$$A(t,s) = \left[\frac{\lambda\sigma\kappa}{(\beta+\kappa)^2} - \frac{\sigma^2\kappa^2}{2(\beta+\kappa)^4}\right](s-t) + \left[\frac{\lambda\sigma\beta}{2(\beta+\kappa)} - \frac{\sigma^2\beta\kappa}{2(\beta+\kappa)^3}\right](s-t)^2 -$$

$$\frac{\sigma^2\beta^2}{6(\beta+\kappa)^2}(s-t)^3 + \left[\frac{\lambda\sigma\kappa}{(\beta+\kappa)^3} - \frac{\sigma^2\beta\kappa}{(\beta+\kappa)^5} - \frac{\sigma^2\kappa^2}{(\beta+\kappa)^5}\right]e^{-(\beta+\kappa)(s-t)} +$$

$$\frac{\sigma^2\kappa^2}{4(\beta+\kappa)^5}e^{-2(\beta+\kappa)(s-t)} - \frac{\sigma^2\beta\kappa}{(\beta+\kappa)^4}(s-t)e^{-(\beta+\kappa)(s-t)} - \frac{\lambda\sigma\kappa}{(\beta+\kappa)^3} + \frac{\sigma^2\beta\kappa}{(\beta+\kappa)^5} + \frac{3\sigma^2\kappa^2}{4(\beta+\kappa)^5}, \tag{A6}$$

$$B(t,s) = \frac{\beta}{\beta+\kappa}(s-t) + \frac{\kappa}{(\beta+\kappa)^2}\left[1 - e^{-(\beta+\kappa)(s-t)}\right], \tag{A7}$$

$$C(t,s) = \frac{\kappa}{\beta+\kappa}(s-t) - \frac{\kappa}{(\beta+\kappa)^2}\left[1 - e^{-(\beta+\kappa)(s-t)}\right]. \tag{A8}$$

Appendix 2

Theorem 1. *Let the characteristic polynomial of A which is a square matrix of order n is. As shown in* equation (A9)

$$c(\lambda) = det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0. \tag{A9}$$

If $D = d/dt$, then each element of the matrix exponential function e^{At} satisfies the linear differential equation: $c(D)y = 0$, and $\Phi(t) = e^{At}$ is the unique solution of the linear matrix differential equations as shown in equation (A10), equation (A11)

$$\Phi^{(n)} + c_{n-1}\Phi^{(n-1)} + \cdots + c_1\Phi' + c_0\Phi = 0, \tag{A10}$$

$$\Phi(0) = I, \Phi'(0) = A, \cdots, \Phi^{(n-1)} = A^{n-1}. \tag{A11}$$

If

$$A = -\mathrm{K}_0 = \begin{pmatrix} -\kappa & \kappa \\ \beta & -\beta \end{pmatrix},$$

then

$$det(\lambda I - A) = \begin{vmatrix} \lambda+\kappa & -\kappa \\ -\beta & \lambda+\beta \end{vmatrix} = \lambda^2 + (\beta+\kappa)\lambda.$$

According to Theorem 1, *$\Phi(t) = e^{At}$ is the unique solution of the linear matrix differential equation*

$$\Phi''(t) + (\beta+\kappa)\Phi'(t) = 0, \Phi(0) = I, (0) = A. \tag{A12}$$

As characteristic values of the matrix A are 0 and $-(\beta+\kappa)$, the general solution of the differential equation (A12) *is:*

$$\Phi(t) = C_1 + C_2e^{-(\beta+\kappa)t}, \tag{A13}$$

where C_1 and C_2 are two constant matrices of order n. The boundary condition $\Phi(0) = I$ and $\Phi'(0) = A$ imply $C_1 = I + \frac{1}{\beta+\kappa}A$ and $C_2 = -\frac{1}{\beta+\kappa}A$. According to Theorem 1, *substituting into* equation (A13) *yields* equation (A14).

$$e^{At} = e^{-\mathrm{K}_0t} = \begin{pmatrix} \frac{\beta}{\beta+\kappa} + \frac{\kappa}{\beta+\kappa}e^{-(\beta+\kappa)t} & \frac{\kappa}{\beta+\kappa}\left(1 - e^{-(\beta+\kappa)t}\right) \\ \frac{\beta}{\beta+\kappa}\left(1 - e^{-(\beta+\kappa)t}\right) & \frac{\kappa}{\beta+\kappa} + \frac{\beta}{\beta+\kappa}e^{-(\beta+\kappa)t} \end{pmatrix}. \tag{A14}$$

Since $e^{A't} = (e^{At})'$ that is derived from the definition of matrix exponential function, we obtain equation (A15)

$$\int_{0}^{\Delta t} e^{-\mathrm{K}_0 u}\Sigma_0\Sigma_0{}' e^{-\mathrm{K}_0{}' u} du = \int_{t-h}^{t} e^{-\mathrm{K}_0 u}\Sigma_0\Sigma_0{}'\left(e^{-\mathrm{K}_0 u}\right)' du. \tag{A15}$$

The details of $e^{-\mathrm{K}_0 u}$ *and* Σ_0 *are known, so simple matrix and integral computations can give* equation (A16)

$$\int_{0}^{\Delta t} e^{-\mathrm{K}_0 u}\Sigma_0\Sigma_0{}' e^{-\mathrm{K}_0{}' u} du = \begin{pmatrix} a_{11}(\Delta t) & a_{12}(\Delta t) \\ a_{21}(\Delta t) & a_{22}(\Delta t) \end{pmatrix}\sigma^2, \tag{A16}$$

where

$$a_{11}(\Delta t) = \frac{\beta^2}{(\beta+\kappa)^2}\Delta t + \frac{2\beta\kappa}{(\beta+\kappa)^3}\left(1 - e^{-(\beta+\kappa)\Delta t}\right) + \frac{\kappa^2}{2(\beta+\kappa)^3}\left(1 - e^{-2(\beta+\kappa)\Delta t}\right),$$

$$a_{22}(\Delta t) = \frac{\beta^2}{(\beta+\kappa)^2}\Delta t - \frac{2\beta^2}{(\beta+\kappa)^3}\left(1 - e^{-(\beta+\kappa)\Delta t}\right) + \frac{\beta^2}{2(\beta+\kappa)^3}\left(1 - e^{-2(\beta+\kappa)\Delta t}\right),$$

$$a_{12}(\Delta t) = a_{21}(\Delta t) = \frac{\beta^2}{(\beta+\kappa)^2}h + \frac{\beta(\kappa-\beta)}{(\beta+\kappa)^3}\left(1 - e^{-(\beta+\kappa)\Delta t}\right) - \frac{\beta\kappa}{2(\beta+\kappa)^3}\left(1 - e^{-2(\beta+\kappa)\Delta t}\right).$$

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