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ORIGINAL PAPER

An approach to decision-making via picture fuzzy soft graphs

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Abstract
Fuzzy soft graphs are effective mathematical tools that are used to model the vagueness of the real world. A fuzzy soft graph is a fusion of the fuzzy soft set and the graph model and is widely used across different fields. In this current research, the concept of picture fuzzy soft graphs is presented by combining the theory of picture soft sets with graphs. The introduction of this new picture fuzzy soft graphs is an emerging concept that can be rather developed into various graph theoretical concepts. Since soft sets are most usable in real-life applications, the newly combined concepts of the picture and fuzzy soft sets will lead to many possible applications in the fuzzy set theoretical area by adding extra fuzziness in analyzing. The notions of picture soft graphs, strong and complete picture soft fuzzy graphs, a few types of product picture fuzzy soft graphs, and regular, totally regular picture fuzzy soft graphs are discussed and validated using real-world scenarios. In addition, an application of decision-making for medical diagnosis in the current COVID scenario using the picture fuzzy soft graph has been illustrated.

Keywords Picture soft graph · Operations · Picture set · Regularity · Fuzzy graphs

1 Introduction

Fuzzy set theory (Zadeh 1965) is an emerging mathematical domain, essential to solving vagueness and incomplete information in real-life situations. It is an extension of the crisp set, where elements have a membership value in the interval [0, 1]. Fuzzy sets and fuzzy logic have potential applications in wide-ranging fields, including mathematics, computer science, engineering, statistics, artificial intelligence, decision-making, image analysis, and pattern recognition (Liu et al. 2020; Zeng et al. 2019; Zhang et al. 2020; Zou et al. 2020; Meng et al. 2020).

Atanassov (1983) extended the fuzzy set to a set that gives membership and non-membership grades for each element. The set where the sum of both these values lies between 0 and 1 is known as an intuitionistic fuzzy set. Neutrosophic sets presented in Smarandache (1998) is a generalization of the theory of fuzzy and intuitionistic fuzzy sets (Zadeh 1965; Atanassov and Gargov 1989) and deal with inconsistent information. The Neutrosophic set is characterized by elements with truth, indeterminacy, and false membership functions that fall within the real unit interval. The concept of a single-valued neutrosophic set, which is a set with elements possessing three membership functions lying in the interval [0, 1], was proposed by Wang et al. (2010).

Molodtsov (1999) developed a novel mathematical concept known as the soft set theory for solving uncertainties. Soft sets have been applied in different domains, such as operation research, Riemann integration, measurement and probability theory. Many researchers have further improved on the soft set theory, notably, the operations on soft sets in Ali et al. (2009), the concept of bijective soft sets and the concepts of relations and

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functions in soft set theory (Babitha and Sunil 2010). Maji et al. (2001a) proposed a combination of soft and fuzzy sets and also combined soft set with intuitionistic and neutrosophic sets in Maji (2013), Maji et al. (2001b). The combined concept of the soft and fuzzy set was studied as fuzzy soft sets which led to the development of soft relation, fuzzy soft relation, and the algebraic structure of soft set theory as in Roy and Maji (2007), Ali (2011), Som (2006).

In recent days, the concept of interval-valued fuzzy priority for decision-making using Intuitionistic fuzzy soft sets in Mohanty (2021), a soft-set in the type-2 environment by Paik and Mondal (2021), fuzzy soft sets have been advanced to hypersoft set, plithogenic hypersoft sets and applied in decision-making by Smarandache (2018), Yolcu and Ozturk (2021), the symmetric cross-entropy of hesitant fuzzy soft sets considering the relative entropy in Suo et al. (2021), topological space on fuzzy bipolar soft sets, fuzzy bipolar soft point and fuzzy bipolar soft interior and closure points were defined in Dizman and Ozturk (2021).

Cuong and Kreinovich (2013) and Cuong (2014) introduced the picture fuzzy set, which was developed from fuzzy and intuitionistic sets and is distinguished by positive, negative and neutral membership functions. Similarity measures have also been proposed in picture fuzzy environments and used in decision making, clustering and pattern recognition (Singh and Ganie 2021). The Picture fuzzy distance measures, based on direct functions, have been applied in decision making (Ganie and Singh 2021). Researchers have studied in-depth the usage of picture fuzzy sets and developed new concepts as in Son (2016), Wei (2018). Picture fuzzy soft sets have been advanced to concepts as generalized picture fuzzy soft sets as an extension of the picture fuzzy soft sets in Khan et al. (2019), b-picture fuzzy soft sets (bPFSS) and generalized b-picture fuzzy soft sets in (GbPFSS) based on the bijective soft sets and their basic properties in Khan et al. (2020).

Graphs are visual representations of objects and their relationships. Real-world problems cause uncertainties in the relationships between objects. Thus, the simple graph model becomes a fuzzy graph model. Akram et al., introduced the concept of fuzzy soft graphs, studied the operations on fuzzy graphs and many other graph-theoretical concepts (Akram 2011, 2013; Akram and Nawaz 2015a, b). The concept of soft sets has been introduced into intuitionistic graphs in Shahzadi and Akram (2017) and merged as intuitionistic fuzzy soft graphs along with the concepts of strong, complete and complement of intuitionistic fuzzy soft graphs. Akram and Shahzadi (2017) introduced neutrosophic soft graphs and also developed the concepts of complete, strong and complement graphs. Kauffman (1973) presented the concept of fuzzy graphs using Zadeh's fuzzy relation. Rosenfeld (1975) defined and developed many fundamental graph-theoretical ideas like cycles, bridges, and connectedness. Operations on fuzzy graphs along with their properties and the concept of regular fuzzy graphs were proposed in Moderson and Nair (2012). Zuo et al. (2019) advanced the concept of the fuzzy graph to picture fuzzy graphs by blending the fuzzy graphs and picture fuzzy sets. Parvathi and Karunambigai (2006) proposed the intuitionistic fuzzy graph which was subsequently extended to the intuitionistic fuzzy hypergraph and its possible applications have been explored in Akram and Dudek (2013).

Broumi et al. (2016) proposed single-valued neutrosophic graphs with examples and properties. The properties of degree, size, and regular single-valued neutrosophic graphs were also examined by Naz et al. (2017). They later progressed into bipolar single-valued neutrosophic graphs, and strong and regular bipolar single-valued neutrosophic graphs (Naz et al. 2018). The recent conceptual developments in picture fuzzy graphs are the shortest path algorithm, newly defined using the picture fuzzy digraphs (Mani et al. 2021), domination in picture fuzzy graphs (MohamedIsmayil and AshaBoslley 2019), using competition in graphs on picture fuzzy environment with applications in the medical field (Das and Ghorai 2020) and a genus of graphs in the picture fuzzy environment (Das et al. 2021). In recent, many researchers are working on the development of the picture sets in Garg and Kaur (2021), Garg et al. (2021), Bibin et al. (2020), Khalil et al. (2019), Riaz et al. (2021), Akram et al. (2020, 2021), Akram and Habib (2019). The introduction of this new Picture fuzzy soft graphs is an emerging new concept that can be rather developed into various graph theoretical concepts. to contribute to the theoretical aspect of fuzzy graph theory, thus we have introduced this picture fuzzy soft graph and explored its properties and established related theorems. Since soft sets are most usable in real-life applications, the newly combined concepts of the picture and fuzzy soft sets will lead to many possible applications in the fuzzy set theoretical area by adding extra fuzziness in analyzing. As a practical application, we have developed a model using this defined graph and applied it in decision making.

In this research, the picture soft set and fuzzy graphs are combined to form a unique mathematical model called picture fuzzy soft graphs. A few significant notions of picture fuzzy soft graphs are also discussed briefly. The paper is structured as follows: Sect. 2 reviews the fundamental concepts and terminologies used in fuzzy graph theory. Section 3 proposes the concept of picture fuzzy soft graphs and Sect. 4 illustrates a model for the application of these picture fuzzy soft graphs.

2 Preliminaries

The elementary concepts which are necessary for the results are discussed.

Definition 1 (Cuong and Kreinovich 2013) Let Y be a non-void set. A picture fuzzy set (PFS) P of Y characterized by positive (PM), neutral (NM) and negative membership (NEM) functions denoted by μ_P, γ_P and σ_P, respectively, are given by $\mu_P : Y$ to [0, 1], $\gamma_P : Y$ to [0, 1] and $\sigma_P : Y$ to [0, 1] such that $0 \le \mu_P(c) + \gamma_P(c) + \sigma_P(c) \le 1$.

Definition 2 (Cuong and Kreinovich 2013) A PFS O is a subset of another PFS T if $\forall \ y \in Y$, $\mu_O(y) \le \mu_T(y)$, $\gamma_O(y) \le \gamma_T(y)$ and $\sigma_O(y) \le \sigma_T(y)$.

Definition 3 (Cuong and Kreinovich 2013) The complement of PFS O over Y is $O^c = \{<y, \sigma_O(y), 1 - \gamma_O(y), \mu_O(y)> \ : y \in Y\}$.

Definition 4 (Cuong and Kreinovich 2013) The union of two PFS O and T is a PFS denoted by $O \cup T$, where the PM, NM and NEM functions are defined as

$$\mu_{O\cup T}(y) = \max\{\mu_O(y), \mu_T(y)\}$$
$$\gamma_{O\cup T}(y) = \max\{\gamma_O(y), \gamma_T(y)\}$$
$$\sigma_{O\cup T}(y) = \min\{\sigma_O(y), \sigma_T(y)\} \, \forall \, y \in Y.$$

Definition 5 (Cuong and Kreinovich 2013) The intersection of two picture fuzzy subsets O & T is also a PFS denoted by $O \cap T$, where the PM, NM and NEM functions are defined as

$$\mu_{O\cap T}(y) = \min\{\mu_O(y), \mu_T(y)\}$$
$$\gamma_{O\cap T}(y) = \min\{\gamma_O(y), \gamma_T(y)\}$$
$$\sigma_{O\cap T}(y) = \max\{\sigma_O(y), \sigma_T(y)\} \, \forall \, y \in Y.$$

Definition 6 (Zuo et al. 2019) A picture fuzzy graph is $G = (V, E)$ with $V = \{v_1, v_2 \ldots v_n\}$ such that μ_P, γ_P and σ_P, respectively, are given by $\mu_1 : V$ to [0, 1], $\gamma_1 : V$ to [0, 1] and $\sigma_1 : V$ to [0, 1] which denote PM, NM and NEM functions, respectively, and $0 \le \mu_1(a) + \gamma_1(a) + \sigma_1(a) \le 1$. $\forall \ a \in V$; $E \subseteq V \times V$ with $\mu_2 : V \times V \to [0, 1]$, $\gamma_2 : V \times V \to [0, 1]$ and $\sigma_2 : V \times V \to [0, 1]$ such that

$$\mu_2(ab) \le \min\{\mu_2(a), \mu_1(b)\}$$
$$\gamma_2(ab) \le \min\{\gamma_2(a), \gamma_1(b)\}$$
$$\sigma_2(ab) \le \max\{\sigma_2(a), \sigma_1(b)\} \, \forall \, ab \in E.$$

Considering the vagueness in modeling and soft computing, soft set theory was introduced by Molodstov.

Let V be the universe and P be all possible parameters related to objects in V. The duo (V, P) is called a soft universe and P(V), the power set.

Definition 7 (Shahzadi and Akram 2017) A duo (F, O) is soft over V, when $O \subseteq P$, and F is a set-valued function $F : O \to P(V)$. A soft set over V is a parametrized family of subsets of V.

Definition 8 (Shahzadi and Akram 2017) Let V be the universe and P the set of parameters with $O \subset P$. The collection of all PFS of V is P(V). The collection (S, O) is named as the picture fuzzy soft set (PFSS) over V, where S is a mapping given by $S : O \to P(V)$.

Definition 9 (Yang et al. 2015) Let (S, O) & (K, T) be PFSS over U. (S, O) is named as picture fuzzy soft subset of (K, T) if $O \subset T$,&

$$\mu_{S(o)}(k) \le \mu_{K(0)}(k)$$
$$\gamma_{S(o)}(k) \le \gamma_{K(0)}(k)$$
$$\sigma_{S(o)}(k) \ge \sigma_{K(0)}(k) \, \forall \, o \in O \text{ and } k \in V.$$

Definition 10 (Yang et al. 2015) Let (S, O) and (K, T) be PFSS over V. The union of (S, O) & (K, B) is a PFSS $(H, D) = (S, O) \cup (K, T)$, where $D = O \cup T$ and PM, NM, NEM of (H, D) are defined by

$$\mu_{H(a)}(w) = \begin{cases} \mu_{S(a)}(w) & \text{if } a \in O - T; \\ \mu_{K(a)}(w) & \text{if } a \in T - O; \\ \max\{\mu_{S(a)}(w), \mu_{K(a)}(w)\} & \text{if } a \in O \cap T \end{cases}$$

$$\gamma_{H(a)}(w) = \begin{cases} \gamma_{S(a)}(w) & \text{if } a \in O - T; \\ \gamma_{K(a)}(w) & \text{if } a \in T - O; \\ \max\{\gamma_{S(a)}(w), \gamma_{K(a)}(w)\} & \text{if } a \in O \cap T \end{cases}$$

$$\sigma_{H(a)}(w) = \begin{cases} \sigma_{S(a)}(w) & \text{if } a \in O - T; \\ \sigma_{K(a)}(w) & \text{if } a \in T - O; \\ \min\{\sigma_{S(a)}(w), \sigma_{K(a)}(w)\}, & \text{if } a \in O \cap T \end{cases}$$

Similarly, the intersection of (S, O) and (K, T) is a PFSS (H, D), in which $D = O \cap T$ & the membership functions are defined by

$$\mu_{H(b)}(t) = \begin{cases} \mu_{S(b)}(t) \text{ if } b \in O - T; \\ \mu_{K(b)}(t) \text{ if } b \in T - O; \\ \min\{\mu_{S(b)}(t), \mu_{K(b)}(t)\} \text{ if } b \in O \cap T \end{cases}$$

$$\gamma_{H(b)}(t) = \begin{cases} \gamma_{S(b)}(t) \text{ if } b \in O - T; \\ \gamma_{K(b)}(t) \text{ if } b \in T - O; \\ \min\{\gamma_{S(b)}(t), \gamma_{K(b)}(t)\} \text{ if } b \in O \cap T \end{cases}$$

$$\sigma_{H(b)}(t) = \begin{cases} \sigma_{S(b)}(t) \text{ if } b \in O - T; \\ \sigma_{K(b)}(t) \text{ if } b \in T - O; \\ \max\{\sigma_{S(b)}(t), \sigma_{K(b)}(t)\} \text{ if } b \in O \cap T \end{cases}$$

Definition 11 (Yang et al. 2015) Let (*S*, *D*) & (*G*, *C*) be PFSS over *V*. The Cartesian product of (*S*, *D*) & (*G*, *C*) is $(S,D) \times (G,C) = (H, D \times C)$. The PM, NM, NEM functions of $(H, D \times C)$ are

$$\mu_{H(D\times C)}(k) = \min\{\mu_{S(o)}(k), \mu_{G(t)}(k)\}$$
$$\gamma_{H(D\times C)}(k) = \min\{\gamma_{S(o)}(k), \gamma_{G(t)}(k)\}$$
$$\sigma_{H(D\times C)}(k) = \max\{\sigma_{S(o)}(k), \sigma_{G(t)}(k)\}.$$

Definition 12 (Yang et al. 2015) Let (*S*, *O*) & (*K*, *T*) be two PFSS over *V*. A picture soft relation (*R*) from (*S*, *O*) to (*K*, *T*) is of the form (*R*, *C*), where $C \subset O \times T$ and $R(g,h) \subset (S,O) \times (K,T)\ \forall\ (g,H) \in C$.

3 Picture fuzzy soft graphs

Let *V* be the universe, *E* the set of all parameters and *P*(*V*) be the collection of all PFSS of *V*. Let *A* be a subset of *V*. The set of all picture fuzzy sets of *V* & *E* are denoted by *P*(*V*) and *P*(*E*).

Definition 13 Let PFSG $G = (G^*, F, K, O)$ is an ordered quadruple with $G^* = (V, E)$, *O* a non-void set of parameters, (*F*, *O*) and (*K*, *O*) are PFSS over *V* and *E*, respectively; (*F*(*o*), *K*(*o*)) for $o \in O$ is a picture fuzzy soft graph of *G* when it satisfies the following conditions:

$$\mu_{K(o)}(xy) \le \min\{\mu_{F(o)}(x), \mu_{F(o)}(y)\}$$
$$\gamma_{K(o)}(xy) \le \min\{\gamma_{F(o)}(x), \gamma_{F(o)}(y)\}$$
$$\sigma_{K(o)}(xy) \le \max\{\sigma_{F(o)}(x), \sigma_{F(o)}(y)\}$$

such that $0 \le \mu_{K(o)}(xy) + \gamma_{K(o)}(xy) + \sigma_{K(o)}(xy) \le 1\ \forall\ o \in O;\ x, y \in V$.

The picture fuzzy graph (PFG) (*F*(*o*), *K*(*o*)) is represented as *S*(*o*) throughout this paper. A PFSG is a collection of PFG. The collection of all PFSG is $PS(G^*)$.

Note 14 $\mu_{K(o)}(xy) = \gamma_{K(o)}(xy) = 0$ & $\sigma_{K(o)}(xy) = 1\ \forall\ xy \in V \times V - E, o \notin O$.

Definition 15 Let $G_1 = (F_1, K_1, O)$ and $G_2 = (F_2, K_2, T)$ be two PFSG of G^* then G_1 is picture fuzzy soft sub graph (PFSSG) of G_2 if $O \subseteq T$ and $S_1(o)$ is a partial sub graph of $S_2(o)\ \forall\ o \in O$.

Example 16 Consider $G^* = (V, E)$ such that $V = \{v_1, v_2, v_3, v_4\}$ & $E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_4, v_3v_4\}$. Let $O = \{e_1, e_2\}$ be the parameters & (*F*, *O*) be a PFSS over *V* with $F : O \to P(V)$ defined by

$$F(e_1) = \{(v_1, 0.5, 0.2, 0.1), (v_2, 0.4, 0.3, 0.2), (v_3, 0.3, 0.4, 0.3), (v_4, 0.2, 0.5, 0.5)\}$$
$$F(e_2) = \{(v_1, .4, 0.3, 0.2), (v_2, 0.3, 0.4, 0.1), (v_3, 0.5, 0.2, 0.3), (v_4, 0.2, 0.1, 0.4)\}$$

Let (*K*, *O*) be a PFSS over *E* with $K : O \to P(E)$ defined by

$$K(e_1) = \{(v_1v_2, 0.3, 0.2, 0.1), (v_1v_3, 0.2, 0.1, 0.2), (v_1v_4, 0.1, 0.1, 0.3)\}$$
$$K(e_2) = \{(v_1v_3, 0.3, 0.2, 0.2), (v_2v_4, 0.2, 0.1, 0.3), (v_3v_4, 0.1, 0.1, 0.2)\}.$$

Clearly $S(e_1) = (F(e_1), K(e_1))$ and $S(e_2) = (F(e_2),$

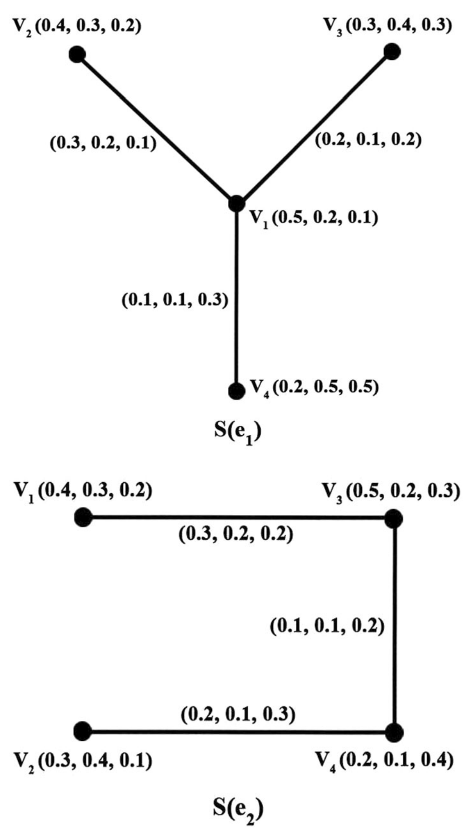

Fig. 1 Picture fuzzy soft graph $G = \{S(e_1), S(e_2)\}$

$K(e_2))$ are PFSG corresponding to e_1 and e_2, respectively, as shown in Fig. 1. Thus $G = (S(e_1), S(e_2))$ is a PFSG of G^*.

Definition 17 A Picture fuzzy soft path P in PFSG $G = (G^*, F, K, O)$ is a sequence of distinct vertices $x_o, x_1, \ldots, x_n$ $(except\ x_o\ \&\ x_1)$ such that $(\mu_{k(o)}(x_{i-1}, x_i), \gamma_{k(o)}(x_{i-1}, x_i), \sigma_{k(o)}(x_{i-1}, x_i)) > 0, \quad i = 1, \ldots n$ Here n denotes the length of the PFSP. The successive pairs are called the edges of PFSP. For example, in Fig. 1, consider $S(e_1)$ then v_2, v_1 is a PFSP.

Definition 18 The diameter of $a, b \in V$, is the length of the longest PFSP joining a to b. The strength of P is defined as

$$\left(\bigwedge_{i=1}^{n} \mu_{k(o)}(x_{i-1}, x_i), \bigwedge_{i=1}^{n} \gamma_{k(o)}(x_{i-1}, x_i), \bigvee_{i=1}^{n} \sigma_{k(o)}(x_{i-1}, x_i)\right).$$

The strength of the PFSP is the weight of the weakest edges and represented by $d(P)$. The strength of connectedness between two vertices a & b is defined as the maximum of the strengths of all PFSP's between a and b and represented by $\mathrm{PFSCONN}_G(a, b)$.

Definition 19 The PFSG $G_1 = (G^*, F_1, K_1, T)$ is called a spanning PFSSG of $G = (G^*, F, K, O)$ if

(i) $T \subseteq O$

(ii)

$$\begin{aligned}\mu_{F_1(z)}(q) &= \mu_{F_(z)}(q),\\ \gamma_{F_1(z)}(q) &= \gamma_{F_(z)}(q),\\ \sigma_{F_1(z)}(q) &= \sigma_{F_(z)}(q)\,\forall z \in O,\ q \in V.\end{aligned}$$

Definition 20 The order of a picture fuzzy soft graph is

$$O(G) = \left(\sum_{e_i \in O}\left(\sum_{u \in V} \mu_{F(e_i)}(u)\right), \sum_{e_i \in O}\left(\sum_{u \in V} \gamma_{F(e_i)}(u)\right), \sum_{e_i \in O}\left(\sum_{u \in V} \sigma_{F(e_i)}(u)\right)\right).$$

The size of picture fuzzy soft graph is

$$S(G) = \left(\sum_{e_i \in O}\left(\sum_{uv \in E} \mu_{K(e_i)}(uv)\right), \sum_{e_i \in O}\left(\sum_{uv \in E} \gamma_{K(e_i)}(uv)\right), \sum_{e_i \in O}\left(\sum_{uv \in E} \sigma_{K(e_i)}(uv)\right)\right).$$

Example 21 Consider a crisp graph $G^* = (V, E)$ such that $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{v_1v_2, v_2v_3, v_2v_4, v_4v_5, v_1v_5, v_3v_5\}$. Let $O = \{e_1, e_2, e_3\}$ be a parameter set and let (F, O) be Picture fuzzy soft set over V with $F : O \to P(V)$ defined by

$$\begin{aligned}F(e_1) &= \{(v_1, 0.4, 0.3, 0.3), (v_2, 0.2, 0.2, 0.3), (v_3, 0.2, 0.1, 0.1),\\ &\quad (v_4, 0.1, 0.3, 0.2), (v_5, 0.3, 0.4, 0.1)\}\\ F(e_2) &= \{(v_1, 0.4, 0.3, 0.2), (v_2, 0.3, 0.2, 0.3), (v_3, 0.4, 0.1, 0.2),\\ &\quad (v_4, 0.1, 0.1, 0.2), (v_5, 0.2, 0.4, 0.1)\}\\ F(e_3) &= \{(v_1, 0.3, 0.4, 0.2), (v_2, 0.3, 0.2, 0.1), (v_3, 0.3, 0.4, 0.1),\\ &\quad (v_4, 0.4, 0.2, 0.1), (v_5, 0.3, 0.2, 0.4)\}.\end{aligned}$$

Let (K, O) be picture fuzzy soft set over E with $K : O \to P(E)$ defined by

$$\begin{aligned}K(e_1) = \{&(v_1v_2, 0.1, 0.2, 0.3), (v_2v_3, 0.2, 0.1, 0.2),\\ &(v_2v_4, 0.1, 0.2, 0.2),\\ &(v_4v_5, 0.1, 0.3, 0.2), (v_1v_5, 0.3, 0.3, 0.2),\\ &(v_3v_5, 0.2, 0.1, 0.1)\}\\ K(e_2) = \{&(v_1v_2, 0.3, 0.2, 0.2), (v_2v_3, 0.3, 0.1, 0.3),\\ &(v_2v_4, 0.1, 0.2, 0.3),\\ &(v_4v_5, 0.1, 0.1, 0.2), (v_1v_5, 0.2, 0.3, 0.2),\\ &(v_3v_5, 0.2, 0.1, 0.2)\}\\ K(e_3) = \{&(v_1v_2, 0.3, 0.2, 0.2), (v_2v_3, 0.3, 0.2, 0.1),\\ &(v_2v_4, 0.2, 0.2, 0.1),\\ &(v_4v_5, 0.3, 0.2, 0.3), (v_1v_5, 0.2, 0.2, 0.3),\\ &(v_3v_5, 0.3, 0.2, 0.3)\}0.\end{aligned}$$

The Picture fuzzy graphs of G are $S(e_1) = (F(e_1), K(e_1))$, $S(e_2) = (F(e_2), K(e_2))$, and $S(e_3) = (F(e_3), K(e_3))$ according to the parameters e_1, e_2 and e_3, respectively. Thus, $G = S(e_1), S(e_2), S(e_3)$ is a picture fuzzy soft graph on O.

The order of Picture fuzzy soft graph is $O(G) = ((0.4 + 0.2 + 0.2 + 0.1 + 0.3) + (0.4 + 0.3 + 0.4 + 0.1 + 0.2) + (0.3 + 0.3 + 0.3 + 0.4 + 0.3), (0.3 + 0.2 + 0.1 + 0.3 + 0.4) + (0.3 + 0.2 + 0.1 + 0.1 + 0.4) + (0.4 + 0.2 + 0.4 + 0.2 + 0.2), (0.3 + 0.3 + 0.1 + 0.2 + 0.1) + (0.2 + 0.3 + 0.2 + 0.2 + 0.1) + (0.2 + 0.1 + 0.1 + 0.1 + 0.4)) = (4.2, 3.8, 2.9)$

The size of the Picture fuzzy soft graph $S(G)$ is $((0.1 + 0.2 + 0.1 + 0.1 + 0.3 + 0.2) + (0.3 + 0.3 + 0.1 + 0.1 + 0.2 + 0.2) + (0.3 + 0.3 + 0.2 + 0.3 + 0.2 + 0.3), (0.2 + 0.1 + 0.2 + 0.3 + 0.3 + 0.1) + (0.2 + 0.1 + 0.1 + 0.1 + 0.3 + 0.1) + (0.2 + 0.2 + 0.2 + 0.2 + 0.2 + 0.2), (0.3 + 0.2 + 0.2 + 0.2 + 0.2 + 0.1) + (0.2 + 0.3 + 0.3 + 0.2 + 0.2 + 0.2) + (0.2 + 0.1 + 0.1 + 0.3 + 0.3 + 0.3)) = (3.8, 3.3, 3.9)$

3.1 Regularity of picture fuzzy soft graphs

Definition 22 Let G be a Picture fuzzy soft graph of G^*. G is called a regular picture fuzzy soft graph if $S(e)$ is a regular picture fuzzy graph. $\forall\ e \in O$.

Example 23 Consider a simple graph $G^* = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let $O = \{e_1, e_2\}$ be a set of parameters. Let (S, O) be a picture fuzzy soft graph, where picture fuzzy graphs $S(e_1), S(e_2)$ corresponding to parameters e_1, e_2, e_3 & e_4, receptively, are

$$S(e_1) = \Big(\{(v_1, .5, .4, .1), (v_2, .4, .3, .2), (v_3, .4, .3, .2), (v_4, .3, .4, .1)\}, \{(v_1v_2, .3, .3, .1), (v_2v_3, .3, .2, .1), (v_3v_4, .3, .3, .1), (v_4v_1, .3, .2, .1)\}\Big)$$

$$S(e_2) = \Big(\{(v_1, .5, .4, .2), (v_2, .4, .3, .3), (v_3, .4, .3, .2), (v_4, .3, .4, .1)\}, \{(v_1v_2, .3, .2, .2), (v_2v_3, .2, .3, .1), (v_3v_4, .3, .2, .2), (v_4v_1, .2, .3, .1)\}\Big)$$

The regular picture fuzzy graph is drawn in Fig. 2.

Definition 24 Let G be a picture fuzzy soft graph of G^*. G is said to be totally regular picture fuzzy soft graph if $S(e)$ is a totally picture fuzzy graph for all $e \in O$.

Example 25 Consider a simple graph $G^* = (V, E)$, where $V = \{a_1, a_2, a_3, a_4\}$ and $E = \{a_1a_2, a_2a_3, a_3a_4, a_1a_3\}$. Let $O = \{e_1, e_2\}$ be a set of parameters. Let $G = (S, O)$ be a picture fuzzy soft graphs, where $S(e_1) = (F(e_1), K(e_1))$ and $S(e_2) = (F(e_2), K(e_2))$ corresponding to the parameters e_1 and e_2, respectively are defined as

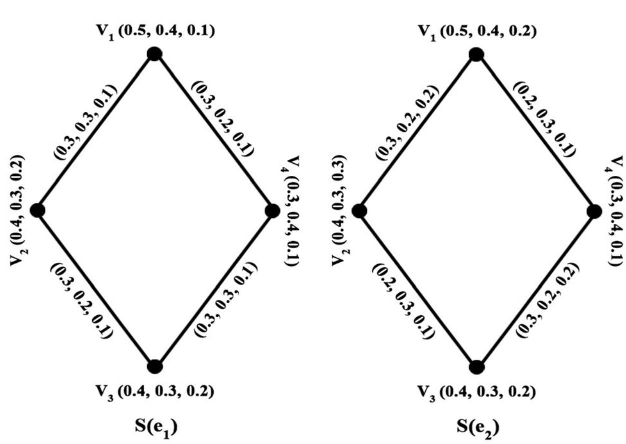

Fig. 2 Regular Picture fuzzy soft graph $G = \{S(e_1), S(e_2)\}$

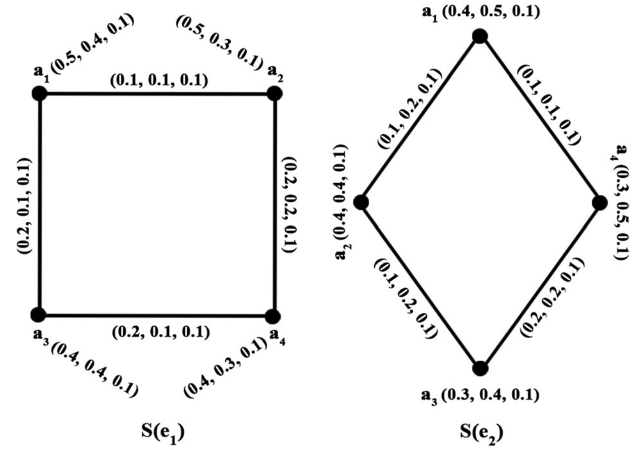

Fig. 3 Totally regular picture fuzzy soft graph $G = \{S(e_1), S(e_2)\}$

$$S(e_1) = (\{(a_1, .5, .4, .1), (a_2, .5, .3, .1), (a_3, .4, .4, .1), a_4, .4, .3, .1)\}, \{(a_1a_2, .1, .1, .1), (a_2a_4, .2, .2, .1), (a_3a_4, .2, .1, .1), (a_3a_1, .2, .1, .1)\})$$

$$S(e_2) = (\{(a_1, .4, .5, .1), (a_2, .3, .5, .1), (a_3, .4, .4, .1), a_4, .3, .4, .1)\}, \{(a_1a_2, .1, .1, .1), (a_2a_4, .2, .2, .1), (a_3a_4, .1, .2, .1), (a_3a_1, .1, .2, .1)\})$$

The totally regular picture fuzzy soft graph is drawn in Fig. 3.

Definition 26 Let G be a Picture fuzzy soft graph on V. Then G is called as perfectly regular picture fuzzy soft graph if $S(e_i)$ is a regular and totally regular picture fuzzy graph for all $e_i \in O$.

Proposition 27 *For a perfectly regular picture fuzzy soft graph $G = (F, K, O)$, F is a constant function.*

Theorem 28 *Let G be a picture fuzzy soft graph. Then G is perfectly regular if and only if*

1. $\sum_{a \neq b} \mu_{K(e_i)}(ab) = \sum_{c \neq b} \mu_{K(e_i)}(bc)$,
$\sum_{a \neq b} \gamma_{K(e_i)}(ab) = \sum_{c \neq b} \gamma_{K(e_i)}(bc)$,
$\sum_{a \neq b} \sigma_{K(e_i)}(ab) = \sum_{c \neq b} \sigma_{K(e_i)}(bc)$.
2. $\mu_{F(e_i)}(a) = \mu_{K(e_i)}(b)$,
$\gamma_{F(e_i)}(a) = \gamma_{K(e_i)}(b)$,
$\sigma_{F(e_i)}(a) = \sigma_{K(e_i)}(b)\ \forall\ a, b \in V,\ e_i \in O$.

Proof Assume G is perfectly regular picture fuzzy soft graph. So G is regular picture fuzzy soft graph. Thus, from definition we have $\deg_\mu(a) = \deg_\mu(c)$, $\deg_\gamma(a) = deg_\gamma(c)$, $\deg_\sigma(a) = \deg_\sigma(c)$, $\forall\ a, c \in V\ e_i \in O$. Then

$$\sum_{a\neq b}\mu_{K(e_i)}(ab)=\sum_{c\neq b}\mu_{K(e_i)}(bc),$$
$$\sum_{a\neq b}\gamma_{K(e_i)}(ab)=\sum_{c\neq b}\gamma_{K(e_i)}(cb),$$
$$\sum_{a\neq b}\sigma_{K(e_i)}(ab)=\sum_{c\neq b}\sigma_{K(e_i)}(bc)\,\forall\, a,b,C\in V,\; e_i\in O.$$

Thus (1) holds. By the proposition (3.14), (2) also holds.

Conversely, suppose that G is a picture fuzzy soft graph such that it satisfies the conditions. From (1), we have

$$\sum_{a\neq b}\mu_{K(e_i)}(ab)=\sum_{c\neq b}\mu_{K(e_i)}(bc),$$
$$\sum_{a\neq b}\gamma_{K(e_i)}(ab)=\sum_{c\neq b}\gamma_{K(e_i)}(bc),$$
$$\sum_{a\neq b}\sigma_{K(e_i)}(ab)=\sum_{c\neq b}\sigma_{K(e_i)}(bc)\,\forall\, a,b,c\in V,\; e_i\in O.$$

$\deg_\mu(a)=\deg_\mu(c)=r_i$, $\deg_\gamma(a)=\deg_\gamma(c)=r_i^{'}$, and $\deg_\sigma(a)=\deg_\sigma(c)=r_i^{''}$, $\forall\; c,a\in V,\; e_i\in O$. This implies $S(e_i)$ is a regular picture fuzzy graph. Thus, G is a regular picture fuzzy soft graph.

From (2), $\mu_{F(e_i)}(a)=\mu_{F(e_i)}(c)=x_i$, $\gamma_{F(e_i)}(a)=\gamma_{F(e_i)}(c)=x_i^{'}$, and $\sigma_{F(e_i)}(a)=\sigma_{F(e_i)}(c)=x_i^{''}$, $\forall\; c,a\in V,\; e_i\in O$.

Thus, F is a constant function

$$\deg_\mu[c]=\deg_\mu(c)+\mu_{F(e_i)}(c)=r_i+x_i,$$
$$\deg_\mu[d]=\deg_\mu(d)+\mu_{F(e_i)}(d)=r_i+x_i.$$
$$\deg_\gamma[c]=\deg_\gamma(c)+\gamma_{F(e_i)}(c)=r_i^{'}+x_i^{'},$$
$$\deg_\gamma[d]=\deg_\gamma(c)+\gamma_{F(e_i)}(d)=r_i^{'}+x_i^{'}.$$
$$\deg_\sigma[c]=\deg_\sigma(c)+\sigma_{F(e_i)}(c)=r_i^{''}+x_i^{''},$$
$$\deg_\sigma[d]=\deg_\sigma(c)+\sigma_{F(e_i)}(d)=r_i^{''}+x_i^{''}.$$
$$\forall\, c,d\in V,\; e_i\in O.$$

$\deg_\mu[c]=\deg_\mu[d]=k_i$, $\deg_\gamma[c]=\deg_\gamma[d]=k_i^{'}$ and $\deg_\sigma[c]=\deg_\sigma[d]=k_i^{''}$. (ie) $\deg[c]=\deg[d]=(k_i,k_i^{'},k_i^{''})$ $\forall\, c,d\in V,\; e_i\in O$. So $S(e_i)$ is a totally regular picture fuzzy graph.

Hence, G is totally regular picture fuzzy soft graph. This implies G is a perfect picture fuzzy soft graph. □

Corollary 29 *If G is a perfectly regular picture fuzzy soft graph and*

$$F=\left(\mu_{F(e_i)}(c),\gamma_{F(e_i)}(c),\sigma_{F(e_i)}(c)\right)=\left(r_i,r_i^{'},r_i^{''}\right)$$

, $\forall\; c\in V,\; e_i\in O$ *is a constant function in* $S(e_i)$ *then* $O(S(e_i))=|V|\left(r_i,r_i^{'},r_i^{''}\right)$.

Theorem 30 *Let* $G=(F,K,O)$ *be a perfectly regular picture fuzzy soft graph. Then size of* $S(e_i)$ *is* $S(S(e_i))=\frac{|V|}{2}(x_i,x_i^{'},x_i^{''})$, *where* $(x_i,x_i^{'},x_i^{''})$ *is the degree of a vertex in* $S(e_i)$ $\forall\; e_i\in O$.

3.2 Operations of picture soft graphs

Definition 31 Let $G_1=(F_1,K_1,O)$ and $G_2=(F_2,K_2,T)$ be two PFSG of $G_1^*=(V_1,E_1)$, $G_2^*=(V_2,E_2)$. The Cartesian Product of G_1, G_2 is a PFSG $G:G_1\times G_2=(F,K,O\times T)$, where $(F=F_1\times F_2,O\times T)$ is a PFSS over $V=V_1\times V_2$, $(K=K_1\times K_2,O\times T)$ is a PFSS over $E=\{((y,z_1),(y,z_2)):\; y\in V_1,(z_1,z_2)\in E_2\}\cup\{((y_1,z),(y_2,z)):z\in V_2,(y_1,y_2)\in E_1\}$ and $(F,K,O\times T)$ are PFSG such that

1.
$$\mu_{F(d,k)}(y,z)=\mu_{F_1(d)}(y)\wedge\mu_{F_2(k)}(z)$$
$$\gamma_{F(d,k)}(y,z)=\gamma_{F_1(d)}(y)\wedge\gamma_{F_2(k)}(z)$$
$$\sigma_{F(d,k)}(y,z)=\sigma_{F_1(d)}(y)\vee\sigma_{F_2(k)}(z)$$
$$\forall\;(y,z)\in V,(d,k)\in O\times T$$

2.
$$\mu_{K(d,k)}\;((y,z_1),(y,z_2))=\mu_{F_1(d)}(y)\wedge\mu_{K_2(k)}(z_1,z_2)$$
$$\gamma_{K(d,k)}\;((y,z_1),(y,z_2))=\gamma_{F_1(d)}(y)\wedge\gamma_{K_2(k)}(z_1,z_2)$$
$$\sigma_{K(d,k)}\;((y,z_1),(y,z_2))=\sigma_{F_1(d)}(y)\vee\sigma_{K_2(k)}(z_1,z_2)$$
$$\forall y\in V_1,(z_1,z_2)\in E_2$$

3.
$$\mu_{K(d,k)}\;((y_1,z),(y_2,z))=\mu_{F_2(k)}(z)\wedge\mu_{K_1(d)}(y_1,y_2)$$
$$\gamma_{K(d,k)}\;((y_1,z),(y_2,z))=\gamma_{F_2(k)}(z)\wedge\gamma_{K_1(d)}(y_1,y_2)$$
$$\sigma_{K(d,k)}\;((y_1,z),(y_2,z))=\sigma_{F_2(k)}(z)\vee\sigma_{K_1(d)}(y_1,y_2)$$
$$\forall z\in V_2,(y_1,y_2)\in E_2$$

$S(d,k)=S_1(d)\times S_2(k)\;\forall\;(d,k)\in O\times T$ are picture fuzzy graphs of G.

Example 32 Let $O=\{e_1,e_2\}$ and $T=\{e_3,e_4\}$ be the set of parameters. Consider two PFSG's $G_1=\{S_1(e_1),S_1(e_2)\}$ and $G_2=\{S_2(e_3),S_2(e_4)\}$ as in Figs. 4, 5, 6, 7.

The Cartesian product of G_1 and G_2 is $G_1\times G_2=S,O\times T$, where $O\times T=(e_1,e_3),(e_1,e_4),(e_2,e_3),(e_2,e_4)$, $S(e_1,e_3)=S_1(e_1)\times S_2(e_3),S(e_1,e_4)=S_1(e_1)\times\; S_2(e_4),S(e_2,e_3)=S_1(e_2)\times S_2(e_3)$ and $S(e_2,e_4)=S_1\;(e_2)\times S_2(e_4)$ are picture fuzzy soft graphs of $G_1\times G_2$. $S(e_1,e_3)=S_1(e_1)\times S_2(e_3)$ is shown in Fig. 8. In the similar way, the Cartesian product of $S(e_1,e_4)$, $S(e_2,e_3)$, $S(e_2,e_4)$ can be drawn.

Theorem 33 *The Cartesian product of two PFSG is a PFSG.*

Proof Let $G_1=(F_1,K_1,O)$, $G_2=(F_2,K_2,T)$ be two PFSG of $G_1^*=(V_1,E_1)$, $G_2^*=(V_2,E_2)$, respectively. Let

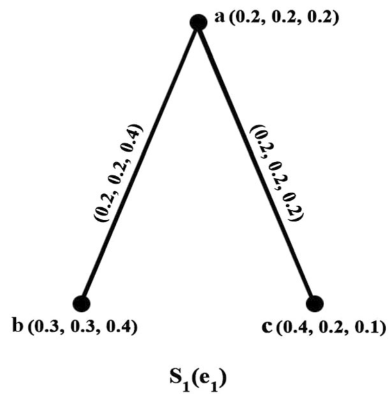

Fig. 4 Picture fuzzy soft graph $S_1(e_1)$

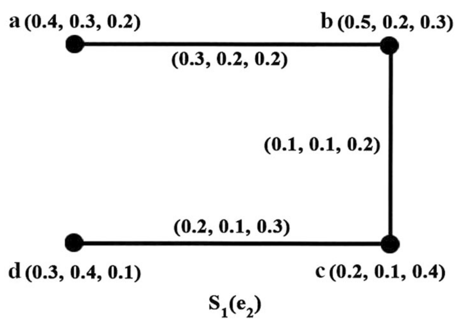

Fig. 5 Picture fuzzy soft graph $S_1(e_2)$

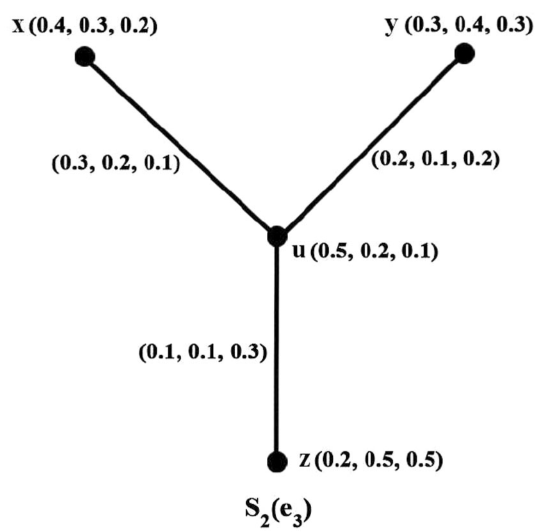

Fig. 6 Picture fuzzy soft graph $S_2(e_3)$

$G = G_1 \times G_2 = (F, K, O \times T)$ be the Cartesian product of G_1 and G_2. We claim that $G = (F, K, O \times T)$ is a PFSG and $(S, O \times T) = \{F_1 \times F_2(a_i, b_j), K_1 \times K_2(a_i, b_j)\}\ \forall\ a_i \in O,\ b_j \in T$ for $i = 1$ to m, $j = 1$ to n are PFSG of G.

Consider

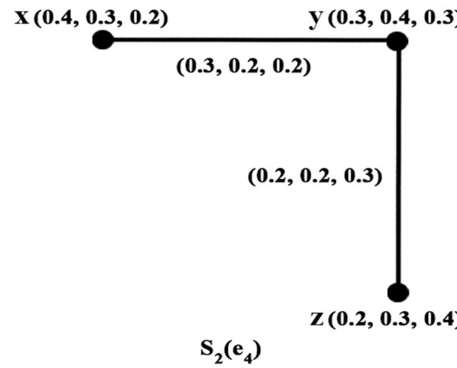

Fig. 7 Picture fuzzy soft graph $S_2(e_4)$

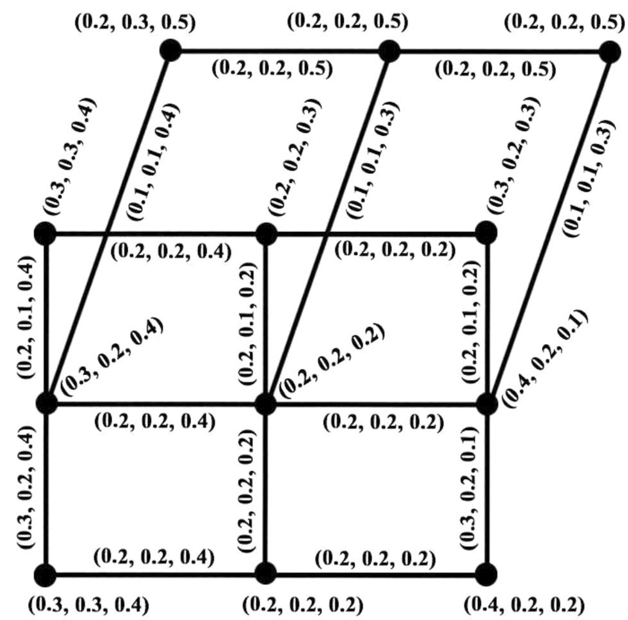

Fig. 8 Cartesian Product $S(e_1, e_3)$

$$
\begin{aligned}
&\mu_{K_{(a_i,b_j)}}((h, o_1), (h, o_2)) \\
&\quad = \min\{\mu_{F_1(a_i)}(h), \mu_{K_2(b_j)}(o_1, o_2)\} \\
&\qquad \text{for } i = 1 \text{ to } m, j = 1 \text{ to } n \\
&\quad \leq \min\{\mu_{F_1(a_i)}(h), \min\{\mu_{F_2(b_j)}(o_1), \mu_{F_2(b_j)}(o_2)\}\} \\
&\quad = \min\{\min\{\mu_{F_1(a_i)}(h), \mu_{F_2(b_j)}(o_1)\}, \\
&\qquad \min\{\mu_{F_1(a_i)}(h), \mu_{F_2(b_j)}(o_2)\}\} \\
&\mu_{K_{(a_i,b_j)}}((h, o_1), (h, o_2)) \leq \min\{(\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(h, o_1), \\
&\qquad (\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(h, o_2)\},
\end{aligned}
$$

$i = 1$ to m, $j = 1$ to n.

$\gamma_{K_{(a_i,b_j)}}((h,o_1),(h,o_2))$
$= \min\{\gamma_{F_1(a_i)}(h), \min\{\gamma_{F_2(b_j)}(o_1), \gamma_{F_2(b_j)}(o_2)\}\}$
$= \min\{\min\{\gamma_{F_1(a_i)}(h), \gamma_{F_2(b_j)}(o_1)\}$
$\min\{\gamma_{F_1(a_i)}(h), \gamma_{F_2(b_j)}(o_2)\}\}$
$\sigma_{K_{(a_i,b_j)}}((h,o_1),(h,o_2))$
$= \max\{\sigma_{F_1(a_i)}(h), \sigma_{F_2(b_j)}(o_1,o_2)\}$
for $i = 1$ to $m, j = 1$ to n
$\leq \max\{\sigma_{F_1(a_i)}(h), \max\{\sigma_{F_2(b_j)}(o_1), \sigma_{F_2(b_j)}(o_2)\}\}$
$= \max\{\max\{\sigma_{F_1(a_i)}(h), \sigma_{F_2(b_j)}(o_1)\},$
$\max\{\sigma_{F_1(a_i)}(h), \sigma_{F_2(b_j)}(o_2)\}\}$
$\sigma_{K_{(a_i,b_j)}}((h,o_1),(h,o_2)) \leq \max\{(\sigma_{F_1(a_i)} \times \sigma_{F_2(b_j)})(h,o_1),$
$(\sigma_{F_1(a_i)} \times \sigma_{F_2(b_j)})(h,o_2)\},$

$i = 1$ to $m, j = 1$ to n.

Similarly,

$\mu_{K_{(a_i,b_j)}}((h_1,o),(h_2,o)) \leq \min\{(\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(h_1,o),$
$(\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(h_2,o)\},$

$i = 1$ to $m, j = 1$ to n.

$\gamma_{K_{(a_i,b_j)}}((h_1,o),(h_2,o)) \leq \min\{(\gamma_{F_1(a_i)} \times \gamma_{F_2(b_j)})(h_1,o),$
$(\gamma_{F_1(a_i)} \times \gamma_{F_2(b_j)})(h_2,o)\},$

$i = 1$ to $m, j = 1$ to n.

$\sigma_{K_{(a_i,b_j)}}((h_1,o),(h_2,o)) \leq \max\{(\sigma_{F_1(a_i)} \times \sigma_{F_2(b_j)})(h_1,o),$
$(\sigma_{F_1(a_i)} \times \sigma_{F_2(b_j)})(h_2,o)\},$

$i = 1$ to $m, j = 1$ to n.

Therefore $G = (F, K, O \times T)$ is PFSG. □

Definition 34 The cross product of G_1 & G_2 is $G = G_1 \circ G_2 = (F, K, O \times T)$, where $(F, O \times T)$ is a PFSS over $V = V_1 \times V_2$, $(K, O \times T)$ is a PFSS over $E = \{((y_1,z_1),((y_2,z_2)) : (y_1,y_2) \in E_1, (z_1,z_2) \in E_2\}$ and $(F, K, O \times T)$ are PFSG such that

1. $\mu_{F(d,h)}(y,z) = \mu_{F_1(d)}(y) \wedge \mu_{F_2(h)}(z)$
$\gamma_{F(d,h)}(y,z) = \gamma_{F_1(d)}(y) \wedge \gamma_{F_2(h)}(z)$
$\sigma_{F(d,h)}(y,z) = \sigma_{F_1(d)}(y) \vee \sigma_{F_2(h)}(z)$
$\forall\ (y,z) \in V, (d,h) \in O \times T.$

2. $\mu_{K(d,h)}\ ((y_1,z_1),(y_2,z_2)) = \mu_{K_1(d)}(y_1,y_2) \wedge \mu_{K_2(h)}(z_1,z_2)$
$\gamma_{K(d,h)}\ ((y_1,z_1),(y_2,z_2)) = \gamma_{K_1(d)}(y_1,y_2) \wedge \gamma_{K_2(h)}(z_1,z_2)$
$\sigma_{K(d,h)}\ ((y_1,z_1),(y_2,z_2)) = \sigma_{K_1(d)}(y_1,y_2) \vee \sigma_{K_2(h)}(z_1,z_2)$
$\forall\ (y_1,y_2) \in V_1, (z_1,z_2) \in E_2$

$S(d,h) = S_1(d) \circ\ S_2(h)\ \forall\ (d,h) \in O \times T$ are PFSG of G.

Example 35 Let $O = \{e_1, e_2\}$ and $T = \{e_3, e_4\}$ be the set of parameters. Consider two PFSG's $G_1 = \{S_1(e_1), S_1(e_2)\}$ and $G_2 = \{S_2(e_3), S_2(e_4)\}$ as in Figs. 4, 5, 6, 7.

The Cross product of G_1 and G_2 is $G_1 \times G_2 = S, O \times T$, where $O \times T = (e_1,e_3),(e_1,e_4),(e_2,e_3),(e_2,e_4), S(e_1, e_3) = S_1(e_1)\ \times\ S_2(e_3), S(e_1,e_4) = S_1(e_1) \times S_2(e_4), S(e_2, e_3) = S_1(e_2) \times\ S_2(e_3)$ and $S(e_2,e_4) = S_1(e_2) \times S_2(e_4)$ are picture fuzzy soft graphs of $G_1 \times G_2$. $S(e_1,e_3) = S_1(e_1) \times S_2(e_3)$ is shown in Fig. 9. In the similar way the cross product of $S(e_1,e_4)$, $S(e_2,e_3)$, $S(e_2,e_4)$ can be drawn.

Theorem 36 *The cross product of two PFSG is also a PFSG.*

Proof Let $G_1 = (F_1, K_1, O)$ & $G_2 = (F_2, K_2, T)$ be PFSG of $G_1^* = (V_1, E_1)$ & $G_2^* = (V_2, E_2)$, respectively. Let the cross product be $G = G_1 \circ G_2 = (F, K, O \times T)$. We claim that $G = (F, K, O \times T)$ is a PFSG and $(S, O \times T) = \{F_1 \circ F_2(a_i, b_j), K_1 \circ K_2(a_i, b_j)\}\ \forall\ a_i$ in O, b_j in T for $i = 1$ to m, $j = 1$ to n are PFSG of G.

Consider

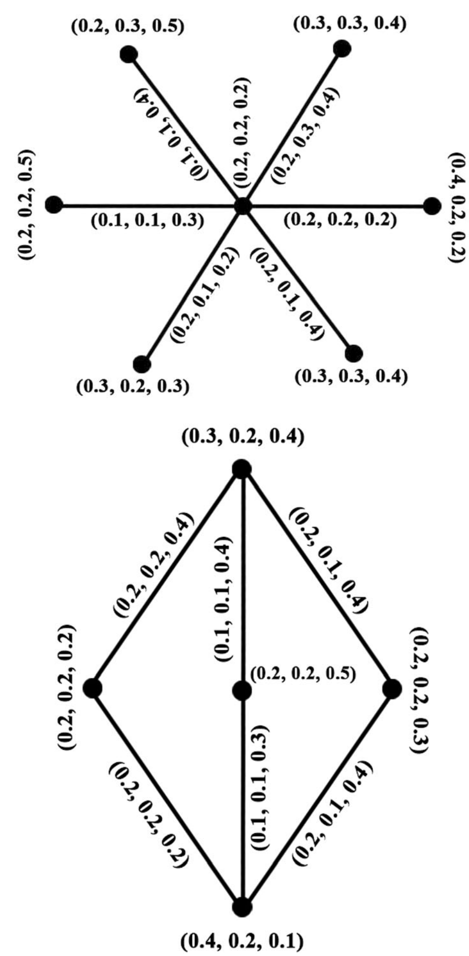

Fig. 9 Cross product $S(e_1, e_3)$

$$\mu_{K_{(a_i,b_j)}}((c_1,t_1),(c_2,t_2)) = \min\{\mu_{K_1(a_i)}(c_1,c_2),\mu_{K_2(b_j)}(t_1,t_2)\}$$

for $i=1$ to m, $j=1$ to n

$$\leq \min\{\min\{\mu_{F_1(a_i)}(c_1),\mu_{F_1(a_i)}(c_2)\}, \min\{\mu_{F_2(b_j)}(t_1),\mu_{F_2(b_j)}(t_2)\}\}$$
$$= \min\{\min\{\mu_{F_1(a_i)}(c_1),\mu_{F_2(b_j)}(t_1)\}, \min\{\mu_{F_1(a_i)}(c_2),\mu_{F_2(b_j)}(t_2)\}\}$$
$$\mu_{K_{(a_i,b_j)}}((c_1,t_1),(c_2,t_2)) \leq \min\{(\mu_{F_1(a_i)}\circ\mu_{F_2(b_j)})(c_1,t_1), (\mu_{F_1(a_i)}\circ\mu_{F_2(b_j)})(c_2,t_2)\}$$

for $i=1$ to $m, j=1$ to n.

$$\gamma_{K_{(a_i,b_j)}}((c_1,t_1),(c_2,t_2)) = \min\{\gamma_{K_1(a_i)}(c_1,c_2),\gamma_{K_2(b_j)}(t_1,t_2)\}$$

for $i=1$ to $m, j=1$ to n.

$$\leq \min\{\min\{\gamma_{F_1(a_i)}(c_1),\gamma_{F_1(a_i)}(c_2)\}, \min\{\gamma_{F_2(b_j)}(t_1),\gamma_{F_2(b_j)}(t_2)\}\}$$
$$= \min\{\min\{\gamma_{F_1(a_i)}(c_1),\gamma_{F_2(b_j)}(t_1)\}\}, \{\min\{\gamma_{F_1(a_i)}(c_2),\gamma_{F_2(b_j)}(t_2)\}\}$$
$$\sigma_{K_{(a_i,b_j)}}((c_1,t_1),(c_2,t_2)) \leq \min\{(\gamma_{F_1(a_i)}\circ\gamma_{F_2(b_j)})(c_1,t_1), \gamma_{F_2(a_i)}\circ\gamma_{F_2(b_j)}(c_2,t_2)\}$$

for $i=1$ to $m, j=1$ to n.

$$\sigma_{K_{(a_i,b_j)}}((c_1,t_1),(c_2,t_2)) = \max\{\sigma_{K_1(a_i)}(c_1,c_2),\sigma_{K_2(b_j)}(t_1,t_2)\}$$

for $i=1$ to $m, j=1$ to n.

$$\leq \max\{\max\{\sigma_{F_1(a_i)}(c_1),\sigma_{F_1(a_i)}(c_2)\}, \max\{\sigma_{F_2(b_j)}(t_1),\sigma_{F_2(b_j)}(t_2)\}\}$$
$$= \max\{\max\{\sigma_{F_1(a_i)}(c_1),\sigma_{F_2(b_j)}(t_1)\}, \max\{\sigma_{F_1(a_i)}(c_2),\sigma_{F_2(b_j)}(t_2)\}\}$$
$$\sigma_{K_{(a_i,b_j)}}((c_1,t_1),(c_2,t_2)) \leq \max\{(\sigma_{F_1(a_i)}\circ\sigma_{F_2(b_j)})(c_1,t_1), \sigma_{F_2(a_i)}\circ\sigma_{F_2(b_j)}(c_2,t_2)\}$$

for $i=1$ to $m, j=1$ to n.

Hence, $G=(F,K,O\times T)$ is a PFSG. □

Definition 37 The lexicographic product of G_1 & G_2 is $G=G_1\bigodot G_2=(F,K,O\times T)$, where $(F,O\times T)$ is PFSS over $V=V_1\times V_2,(K,O\times T)$ is a PFSS over $E=\{((c,h_1),((c,h_2)):c\in V_1,(h_1,h_2)\in E_2\} \cup\{((c_1,h_1),((c_2,h_2)):(c_1,h_1)\in E_1,(c_2,h_2)\in E_2\}$ and $(F,K,O\times T)$ are PFSG such that

1. $$\mu_{F(p,q)}(c,h)=\mu_{F_1(p)}(c)\wedge\mu_{F_2(q)}(h)$$
$$\gamma_{F(p,q)}(c,h)=\gamma_{F_1(p)}(c)\wedge\gamma_{F_2(q)}(h)$$
$$\sigma_{F(p,q)}(c,h)=\sigma_{F_1(p)}(c)\vee\sigma_{F_2(q)}(h)$$
$$\forall(c,h)\in V,(p,q)\in O\times T.$$

2. $$\mu_{K(p,q)}\ ((c,h_1),(c,h_2))=\mu_{F_1(p)}(c)\wedge\mu_{K_2(q)}(h_1,h_2)$$
$$\gamma_{K(p,q)}\ ((c,h_1),(c,h_2))=\gamma_{F_1(p)}(c)\wedge\gamma_{K_2(q)}(h_1,h_2)$$
$$\sigma_{K(p,q)}\ ((c,h_1),(c,h_2))=\sigma_{F_1(p)}(c)\vee\sigma_{K_2(q)}(h_1,h_2)$$
$$\forall\ c\in V_1,(h_1,h_2)\in E_2.$$

3. $$\mu_{K(p,q)}\ ((c_1,h_1),(c_2,h_2)) =\mu_{K_1(p)}(c_1,c_2)\wedge\mu_{K_2(q)}(h_1,h_2)$$
$$\gamma_{K(p,q)}\ ((c_1,h_1),(c_2,h_2)) =\gamma_{K_1(p)}(c_1,c_2)\wedge\gamma_{K_2(q)}(h_1,h_2)$$
$$\sigma_{K(p,q)}\ ((c_1,h_1),(c_2,h_2)) =\sigma_{K_1(p)}(c_1,c_2)\vee\sigma_{K_2(q)}(h_1,h_2)$$
$$\forall(c_1,c_2)\in E_1,(h_1,h_2)\in E_2.$$

Example 38 Consider the graphs G_1 and G_2 from example 1. The lexico product of $S(e_1,e_3)$ is given in Fig. 10. In the similar way, the lexico product of $S(e_1,e_4)$, $S(e_2,e_3)$, $S(e_2,e_4)$ can be drawn.

Theorem 39 *The lexicographic product of two PFSG is a PFSG.*

Proof $G_1=(F_1,K_1,O)$ and $G_2=(F_2,K_2,T)$ be PFSG of $G_1^*=(V_1,E_1)$ and $G_2^*=(V_2,E_2)$, respectively. Let $G_1\bigodot G_2=(F,K,O\times T)$, be composition of G_1 and G_2. We claim that $G_1\bigodot G_2=G=(F,K,O\times T)$ is a PFSG and $(S,O\times T)=\{F_1(a_i)\bigodot F_2(b_j),K_1(a_i)\bigodot K_2(a_i)\}$ $\forall$ $a_i\in O,b_j\in T$ for $i=1$ to m, $j=1$ to n are PFG of G. Let $q\in V_1$ and $(w_1,w_2)\in E_2$, we have

$$\mu_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) =\min\{\mu_{F_1(a_i)}(q),\mu_{F_2(b_j)}(w_1,w_2)\}$$

for $i=1$ to m, $j=1$ to n.

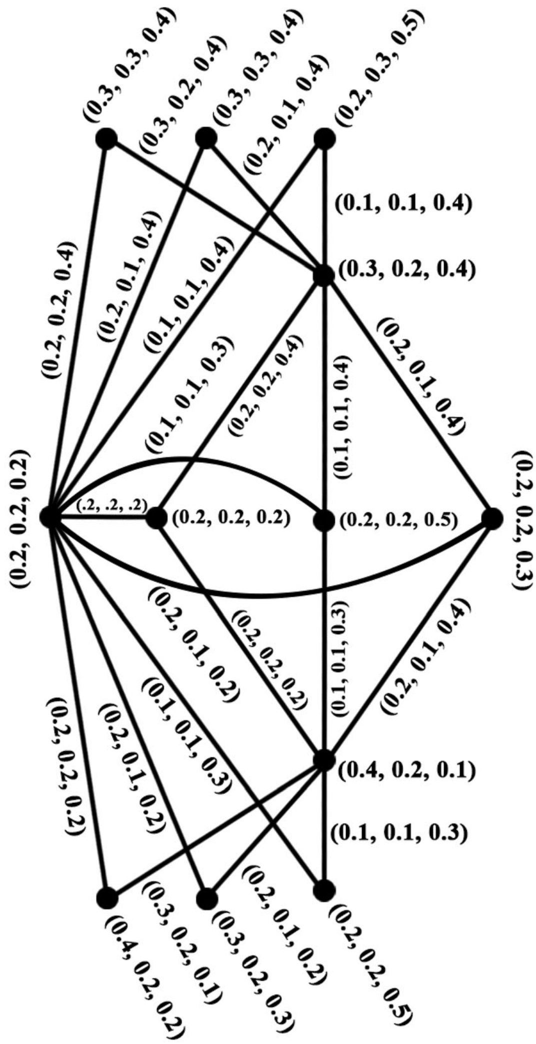

Fig. 10 Lexico product $S(e_1, e_3)$

$$\mu_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) \le \min\{\mu_{F_1(a_i)}(q), \min\{\mu_{F_2(b_j)}(w_1, \mu_{F_2(b_j)}(w_2)\}\} = \min\{\min\{\mu_{F_1(a_i)}(q), \mu_{F_2(b_j)}(w_1)\}, \min\{\mu_{F_1(a_i)}(q), \mu_{F_2(b_j)}(w_2)\}\} = \min\{(\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(q,w_1), (\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(q,w_2)\}$$

$$\mu_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) \le \min\{\mu_{F(a_i,b_j)}(q,w_1), \mu_{F(a_i,b_j)}(q,w_2)\}$$

$$\gamma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) = \min\{\gamma_{F_1(a_i)}(q), \gamma_{K_2(b_j)}(w_1,w_2)\}$$

for $i = 1$ to m, $j = 1$ to n.

$$\gamma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) \le \min\{\min\{\gamma_{F_1(a_i)}(q), \min\{\gamma_{F_2(b_j)}(w_1), \gamma_{F_2(b_j)}(w_2)\}\}\} = \min\{\min\{\gamma_{F_1(a_i)}(q), \gamma_{F_2(b_j)}(w_1)\}, \min\{\gamma_{F_1(a_i)}(q), \gamma_{F_2(b_j)}(w_2)\}\} = \min\{(\gamma_{F_1(a_i)} \times \gamma_{F_2(b_j)})(q,w_1), (\gamma_{F_1(a_i)} \times \gamma_{F_2(b_j)})(q,w_2)\}$$

$$\mu_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) \le \min\{\mu_{F(a_i,b_j)}(q,w_1), \mu_{F(a_i,b_j)}(q,w_2)\}$$

$$\sigma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) = \max\{\sigma_{F_1(a_i)}(q), \sigma_{K_2(b_j)}(w_1,w_2)\}$$

for $i = 1$ to m, $j = 1$ to n.

$$\sigma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) \le \max\{\sigma_{F_1(a_i)}(q), \max\{\sigma_{F_2(b_j)}(w_1), \sigma_{F_2(b_j)}(w_2)\}\} = \min\{\max\{\sigma_{F_1(a_i)}(q), \sigma_{F_2(b_j)}(w_1)\}, \max\{\sigma_{F_1(a_i)}(q), \sigma_{F_2(b_j)}(w_2)\}\} = \max\{(\sigma_{F_1(a_i)} \times \sigma_{F_2(b_j)})(q,w_1), (\sigma_{F_1(a_i)} \times \sigma_{F_2(b_j)})(q,w_2)\}$$

$$\sigma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) \le \max\{\sigma_{F(a_i,b_j)}(q,w_1), \sigma_{F(a_i,b_j)}(q,w_2)\}.$$

Consider

$$\mu_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) = \min\{\mu_{K_1(a_i)}(c_1,c_2), \mu_{K_2(b_j)}(h_1,h_2)\}$$

for $i = 1$ to m, $j = 1$ to n

$$\le \min\{\min\{\mu_{F_1(a_i)}(c_1), \mu_{F_1(a_i)}(c_2)\}, \min\{\mu_{F_2(b_j)}(h_1), \mu_{F_2(b_j)}(h_2)\}\} = \min\{\min\{\mu_{F_1(a_i)}(c_1), \mu_{F_2(b_j)}(h_1)\}, \min\{\mu_{F_1(a_i)}(c_2), \mu_{F_2(b_j)}(h_2)\}\}$$

$$\mu_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) \le \min\{\mu_{F(a_i,b_j)}(c_1,h_1), \mu_{F(a_i,b_j)}(c_2,h_2)\}$$

for $i = 1$ to m, $j = 1$ to n.

$$\gamma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) = \min\{\gamma_{K_1(a_i)}(c_1,c_2), \gamma_{K_2(b_j)}(h_1,h_2)\}$$

for $i = 1$ to m, $j = 1$ to n

$$\leq \min\{\min\{\gamma_{F_1(a_i)}(c_1), \gamma_{F_1(a_i)}(c_2)\}, \min\{\gamma_{F_2(b_j)}(h_1), \gamma_{F_2(b_j)}(h_2)\}\}$$
$$= \min\{\min\{\gamma_{F_1(a_i)}(c_1), \gamma_{F_2(b_j)}(h_1)\}\}, \{\min\{\gamma_{F_1(a_i)}(c_2), \gamma_{F_2(b_j)}(h_2)\}\}$$
$$\gamma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) \leq \min\{\gamma_{F(a_i,b_j)}(c_1,h_1), \gamma_{F(a_i,b_j)}(c_2,h_2)\}$$

for $i = 1$ to m, $j = 1$ to n.

$$\sigma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) \leq \min\{(\gamma_{F_1(a_i)} \circ \gamma_{F_2(b_j)})(c_1,h_1), (\gamma_{F_2(a_i)} \circ \gamma_{F_2(b_j)})(c_2,h_2)\}$$

for $i = 1$ to m, $j = 1$ to n

$$\sigma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) = \max\{\sigma_{K_1(a_i)}(c_1,c_2), \sigma_{K_2(b_j)}(h_1,h_2)\}$$

for $i = 1$ to m, $j = 1$ to n

$$\leq \max\{\max\{\sigma_{F_1(a_i)}(c_1), \sigma_{F_1(a_i)}(c_2)\}, \max\{\sigma_{F_2(b_j)}(h_1), \sigma_{F_2(b_j)}(h_2)\}\}$$
$$= \max\{\max\{\sigma_{F_1(a_i)}(c_1), \sigma_{F_2(b_j)}(h_1)\}, \max\{\sigma_{F_1(a_i)}(c_2), \sigma_{F_2(b_j)}(h_2)\}\}$$
$$\sigma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) \leq \max\{\sigma_{F(a_i,b_j)}(c_1,h_1), \mu_{F(a_i,b_j)}(c_2,h_2)\}$$

for $i = 1$ to m, $j = 1$ to n.

Hence, the claim. □

Definition 40 The complement of PFSG $G = (F, K, O)$ is represented by $G^c = (F^c, K^c, O^c)$ and is defined as

1. $O^c = O$.
2. $F^c(e) = F(e)$.
3. $\mu_{K^c(e)}(u,v) = \mu_{F(e)}(u) \wedge \mu_{F(e)}(v) - \mu_{K(e)}(u,v)$.
4. $\gamma_{K^c(e)}(u,v) = \gamma_{F(e)}(u) \wedge \gamma_{F(e)}(v) - \gamma_{K(e)}(u,v)$.
5. $\sigma_{K^c(e)}(u,v) = \sigma_{F(e)}(u) \vee \sigma_{F(e)}(v) - \sigma_{K(e)}(u,v)$.

Example 41 Consider a graph $G^* = (V, E)$, where $V = \{a_1, a_2, a_3, a_4\}$ and $E = \{a_1a_2, a_2a_3, a_3a_4\}$. Let $O = \{e_1\}$ and let (F, O), (K, O) be the picture fuzzy soft sets over V, E correspondingly and functions $F : O \to P(V)$, $K : O \to P(E)$ be given by

$$F(e_1) = \{(a_1, .4, .3, .2), (a_2, .3, .4, .3), (a_3, .2, .3, .4), (a_4, .3, .2, .3)\}$$
$$K(e_1) = \{(a_1a_2, .3, .2, .2), (a_3a_4, .1, .1, .3), (a_2a_3, .2, .2, .3)\}.$$

The PFSG $G = \{S(e_1)\}$ is shown in Fig. 11. The complement of PFSG is in Fig. 12.

Definition 42 The strong product of G_1, G_2 is a PFSG $G = G_1 \bigotimes G_2 = (F, K, O \times T)$, where $(F, O \times T)$ is a PFSS over $V = V_1 \times V_2$, $(K, O \times T)$ is a PFSS over $E = \{((k,b_1),((k,b_2)) : k \in V_1, (b_1,b_2) \in E_2\} \cup \{((k_1,b),(k_2,b)) : b \in V_2, (k_1,k_2) \in E_1\} \cup \{((k_1,b_1),(k_2,b_2)) : (k_1,k_2) \in k_2, (b_1,b_2) \in E_2\}$ and $(F, K, O \times T)$ are PFSG such that

1. $$\mu_{F(h,g)}(k,b) = \mu_{F_1(h)}(k) \wedge \mu_{F_2(g)}(b)$$
$$\gamma_{F(h,g)}(k,b) = \gamma_{F_1(h)}(k) \wedge \gamma_{F_2(g)}(b)$$
$$\sigma_{F(h,g)}(k,b) = \sigma_{F_1(h)}(k) \vee \sigma_{F_2(g)}(b)$$
$$\forall\ (k,b) \in V, (h,g) \in O \times T.$$

2. $$\mu_{K(h,g)}\ ((k,b_1),(k,b_2)) = \mu_{F_1(h)}(k) \wedge \mu_{F_2(g)}(b_1,b_2)$$
$$\gamma_{K(h,g)}\ ((k,b_1),(k,b_2)) = \gamma_{F_1(h)}(k) \wedge \gamma_{F_2(g)}(b_1,b_2)$$
$$\sigma_{K(h,g)}\ ((k,b_1),(k,b_2)) = \sigma_{F_1(h)}(k) \vee \sigma_{F_2(g)}(b_1,b_2)$$
$$\forall\ k \in V_1, (b_1,b_2) \in E_2.$$

3. $$\mu_{K(h,g)}\ ((k_1,b),(k_2,b)) = \mu_{F_2(g)}(b) \wedge \mu_{F_1(h)}(k_1,k_2)$$
$$\gamma_{K(h,g)}\ ((k_1,b),(k_2,b)) = \gamma_{F_2(g)}(b) \wedge \gamma_{F_1(h)}(k_1,k_2)$$
$$\sigma_{K(h,g)}\ ((k_1,b),(k_2,b)) = \sigma_{F_2(g)}(b) \vee \sigma_{F_1(h)}(k_1,k_2)$$
$$\forall\ b_1 \in V_2, (k_1,k_2) \in E_1.$$

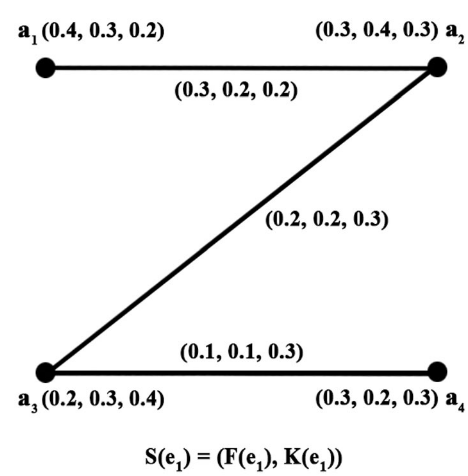

Fig. 11 Picture fuzzy soft graph $G = \{S(e_1)\}$

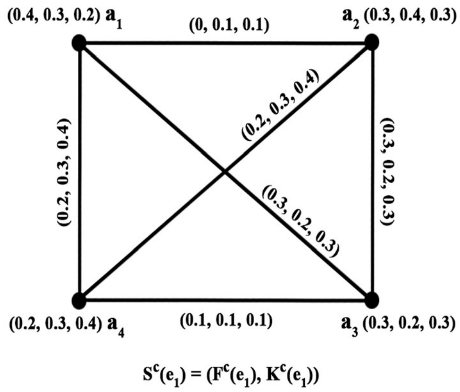

Fig. 12 Complement picture fuzzy soft graph $G^c = \{S^c(e_1)\}$

4. $\mu_{K(h,g)}\ ((k_1,b_1),(k_2,b_2))$
$= \mu_{K_1(h)}(k_1,k_2) \wedge \mu_{K_2(g)}(b_1,b_2)$
$\gamma_{K(h,g)}\ ((k_1,b_1),(k_2,b_2))$
$= \gamma_{K_1(h)}(k_1,k_2) \wedge \gamma_{K_2(g)}(b_1,b_2)$
$\sigma_{K(h,g)}\ ((k_1,b_1),(k_2,b_2))$
$= \sigma_{K_1(h)}(k_1,k_2) \vee \sigma_{K_2(g)}(b_1,b_2)$
$\forall\ (k_1,k_2) \in E_1, (b_1,b_2) \in E_2.$

$S(h,g) = S_1(h) \bigotimes S_2(g)\ \forall\ (h,g) \in O \times T$ are PFSG of G.

Example 43 Consider the graphs G_1 and G_2 from example 1. The strong product of $S(e_1,e_3)$ is given in Fig. 13. In the similar way the strong product of $S(e_1,e_4)$, $S(e_2,e_3)$, $S(e_2,e_4)$ can be drawn.

Theorem 44 *The strong product of two PFSG is also a PFSG.*

Proof $G_1 = (F_1, K_1, O)$ and $G_2 = (F_2, K_2, T)$ be PFSG of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Let $G_1 \bigotimes G_2 = (F, K, O \times T)$, be composition of G_1 and G_2. We claim that $G_1 \bigotimes G_2 = G = (F, K, O \times T)$ is a PFSG and $(S, O \times T) = \{F_1(a_i) \bigotimes F_2(b_j), K_1(a_i) \bigotimes K_2(a_i)\}\ \ \forall$ $a_i \in O, b_j \in T$ for $i = 1$ to m, $j = 1$ to n are PFG of G. Let $h \in V_1$ and $(g_1, g_2) \in E_2$, we have

$$\mu_{K_{(a_i,b_j)}}((h,g_1),(h,g_2)) = \min\{\mu_{F_1(a_i)}(h), \mu_{F_2(b_j)}(g_1,g_2)\}$$

for $i = 1$ to m, $j = 1$ to n.

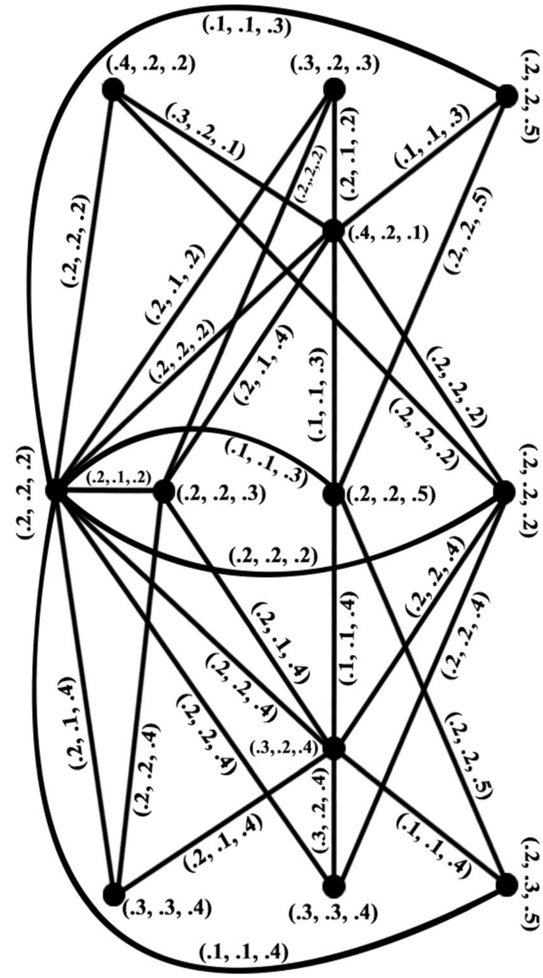

Fig. 13 Strong product $G = \{S(e_1, e_3)\}$

$$\mu_{K_{(a_i,b_j)}}((h,g_1),(h,g_2))$$
$$\leq \min\{\mu_{F_1(a_i)}(h), \min\{\mu_{F_2(b_j)}(g_1), \mu_{F_2(b_j)}(g_2)\}\}$$
$$= \min\{\min\{\mu_{F_1(a_i)}(h), \mu_{F_2(b_j)}(g_1)\},$$
$$\min\{\mu_{F_1(a_i)}(h), \mu_{F_2(b_j)}(g_2)\}\}$$
$$= \min\{(\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(h,g_1),$$
$$(\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(h,g_2)\}\mu_{K_{(a_i,b_j)}}((h,g_1),(h,g_2))$$
$$\leq \min\{\mu_{F(a_i,b_j)}(h,g_1), \mu_{F(a_i,b_j)}(h,g_2)\}$$

$$\gamma_{K_{(a_i,b_j)}}((h,g_1),(h,g_2)) = \min\{\gamma_{F_1(a_i)}(h), \gamma_{K_2(b_j)}(g_1,g_2)\}$$

for $i = 1$ to m, $j = 1$ to n.

$$\gamma_{K_{(a_i,b_j)}}((h,g_1),(h,g_2)) \leq \min\{\min\{\gamma_{F_1(a_i)}(h), \min\{\gamma_{F_2(b_j)}(g_1),\gamma_{F_2(b_j)}(g_2)\}\}\}$$
$$= \min\{\min\{\gamma_{F_1(a_i)}(h),\gamma_{F_2(b_j)}(g_1)\}, \min\{\gamma_{F_1(a_i)}(h),\gamma_{F_2(b_j)}(g_2)\}\}$$
$$= \min\{(\gamma_{F_1(a_i)}\times\gamma_{F_2(b_j)})(h,g_1), (\gamma_{F_1(a_i)}\times\gamma_{F_2(b_j)})(h,g_2)\}\mu_{K_{(a_i,b_j)}}((h,g_1),(h,g_2))$$
$$\leq \min\{\mu_{F(a_i,b_j)}(h,g_1),\mu_{F(a_i,b_j)}(h,g_2)\}$$
$$\sigma_{K_{(a_i,b_j)}}((h,g_1),(h,g_2)) = \max\{\sigma_{F_1(a_i)}(h),\sigma_{K_2(b_j)}(g_1,g_2)\}$$

for $i=1$ to m, $j=1$ to n.

$$\sigma_{K_{(a_i,b_j)}}((h,g_1),(h,g_2)) \leq \max\{\sigma_{F_1(a_i)}(h), \max\{\sigma_{F_2(b_j)}(g_1),\sigma_{F_2(b_j)}(g_2)\}\}$$
$$= \min\{\max\{\sigma_{F_1(a_i)}(h),\sigma_{F_2(b_j)}(g_1)\}, \max\{\sigma_{F_1(a_i)}(h),\sigma_{F_2(b_j)}(g_2)\}\}$$
$$= \max\{(\sigma_{F_1(a_i)}\times\sigma_{F_2(b_j)})(h,g_1), (\sigma_{F_1(a_i)}\times\sigma_{F_2(b_j)})(h,g_2)\}\sigma_{K_{(a_i,b_j)}}((h,g_1),(h,g_2))$$
$$\leq \max\{\sigma_{F(a_i,b_j)}(h,g_1),\sigma_{F(a_i,b_j)}(h,g_2)\}.$$

Similarly, for any $w \in V_2$ and $(q_1,q_2) \in E_1$, we have

$$\mu_{K_{(a_i,b_j)}}((q_1,w),(q_2,w)) \leq \min\{\mu_{F(a_i,b_j)}(q_1,w),\mu_{F(a_i,b_j)}(q_2,w)\}$$
$$\gamma_{K_{(a_i,b_j)}}((q_1,w),(q_2,w) \leq \min\{\gamma_{F(a_i,b_j)}(q_1,w),\gamma_{F(a_i,b_j)}(q_2,w)\}$$
$$\sigma_{K_{(a_i,b_j)}}((q_1,w),(q_2,w)) \leq \min\{\sigma_{F(a_i,b_j)}(q_1,w),\sigma_{F(a_i,b_j)}(q_2,w)\}.$$

Consider

$$\mu_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) = \min\{\mu_{K_1(a_i)}(c_1,c_2),\mu_{K_2(b_j)}(h_1,h_2)\}$$

for $i=1$ to m, $j=1$ to n.

$$\leq \min\{\min\{\mu_{F_1(a_i)}(c_1),\mu_{F_1(a_i)}(c_2)\}, \min\{\mu_{F_2(b_j)}(h_1),\mu_{F_2(b_j)}(h_2)\}\}$$
$$= \min\{\min\{\mu_{F_1(a_i)}(c_1),\mu_{F_2(b_j)}(h_1)\}, \min\{\mu_{F_1(a_i)}(c_2),\mu_{F_2(b_j)}(h_2)\}\}$$
$$\mu_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) \leq \min\{\mu_{F(a_i,b_j)}(c_1,h_1),\mu_{F(a_i,b_j)}(c_2,h_2)\}$$

for $i=1$ to m, $j=1$ to n.

$$\gamma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) = \min\{\gamma_{K_1(a_i)}(c_1,c_2),\gamma_{K_2(b_j)}(h_1,h_2)\}$$

for $i=1$ to m, $j=1$ to n.

$$\leq \min\{\min\{\gamma_{F_1(a_i)}(c_1),\gamma_{F_1(a_i)}(c_2)\}, \min\{\gamma_{F_2(b_j)}(h_1),\gamma_{F_2(b_j)}(h_2)\}\}$$
$$= \min\{\min\{\gamma_{F_1(a_i)}(c_1),\gamma_{F_2(b_j)}(h_1)\}\}, \{\min\{\gamma_{F_1(a_i)}(c_2),\gamma_{F_2(b_j)}(h_2)\}\}$$
$$\gamma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) \leq \min\{\gamma_{F(a_i,b_j)}(c_1,h_1),\gamma_{F(a_i,b_j)}(c_2,h_2)\}$$

for $i=1$ to m, $j=1$ to n.

$$\sigma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) \leq \min\{(\gamma_{F_1(a_i)}\circ\gamma_{F_2(b_j)})(c_1,h_1),\gamma_{F_2(a_i)}\circ\gamma_{F_2(b_j)}(c_2,h_2)\}$$

for $i=1$ to m, $j=1$ to n.

$$\sigma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) = \max\{\sigma_{K_1(a_i)}(c_1,c_2),\sigma_{K_2(b_j)}(h_1,h_2)\}$$

for $i=1$ to m, $j=1$ to n.

$$\leq \max\{\max\{\sigma_{F_1(a_i)}(c_1),\sigma_{F_1(a_i)}(c_2)\}, \max\{\sigma_{F_2(b_j)}(h_1),\sigma_{F_2(b_j)}(h_2)\}\}$$
$$= \max\{\max\{\sigma_{F_1(a_i)}(c_1),\sigma_{F_2(b_j)}(h_1)\}, \max\{\sigma_{F_1(a_i)}(c_2),\sigma_{F_2(b_j)}(h_2)\}\}$$
$$\sigma_{K_{(a_i,b_j)}}((c_1,h_1),(c_2,h_2)) \leq \max\{\sigma_{F(a_i,b_j)}(c_1,h_1),\mu_{F(a_i,b_j)}(c_2,h_2)\}$$

for $i=1$ to m, $j=1$ to n.

Hence, $G=(F,K,O\times T)$ is a PFSG. □

Definition 45 The composition of G_1, and G_2 is $G=G_1[G_2]=(F,K,O\times T)$, where $(F,O\times T)$ is a PFSS over $V=V_1\times V_2$, $(K,O\times T)$ is a PFSS over $E=\{((w,d_1),((w,d_2)) : w\in V_1,(d_1,d_2)\in E_2\}\cup$ $\{((w_1,d),(w_2,d)) : d\in V_2,(w_1,w_2)\in E_1\}\cup$ $\{((w_1,d_1),(w_2,d_2)) : (w_1,w_2)\in E_1, d_1\neq d_2\}$ and $(F,K,O\times T)$ are PFSG such that

1. $$\mu_{F(t,g)}(w,d)=\mu_{F_1(t)}(w)\wedge\mu_{F_2(g)}(d)$$
$$\gamma_{F(t,g)}(w,d)=\gamma_{F_1(t)}(w)\wedge\gamma_{F_2(g)}(d)$$
$$\sigma_{F(t,g)}(w,d)=\sigma_{F_1(t)}(w)\vee\sigma_{F_2(g)}(d)$$
$$\forall\ (w,d)\in V,(t,g)\in O\times T.$$

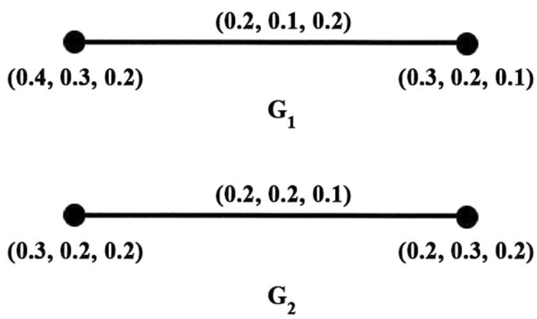

Fig. 14 Picture fuzzy soft graph G_1 and G_2

2. $$\mu_{K(t,g)}\ ((w,d_1),(w,d_2)) = \mu_{F_1(t)}(w) \wedge \mu_{K_2(g)}(d_1,d_2)$$
$$\gamma_{K(t,g)}\ ((w,d_1),(w,d_2)) = \gamma_{F_1(t)}(w) \wedge \gamma_{K_2(g)}(d_1,d_2)$$
$$\sigma_{K(t,g)}\ ((w,d_1),(w,d_2)) = \sigma_{F_1(t)}(w) \vee \sigma_{K_2(g)}(d_1,d_2)$$
$$\forall\ w \in V_1, (s_1,s_2) \in E_2.$$

3. $$\mu_{K(t,g)}\ ((w_1,d),(w_2,d)) = \mu_{F_2(g)}(d) \wedge \mu_{F_1(t)}(w_1,w_2)$$
$$\gamma_{K(t,g)}\ ((w_1,d),(w_2,d)) = \gamma_{F_2(g)}(d) \wedge \gamma_{F_1(t)}(w_1,w_2)$$
$$\sigma_{K(t,g)}\ ((w_1,d),(w_2,d)) = \sigma_{F_2(g)}(d) \vee \sigma_{F_1(t)}(w_1,w_2)$$
$$\forall\ d \in V_2, (w_1,w_2) \in E_1.$$

4. $$\mu_{K(t,g)}\ ((u_1,d_1),(w_2,d_2)) = \mu_{K_1(t)}(w_1,w_2) \wedge \mu_{F_2(g)}(d_1) \wedge \mu_{F_2(b)}(d_2)$$
$$\gamma_{K(t,g)}\ ((u_1,d_1),(w_2,d_2)) = \gamma_{K_1(t)}(w_1,w_2) \wedge \gamma_{F_2(g)}(d_1) \wedge \mu_{F_2(b)}(d_2)$$
$$\sigma_{K(t,g)}\ ((w_1,d_1),(w_2,d_2)) = \sigma_{K_1(t)}(w_1,w_2) \vee \sigma_{F_2(g)}(d_1) \vee \mu_{F_2(b)}(d_2)$$
$$\forall\ (w_1,w_2) \in E_1, d_1 \neq d_2.$$

$S(t,g) = S_1(t)\ [S_2(g)]$ for all $(t,g) \in O \times T$ are PFSG of G.

Example 46 The PFSG G_1 and G_2 is given in Fig. 14. The composition of G_1 and G_2 is drawn in Fig. 15.

Theorem 47 *If G_1 and G_2 are PFSG, then $G_1[G_2]$ is a PFSG.*

Proof Let $G_1 = (F_1, K_1, O)$ and $G_2 = (F_2, K_2, T)$ be PFSG of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Let $G_1[G_2] = (F, K, O \times T)$ be the composition of G_1 and G_2. We claim that $G_1[G_2] = G = (F, K, O \times T)$ is a PFSG and $(S, O \times T) = \{F_1(a_i)[F_2(b_j)], K_1(a_i)[K_2(a_i)]\}\ \forall\ a_i \in O, b_j \in T$ for $i = 1$ to m, $j = 1$ to n are PFG of G. Let $q \in V_1$ and $(w_1, w_2) \in E_2$, we have

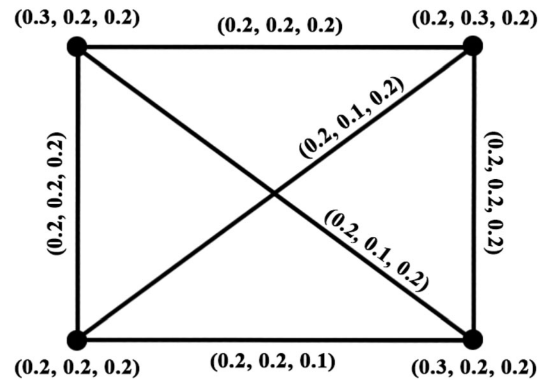

Fig. 15 Composition of G_1 and G_2

$$\mu_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) = \min\{\mu_{F_1(a_i)}(q), \mu_{F_2(b_j)}(w_1,w_2)\} \quad \text{for } i = 1 \text{ to } m, j = 1 \text{ to } n$$
$$\mu_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) \leq \min\{\mu_{F_1(a_i)}(q), \min\{\mu_{F_2(b_j)}(w_1, \mu_{F_2(b_j)}(w_2)\}\}$$
$$= \min\{\min\{\mu_{F_1(a_i)}(q), \mu_{F_2(b_j)}(w_1)\}, \min\{\mu_{F_1(a_i)}(q), \mu_{F_2(b_j)}(w_2)\}\}$$
$$= \min\{(\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(q,w_1), (\mu_{F_1(a_i)} \times \mu_{F_2(b_j)})(q,w_2)\}\mu_{K_{(a_i,b_j)}}((q,w_1),(q,w_2))$$
$$\leq \min\{\mu_{F(a_i,b_j)}(q,w_1), \mu_{F(a_i,b_j)}(q,w_2)\}$$
$$\gamma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) = \min\{\gamma_{F_1(a_i)}(q), \gamma_{K_2(b_j)}(w_1,w_2)\}$$

for $i = 1$ to m, $j = 1$ to n.

$$\gamma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) \leq \min\{\min\{\gamma_{F_1(a_i)}(q), \min\{\gamma_{F_2(b_j)}(w_1), \gamma_{F_2(b_j)}(w_2)\}\}\}$$
$$= \min\{\min\{\gamma_{F_1(a_i)}(q), \gamma_{F_2(b_j)}(w_1)\}, \min\{\gamma_{F_1(a_i)}(q), \gamma_{F_2(b_j)}(w_2)\}\}$$
$$= \min\{(\gamma_{F_1(a_i)} \times \gamma_{F_2(b_j)})(q,w_1), (\gamma_{F_1(a_i)} \times \gamma_{F_2(b_j)})(q,w_2)\}\mu_{K_{(a_i,b_j)}}((q,w_1),(q,w_2))$$
$$\leq \min\{\mu_{F(a_i,b_j)}(q,w_1), \mu_{F(a_i,b_j)}(q,w_2)\}$$
$$\sigma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) = \max\{\sigma_{F_1(a_i)}(q), \sigma_{K_2(b_j)}(w_1,w_2)\}$$

for $i = 1$ to m, $j = 1$ to n.

$$\sigma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2)) \le \max\{\sigma_{F_1(a_i)}(q), \max\{\sigma_{F_2(b_j)}(w_1), \sigma_{F_2(b_j)}(w_2)\}\}$$
$$= \min\{\max\{\sigma_{F_1(a_i)}(q), \sigma_{F_2(b_j)}(w_1)\}, \max\{\sigma_{F_1(a_i)}(q), \sigma_{F_2(b_j)}(w_2)\}\}$$
$$= \max\{(\sigma_{F_1(a_i)} \times \sigma_{F_2(b_j)})(q,w_1), (\sigma_{F_1(a_i)} \times \sigma_{F_2(b_j)})(q,w_2)\}\sigma_{K_{(a_i,b_j)}}((q,w_1),(q,w_2))$$
$$\le \max\{\sigma_{F(a_i,b_j)}(q,w_1), \sigma_{F(a_i,b_j)}(q,w_2)\}.$$

Similarly, for any $w \in V_2$ and $(q_1,q_2) \in E_1$, we have

$$\mu_{K_{(a_i,b_j)}}((q_1,w),(q_2,w)) \le \min\{\mu_{F(a_i,b_j)}(q_1,w), \mu_{F(a_i,b_j)}(q_2,w)\}$$
$$\gamma_{K_{(a_i,b_j)}}((q_1,w),(q_2,w)) \le \min\{\gamma_{F(a_i,b_j)}(q_1,w), \gamma_{F(a_i,b_j)}(q_2,w)\}$$
$$\sigma_{K_{(a_i,b_j)}}((q_1,w),(q_2,w)) \le \min\{\sigma_{F(a_i,b_j)}(q_1,w), \sigma_{F(a_i,b_j)}(q_2,w)\}.$$

Let $(q_1,w),(q_2,w) \in E_1$, and $w_1 \ne w_2$. Then we have

$$\mu_{K_{(a_i,b_j)}}((q_1,w),(q_2,w)) = \min\{\mu_{K_1(a_i)}(q_1,q_2), \mu_{F_2(b_j)}(w_1), \mu_{F_2(b_j)}(w_2)\}$$
$$\le \{\min\{\max\{\mu_{F_1(a_i)}(q_1), \mu_{F_1(a_i)}(q_2)\}, \{\mu_{F_2(b_j)}(w_1), \mu_{F_2(b_j)}(w_2)\}\}\}$$
$$= \min\{\min\{\mu_{F_1(a_i)}(q_1), \mu_{F_2(b_j)}(w_2)\}, \min\{\mu_{F_1(a_i)}(q_2), \mu_{F_2(b_j)}(w_2)\}\}$$
$$\mu_{K_{(a_i,b_j)}}((q_1,w_1),(q_2,w_2)) \le \min\{\mu_{F(a_i,b_j)}(q_1,w_1), \mu_{F(a_i,b_j)}(q_2,w_2)\}$$
$$\gamma_{K_{(a_i,b_j)}}((q_1,w_1),(q_2,w_2)) = \min\{\gamma_{K_1(a_i)}(q_1,w_1), \gamma_{F_2(b_j)}(w_1), \gamma_{F_2(b_j)}(w_2)\}$$
$$\le \min\{\min\{\gamma_{F_1(a_i)}(q_1), \mu_{F_1(a_i)}(q_2)\}, \gamma_{F_2(b_j)}(w_1), \gamma_{F_2(b_j)}(w_1), \gamma_{F_2(b_j)}(w_2)\}$$
$$= \min\{\min\{\gamma_{F_1(a_i)}(q_1), \gamma_{F_2(b_j)}(w_1)\}, \min\{\gamma_{F_1(a_i)}(q_2), \gamma_{F_2(b_j)}(w_2)\}\}$$
$$\gamma_{K_{(a_i,b_j)}}((q_1,w_1),(q_2,w_2)) \le \min\{\gamma_{F(a_i,b_j)}(q_1,w_1), \gamma_{F(a_i,b_j)}(q_2,w_2)\}$$
$$\sigma_{K_{(a_i,b_j)}}((q_1,w_1),(q_2,w_2)) = \max\{\sigma_{K_1(a_i)}(q_1,q_2), \sigma_{F_2(b_j)}(w_1), \sigma_{F_2(b_j)}(w_2)\}$$
$$\le \max\{\max\{\sigma_{F_1(a_i)}(q_1), \sigma_{F_1(a_i)}(q_2)\}, \sigma_{F_2(b_j)}(w_1), \sigma_{F_2(b_j)}(w_1), \gamma_{F_2(b_j)}(w_2)\}$$
$$= \max\{\max\{\sigma_{F_1(a_i)}(q_1), \sigma_{F_2(b_j)}(w_1)\}, \min\{\sigma_{F_1(a_i)}(q_2), \sigma_{F_2(b_j)}(w_2)\}\}$$
$$\sigma_{K_{(a_i,b_j)}}((q_1,w_1),(q_2,w_2)) \le \max\{\sigma_{F(a_i,b_j)}(q_1,w_1), \sigma_{F(a_i,b_j)}(q_2,w_2)\}.$$

Hence, the claim. □

Definition 48 If $G_1 = (F_1, K_1, O)$ and $G_2 = (F_2, K_2, T)$ are two PFSG, the intersection of G_1 and G_2 is a PFSG $G = G_1 \cap G_2 = (F, K, O \cap T)$, where $(F, O \cap T)$ is a PFSS over $V = V_1 \cap V_2, (K, O \cap T)$ is a PFSS over $E = E_1 \cap E_2$, and the PM, NM, NEM functions of $G\, \forall\, r,t \in V$ are PFSG such that

$$\mu_{F(d)}(t) = \begin{cases} \mu_{F_1(d)}(t) & \text{if} d \in O - T; \\ \mu_{F_2(d)}(t) & \text{if} d \in T - O; \\ \mu_{F_1(d)}(t) \wedge \mu_{F_2(d)}(t) & \text{if} d \in O \cap T \end{cases}$$
$$\gamma_{F(d)}(t) = \begin{cases} \gamma_{F_1(d)}(t) \text{if} d \in O - T; \\ \gamma_{F_2(d)}(t) \text{if} d \in T - O; \\ \gamma_{F_1(d)}(t) \wedge \gamma_{F_2(d)}(t) \quad \text{if} d \in O \cap T \end{cases}$$
$$\sigma_{F(d)}(t) = \begin{cases} \sigma_{F_1(d)}(t) \text{if} d \in O - T; \\ \sigma_{F_2(d)}(t) \text{if} d \in T - O; \\ \sigma_{F_1(d)}(t) \vee \sigma_{F_2(d)}(t) \quad \text{if} d \in O \cap T \end{cases}$$
$$\mu_{K(d)}(rt) = \begin{cases} \mu_{K_1(d)}(rt) \text{if} d \in O - T; \\ \mu_{K_2(d)}(rt) \text{if} d \in T - O; \\ \mu_{K_1(d)}(rt) \wedge \mu_{k_2(d)}(rt) \text{if} d \in O \cap T \end{cases}$$
$$\gamma_{K(d)}(rt) = \begin{cases} \gamma_{K_1(d)}(rt) \text{if} d \in O - T; \\ \gamma_{K_2(d)}(rt) \text{if} d \in T - O; \\ \gamma_{K_1(d)}(rt) \wedge \gamma_{K_2(d)}(rt) \text{if} d \in O \cap T \end{cases}$$
$$\sigma_{K(d)}(rt) = \begin{cases} \sigma_{K_1(d)}(rt) \text{if} d \in O - T; \\ \sigma_{K_2(d)}(rt) \text{if} d \in T - O; \\ \sigma_{K_1(d)}(rt) \vee \sigma_{K_2(d)}(rt) \text{if} d \in O \cap T \\ . \end{cases}$$

Example 49 Let $G_1 = (F_1, K_1, O)$ and $G_2 = (F_2, K_2, T)$ are two PFSG as in Figs. 16 and 17. The intersection of G_1 and G_2 is $G = G_1 \cap G_2 = (F, K, O \cap T)$, where $(F, O \cap T)$ is a PFSS over $V = V_1 \cap V_2, (K, O \cap T)$ is a PFSS over $E = E_1 \cap E_2$, which is given in Fig. 18.

Definition 50 A PFSG G is a complete PFSG if $S(e)$ is a complete PFG of G for all $e\ \in O$,

$$\mu_{K(e)}(uv) = \min\{\mu_{F(e)}(u), \mu_{F(e)}(v)\}$$
$$\gamma_{K(e)}(uv) = \min\{\gamma_{F(e)}(u), \gamma_{F(e)}(v)\}$$
$$\sigma_{K(e)}(uv) = \max\{\sigma_{F(e)}(u), \sigma_{F(e)}(v)\}$$
$$\forall\, u,v \in V, e \in O.$$

Example 51 Consider the simple graph $G^* = (V, E)$, where $V = \{b_1, b_2, b_3, b_4\}$ and $E = \{b_1b_2, b_1b_3, b_1b_4, b_1b_5,$

$b_2b_5, b_2b_3, b_3b_5, b_4b_5, b_2b_4, b_3b_4\}$. Let $O = \{e_1, e_2, e_3\}$ and (F, O), (K, O) be the picture fuzzy soft set over V and E correspondingly with functions $F : O \rightarrow P(V)$, $K : O \rightarrow P(E)$ defined by

$$F(e_1) = \{(b_1, .5, .2, .2), (b_2, .4, .3, .2), (b_3, .3, .3, .2), (b_4, .6, .2, .1)\}$$
$$F(e_2) = \{(b_1, .6, .2, .2), (b_3, .5, .3, .1), (b_4, .4, .3, .2)\}$$
$$F(e_3) = \{(b_1, .5, .2, .2), (b_2, .4, .3, .2), (b_3, .3, .3, .2), (b_4, .6, .2, .1), (b_5, .5, .3, .1)\}$$
$$K(e_1) = \{(b_1b_2, .4, .2, .2), (b_1b_3, .3, .2, .2), (b_1b_4, .5, .2, .2), (b_2b_3, .3, .3, .2), (b_2b_4, .4, .2, .2), (b_3b_4, .3, .2, .2)\}$$
$$K(e_2) = \{(b_1b_3, .5, .2, .2), (b_1b_4, .4, .2, .2), (b_3b_4, .4, .3, .2)\}$$
$$K(e_3) = \{K(e_1), (b_1b_5, .5, .2, .2), (b_2b_5, .4, .3, .2), (b_3b_5, .3, .3, .2), (b_4b_5, .5, .2, .1)\}.$$

The complete picture fuzzy soft graph is given in Fig. 19.

Definition 52 If $G_1 = (F_1, K_1, O)$ and $G_2 = (F_2, K_2, T)$ are two PFSG, the union of $G = G_1 \cup G_2 = (F, K, O \cup T)$ is a PFSG, where $(F, O \cup T)$ is a PFSS over $V = V_1 \cup V_2$, $(K, O \cup T)$ is a PFSS over $E = E_1 \cup E_2$, and the PM, NM, NEM functions of $G \ \forall\, t, r \in V$ are defined by

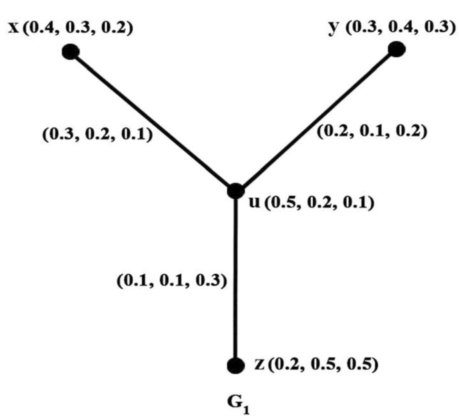

Fig. 16 G_1

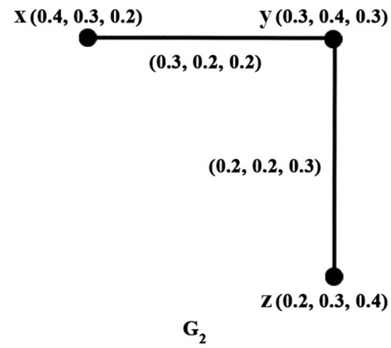

Fig. 17 G_2

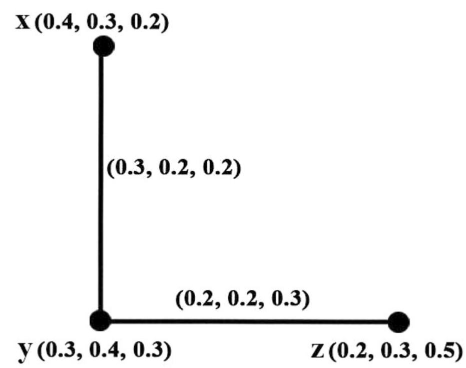

Fig. 18 Intersection of G_1 and G_2

$$\mu_{F(w)}(r) = \begin{cases} \mu_{F_1(w)}(r) \text{if} \quad w \in O - T; \\ \mu_{F_2(w)}(r) \text{if} \quad w \in T - O; \\ \mu_{F_1(w)}(r) \vee \mu_{F_2(w)}(r) \text{if} \quad w \in O \cap T \end{cases}$$

$$\gamma_{F(w)}(r) = \begin{cases} \gamma_{F_1(w)}(r) \text{if} \quad w \in O - T; \\ \gamma_{F_2(w)}(r) \text{if} \quad w \in T - O; \\ \gamma_{F_1(w)}(r) \vee \gamma_{F_2(w)}(r) \text{if} \quad w \in O \cap T \end{cases}$$

$$\sigma_{F(w)}(r) = \begin{cases} \sigma_{F_1(w)}(r) \text{if} \quad w \in O - T; \\ \sigma_{F_2(w)}(r) \text{if} \quad w \in T - O; \\ \sigma_{F_1(w)}(r) \wedge \sigma_{F_2(w)}(r) \text{if} \quad w \in O \cap T \end{cases}$$

$$\mu_{K(w)}(tr) = \begin{cases} \mu_{K_1(w)}(tr) \text{if } w \in O - T; \\ \mu_{K_2(w)}(tr) \text{if } w \in T - O; \\ \mu_{K_1(w)}(tr) \vee \mu_{K_2(w)}(tr) \quad \text{if } w \in O \cap T \end{cases}$$

$$\gamma_{K(w)}(tr) = \begin{cases} \gamma_{K_1(w)}(tr) \text{if } w \in O - T; \\ \gamma_{K_2(w)}(tr) \text{if } w \in T - O; \\ \gamma_{K_1(w)}(tr) \vee \gamma_{K_2(w)}(tr) \quad \text{if } w \in O \cap T \end{cases}$$

$$\sigma_{K(w)}(tr) = \begin{cases} \sigma_{K_1(w)}(tr) \text{if } w \in O - T; \\ \sigma_{K_2(w)}(tr) \text{if } w \in T - O; \\ \sigma_{K_1(w)}(tr) \wedge \sigma_{K_2(w)}(tr) \quad \text{if } w \in O \cap T \end{cases}$$

Example 53 Let $G_1 = (F_1, K_1, O)$ and $G_2 = (F_2, K_2, T)$ are two PFSG as in Figs. 16 and 17. The union of G_1 and G_2 is drawn in Fig. 20.

Definition 54 A PFSG G is a strong PFSG if $S(e)$ is a strong PFG for all $e \in O$.

Example 55 Let G^* be a graph with $V = \{c_1, c_2, c_3, c_4\}$ and $E = \{c_1c_2, c_1c_3, c_1c_4, c_2c_3, c_2c_4, c_3c_4\}$ and $O = \{e_1, e_2, e_3\}$ let (F, O), (K, O) be picture fuzzy soft set over V, E with functions $F : O \to P(V)$, $K : O \to P(E)$ correspondingly and defined by,

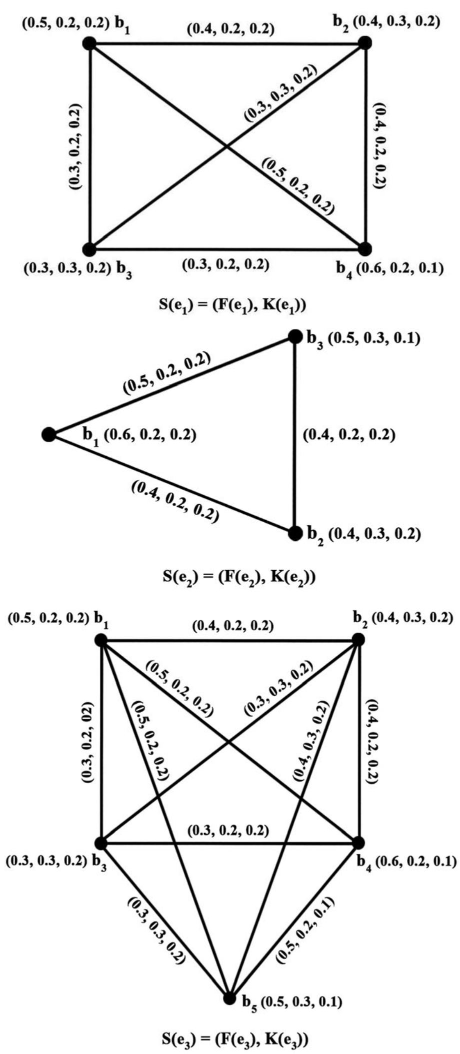

Fig. 19 Complete picture fuzzy soft graph $G = \{S(e_1), S(e_2), S(e_3)\}$

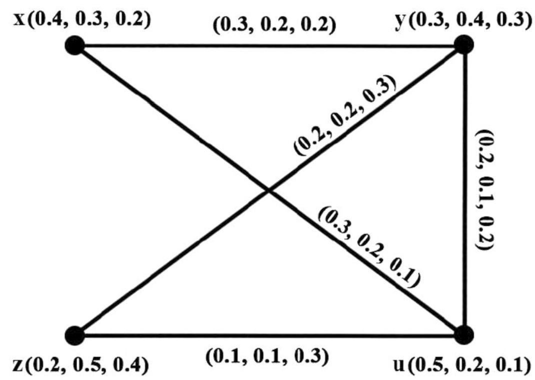

Fig. 20 Union of G_1 and G_2

$$F(e_1) = \{(c_1, .2, .2, .2), (c_2, .3, .3, .4), (c_3, .4, .2, .1)\}$$
$$F(e_2) = \{(c_1, .4, .2, .3), (c_2, .5, .3, .2), (c_3, .4, .1, .1), (c_4, .3, .3, .2)\}$$
$$F(e_3) = \{(c_1, .5, .3, .2), (c_2, .4, .1, .1), (c_3, .3, .4, .2), (c_4, .4, .2, .4)\}$$
$$K(e_1) = \{(c_1c_2, .2, .2, .4), (c_1c_3, .2, .2, .2)\}$$
$$K(e_2) = \{(c_1c_3, .4, .1, .3), (c_1c_4, .3, .2, .3), (c_1c_2, .4, .2, .3), (c_2c_4, .3, .3, .3)\}$$
$$K(e_3) = \{(c_1c_4, .4, .2, .4), (c_1c_3, .3, .3, .2), (c_2c_4, .4, .1, .4)\}$$

The strong picture fuzzy soft graph is given in Fig. 21.

4 Application

4.1 Algorithm

An algorithm for decision making using the proposed Picture fuzzy soft graphs. The notation A denote the attributes, F and K are the mapping from A to $P(V)$ and $P(E)$ and S_e denote the Picture fuzzy soft graph. The algorithm proposed for this PFSG is as follows:

Step 1: Consider the picture fuzzy soft sets (F, A) and (K, A) according to the attributes in A.

Step 2: Draw the PFSG S_e corresponding to each attribute $(e \in A)$ for the considered problem.

Step 3: Calculate the resultant PFSG $(S(e))$ by taking intersection of the PFSG's S_e for each attribute and the adjacency matrix of the resultant matrix $S(e)$.

Step 4: Calculate the score values for the $S(e)$ using the score function

$$\frac{1 + \text{positive} \ - 2(\text{neutral}) \ - \text{negative}}{2}$$

and choice value by adding the score values of a particular element of the universal set.

Step 5: Obtain the optimal decision by choosing the maximum of the calculated choice values.

4.2 Illustration

The coronavirus family causes illnesses ranging from the common cold to more severe diseases such as severe acute respiratory syndrome (SARS) and the Middle East respiratory syndrome (MERS), according to the WHO. The pandemic is affecting different people in different ways. While some try to adapt to working online, homeschooling their children and ordering food via Instacart, others have no choice but to be exposed to the virus while keeping society functioning. We all have been affected by the current COVID-19 pandemic. However, the impact of the pandemic and its consequences are felt differently depending on our status as individuals and as members of society. Common signs of infection include fever, coughing and breathing difficulties. In severe cases, it can cause pneumonia, multiple organ failure and death. The incubation period of COVID-19 is thought to be between one and 14 days. It is contagious before symptoms appear, which is why so many people get infected. For almost a year, the pandemic has locked us all up, and we are still suffering and afraid of COVID-19. For the medical team, treating all the patients is a difficult task. An important decision-making process in the medical team is the selection of the sickest individual to give treatment. It may even be fatal if something is delayed in selecting the treatment for the patients. The main objective is to select and treat patients who are at high risk of COVID to prevent them from becoming more affected. For the treatment of a patient with a high risk of a virus, we propose a decision-making algorithm. To test the possibility of COVID-19, let us consider a set of six patients. Since it is a difficult process and consumes time to choose the most affected individual. Let $V = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ the set of six-person be considered as the Universal set and $A = \{e_1, e_2\}$ be the set of parameters that characterize the risk for patients, the attributes e_1 and e_2 stands for the symptoms and the illness they already have in their system. Consider the picture fuzzy soft set (F, A) over V which describes the "impact of the virus on patients" corresponding to the given parameters. (K, A) is a picture soft set over $E = \{P_1P_2, P_1P_3, P_1P_4, P_1P_5, P_1P_6, P_2P_3, P_2P_4, P_2P_5, P_2P_6, P_3P_4, P_3P_5, P_3P_6, P_4P_5, P_4P_6, P_5P_6\}$ describe the degree of positive, neutral & negative of the relation between patients corresponding to parameters e_1 & e_2. The PFSG's S_{e_1} & S_{e_2} corresponding to attributes 'symptoms' & 'illness', respectively, in Figs. 22 and 23, respectively.

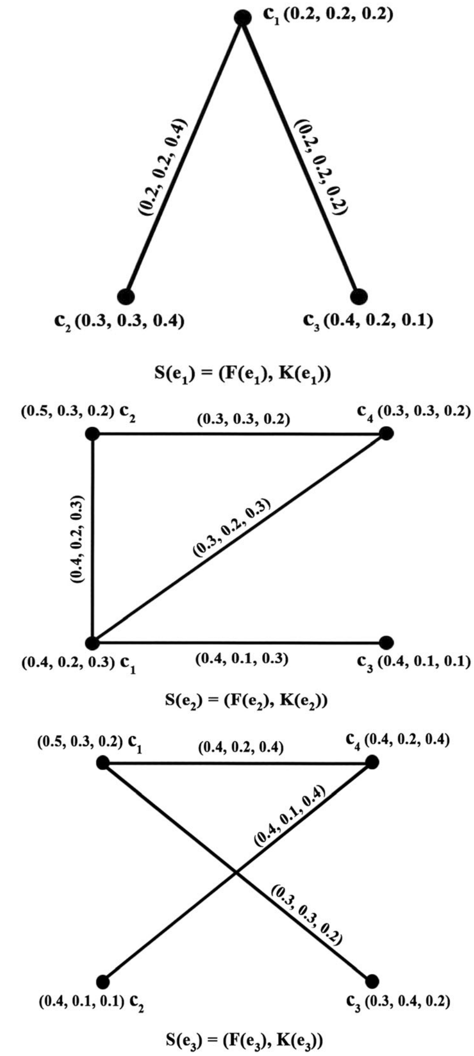

Fig. 21 Strong picture fuzzy soft graph $G = \{S(e_1), S(e_2), S(e_3)\}$

$F(e_1) = \{P_1(0.6, 0.2, 0.1), P_2(0.5, 0.2, 0.3),$
$P_3(0.3, 0.4, 0.3), P_4(0.4, 0.3, 0.2),$
$P_5(0.7, 0.1, 0.1), P_6(0.8, 0.1, 0.1)\}.$
$K(e_1) = \{P_1P_3(.3, .2, .2), P_1P_4(.4, .2, .2),$
$P_1P_5(.6, .1, .1), P_3P_6(.3, .1, .2),$
$P_4P_2(.4, .2, .3), P_4P_6(.4, .1, .2),$
$P_2P_6(.5, .1, .2), P_2P_5(.5, .1, .3),$
$P_5P_6(.7, .1, .1)\}.$

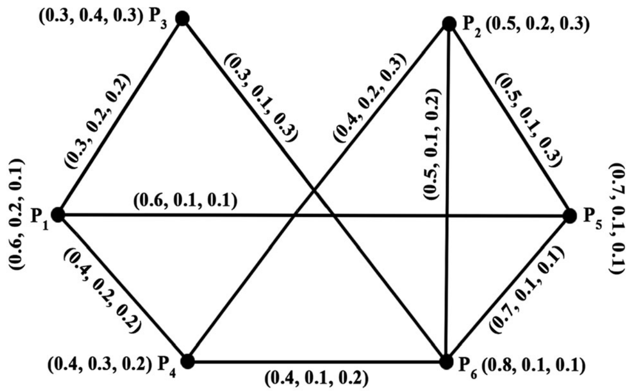

Fig. 22 Picture fuzzy soft graph $S(e_1)$

$F(e_2) = \{P_1(0.8, 0.1, 0.1), P_2(0.7, 0.2, 0.1),$
$P_3(0.5, 0.3, 0.2), P_4(0.4, 0.2, 0.4),$
$P_5(0.3, 0.3, 0.3), P_6(0.6, 0.2, 0.2)\}.$
$K(e_2) = \{P_1P_2(0.7, 0.1, 0.1), P_1P_5(0.3, 0.1, 0.3),$
$P_1P_4(0.4, 0.1, 0.4), P_1P_6(0.6, 0.1, 0.2),$
$P_2P_3(0.5, 0.2, 0.2), P_2P_4(0.4, 0.2, 0.3),$
$P_3P_4(0.4, 0.2, 0.4), P_3P_5(0.3, 0.2, 0.3),$
$P_4P_6(0.4, 0.1, 0.3), P_6P_5(0.3, 0.1, 0.3),$
$P_5P_4(0.3, 0.2, 0.4)\}.$

By taking the intersection of PFSG's, S_{e_1} & S_{e_2} we have a resultant PFSG $S(e)$. The adjacency matrix of resultant PFSG is

$$\begin{bmatrix} (0,0,0) & (0,0,.1) & (0,0,.2) & (.4,.1,.4) & (.3,.1,.3) & (0,0,.2) \\ (0,0,.1) & (0,0,0) & (0,0,.2) & (.4,.2,.3) & (0,0,.3) & (0,0,.2) \\ (0,0,.2) & (0,0,.2) & (0,0,0) & (0,0,.4) & (0,0,.3) & (0,0,.3) \\ (.4,.1,.4) & (.4,.2,.3) & (0,0,.4) & (0,0,0) & (0,0,.4) & (.4,.1,.3) \\ (.3,.1,.3) & (0,0,.3) & (0,0,.3) & (0,0,.4) & (0,0,0) & (.3,.1,.3) \\ (0,0,.2) & (0,0,.2) & (0,0,.3) & (.4,.1,.3) & (.3,.1,.3) & (0,0,0) \\ . & & & & & \end{bmatrix}$$

The score values of the resultant PFSG $S(e)$ is computed with score function

$$\frac{1 + \text{positive} \ - 2(\text{neutral}) \ - \text{negative}}{2}$$

and choice values is given in Table 1.

P_1 has highest risk, first treatment is given for P_1. From Table 1, it follows that the maximum choice value is $P_1 = 2.55$ and so the optimal decision is to select patient 1 that he/she has a high risk of COVID-19.

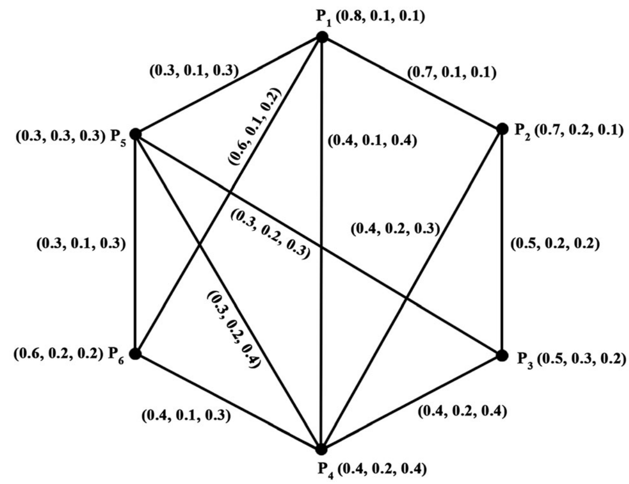

Fig. 23 Picture fuzzy soft graph $S(e_2)$

4.3 Comparison analysis

The proposed model is better than intuitionistic fuzzy models because of the relaxed fuzziness but not more than neutrosophic models and the intuitionistic values can be deduced from picture fuzzy models by just considering the positive and negative membership values. The limitation would be it is an advanced level but little strict criteria in fuzziness compared to neutrosophic fuzzy soft models because of the difference in the sum of the membership values. The proposed method is compared to the standard model in Akram and Shahzadi (2017) and the results are the same though they differ in the models proposed.

5 Conclusion

Picture fuzzy set is a new concept that is a fusion of fuzzy and intuitionistic sets symbolized by positive, negative and neutral degrees. The introduction of this new picture fuzzy soft graphs is an emerging new concept that can be rather developed into various graph theoretical concepts. Our goal was to contribute to the theoretical aspect of fuzzy graph theory we have introduced this Picture fuzzy soft graph and explored its properties and established related theorems. The picture fuzzy soft graphs have been introduced by applying the picture fuzzy soft sets to fuzzy graphs. The PFSG have been defined along with a few of its basic properties and some operations as strong product, lexicographic, cross-product and composition of PFSG have been defined with good examples. The order, size of PFSG, regular, totally regular and perfectly regular picture fuzzy soft graphs have been defined with suitable examples. Since soft sets are most usable in real-life applications, the newly combined concepts of the picture and fuzzy soft sets

Table 1 The score values of $S(e)$ with choice values

	P_1	P_2	P_3	P_4	P_5	P_6	Choice value
P_1	0.5	0.45	0.4	0.4	0.4	0.4	2.55
P_2	0.45	0.5	0.4	0.35	0.35	0.4	2.45
P_3	0.4	0.4	0.5	0.3	0.35	0.35	2.3
P_4	0.4	0.35	0.3	0.5	0.3	0.45	2.3
P_5	0.4	0.35	0.35	0.3	0.5	0.4	2.3
P_6	0.4	0.4	0.35	0.45	0.4	0.5	2.45

will lead to many possible applications in the fuzzy set theoretical area by adding extra fuzziness in analyzing. As a practical application, we have developed a model using this defined graph and applied it in decision making. We have also briefly discussed the application of picture soft fuzzy graphs in decision making for medical diagnosis in the current COVID scenario. In future this work may be extended to the concepts as picture fuzzy irregular graphs and planarity ideas can be explored. Furthermore, many real-life applications can be explored by extending this work to studies on the labelling and energy of PFSG.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest regarding the publication of the research article.

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