

SCIENTIFIC REPORTS



OPEN

Evolution equation for quantum coherence

Ming-Liang Hu¹ & Heng Fan^{2,3}

Received: 26 April 2016

Accepted: 13 June 2016

Published: 07 July 2016

The estimation of the decoherence process of an open quantum system is of both theoretical significance and experimental appealing. Practically, the decoherence can be easily estimated if the coherence evolution satisfies some simple relations. We introduce a framework for studying evolution equation of coherence. Based on this framework, we prove a simple factorization relation (FR) for the l_1 norm of coherence, and identified the sets of quantum channels for which this FR holds. By using this FR, we further determine condition on the transformation matrix of the quantum channel which can support permanently freezing of the l_1 norm of coherence. We finally reveal the universality of this FR by showing that it holds for many other related coherence and quantum correlation measures.

Quantum coherence, an embodiment of the superposition principle of states, lies at the heart of quantum mechanics, and is also a major concern of quantum optics¹. Physically, coherence constitutes the essence of quantum correlations (e.g., entanglement² and quantum discord³) in bipartite and multipartite systems which are indispensable resources for quantum communication and computation tasks. It also finds support in the promising subject of thermodynamics^{4–8} and quantum biology⁹.

Clarifying the decoherence mechanism of a noisy system is an important research direction of quantum mechanics. But due to the lack of rigorous coherence measures, studies in this subject were usually limited to the qualitative analysis. Sometimes, coherence behaviors were also analyzed indirectly via various quantum correlation measures³. However, coherence and quantum correlations are in fact different. Very recently, the characterization and quantification of quantum coherence from a mathematically rigorous and physically meaningful perspective has been achieved¹⁰. This sets the stage for quantitative analysis of coherence, which were carried out mainly around the identification of various coherence monotones^{11–16} and their calculation¹⁷. Some other progresses about coherence quantifiers include their connections with quantum correlations^{18–20}, their behaviors in noisy environments^{21,22}, their local and nonlocal creativity^{23,24}, their distillation^{25,26} and the role they played in the fundamental issue of quantum mechanics^{27–30}.

One major goal of quantum theory is to find effective ways of maintaining the amount of coherence within a system. The reason is twofold. First, coherence represents a basic feature of quantum states, and underpins all forms of quantum correlations¹. Second, coherence itself is a precious resource for many new quantum technologies, but the unavoidable interaction of quantum devices with the environment often decoheres the input states and induces coherence loss, hence damage the superiority of these quantum technologies³¹.

Looking for general law determining the evolution equation of coherence can facilitate the design of effective coherence preservation schemes. Remarkably, the evolution equations for certain entanglement monotones (or their bounds)^{32–40} and geometric discords⁴¹ were found to obey the factorization relation (FR) for specific initial states. Then, it is natural to ask whether there exists similar FR for various coherence monotones. In this work, we aimed at solving this problem. We first classify the general d -dimensional states into different families, and then prove a FR which holds for them. By employing this FR, we further identified condition on the quantum channel for freezing coherence. We also showed that this FR applies to many other coherence and correlation measures. These results are hoped to add another facet to the already rich theory of decoherence, and shed light on revealing the interplay between structures of quantum channel and geometry of the state space, as well as how they determine quantum correlation behaviors of an open system.

Results

Coherence measures. By establishing rigorously the sets \mathcal{I} of incoherent states which are diagonal in the reference basis $\{|i\rangle\}_{i=1,\dots,d}$, and incoherent operations Λ specified by the Kraus operators $\{E_i\}$ which map $\delta \in \mathcal{I}$

¹School of Science, Xi'an University of Posts and Telecommunications, Xi'an 710121, China. ²Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China.

³Collaborative Innovation Center of Quantum Matter, Beijing 100190, China. Correspondence and requests for materials should be addressed to M.-L.H. (email: mingliang0301@163.com)

into \mathcal{I} , Baumgratz *et al.*¹⁰ presented the defining properties for an information-theoretic coherence measure C : (1) $C(\rho) \geq 0$ for all states ρ , and $C(\delta) = 0$ iff $\delta \in \mathcal{I}$. (2) Monotonicity under the actions of Λ , $C(\rho) \geq C(\Lambda(\rho))$. (3) Monotonicity under selective incoherent operations on average, i.e., $C(\rho) \geq \sum_l p_l C(\rho_l)$, where $\rho_l = E_l \rho E_l^\dagger / p_l$, and $p_l = \text{Tr}(E_l \rho E_l^\dagger)$ is the probability of obtaining the outcome l . (4) Convexity, $\sum_l p_l C(\rho_l) \geq C(\sum_l p_l \rho_l)$, with $p_l \geq 0$ and $\sum_l p_l = 1$.

There are several coherence measures satisfying the above conditions. They are the l_1 norm and relative entropy¹⁰, the Uhlmann fidelity¹², the intrinsic randomness¹⁴, and the robustness of coherence⁴². In this work, we concentrate mainly on the l_1 norm of coherence, which is given by $C(\rho) = \sum_{i \neq j} |\langle i | \rho | j \rangle|$ in the basis $\{|i\rangle\}_{i=1, \dots, d}$ ¹⁰, and will mention other coherence measures if necessary.

FR for quantum coherence. Consider a general d -dimensional state in the Hilbert space \mathcal{H} , with the density matrix

$$\rho = \frac{1}{d} \mathbb{I}_d + \frac{1}{2} \vec{x} \cdot \vec{X}, \tag{1}$$

where \mathbb{I}_d is the $d \times d$ identity matrix, $\vec{x} = (x_1, x_2, \dots, x_{d^2-1})$, $\vec{X} = (X_1, X_2, \dots, X_{d^2-1})$, $x_i = \text{Tr}(\rho X_i)$, and $X_i \propto T_i$. Here, $\{T_i\}$ are generators of the Lie algebra $\text{SU}(d)$. They can be represented by the $d \times d$ traceless Hermitian matrices which satisfy $T_i T_j = \delta_{ij} \mathbb{I}_d / 2d + \sum_{k=1}^{d^2-1} (f_{ijk} + d_{ijk}) T_k / 2$, with f_{ijk} (d_{ijk}) being the structure constants that are completely antisymmetric (symmetric) in all indices^{43,44}. If one arranges $\vec{X} = \{u_{12}, v_{12}, \dots, u_{d-1,d}, v_{d-1,d}, w_1, \dots, w_{d-1}\}$, then

$$\begin{aligned} u_{jk} &= |j\rangle\langle k| + |k\rangle\langle j|, \quad v_{jk} = -i(|j\rangle\langle k| - |k\rangle\langle j|), \\ w_l &= \sqrt{\frac{2}{l(l+1)}} \sum_{j=1}^l (|j\rangle\langle j| - |l+1\rangle\langle l+1|), \end{aligned} \tag{2}$$

where $j, k \in \{1, 2, \dots, d\}$ with $j < k$, and $l \in \{1, 2, \dots, d-1\}$. Clearly, $\{X_i\}$ satisfy $\text{Tr}(X_i^\dagger X_i) = 2\delta_{ii}$. Moreover, the notation i appeared in v_{jk} is the imaginary unit.

For ρ represented as Eq. (1), $C_{l_1}(\rho)$ can be derived as

$$C_{l_1}(\rho) = \sum_{r=1}^{d_0} \sqrt{x_{2r-1}^2 + x_{2r}^2}, \tag{3}$$

where $d_0 = (d^2 - d) / 2$, and x_i related to w_l which is diagonal in the basis $\{|i\rangle\}_{i=1, \dots, d}$ do not contribute to $C_{l_1}(\rho)$.

To investigate evolution equation of coherence, we suppose the system S of interest interacts with its environment E , then by considering S and E as a whole for which their evolution is unitary, the reduced density matrix for S is obtained by tracing out the environmental degrees of freedom, $\rho(t) = \text{Tr}_E\{U(t)(\rho(0) \otimes \rho_E(0))U^\dagger(t)\}$. In terms of the master equation description, the equation of motion of ρ can be written in a local-in-time form³¹

$$\dot{\rho}(t) = \mathcal{L}\rho(t), \tag{4}$$

with \mathcal{L} being the Louville super-operator which may be time independent or time dependent.

As it has been shown that for any master equation which is local in time, whether Markovian, non-Markovian, of Lindblad form or not, one can always construct a linear map which gives $\rho(t) = \mathcal{E}[\rho(0)]$ (the opposite case may not always be true), and the linear map can be expressed in the Kraus-type representations⁴⁵. If the map \mathcal{E} is completely positive and trace preserving (CPTP), then one can explicitly construct the Kraus operators $\{E_\mu\}$ such that

$$\mathcal{E}(\rho) = \sum_\mu E_\mu \rho E_\mu^\dagger = \frac{1}{d} \mathbb{I}_d + \frac{1}{2} \vec{x}' \cdot \vec{X}, \tag{5}$$

where elements of \vec{x}' for $\mathcal{E}(\rho)$ are given by $x'_i = \text{Tr}[\mathcal{E}(\rho)X_i]$.

For convenience of later discussion, we turn to the Heisenberg picture to describe \mathcal{E} via the map $\mathcal{E}^\dagger(X_i) = \sum_\mu E_\mu^\dagger X_i E_\mu$, which gives $x'_i = \text{Tr}[\rho \mathcal{E}^\dagger(X_i)]$. As an Hermitian operator \mathcal{O} on $\mathbb{C}^{d \times d}$ can always be decomposed as $\mathcal{O} = \sum_{i=0}^{d^2-1} r_i X_i$ ($r_i \in \mathbb{R}$), $\mathcal{E}^\dagger(X_i)$ can be further characterized by the transformation matrix T defined via

$$\mathcal{E}^\dagger(X_i) = \sum_{j=0}^{d^2-1} T_{ij} X_j, \tag{6}$$

where $T_{ij} = \text{Tr}[\mathcal{E}^\dagger(X_i)X_j] / 2$, and here we denote by $X_0 = \sqrt{2/d} \mathbb{I}_d$. Clearly, $T_{00} = 1$, and $T_{0j} = 0$ for $j \geq 1$. This further gives $x'_i = \sum_{j=0}^{d^2-1} T_{ij} x_j$.

To present our central result, we first classify the states ρ into different families: $\rho = \{\rho^{\hat{n}}\}$, with

$$\rho^{\hat{n}} = \frac{1}{d} \mathbb{I}_d + \frac{1}{2} \chi^{\hat{n}} \cdot \vec{X}, \tag{7}$$

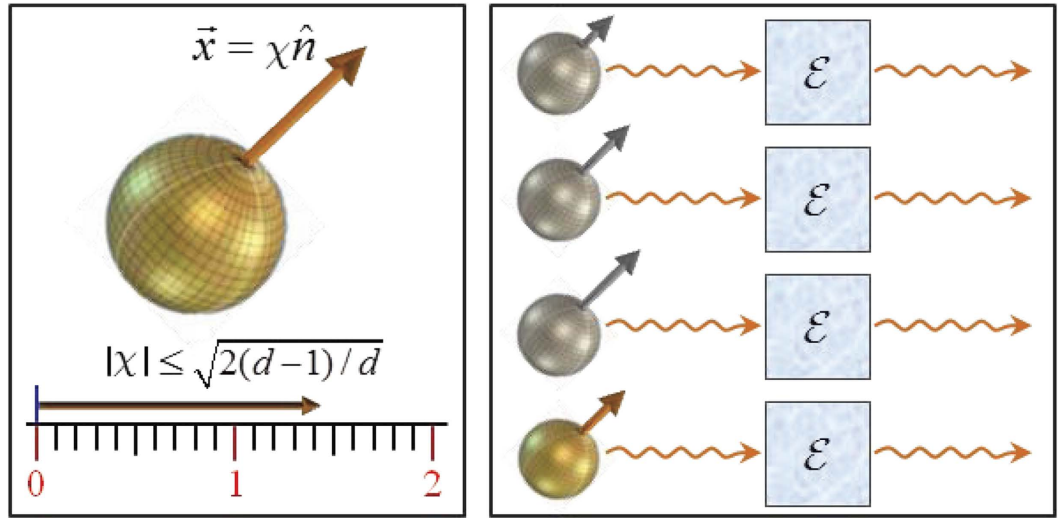


Figure 1. States of the same family $\rho^{\hat{n}}$ are represented by the characteristic vectors \vec{x} along the same or opposite directions (left). When $\rho_p^{\hat{n}}$ traverse a quantum channel \mathcal{E} (right), their decoherence process can be described qualitatively by that of $\rho_p^{\hat{n}}$ with the unit vector \hat{n} (the bottommost golden one).

and $\hat{n} = (n_1, n_2, \dots, n_{d^2-1})$ is a unit vector in \mathbb{R}^{d^2-1} , while χ is a parameter satisfying $|\chi| \leq \sqrt{2(d-1)/d}$ as $\text{Tr}(\rho^{\hat{n}})^2 = |\chi|^2/2 + 1/d$. By this classification scheme, different families of states are labeled by different unit vectors \hat{n} , while states belong to the same family are characterized by a common \hat{n} , and can be distinguished by different multiplicative factors χ (see Fig. 1). That is to say, $\rho^{\hat{n}}$ represents states with the characteristic vectors \vec{x} along the same or completely opposite directions but possessing different lengths.

While $\rho^{\hat{n}}$ is fully described by $\chi\hat{n}$, and the action of \mathcal{E} on it can be written equivalently as the map: $\chi\hat{n} \mapsto \mathcal{E}(\chi\hat{n})$, a measure Q may only be function of $\chi\hat{n}^s$, i.e., $Q(\rho^{\hat{n}}) = Q(\chi\hat{n}^s)$, with $\hat{n}^s = (n_{k_1}, n_{k_2}, \dots, n_{k_\alpha})$ ($\alpha \leq d^2 - 1$). Then as one can always make $Q_{\max} \geq 1$ (otherwise, one can normalize it by multiplying a constant to it), we have the following lemma.

Lemma 1. For any quantum measure of $Q(\rho^{\hat{n}}) = Q(\chi\hat{n}^s)$ that can be factorized as $Q(\chi\hat{n}^s) = f(\chi)g(\hat{n}^s)$, and quantum channel \mathcal{E} that gives the map $\mathcal{E}(\chi\hat{n}^s) = \chi\mathcal{E}(\hat{n}^s)$, the FR

$$Q[\mathcal{E}(\rho^{\hat{n}})] = Q(\rho^{\hat{n}})Q[\mathcal{E}(\rho_p^{\hat{n}})], \tag{8}$$

holds, where $f(\chi)$ and $g(\hat{n}^s)$ are functionals of χ and \hat{n}^s , respectively, and $\rho_p^{\hat{n}} = \mathbb{I}_d/d + \chi_p \hat{n} \cdot \vec{X}/2$ is the probe state, with χ_p solution of the equation $f(\chi_p)g(\hat{n}^s) = 1$.

The proof is given in Methods. Equipped with this lemma, we are now in position to present our central result.

Theorem 1. If the transformation matrix elements $T_{k0} = 0$ for $k \in \{1, 2, \dots, d^2 - d\}$, then the evolution of $C_{l_1}[\mathcal{E}(\rho^{\hat{n}})]$ obeys the following FR

$$C_{l_1}[\mathcal{E}(\rho^{\hat{n}})] = C_{l_1}(\rho^{\hat{n}})C_{l_1}[\mathcal{E}(\rho_p^{\hat{n}})], \tag{9}$$

with $\rho_p^{\hat{n}}$ the probe state, and $\chi_p = 1/\sum_{r=1}^{d_0} (n_{2r-1}^2 + n_{2r}^2)^{1/2}$.

The proof is left to the Methods. Here, we further show an implication of it. As $T_{k0} = 0$ for $k \in \{1, 2, \dots, d^2 - d\}$, we have $\mathcal{E}^\dagger(X_k) = \sum_{j=1}^{d^2-1} T_{kj}X_j$, hence $\text{Tr}[\mathcal{E}^\dagger(X_k)] = 0$. On the other hand, $\text{Tr}[\mathcal{E}^\dagger(X_k)] = \text{Tr}(X_k \sum_{\mu} E_{\mu} E_{\mu}^\dagger)$. This, together with Eq. (2), requires that all the nondiagonal elements of $A = \sum_{\mu} E_{\mu} E_{\mu}^\dagger$ must be zero.

Corollary 1. If the operator $A = \sum_{\mu} E_{\mu} E_{\mu}^\dagger$ is diagonal, then the evolution of $C_{l_1}[\mathcal{E}(\rho^{\hat{n}})]$ obeys the FR (9).

This corollary means that in addition to the usual completeness condition $\sum_{\mu} E_{\mu}^\dagger E_{\mu} = \mathbb{I}_d$ of the CPTP map³¹, the FR (9) further requires $\sum_{\mu} E_{\mu} E_{\mu}^\dagger$ to be diagonal. We denote this kind of channels \mathcal{E}_F . Clearly, they include the unital channel \mathcal{E}_U [i.e., $\mathcal{E}_U(\mathbb{I}_d/d) = \mathbb{I}_d/d$] as a special case.

From a geometric perspective, Theorem 1 indicates that for all states of the same family $\rho^{\hat{n}}$, namely, states with the characteristic vectors \vec{x} along the same or opposite directions, their coherence dynamics measured by the l_1 norm can be represented qualitatively by that of the probe state $\rho_p^{\hat{n}}$, as the magnitude of $C_{l_1}[\mathcal{E}_F(\rho^{\hat{n}})]$ equals the product of the initial coherence $C_{l_1}(\rho^{\hat{n}})$ and the evolved coherence $C_{l_1}[\mathcal{E}_F(\rho_p^{\hat{n}})]$. This simplifies greatly the assessment of the decoherence process of an open system. Moreover, the FR (9) provides a strong link between amount of the coherence loss of a system and structures of the applied quantum channels. Particularly, as $\rho^{\hat{n}}$ with the vectors \vec{x} along the same or opposite directions fulfill the same decoherence law, the approach adopted here may offer a route for better understanding the interplay between geometry of the state space and various aspects of its

quantum features. It might also provides a deeper insight into the effects of gate operation in quantum computing and experimental generation of coherent resources in noisy environments, as $\mathcal{E}_F(\rho^{\hat{n}})$ can specify the actions of environments, of measurements, or of both on the states $\rho^{\hat{n}}$.

When some restrictions are imposed on the quantum channels, the FR (9) can be further simplified.

Corollary 2. *If a channel \mathcal{E} yields $\mathcal{E}^\dagger(X_k) = q(t)X_k$ for $\{X_k\}_{k=k_1, \dots, k_\beta}$ ($\beta \leq d^2 - d$), with $q(t)$ containing information on \mathcal{E} 's structure, then the FR*

$$C_{l_1}[\mathcal{E}(\rho)] = |q(t)|C_{l_1}(\rho), \tag{10}$$

holds for the family of states $\rho = \mathbb{I}_d/d + \sum_{k=k_1}^{k_\beta} x_k X_k/2 + \sum_{l=d^2-d+1}^{d^2-1} x_l X_l/2$.

The proof of this corollary is direct. As $\mathcal{E}^\dagger(X_k) = q(t)X_k$, the parameters x'_k for $\mathcal{E}(\rho)$ are given by $x'_k = q(t)x_k$. Therefore, by Eq. (3) we obtain $C_{l_1}[\mathcal{E}(\rho)] = |q(t)|C_{l_1}(\rho)$. Clearly, its evolution is solely determined by the product of the initial coherence and a noise parameter $|q(t)|$.

There are many quantum channels satisfying the condition of Corollary 2. For instance, the Pauli channel \mathcal{E}_{PL} and Gell-Mann channel \mathcal{E}_{GM} given in ref. 41, and the generalized amplitude damping channel \mathcal{E}_{GAD} ³¹. Notably, \mathcal{E}_{PL} covers the bit flip, phase flip, bit-phase flip, phase damping, and depolarizing channels which embody typical noisy sources in quantum information, while \mathcal{E}_{GAD} covers the structured reservoirs with Lorentzian and Ohmic-type spectral densities.

One can also construct quantum channel \mathcal{E}_G under the action of which $C_{l_1}[\mathcal{E}_G(\rho)]$ obeys the FR (10) for arbitrary initial state. The Kraus operators describing \mathcal{E}_G are given by

$$\begin{aligned} E_0^G &= \frac{1}{d} \sqrt{1 + (d^2 - d)q + (d - 1)q_0} \mathbb{I}_d, \\ E_k^G &= \sqrt{\frac{1 - q_0}{2d}} X_k, \quad E_l^G = \sqrt{\frac{1 - dq + (d - 1)q_0}{2d}} X_l, \end{aligned} \tag{11}$$

with $k \in \{1, \dots, d^2 - d\}$, and $l \in \{d^2 - d + 1, \dots, d^2 - 1\}$, while q and q_0 are time-dependent noisy parameters. Clearly, \mathcal{E}_G reduces to the depolarizing channel when $q_0 = q$.

N-qubit case. A general N -qubit state can be written as $\rho_N = \mathbb{I}_{2^N}/2^N + \vec{y} \cdot \vec{Y}/2$, with $\vec{Y} = (Y_1, Y_2, \dots, Y_{4^N-1})$, and

$$Y_j = 2^{(1-N)/2} \sigma_{j_1} \otimes \sigma_{j_2} \otimes \dots \otimes \sigma_{j_N}, \tag{12}$$

here, $\sigma_0 = \mathbb{I}_2$, and $\sigma_{1,2,3}$ are the usual Pauli matrices, while j_k takes the possible values of $\{0, 1, 2, 3\}$ other than the special case $j_k = 0$ for all k . In the Methods section, we have proved that for every family of the N -qubit states $\rho_N^{\hat{m}} = \mathbb{I}_{2^N}/2^N + \chi \hat{m} \cdot \vec{Y}/2$, with $\hat{m} = (m_1, m_2, \dots, m_{4^N-1})$ being a given unit vector, one can construct an auxiliary channel \mathcal{E}_{aux} such that $\rho_N^{\hat{m}} = \mathcal{E}_{aux}(\rho_N)$. This, together with Eq. (9), gives:

Corollary 3. *For any N -qubit state ρ_N , there exists an auxiliary channel \mathcal{E}_{aux} such that*

$$C_{l_1}[\mathcal{E}_F \mathcal{E}_{aux}(\rho_N)] = C_{l_1}[\mathcal{E}_{aux}(\rho_N)] C_{l_1}[\mathcal{E}_F(\rho_N^{\hat{m}})], \tag{13}$$

with $\rho_{N,p}^{\hat{m}} = \mathbb{I}_{2^N}/2^N + \chi_p \hat{m} \cdot \vec{Y}/2$, $\chi_p = 1/\sum_{r=1}^{d_0} (\bar{m}_{2r-1}^2 + \bar{m}_{2r}^2)^{1/2}$, $d_0 = (4^N - 2^N)/2$, and $\bar{m}_i = \sum_j a_{ij} m_j$, with a_{ij} being determined by the transformation between $\{Y_j\}$ and $\{X_i\}$: $X_i = \sum_j a_{ij} Y_j$.

This corollary generalizes the FR (9) for the N -qubit states. It shows that coherence of the evolved state under the actions of two cascaded channels $\mathcal{E}_F \mathcal{E}_{aux}$ is determined by the product of the coherence for the evolved probe state under the action of \mathcal{E}_F and the coherence for the generated state by \mathcal{E}_{aux} . As every Y_j can always be decomposed as linear combinations of the generators $\{X_i\}$, the above result applies also to the qudit states with $d = 2^N$. As an explicit example, the transformation between $\{Y_j\}$ and $\{X_i\}$ for $N = 2$ is given in the Methods section, from which \mathcal{E}_{aux} and $\{a_{ij}\}$ can be constructed directly.

Frozen coherence. By Theorem 1 we can also derive conditions on the quantum channel for which the l_1 norm of coherence is frozen. To elucidate this, we return to Eq. (9), from which one can see that $C_{l_1}[\mathcal{E}(\rho^{\hat{n}})]$ is frozen if the coherence of the probe state remains constant 1 during the evolution, i.e., $C_{l_1}[\mathcal{E}(\rho_p^{\hat{n}})] \equiv 1$. For later use, we denote by T^S the submatrix of T consisting T_{ij} with i ranging from 1 to $d^2 - d$ and j from 1 to $d^2 - 1$. Then by Theorem 1 and the reasoning in its proof, we obtain the fourth corollary.

Corollary 4. *If $T_{k0} = 0$ for $k \in \{1, 2, \dots, d^2 - d\}$, and T^S is a rectangular block diagonal matrix, with the main diagonal blocks*

$$T_r^S = \begin{pmatrix} T_{2r-1, 2r-1} & T_{2r-1, 2r} \\ T_{2r, 2r-1} & T_{2r, 2r} \end{pmatrix} (r \in \{1, \dots, d_0\}), \tag{14}$$

being orthogonal matrices, i.e., $(T_r^S)^T T_r^S = \mathbb{I}_2$, the l_1 norm of coherence for $\rho^{\hat{n}}$ will be frozen during the entire evolution.

The proof is given in Methods. It enables one to construct channels \mathcal{E} for which the l_1 norm of coherence is frozen. As an explicit example, we consider the one-qubit case, with \mathcal{E} being described by $E_i = \sum_{j=0}^3 \varepsilon_{ij} \sigma_j$, $i \in \{0, 1, 2, 3\}$ and $\varepsilon_{ij} \in \mathbb{C}$. Then by Corollary 4, one can obtain that when $\varepsilon_{i0} = \varepsilon_{i3} = 0$, and $\sum_i |\varepsilon_{i1} \pm i\varepsilon_{i2}|^2 = [\sum_i (|\varepsilon_{i1}|^2 - |\varepsilon_{i2}|^2)]^2 + 4[\sum_i \text{Re}(\varepsilon_{i1}^* \varepsilon_{i2})]^2 = 1$, or when $\varepsilon_{i1} = \varepsilon_{i2} = 0$, and $\sum_i |\varepsilon_{i0} \pm \varepsilon_{i3}|^2 = [\sum_i (|\varepsilon_{i0}|^2 - |\varepsilon_{i3}|^2)]^2 + 4[\sum_i \text{Im}(\varepsilon_{i3}^* \varepsilon_{i0})]^2 = 1$, with $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ representing, respectively, the real and imaginary parts of a number, the l_1 norm of coherence will be frozen. There are a host of $\{\varepsilon_{ij}\}$ that fulfill the requirements, e.g., $\varepsilon_{01} = q(t)$, $\varepsilon_{02} = \pm \sqrt{1 - q^2(t)}$, $\varepsilon_{k1} = \varepsilon_{k2} = 0$, or $\varepsilon_{00} = q(t)$, $\varepsilon_{03} = \pm i\sqrt{1 - q^2(t)}$, $\varepsilon_{k0} = \varepsilon_{k3} = 0$, with $k \in \{1, 2, 3\}$, and $q(t)$ contains the information on \mathcal{E} 's structure and its coupling with the system.

Moreover, for certain special initial states, the freezing condition presented in Corollary 4 may be further relaxed. In fact, for $\rho^{\hat{n}}$ with certain $n_{2r-1} = 0$ (or $n_{2r} = 0$), $(T_r^S)^T T_r^S = \mathbb{I}_2$ simplifies to $T_{2r-1,2r}^2 + T_{2r,2r}^2 = 1$ (or $T_{2r-1,2r-1}^2 + T_{2r,2r-1}^2 = 1$). For instance, when considering the channel $\mathcal{E}_{\text{PL}}^{41}$, the l_1 norm of coherence for $\rho^{\hat{n}}$ with $n_2 = 0$ is frozen during the entire evolution when $q_1 = 1$ (i.e., the bit flip channel). Similarly, for $\rho^{\hat{n}}$ with $n_1 = 0$, it is frozen when $q_2 = 1$ (i.e., the bit-phase flip channel). These are in fact the results obtained in ref. 21. Needless to say, when $(T_r^S)^T T_r^S = \mathbb{I}_2$, the l_1 norm of coherence is also frozen for $\rho^{\hat{n}}$ with certain $n_{2r-1} = 0$ or $n_{2r} = 0$.

Outlook. The FR (9) presented here can be of direct relevance to other issues of quantum theory. For example, the l_1 norm of coherence is a monotone of the entanglement-based coherence measure for one-qubit states¹². Its logarithmic form $\log_2[dC_{l_1}(\rho)]$ is lower bounded by the relative entropy of coherence $C_r(\rho)$ which has a clear physical interpretation, while $C_{l_1}(\rho) \geq C_r(\rho)$ for arbitrary ρ has also been conjectured⁴⁶. Further study shows that $C_{l_1}(\rho)$ also bounds the robustness of coherence, i.e., $(d - 1)^{-1} C_{l_1}(\rho) \leq C_{\text{RoC}}(\rho) \leq C_{l_1}(\rho)$ ⁴². It is also connected to the success probability of state discrimination in interference experiments²⁹ and the negativity of quantumness^{21,47}. Thus, our results provide a route for inspecting the interrelations between decay behaviors of coherence, quantumness, and entanglement.

The FR also applies to other related coherence measures, as well as quantum correlations which are relevant to coherence. Some examples are as follows (see Methods section for their proof): (i) the coherence concurrence for one-qubit states¹⁴, and the trace norm coherence for one-qubit and certain qutrit states^{13,46}; (ii) the genuine quantum coherence (GQC) defined via the Schatten p -norm for all states⁴⁸, which is related to quantum thermodynamics and the resource theory of asymmetry; (iii) the robustness of coherence for the one-qubit states and d -dimensional states with X -shaped density matrix, and its lower bound $\|\rho - \Delta(\rho)\|_2^2$ which is a measure of the GQC for all states⁴²; (iv) the K coherence defined based on the Wigner-Yanase skew information¹¹, although it is problematic in the framework of coherence by Baumgratz *et al.*⁴⁹, it may be a proper measure of the GQC⁴⁸; (v) the purity of a state which is complementary with quantum coherence²⁸; (vi) the geometric discord⁵⁰⁻⁵⁴ and measurement-induced nonlocality^{55,56}; (vii) the maximum Bell-inequality violation⁵⁷, and average fidelity of remote state preparation⁵⁸ and quantum teleportation⁵⁹. All these manifest the universality of the FR formulated in this paper, and will certainly deepen our understanding of the already rich and appealing subject of quantum channels or the CPTP maps.

Recently, Jing *et al.* studied quantum speed limits to the rate of change of quantumness measured by the non-commutativity of the algebra of observables⁶⁰. We note that the coherence quantifiers can also be considered as a measure of quantumness, but it is different from the notion of quantumness considered in ref. 60 and references therein, although they both characterize global quantum nature of a state, and are intimately related to quantum correlations such as discord. The coherence monotones characterize quantumness of a single state. It is basis dependent, and vanishes for the diagonal states. The quantumness based on the non-commutativity relations measures the relative quantumness of two states. It is basis independent, and vanishes only for the maximally mixed states. Of course, it is as well crucial to study evolution equation of it in future work.

Discussion

We have established a simple FR for the evolution equation of the l_1 norm of coherence, which is of practical relevance for assessing coherence loss of an open quantum system. For a general d -dimensional state, we determined condition such that this FR holds. The condition can be described as a restriction on the transformation matrix, or on the operator $\sum_{\mu} E_{\mu} E_{\mu}^{\dagger}$, of the quantum channel. By introducing an auxiliary channel, we further presented a more general relation which applies to any N -qubit state. With the help of the FR, we have also determined a condition the transformation matrix should satisfy such that the l_1 norm of coherence for a general state is dynamically frozen, and constructed explicitly the desired channels for one-qubit states. Finally, we showed that the FR holds for many other related coherence and quantum correlation measures. We hope these results may help in understanding the interplay between structure of the quantum channel, geometry of the state space, and decoherence of an open system, as well as their combined effects on decay behaviors of various quantum correlations.

Methods

Proof of Lemma 1. As \mathcal{E} gives the map $\mathcal{E}(\chi^{\hat{n}^s}) = \chi \mathcal{E}(\hat{n}^s)$, and $Q(\rho^{\hat{n}}) = Q(\chi^{\hat{n}^s})$ fulfills $Q(\chi^{\hat{n}^s}) = f(\chi)g(\hat{n}^s)$, we have

$$\begin{aligned} Q[\mathcal{E}(\rho^{\hat{n}})] &= Q[\chi\mathcal{E}(\hat{n}^s)] = f(\chi)g[\mathcal{E}(\hat{n}^s)], \\ Q[\mathcal{E}(\rho_p^{\hat{n}})] &= Q[\chi_p\mathcal{E}(\hat{n}^s)] = f(\chi_p)g[\mathcal{E}(\hat{n}^s)]. \end{aligned} \tag{15}$$

Hence, it is evident that $Q[\mathcal{E}(\rho^{\hat{n}})] = Q(\rho^{\hat{n}})Q[\mathcal{E}(\rho_p^{\hat{n}})]$ when $f(\chi_p)g(\hat{n}^s) = 1$.

If $Q_{\max} \geq 1$, the equation $f(\chi_p)g(\hat{n}^s) = 1$ with respect to χ_p is always solvable as $f(\chi_p)g(\hat{n}^s) = Q(\rho_p^{\hat{n}})$. If $Q_{\max} < 1$, one can normalize it by simply introducing a constant N such that $Q'_{\max} = NQ_{\max} = 1$, with Q' obeying the FR of Eq. (8).

Proof of Theorem 1. First, by using Eq. (3) and the fact that $\vec{x} = \chi\hat{n}$, we obtain

$$C_{l_1}(\rho^{\hat{n}}) = \chi \sum_{r=1}^{d_0} \sqrt{n_{2r-1}^2 + n_{2r}^2}. \tag{16}$$

which corresponds to $C_{l_1}(\rho^{\hat{n}}) = f(\chi)g(\hat{n}^s)$, with $f(\chi) = \chi$ and $g(\hat{n}^s) = \sum_{r=1}^{d_0} (n_{2r-1}^2 + n_{2r}^2)^{1/2}$.

Second, when the transformation matrix elements $T_{k0} = 0$ for $k \in \{1, 2, \dots, d^2 - d\}$, we have

$$x'_k = \chi \sum_{j=1}^{d^2-1} T_{kj} n_j = \chi \mathcal{E}(n_k), \tag{17}$$

and therefore $\mathcal{E}(\chi\hat{n}^s) = \chi\mathcal{E}(\hat{n}^s)$.

From Eqs (16) and (17) one can see that both the l_1 norm of coherence and the quantum channel \mathcal{E} fulfill the requirements of Lemma 1, and the probe state $\rho_p^{\hat{n}} = \mathbb{I}_d/d + \chi_p \hat{n} \cdot \vec{X}/2$, with χ_p being solution of the equation $f(\chi_p)g(\hat{n}^s) = 1$, which can be solved as $\chi_p = 1/\sum_{r=1}^{d_0} (n_{2r-1}^2 + n_{2r}^2)^{1/2}$. This completes the proof.

Proof of Corollary 3. Suppose \mathcal{E}_{aux} is described by the Kraus operators $E_{\mu}^{\text{aux}} = \sqrt{\varepsilon_{\mu}} Y_{\mu}$, with $\mu \in \{0, 1, \dots, 4^N - 1\}$. Then, by employing the anticommutation relation of the Pauli operators $\sigma_{1,2,3}$, we obtain

$$\mathcal{E}_{\text{aux}}^{\dagger}(Y_{\nu}) = \sum_{\mu=0}^{4^N-1} c_{\nu\mu} \varepsilon_{\mu} Y_{\nu}, \tag{18}$$

where $c_{\nu\mu} = 2^{1-N} (-1)^{\sum_{k=1}^N \xi_k^{\nu\mu}}$, with $\xi_k^{\nu\mu} = 0$ if $v_k \mu_k (v_k - \mu_k) = 0$, and $\xi_k^{\nu\mu} = 1$ otherwise. This formula is equivalent to $\mathcal{E}_{\text{aux}}^{\dagger}(Y_{\nu}) = q_{\nu} Y_{\nu}$, with $q_{\nu} = \sum_{\mu} c_{\nu\mu} \varepsilon_{\mu}$ encoding the information of \mathcal{E}_{aux} .

To solve ε_{μ} , we define coefficient matrix $\hat{c} = (c_{\nu\mu})$, and column vectors $\hat{\varepsilon} = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{4^N-1})^T$, $\hat{q} = (1, q_1, \dots, q_{4^N-1})^T$, then $\mathcal{E}_{\text{aux}}^{\dagger}(Y_{\nu}) = q_{\nu} Y_{\nu}$ becomes $\hat{c}\hat{\varepsilon} = \hat{q}$, hence ε can be derived as $\hat{\varepsilon} = \hat{c}^{-1}\hat{q}$, with c^{-1} denoting the inverse matrix of c . Finally, by choosing $q_{\nu} = \chi m_{\nu}/y_{\nu}$, we obtain $\rho_N^{\hat{m}} = \mathcal{E}_{\text{aux}}(\rho_N)$, thus completes the proof.

The transformation between generators $\{Y_j\}$ for the two-qubit states and $\{X_j\}$ for the qudit states with $d = 4$ are as follows:

$$\begin{aligned} Y_{1,13} &= \frac{1}{\sqrt{2}}(X_1 \pm X_{11}), & Y_{2,14} &= \frac{1}{\sqrt{2}}(X_2 \pm X_{12}), & Y_{4,7} &= \frac{1}{\sqrt{2}}(X_3 \pm X_9), \\ Y_{5,10} &= \frac{1}{\sqrt{2}}(X_7 \pm X_5), & Y_{9,6} &= \frac{1}{\sqrt{2}}(X_6 \pm X_8), & Y_{8,11} &= \frac{1}{\sqrt{2}}(X_4 \pm X_{10}), \\ Y_{12} &= \frac{\sqrt{6}}{3}X_{14} + \frac{1}{\sqrt{3}}X_{15}, & Y_{3,15} &= \frac{1}{\sqrt{2}}X_{13} \mp \frac{1}{\sqrt{6}}X_{14} \pm \frac{1}{\sqrt{3}}X_{15}, \end{aligned} \tag{19}$$

where $\vec{X} = \{u_{12}, v_{12}, u_{13}, v_{13}, \dots, u_{34}, v_{34}, w_1, w_2, w_3\}$, and elements $Y_j = 2^{-1/2} \sigma_{j_1} \otimes \sigma_{j_2}$ of \vec{Y} are arranged with $(j_1 j_2)$ in the sequence (01), (02), (03), (10), (11), (12), (13), ..., (33).

Proof of Corollary 4. As the submatrix T^S is rectangular block diagonal, the elements T_{ij} in the off-diagonal blocks are all zero. This, together with $T_{k0} = 0$ for $k \in \{1, 2, \dots, d^2 - d\}$, yields

$$\begin{aligned} n'_{2r-1} &= \sum_{j=0}^{d^2-1} T_{2r-1,j} n_j = T_{2r-1,2r-1} n_{2r-1} + T_{2r-1,2r} n_{2r}, \\ n'_{2r} &= \sum_{j=0}^{d^2-1} T_{2r,j} n_j = T_{2r,2r-1} n_{2r-1} + T_{2r,2r} n_{2r}, \end{aligned} \tag{20}$$

for $r \in \{1, 2, \dots, d_0\}$. Moreover, the requirement that $(T_r^S)^T T_r^S = \mathbb{I}_2$ yields

$$\begin{aligned} T_{2r-1,2r-1}^2 + T_{2r,2r-1}^2 &= T_{2r-1,2r}^2 + T_{2r,2r}^2 = 1, \\ T_{2r-1,2r-1} T_{2r-1,2r} + T_{2r,2r-1} T_{2r,2r} &= 0. \end{aligned} \tag{21}$$

By using the above two equations, it is straightforward to see that $n_{2r-1}'^2 + n_{2r}'^2 = n_{2r-1}^2 + n_{2r}^2$, and therefore from Eq. (16) we have $C_{l_1}[\mathcal{E}(\rho_p^{\hat{n}})] \equiv 1$. This, together with Theorem 1, implies $C_{l_1}[\mathcal{E}(\rho^{\hat{n}})] = C_{l_1}(\rho^{\hat{n}})$, and hence completes the proof.

Frozen coherence of one qubit. Suppose the required channel \mathcal{E} is described by the Kraus operators $E_i = \sum_{j=0}^3 \varepsilon_{ij} \sigma_j$, with $i \in \{0, 1, 2, 3\}$, and the values of $\varepsilon_{ij} \in \mathbb{C}$ should satisfy certain constraints such that the requirement of Corollary 4 is satisfied. First, the completeness condition of the CPTP map, namely, $\sum_i E_i^\dagger E_i = \mathbb{I}^{31}$, requires

$$\sum_i (|\varepsilon_{i0} \pm \varepsilon_{i3}|^2 + |\varepsilon_{i1} \pm i\varepsilon_{i2}|^2) = 1, \tag{22}$$

$$\sum_i [\text{Re}(\varepsilon_{i0}^* \varepsilon_{i1}) + \text{Im}(\varepsilon_{i3}^* \varepsilon_{i2}) - i \text{Re}(\varepsilon_{i0}^* \varepsilon_{i2}) + i \text{Im}(\varepsilon_{i3}^* \varepsilon_{i1})] = 0,$$

where ε_{ij}^* represents conjugation of ε_{ij} , and the notation i before ε_{i2} , $\text{Re}(\cdot)$, and $\text{Im}(\cdot)$ is the imaginary unit.

Second, Corollary 4 requires $T_{10} = T_{20} = 0$, and T^S to be a rectangular block diagonal matrix which corresponds to $T_{13} = T_{23} = 0$. This yields

$$\sum_i E_i^\dagger \sigma_1 E_i = T_{11} \sigma_1 + T_{12} \sigma_2, \quad \sum_i E_i^\dagger \sigma_2 E_i = T_{21} \sigma_1 + T_{22} \sigma_2, \tag{23}$$

from which one can obtain

$$\sum_i [(\varepsilon_{i0} + \varepsilon_{i3})(\varepsilon_{i1}^* - i\varepsilon_{i2}^*)] = 0, \quad \sum_i [(\varepsilon_{i0}^* - \varepsilon_{i3}^*)(\varepsilon_{i1} - i\varepsilon_{i2})] = 0, \tag{24}$$

and

$$T_{11,22} = \sum_i [|\varepsilon_{i0}|^2 \pm |\varepsilon_{i1}|^2 \mp |\varepsilon_{i2}|^2 - |\varepsilon_{i3}|^2], \quad T_{12,21} = 2 \sum_i [\text{Re}(\varepsilon_{i1}^* \varepsilon_{i2}) \mp \text{Im}(\varepsilon_{i3}^* \varepsilon_{i0})]. \tag{25}$$

By comparing Eqs (22) and (24), one can note that the equalities are satisfied when $\varepsilon_{i0} = \varepsilon_{i3} = 0$, $\sum_i |\varepsilon_{i1} \pm i\varepsilon_{i2}|^2 = 1$, or when $\varepsilon_{i1} = \varepsilon_{i2} = 0$, $\sum_i |\varepsilon_{i0} \pm \varepsilon_{i3}|^2 = 1$. Under these two constraints, Eq. (25) simplifies, respectively, to

$$T_{11} = -T_{22} = \sum_i (|\varepsilon_{i1}|^2 - |\varepsilon_{i2}|^2), \quad T_{12} = T_{21} = 2 \sum_i \text{Re}(\varepsilon_{i1}^* \varepsilon_{i2}), \tag{26}$$

and

$$T_{11} = T_{22} = \sum_i (|\varepsilon_{i0}|^2 - |\varepsilon_{i3}|^2), \quad T_{12} = -T_{21} = -2 \sum_i \text{Im}(\varepsilon_{i3}^* \varepsilon_{i0}). \tag{27}$$

Finally, the requirement that $(T_r^S)^T T_r^S = \mathbb{I}_2$, corresponds to

$$|T_{11}|^2 + |T_{21}|^2 = 1, \quad |T_{12}|^2 + |T_{22}|^2 = 1, \quad T_{11}^* T_{12} + T_{21}^* T_{22} = 0, \tag{28}$$

and from Eqs (26) and (27), one can see that the third equality of Eq. (28) is always satisfied, while the first two equalities are equivalent. Therefore, to freeze the l_1 norm of coherence, ε_{ij} should satisfy one of the following two conditions:

(i) $\varepsilon_{i0} = \varepsilon_{i3} = 0$ for $i \in \{0, 1, 2, 3\}$, and

$$\sum_i |\varepsilon_{i1} \pm i\varepsilon_{i2}|^2 = [\sum_i (|\varepsilon_{i1}|^2 - |\varepsilon_{i2}|^2)]^2 + 4[\sum_i \text{Re}(\varepsilon_{i1}^* \varepsilon_{i2})]^2 = 1, \tag{29}$$

(ii) $\varepsilon_{i1} = \varepsilon_{i2} = 0$ for $i \in \{0, 1, 2, 3\}$, and

$$\sum_i |\varepsilon_{i0} \pm \varepsilon_{i3}|^2 = [\sum_i (|\varepsilon_{i0}|^2 - |\varepsilon_{i3}|^2)]^2 + 4[\sum_i \text{Im}(\varepsilon_{i3}^* \varepsilon_{i0})]^2 = 1. \tag{30}$$

Other measures fulfilling the FR. (i) The coherence concurrence for the one-qubit states¹⁴, and the trace norm coherence for the one-qubit and certain qutrit states^{13,46}, coincide with the l_1 norm of coherence. Hence, the FR applies to them.

(ii) For the GQC measure $\tilde{G}_D(\rho) = \|\rho - \Delta(\rho)\|_p$ presented in ref. 48, we have

$$\tilde{G}_D(\rho) = \frac{1}{2} \|\chi \hat{n} \cdot \bar{X} - \Delta(\chi \hat{n} \cdot \bar{X})\|_p = \frac{1}{2} \chi \|\hat{n} \cdot \bar{X} - \Delta(\hat{n} \cdot \bar{X})\|_p, \tag{31}$$

where $\Delta(\rho) = \sum_i \langle i|\rho|i\rangle |i\rangle\langle i|$ denotes full dephasing of ρ in the basis $\{|i\rangle\}_{i=1,\dots,d}$. Thus, $\tilde{G}_D(\rho) = f(\chi)g(\hat{n})$, with $f(\chi) = \chi/2$, and $g(\hat{n}) = \|\hat{n} \cdot \bar{X} - \Delta(\hat{n} \cdot \bar{X})\|_p$.

For the GQC measure $G_D(\rho) = \min_{\delta \in \mathcal{I}} \|\rho - \delta\|_2$, the FR also holds as the optimal δ is given by $\Delta(\rho)$ ⁴⁸.

(iii) $C_{\text{RoC}}(\rho)$ for the one-qubit states and d -dimensional states with X -shaped density matrix, equals to the l_1 norm of coherence, and thus the FR holds.

(iv) The K coherence is defined as $\mathcal{I}(\rho, K) = \frac{1}{2} \text{Tr}\{[\sqrt{\rho}, K]^2\}^{11}$. As $\sqrt{\rho}$ can be decomposed as $\sqrt{\rho} = \sum_{i=0}^{d^2-1} \gamma_i X_i^{53}$, $\mathcal{I}(\rho, K)$ is a function of $\chi \hat{n} = \vec{\gamma}$, i.e., $\mathcal{I}(\rho, K) = \mathcal{I}(\chi \hat{n}, K)$, with $\vec{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_{d^2-1})$. Then by using $[X_0, K] = 0$, we obtain

$$\mathcal{I}(\rho, K) = -\frac{1}{2} \text{Tr}\{[\vec{\gamma} \cdot \vec{X}, K]^2\} = -\frac{1}{2} \text{Tr}\{[\chi \hat{n} \cdot \vec{X}, K]^2\} = -\frac{1}{2} \chi^2 \text{Tr}\{[\hat{n} \cdot \vec{X}, K]^2\}, \quad (32)$$

thus $\mathcal{I}(\chi \hat{n}, K) = f(\chi)g(\hat{n}, K)$, with $f(\chi) = -\chi^2/2$, and $g(\hat{n}, K) = \text{Tr}\{[\hat{n} \cdot \vec{X}, K]^2\}$.

(v) For the quantifier $\mathcal{P}(\rho) = P(\rho) - 1/d = |\vec{x}|^2/2$ which is a monotonic function of the purity $P(\rho) = \text{Tr}\rho^2$ of a state, we have the FR $\mathcal{P}[\mathcal{E}(\rho)] = \mathcal{P}(\rho)\mathcal{P}[\mathcal{E}(\rho_p)]$, with ρ_p being the probe state for which $|\vec{x}_p| = \sqrt{2}$.

(vi) The general form of geometric quantum correlation measure can be written as

$$D_p(\rho) = \text{opt}_{\Pi^A \in \mathcal{M}} \|\rho - \Pi^A(\rho)\|_p^p, \quad (33)$$

where $\|\rho\|_p = [\text{Tr}(\rho^\dagger \rho)^{p/2}]^{1/p}$ denotes the Schatten p -norm, and opt represents the optimization over some class \mathcal{M} of the local measurements Π^A . This definition covers the geometric discord^{50–52} and measurement-induced nonlocality^{55,56}. For these measures, as $\sum_k \Pi_k^A \Pi_k^A = \mathbb{I}_A$, we have

$$\|\rho - \Pi^A(\rho)\|_p = \frac{1}{2} \|\chi \hat{n} \cdot \vec{X} - \Pi^A(\chi \hat{n} \cdot \vec{X})\|_p = \frac{1}{2} \chi \|\hat{n} \cdot \vec{X} - \Pi^A(\hat{n} \cdot \vec{X})\|_p, \quad (34)$$

then by comparing with Lemma 1, we obtain $f(\chi) = (\chi/2)^p$, and $g(\hat{n}) = \text{opt}_{\Pi^A} \|\hat{n} \cdot \vec{X} - \Pi^A(\hat{n} \cdot \vec{X})\|_p^p$, i.e., the FR holds.

If ρ in Eq. (33) is replaced by $\sqrt{\rho}$, then one obtains the Hellinger distance discord for $p = 2^{53,54}$. As $\sqrt{\rho} = \sum_{ij} \gamma_{ij} X_i^A \otimes X_j^B$, with $\{X_i^A\}_{i=0,1,\dots,d_A^2-1}$ and $\{X_j^B\}_{j=0,1,\dots,d_B^2-1}$ being the sets of Hermitian operators which constitute the orthonormal operator bases for the Hilbert space \mathcal{H}_A and \mathcal{H}_B^{53} , and $\sum_k \Pi_k^A X_0^A \Pi_k^A = X_0^A$, the FR also holds for it.

(vii) For two-qubit states, the maximum Bell-inequality violation $B_{\text{max}}(\rho)^{57}$, remote state preparation fidelity $F_{\text{rsp}}(\rho)^{58}$, and $N_{\text{qt}}(\rho)$ which is a monotone of the average teleportation fidelity $F_{\text{qt}}(\rho) = 1/2 + N_{\text{qt}}(\rho)/6^{59}$, are given by

$$B_{\text{max}}(\rho) = 2\sqrt{E_1 + E_2}, \quad F_{\text{rsp}}(\rho) = \frac{1}{2}(E_2 + E_3), \quad N_{\text{qt}}(\rho) = \sqrt{E_1 + E_2 + E_3}, \quad (35)$$

where $E_1 \geq E_2 \geq E_3$ are eigenvalues of the 3×3 matrix $T^T T$, and $T_{ij} = \text{Tr}(\rho \sigma_i \otimes \sigma_j)$. This gives $E_i(\chi \hat{n}) = \chi^2 E_i(\hat{n})$ for $i \in \{1, 2, 3\}$, which implies that all measures of Eq. (35) satisfy the requirement of Lemma 1.

References

- Ficek, Z. & Swain, S. *Quantum interference and coherence: theory and experiments* (Springer Series in Optical Sciences, 2005).
- Horodecki, R., Horodecki, P., Horodecki, M. & Horodecki, K. Quantum entanglement. *Rev. Mod. Phys.* **81**, 865 (2009).
- Modi, K., Brodutch, A., Cable, H., Paterek, T. & Vedral, V. The classical-quantum boundary for correlations: discord and related measures. *Rev. Mod. Phys.* **84**, 1655 (2012).
- Åberg, J. Catalytic coherence. *Phys. Rev. Lett.* **113**, 150402 (2014).
- Lastaglio, M., Jennings, D. & Rudolph, T. Description of quantum coherence in thermodynamic processes requires constraints beyond free energy. *Nat. Commun.* **6**, 6383 (2015).
- Narasimhachar, V. & Gour, G. Low-temperature thermodynamics with quantum coherence. *Nat. Commun.* **6**, 7689 (2015).
- Lastaglio, M., Korzekwa, K., Jennings, D. & Rudolph, T. Quantum coherence, time-translation symmetry, and thermodynamics. *Phys. Rev. X* **5**, 021001 (2015).
- Gour, G., Müller, M. P., Narasimhachar, V., Spekkens, R. W. & Halpern, N. Y. The resource theory of informational nonequilibrium in thermodynamics. *Phys. Rep.* **583**, 1 (2015).
- Lambert, N. *et al.* Quantum biology. *Nat. Phys.* **9**, 10 (2013).
- Baumgratz, T., Cramer, M. & Plenio, M. B. Quantifying coherence. *Phys. Rev. Lett.* **113**, 140401 (2014).
- Girolami, D. Observable measure of quantum coherence in finite dimensional systems. *Phys. Rev. Lett.* **113**, 170401 (2014).
- Streltsov, A., Singh, U., Dhar, H. S., Bera, M. N. & Adesso, G. Measuring quantum coherence with entanglement. *Phys. Rev. Lett.* **115**, 020403 (2015).
- Shao, L. H., Xi, Z., Fan, H. & Li, Y. Fidelity and trace-norm distances for quantifying coherence. *Phys. Rev. A* **91**, 042120 (2015).
- Zhang, Y. R., Shao, L. H., Li, Y. & Fan, H. Quantifying coherence in infinite dimensional systems. *Phys. Rev. A* **93**, 012334 (2016).
- Yuan, X., Zhou, H., Cao, Z. & Ma, X. Intrinsic randomness as a measure of quantum coherence. *Phys. Rev. A* **92**, 022124 (2015).
- Rastegin, A. E. Quantum-coherence quantifiers based on the Tsallis relative α entropies. *Phys. Rev. A* **93**, 032136 (2016).
- Pires, D. P., Céleri, L. C. & Soares-Pinto, D. O. Geometric lower bound for a quantum coherence measure. *Phys. Rev. A* **91**, 042330 (2015).
- Yao, Y., Xiao, X., Ge, L. & Sun, C. P. Quantum coherence in multipartite systems. *Phys. Rev. A* **92**, 022112 (2015).
- Xi, Z., Li, Y. & Fan, H. Quantum coherence and correlations in quantum system. *Sci. Rep.* **5**, 10922 (2015).
- Ma, J., Yadin, B., Girolami, D., Vedral, V. & Gu, M. Converting coherence to quantum correlations. *Phys. Rev. Lett.* **116**, 160407 (2016).
- Bromley, T. R., Cianciaruso, M. & Adesso, G. Frozen quantum coherence. *Phys. Rev. Lett.* **114**, 210401 (2015).
- Zhang, Y. J., Han, W., Xia, Y. J., Yu, Y. M. & Fan, H. Role of initial system-bath correlation on coherence trapping. *Sci. Rep.* **5**, 13359 (2015).
- Mani, A. & Karimipour, V. Cohering and de-cohering power of quantum channels. *Phys. Rev. A* **92**, 032331 (2015).
- Hu, X., Milne, A., Zhang, B. & Fan, H. Quantum coherence of steered states. *Sci. Rep.* **6**, 19365 (2016).
- Chitambar, E. *et al.* Assisted distillation of quantum coherence. *Phys. Rev. Lett.* **116**, 070402 (2016).

26. Winter, A. & Yang, D. Operational resource theory of coherence. *Phys. Rev. Lett.* **116**, 120404 (2016).
27. Cheng, S. & Hall, M. J. W. Complementarity relations for quantum coherence. *Phys. Rev. A* **92**, 042101 (2015).
28. Singh, U., Bera, M. N., Dhar, H. S. & Pati, A. K. Maximally coherent mixed states: complementarity between maximal coherence and mixedness. *Phys. Rev. A* **91**, 052115 (2015).
29. Bera, M. N., Qureshi, T., Siddiqui, M. A. & Pati, A. K. Duality of quantum coherence and path distinguishability. *Phys. Rev. A* **92**, 012118 (2015).
30. Du, S., Bai, Z. & Guo, Y. Conditions for coherence transformations under incoherent operations. *Phys. Rev. A* **91**, 052120 (2015).
31. Nielsen, M. A. & Chuang, I. L. *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
32. Konrad, T. *et al.* Evolution equation for quantum entanglement. *Nat. Phys.* **4**, 99 (2008).
33. Faras, O. J., Latune, C. L., Walborn, S. P., Davidovich, L. & Ribeiro, P. H. S. Determining the dynamics of entanglement. *Science* **324**, 1414 (2009).
34. Tiersch, M., de Melo, F. & Buchleitner, A. Entanglement evolution in finite dimensions. *Phys. Rev. Lett.* **101**, 170502 (2008).
35. Gour, G. Evolution and symmetry of multipartite entanglement. *Phys. Rev. Lett.* **105**, 190504 (2010).
36. Li, Z. G., Fei, S. M., Wang, Z. D. & Liu, W. M. Evolution equation of entanglement for bipartite systems. *Phys. Rev. A* **79**, 024303 (2009).
37. Li, Z. G., Zhao, M. J., Fei, S. M. & Liu, W. M. Evolution equation for entanglement of assistance. *Phys. Rev. A* **81**, 042312 (2010).
38. Yu, C. S., Yi, X. X. & Song, H. S. Evolution of entanglement for quantum mixed states. *Phys. Rev. A* **78**, 062330 (2008).
39. Liu, Z. & Fan, H. Dynamics of the bounds of squared concurrence. *Phys. Rev. A* **79**, 032306 (2009).
40. Mirafzali, S. Y., Sargolzhai, I., Ahanj, A., Javidan, K. & Sarbishaei, M. Factorization law for two lower bounds of concurrence. *Phys. Rev. A* **82**, 032321 (2010).
41. Hu, M. L. & Fan, H. Evolution equation for geometric quantum correlation measures. *Phys. Rev. A* **91**, 052311 (2015).
42. Napoli, C. *et al.* Robustness of coherence: an operational and observable measure of quantum coherence. *Phys. Rev. Lett.* **116**, 150502 (2016).
43. Byrd, M. S. & Khaneja, N. Characterization of the positivity of the density matrix in terms of the coherence vector representation. *Phys. Rev. A* **68**, 062322 (2003).
44. Kimura, G. The Bloch vector for N-level systems. *Phys. Lett. A* **314**, 339 (2003).
45. Andersson, E., Cresser, J. D. & Hall, M. J. W. Finding the Kraus decomposition from a master equation and vice versa. *J. Mod. Opt.* **54**, 1695 (2007).
46. Rana, S., Parashar, P. & Lewenstein, M. Trace-distance measure of coherence. *Phys. Rev. A* **93**, 012110 (2016).
47. Nakano, T., Piani, M. & Adesso, G. Negativity of quantumness and its interpretations. *Phys. Rev. A* **88**, 012117 (2013).
48. Streltsov, A. Genuine quantum coherence. *arXiv:1511.08346* (2015).
49. Du, S. & Bai, Z. The Wigner-Yanase information can increase under phase sensitive incoherent operations. *Ann. Phys.* **359**, 136 (2015).
50. Dakić, B., Vedral, V. & Brukner, Č. Necessary and sufficient condition for nonzero quantum discord. *Phys. Rev. Lett.* **105**, 190502 (2010).
51. Luo, S. & Fu, S. Geometric measure of quantum discord. *Phys. Rev. A* **82**, 034302 (2010).
52. Paula, F. M., de Oliveira, T. R. & Sarandy, M. S. Geometric quantum discord through the Schatten 1-norm. *Phys. Rev. A* **87**, 064101 (2013).
53. Chang, L. & Luo, S. Remedying the local ancilla problem with geometric discord. *Phys. Rev. A* **87**, 062303 (2013).
54. Girolami, D., Tufarelli, T. & Adesso, G. Characterizing nonclassical correlations via local quantum uncertainty. *Phys. Rev. Lett.* **110**, 240402 (2013).
55. Luo, S. & Fu, S. Measurement-induced nonlocality. *Phys. Rev. Lett.* **106**, 120401 (2011).
56. Hu, M. L. & Fan, H. Measurement-induced nonlocality based on the trace norm. *New J. Phys.* **17**, 033004 (2015).
57. Horodecki, R., Horodecki, P. & Horodecki, M. Violating Bell inequality by mixed spin-1/2 states: necessary and sufficient condition. *Phys. Lett. A* **200**, 340 (1995).
58. Dakić, B. *et al.* Quantum discord as resource for remote state preparation. *Nat. Phys.* **8**, 666 (2012).
59. Horodecki, R., Horodecki, M. & Horodecki, P. Teleportation, Bell's inequalities and inseparability. *Phys. Lett. A* **222**, 21 (1996).
60. Jing, J., Wu, L. A. & del Campo, A. Fundamental speed limits to the generation of quantumness. *arXiv:1510.01106* (2015).

Acknowledgements

This work was supported by NSFC (Grant Nos 11205121, 91536108), New Star Project of Science and Technology of Shaanxi Province (Grant No. 2016KJXX-27), Doctoral Fund of XUPT (Grant No. ZL2015), and CAS (Grant No. XDB01010000).

Author Contributions

M.-L.H. contributed the idea and performed the calculations. M.-L.H. and H.F. wrote the paper. All authors reviewed the manuscript and agreed with the submission.

Additional Information

Competing financial interests: The authors declare no competing financial interests.

How to cite this article: Hu, M.-L. and Fan, H. Evolution equation for quantum coherence. *Sci. Rep.* **6**, 29260; doi: 10.1038/srep29260 (2016).



This work is licensed under a Creative Commons Attribution 4.0 International License. The images or other third party material in this article are included in the article's Creative Commons license, unless indicated otherwise in the credit line; if the material is not included under the Creative Commons license, users will need to obtain permission from the license holder to reproduce the material. To view a copy of this license, visit <http://creativecommons.org/licenses/by/4.0/>