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A heterogeneous continuous age-structured model of mumps with vaccine

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ABSTRACT

In classical mumps models, individuals are generally assumed to be uniformly mixed (homogeneous), ignoring population heterogeneity (preference, activity, etc.). Age is the key to catching mixed patterns in developing mathematical models for mumps. A continuous heterogeneous age-structured model for mumps with vaccines has been developed in this paper. The stability of age-structured models is a difficult question. An explicit formula of R_0 was defined for the various mixing modes (isolation, proportional and heterogeneous mixing) with or without the vaccine. The results show that the endemic steady state is unique and locally stable if $R_0 > 1$ without any additional conditions. A number of numerical examples are given to support the theory.

1. Introduction

Mumps is an infection of the RNA virus, which is primarily transmitted through contact with breathing droplets (CDC, 2023; ChinaCDC, 2023; Latner & Hickman, 2015; Sane et al., 2014). The best way to reduce children's risk is through mumps vaccine. Since its introduction in 1967, there has been a decline in mumps cases (CDC, 2023). However, the mumps vaccine is not effeetive in addressing the high level of mumps antibodies alone. Most developed countries have already implemented a two-dose vaccination program against mumps, but even such measures proved ineffective in controlling mumps transmission (Barskey et al., 2012; Connell et al., 2020; Hanna-Wakim et al., 2008; van der Maas et al., 2016). Vaccine failure is a major reason for mumps relapse (Hamami et al., 2017; Liu et al., 2018; Vygen et al., 2016). A number of mathematic models have been established for the spread of mumps (Hamami et al., 2017; Kitano, 2019; Li et al., 2018; Liu et al., 2018;

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Magpantay, 2017; Qu et al., 2017; Zhao et al., 2017; Zhou et al., 2019) and its influence on the epidemic (Deeks et al., 2011; Gomes et al., 2004; Magpantay et al., 2014, 2016; Takla et al., 2014; van Boven et al., 2013).

The first simplified mathematics model for mumps was the homogenous classic model, which did not consider the effects of age or vaccination (Li et al., 2018; Qu et al., 2017). In classical infectious disease models, individuals are assumed to be uniformly mixed (homogenous), ignoring population heterogeneity (priority, activity, etc.). This hypothesis might lead to the deviation of the R_0 (basic reproduction number). Age is crucial in capturing mixing patterns when constructing mathematical models. Individuals in different age groups show significant differences in key characteristics like birth, mortality and contact rate. For example, children who suffer from childhood diseases are more likely to encounter their peers. Children are most at risk for malaria, leading to high mortality. The highest rate of HIV infection was between 20 and 45 years of age. Therefore, the study of heterogeneous mathematical models based on age plays an important role in understanding disease transmission and epidemic mechanism.

In our latest paper (Azimaqin et al., 2022), we developed a discrete age-structure model for mumps, which is theoretically difficult to analyze. In this paper, a heterogeneous and continuous age-structured model of mumps was constructed, and theoretical analysis was performed without periodical parameters.

There are nine sections in this paper. In the next section, we present a heterogeneous age-structured mumps model with vaccine. The normalized system and the Cauchy problem are established. In section 3, an explicitly computed formula of R_0 for heterogeneous preferential mixing case is defined. In section 4 and 5, the uniqueness, local and global stabilities of steady states are obtained In section 6, numerical examples are given in support of the theoretical results. In the end, the paper is summarized and discussed briefly in last Section.

2. Heterogeneous model

The overall population can be divided into 10 classes: susceptible (S), exposed (E), severely infected (I), mildly infected (L), recovered (R), vaccinated (V), vaccinated susceptible ($\widehat{S}$), vaccinated exposed ($\widehat{V}$), vaccinated severely infected ($\widehat{I}$), vaccinated mildly infected ($\widehat{L}$). The number of whole population is

$$N(t,a) \triangleq [S+V+\widehat{S}+E+I+L+\widehat{V}+\widehat{I}+\widehat{L}+R](t,a).$$

The following continuous model of mumps can be obtained from the flow chart (Fig. 1) and the discrete age-structured model (4) in (Azimaqin et al., 2022):

$$\begin{cases}
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})S = -(\lambda(t,a)+q(a))S,\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})V = -(\theta_2(a)+\theta_3\lambda(t,a)+q(a))V,\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widehat{S} = \theta_2(a)V-(\lambda(t,a)+q(a))\widehat{S},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})E = \lambda(t,a)S-(\delta+q(a))E,\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widehat{V} = \lambda(t,a)(\theta_3 V+\widehat{S})-(\delta+q(a))\widehat{V},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})I = k_1\delta E-(\gamma+q(a))I,\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widehat{I} = k_2\delta\widehat{V}-(\gamma+q(a))\widehat{I},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})L = (1-k_1)\delta E-(\gamma+q(a))L,\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widehat{L} = (1-k_2)\delta\widehat{V}-(\gamma+q(a))\widehat{L},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})R = \gamma(I+\widehat{I}+L+\widehat{L})-q(a)R,
\end{cases} \tag{2.1}$$

with boundary condition

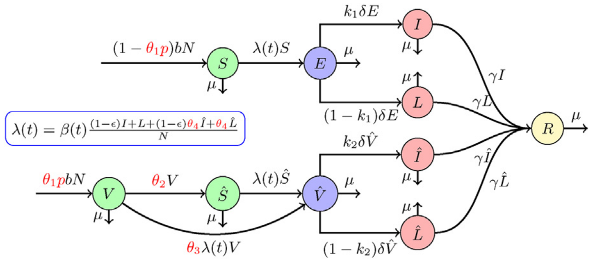

Fig. 1. Flowchart of mumps.

$$S(t,0) = (1-\theta_1 p)\int_0^{\hat{a}} \mu(\sigma)N(t,\sigma)\mathrm{d}\sigma, V(t,0) = \theta_1 p\int_0^{\hat{a}} \mu(\sigma)N(t,\sigma)\mathrm{d}\sigma,$$
$$X(t,0) = 0, X = (\widehat{S}, E, \widehat{V}, I, \widehat{I}, L, \widehat{L}, R),$$

and initial condition

$$X(0,a) = X_0(a), X = (S, V, \widehat{S}, E, \widehat{V}, I, \widehat{I}, L, \widehat{L}, R).$$

where δ is the incubation rate (from virus invasion to before clinical symptoms appear), γ is recover rate, k_1 is the rate moving from E to I, k_2 is the rate moving from $\widehat{V}$ to $\widehat{I}$, $\mu(a)$ is the birth rate, $q(a)$ is the drop out rate of school of children at the age a. There are four parameters associated with the vaccine: p is vaccine coverage for children, θ_1 is a primary vaccine failure (the fraction of vaccine that fails to provide any initial vaccine protection), $\theta_2(a)$ is vaccine wane (vaccine protection ceases after some time), θ_3 is vaccine leakiness (vaccine reduces the potential for infection, but does not eliminate), please refer to (Azimaqin et al., 2022; Magpantay, 2017). The infection rate is

$$\lambda(t,a) = \int_0^{\hat{a}} \beta(a,\sigma)\frac{(1-\epsilon)I(t,\sigma) + (1-\epsilon)\theta_4\widehat{I}(t,\sigma) + L(t,\sigma) + \theta_4\widehat{L}(t,\sigma)}{N(t,\sigma)}\mathrm{d}\sigma, \tag{2.2}$$

where ϵ is isolation rate (some serious infections are isolated because of the high infection rate of mumps) and θ_4 is relative infectivity (transmission rate of a vaccinated person to that of an unvaccinated). Function

$$\beta(a,\sigma) = \beta c(a)[\eta(a)d(a,\sigma) + (1-\eta(a))m(t,\sigma)], \tag{2.3}$$

is transmission rate between the members in age a contacting with individuals in age σ, where β is the per contact infection rate, $c(a)$ is the average physical contact rate in age group a (activity) that can be measured from empirical data (see (Azimaqin et al., 2022; Glasser et al., 2012; Grijalva et al., 2015; Mossong et al., 2008)), $\eta(a) \in [0,1]$ is the preferences (rate of contacts reserved in one's own group). Function $m(t, b)$ is a activity-weighted proportional mixing described by

$$m(t,\sigma) = \frac{(1-\eta(\sigma))c(\sigma)N(t,\sigma)}{\int_0^{\hat{a}}(1-\eta(\sigma))c(\sigma)N(t,\sigma)\mathrm{d}\sigma}.$$

The density function at age a can be described by the Gaussian kernel function (Dirac delta function) $d(a, b)$ (see (Feng et al., 2015; Glasser et al., 2012)) as

$$d(a,b) = \frac{1}{\sqrt{2\pi}\sigma(a)}\exp[-\frac{(b-a)^2}{2\sigma^2(a)}].$$

This kind of transmission function $\beta(a, b)$ is called heterogeneous preferential mixing and has five special types (see (Cui et al., 2019; Feng et al., 2015; Glasser et al., 2012)).

(i) **Proportional Mixing:** $\eta(a) = 0$ for all ages $a \in [0, \hat{a}]$;
(ii) **Isolated Mixing:** $\eta(a) = 1$ for all ages $a \in [0, \hat{a}]$;

(iii) **Preferential Mixing:** $0 < \eta(a) < 1$ for some age $a \in [0, \widehat{a}]$;
(iv) **Homogeneous Preferential Mixing:** $\eta(a) = \eta$ for all ages $a \in [0, \widehat{a}]$;
(v) **Heterogeneous Preferential Mixing:** if $\eta(a) \neq \eta(b)$ for some age $a \neq b, a, b \in [0, \widehat{a}]$.

It is called **Separable mixing** if $\beta(a, b) = \beta_1(a)\beta_2(b)$ (see (Huang, Kang, Lu, Ruan, & Zhuo, 2022; Inaba, 1990; Khan & Zaman, 2018; Kuniya, 2019; Tian & Wang, 2020)). Separable mixing means there is no direct relationship between a person who is infected at the age of b and a susceptible person at a. For the separable case, the existence and stability results of the age-structured model can be given by the explicit formula of R_0, see (Huang, Kang, Lu, Ruan, & Zhuo, 2022; Okuwa et al., 2019). Defining the explicit formula of R_0 and analyzing stability of the heterogeneous age-structured model is a challenging problem.

Theoretical analysis of system (2.1) requires the following assumptions based on the biological significance of parameters. Proportional Mixing is a kind of separable mixing.

Hypothesis 1. We assume that

(i) $\mu(a), \theta_2(a), \eta(a) \in L^\infty_+(0, A), c(a) \in C[0, \widehat{a}]$.

Without losing generality, it may be assumed that the host population density $N(t, a)$ has already reached population steady state, that is

$$[S + V + \widehat{S} + E + I + L + \widehat{V} + \widehat{I} + \widehat{L} + R](t, a) = N(a).$$

By adding the equations in system (2.1), we can obtain that

$$\begin{cases} \dot{N} = -q(a)N(a), \\ N(0) = \displaystyle\int_0^{\hat{a}} \mu(s)N(s)\mathrm{d}s. \end{cases} \tag{2.4}$$

Solving the system, we obtain

$$N(a) = N(0)\mathrm{e}^{-\int_0^a q(\sigma)\mathrm{d}\sigma}. \tag{2.5}$$

Substituting (2.5) into (2.4), we have

$$1 = \int_0^{\hat{a}} \mu(a)\mathrm{e}^{-\int_0^a q(\sigma)\mathrm{d}\sigma}\mathrm{d}a.$$

Let

$$x(t, a) = \frac{X(t, a)}{N(a)}, X = (S, V, \widehat{S}, E, \widehat{V}, I, \widehat{I}, L, \widehat{L}, R), \quad x = (s, v, \widehat{s}, e, \widehat{v}, i, \widehat{i}, l, \widehat{l}, r),$$

then

$$(s + v + \widehat{s} + e + \widehat{e} + i + \widehat{i} + l + \widehat{l} + r)(t, a) = 1,$$

and system (2.1) becomes standard system

$$\begin{cases} (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})s = -\lambda(t,a)s, \\ (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})v = -(\theta_2(a)+\theta_3\lambda(t,a))v, \\ (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widehat{s} = \theta_2(a)v-\lambda(t,a)\widehat{s}, \\ (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})e = \lambda(t,a)s-\delta e, \\ (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widehat{e} = \lambda(t,a)(\theta_3 v+\widehat{s})-\delta\widehat{e}, \\ (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})i = k_1\delta e-\gamma i, \\ (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widehat{i} = k_2\delta\widehat{e}-\gamma\widehat{i}, \\ (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})l = (1-k_1)\delta e-\gamma l, \\ (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widehat{l} = (1-k_2)\delta\widehat{e}-\gamma\widehat{l}, \\ (\frac{\partial}{\partial a}+\frac{\partial}{\partial t})r = \gamma(i+v+l+\widehat{l}), \end{cases} \tag{2.6}$$

with boundary condition

$$s(t,0)=(1-\theta_1 p), v(t,0)=\theta_1 p, x(t,0)=0, x=(\widehat{s},e,\widehat{e},i,\widehat{i},l,\widehat{l},r),$$

and initial condition

$$x(0,a)=x_0(a), x=(s,v,\widehat{s},e,\widehat{e},i,\widehat{i},l,\widehat{l},r).$$

The age-specific infection rate (2.2) is

$$\lambda(t,a)=\beta c(a)\int_0^{\hat{a}}K(a,s)h(t,s)\mathrm{d}s, \tag{2.7}$$

where

$$K(a,s)=\eta(a)d(a,s)+(1-\eta(a))m(s), \qquad h=(1-\epsilon)i+(1-\epsilon)\theta_4\widehat{i}+l+\theta_4\widehat{l},$$

and

$$m(s)=\frac{(1-\eta(s))c(s)N(s)}{\int_0^{\hat{a}}(1-\eta(s))c(s)N(s)\mathrm{d}s}.$$

Define the state space for (i, r)-system as

$$B=\{(v,\widehat{s},e,\widehat{e},i,\widehat{i},l,\widehat{l},r)\in E^9=L^1_+\times L^1_+\times\cdots\times L^1_+|0\le v,\widehat{s},e,\widehat{e},i,\widehat{i},l,\widehat{l},r\le 1\}.$$

Then, B is a convex closed subset of Banach space E^9. Define operators $\mathcal{A}$ and $\mathcal{F}$ on E^9 by

$$(\mathcal{A}\phi)(a)=-\big(\phi_1'(a),\phi_2'(a),\phi_3'(a),\phi_4'(a),\phi_5'(a),\phi_6'(a),\phi_7'(a),\phi_8'(a),\phi_9'(a)\big)^T,$$

and

$$\mathcal{F}(\phi)(a) = \begin{pmatrix} -(\theta_2(a) + \theta_3\lambda(a))\phi_1(a), \\ \theta_2(a)\phi_1(a) - \lambda(a)\phi_2(a), \\ \lambda(a)\left(1 - \sum_{k=1}^{9}\phi_k\right) - \delta\phi_3(a), \\ \lambda(a)(\theta_3\phi_1(a) + \phi_2(a)) - \delta\phi_4(a), \\ k_1\delta\phi_3(a) - \gamma\phi_5(a), \\ k_2\delta\phi_4(a) - \gamma\phi_6(a), \\ (1-k_1)\delta\phi_3(a) - \gamma\phi_7(a), \\ (1-k_2)\delta\phi_4(a) - \gamma\phi_8(a), \\ \gamma(\phi_5(a) + \phi_6(a) + \phi_7(a) + \phi_8(a)), \end{pmatrix}$$

where $\phi_k \in D(\mathcal{A}) = \{\phi \in E^9 | \phi \in AC[0, \widehat{a}], \phi(0) = 0\}$, $AC[0, \widehat{a}]$ is absolutely continuous on $[0, \widehat{a}]$. Then the abstract semilinear Cauchy problem of system (2.6)

$$\dot{u}(t) = \mathcal{A}u(t) + \mathcal{F}(u(t)), u(0) = u_0, \tag{2.8}$$

where operator $\mathcal{A}$ is an infinitesimal generator of C_0-semigroup $\{T(t)\}_{t>0} = \{e^{t\mathcal{A}}\}_{t>0}$, operator $\mathcal{F}$ is Lipschitz continuously on E^9. Similar to previous analyses (Huang, Kang, Lu, & et al, 2022; Inaba, 1990; Webb, 1985), we can obtain that system (2.8) has a unique solution on E^9.

3. Basic reproduction number

Assume that $M^* = (s^*, v^*, \widehat{s}^*, e^*, \widehat{e}^*, i^*, \widehat{i}^*, l^*)$ is the nontrivial ESS (endemic steady state) of system (2.6), then M^* satisfying following system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}a}s^* = -\lambda^* s^*, \\ \frac{\mathrm{d}}{\mathrm{d}a}v^* = -(\theta_2(a) + \theta_3\lambda^*)v^*, \\ \frac{\mathrm{d}}{\mathrm{d}a}\widehat{s}^* = \theta_2(a)v^* - \lambda^*\widehat{s}^*, \\ \frac{\mathrm{d}}{\mathrm{d}a}e^* = \lambda^* s^* - \delta e^*, \\ \frac{\mathrm{d}}{\mathrm{d}a}\widehat{e}^* = \lambda^*(\theta_3 v^* + \widehat{s}^*) - \delta\widehat{e}^*, \\ \frac{\mathrm{d}}{\mathrm{d}a}i^* = k_1\delta e^* - \gamma i^*, \\ \frac{\mathrm{d}}{\mathrm{d}a}\widehat{i}^* = k_2\delta\widehat{e}^* - \gamma\widehat{i}^*, \\ \frac{\mathrm{d}}{\mathrm{d}a}l^* = (1-k_1)\delta e^* - \gamma l^*, \\ \frac{\mathrm{d}}{\mathrm{d}a}\widehat{l}^* = (1-k_2)\delta\widehat{e}^* - \gamma\widehat{l}^*. \end{cases} \tag{3.1}$$

with boundary condition

$$s^*(0) = (1-\theta_1 p), v^*(0) = \theta_1 p, x^*(0) = 0, x = (\widehat{s}, e, \widehat{e}, i, \widehat{i}, l, \widehat{l}, r),$$

where

$$\lambda^*(a) = \beta c(a)\int_0^{\hat{a}} K(a, \sigma)h^*(\sigma)\mathrm{d}\sigma, \tag{3.2}$$

and

$$h^* = (1-\epsilon)i^* + (1-\epsilon)\theta_4\widehat{i}^* + l^* + \theta_4\widehat{l}^*. \tag{3.3}$$

Solving system (3.1), we have

$$\begin{cases} s^* = (1-\theta_1 p)\mathrm{e}^{-\int_0^a \lambda^*(\sigma)\mathrm{d}\sigma} \\ v^* = \theta_1 p \mathrm{e}^{-\int_0^a (\theta_2(\sigma)+\theta_3\lambda^*(\sigma))\mathrm{d}\sigma}, \\ \widehat{s}^* = \theta_1 p \int_0^a \theta_2(\sigma)\mathrm{e}^{-\int_0^\sigma (\theta_2(w)+\theta_3\lambda^*(w))\mathrm{d}w}\mathrm{e}^{-\int_0^\sigma \lambda^*(w)\mathrm{d}w}\mathrm{d}\sigma, \\ e^* = \int_0^a \lambda^*(\sigma)s^*(\sigma)\mathrm{e}^{-\delta(a-\sigma)}\mathrm{d}\sigma, \\ \widehat{e}^* = \int_0^a \lambda^*(\sigma)(\theta_3 v^*(\sigma)+\widehat{s}^*(\sigma))\mathrm{e}^{-\delta(a-\sigma)}\mathrm{d}\sigma, \\ i^* = k_1\delta\int_0^a e^*(\sigma)\mathrm{e}^{-\gamma(a-\sigma)}\mathrm{d}\sigma, \\ \widehat{i}^* = k_2\delta\int_0^a \widehat{e}^*(\sigma)\mathrm{e}^{-\gamma(a-\sigma)}\mathrm{d}\sigma, \\ l^* = (1-k_1)\delta\int_0^a e^*(\sigma)\mathrm{e}^{-\gamma(a-\sigma)}\mathrm{d}\sigma, \\ \widehat{l}^* = (1-k_2)\delta\int_0^a \widehat{e}^*(\sigma)\mathrm{e}^{-\gamma(a-\sigma)}\mathrm{d}\sigma. \end{cases} \tag{3.4}$$

Substituting the expression of $i^*, \widehat{i}^*, l^*, \widehat{l}^*$ into (3.3) and (3.2), we get

$$\begin{aligned} \lambda^*(a) &= (1-\epsilon k_1)(1-\theta_1 p)\delta\int_0^a\int_0^\sigma \lambda^*(\tau)e^{-\int_0^\tau \lambda^*(w)\mathrm{d}w}e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\int_0^a\int_0^\sigma \lambda^*(\tau)e^{-\int_0^\tau(\theta_2(w)+\theta_3\lambda^*(w))\mathrm{d}w}e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+\int_0^a\int_0^\sigma \lambda^*(\tau)\int_0^\tau \theta_2(w)e^{-\int_0^w(\theta_2(\rho)+\theta_3\lambda^*(\rho))\mathrm{d}\rho}e^{-\int_w^\tau \lambda^*(\rho)\mathrm{d}\rho}\mathrm{d}w\mathrm{e}^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma\Big]. \end{aligned} \tag{3.5}$$

Then get a fixed point problem

$$\phi(a) = \Phi(\phi)(a) \tag{3.6}$$

about the $\phi \triangleq \lambda^*$, where

$$\Phi(\phi)(a) \triangleq \beta c(a)\int_0^{\hat{a}} K(a,\sigma)\widehat{\Phi}(\phi)(\sigma)\mathrm{d}\sigma,$$

and

$$\begin{aligned} \widehat{\Phi}(\phi)(a) &\triangleq (1-\epsilon k_1)(1-\theta_1 p)\delta\int_0^a\int_0^\sigma \phi(\tau)e^{-\int_0^\tau \phi(w)\mathrm{d}w}e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\int_0^a\int_0^\sigma \phi(\tau)e^{-\int_0^\tau(\theta_2(w)+\theta_3\phi(w))\mathrm{d}w}e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+\int_0^a\int_0^\sigma \phi(\tau)\int_0^\tau \theta_2(w)e^{-\int_0^w(\theta_2(\rho)+\theta_3\phi(\rho))\mathrm{d}\rho}e^{-\int_w^\tau \phi(\rho)\mathrm{d}\rho}\mathrm{d}w\mathrm{e}^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma\Big]. \end{aligned}$$

The Fréchet derivative of Operator Φ about ϕ at zero is $K \triangleq \Phi'[0]$,

$$K(\phi)(a) = \beta c(a)\int_0^{\hat{a}} K(a,\sigma)\widehat{K}(\phi)(\sigma)\mathrm{d}\sigma, \tag{3.7}$$

where

$$\begin{aligned}\widehat{K}(\phi)(a) \quad &\triangleq (1-\epsilon k_1)(1-\theta_1 p)\delta\int_0^a\int_0^\sigma \phi(\tau)e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\int_0^a\int_0^\sigma \phi(\tau)e^{-\int_0^\tau \theta_2(w)\mathrm{d}w}e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+\int_0^a\int_0^\sigma \phi(\tau)\Big(1-e^{-\int_0^\tau \theta_2(w)\mathrm{d}w}\Big)e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma\Big].\end{aligned}$$

Operator K is the next generation operator (the age distribution of secondary cases) of the system (see (Inaba, 2017; Okuwa et al., 2019)). The spectral radius $r(K)$ equals to $R_0 \triangleq r(K)$ (see (Inaba, 2017; Okuwa et al., 2019)).

Next, we define the R_0 of the heterogeneous preferential mixing case of system (2.6). It can be seen that

$$r(a)\triangleq\beta\int_0^{\hat{a}} K(a,\sigma)\widehat{K}(c)(\sigma)\mathrm{d}\sigma$$

is an eigenvalues corresponding to the eigenvectors $c(a)$ of operator K, where

$$\begin{aligned}\widehat{K}(c)(a) \quad &= (1-\epsilon k_1)(1-\theta_1 p)\delta\int_0^a\int_0^\sigma c(\tau)e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\int_0^a\int_0^\sigma c(\tau)e^{-\int_0^\tau \theta_2(w)\mathrm{d}w}e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+\int_0^a\int_0^\sigma c(\tau)\Big(1-e^{-\int_0^\tau \theta_2(w)\mathrm{d}w}\Big)e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(a-\sigma)}\mathrm{d}\sigma\Big].\end{aligned}$$

That is

$$K(c)(a) = \left[\beta\int_0^{\hat{a}} K(a,\sigma)\widehat{K}(c)(\sigma)\mathrm{d}\sigma\right]c(a).$$

Then, the maximum eigenvalue $r(a)$ is the $\boldsymbol{R_0}$ of system (2.6), that is

$$\begin{aligned}R_0 \quad &= (1-\epsilon k_1)(1-\theta_1 p)\delta\beta\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma c(\tau)e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b \\ &+(1-\epsilon k_2)\theta_1 p\theta_4\delta\beta\Big[\theta_3\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma c(\tau)e^{-\int_0^\tau \theta_2(w)\mathrm{d}w}e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b \\ &+\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma c(\tau)\Big(1-e^{-\int_0^\tau \theta_2(w)\mathrm{d}w}\Big)e^{-\delta(\sigma-\tau)}\mathrm{d}\tau e^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\Big].\end{aligned} \tag{3.8}$$

R_0 can be explained under different mixed modes and vaccine.

(i) By taking $\eta(a)\equiv 0$, we can define the R_0 of **proportional mixing case** (see (Wang et al., 2019)):

$$R_{0p} = R'_{0p} + R''_{0p} + R'''_{0p}, \tag{3.9}$$

where $R'_{0p}, R''_{0p}, R'''_{0p}$ is the R_0 corresponding to the no vaccine ($p=0$), with vaccine but no vaccine wane ($p\neq 0$, $\theta_2(a)\equiv 0$), with vaccine and vaccine wane ($p\neq 0$, $\theta_2(a)\neq 0$) as

$$R'_{0p} = (1-\epsilon k_1)(1-\theta_1 p)\delta\beta \int_0^{\hat{a}} m(b) \int_0^b \int_0^\sigma c(\tau) e^{-\delta(\sigma-\tau)} \mathrm{d}\tau e^{-\gamma(b-\sigma)} \mathrm{d}\sigma \mathrm{d}b,$$

$$R''_{0p} = (1-\epsilon k_2)\theta_1 p \theta_4 \delta\beta\theta_3 \int_0^{\hat{a}} m(b) \int_0^b \int_0^\sigma c(\tau) e^{-\int_0^\tau \theta_2(w)\mathrm{d}w} e^{-\delta(\sigma-\tau)} \mathrm{d}\tau e^{-\gamma(b-\sigma)} \mathrm{d}\sigma \mathrm{d}b,$$

$$R'''_{0p} = (1-\epsilon k_2)\theta_1 p \theta_4 \delta\beta \int_0^{\hat{a}} m(b) \int_0^b \int_0^\sigma c(\tau) \left(1 - e^{-\int_0^\tau \theta_2(w)\mathrm{d}w}\right) e^{-\delta(\sigma-\tau)} \mathrm{d}\tau e^{-\gamma(b-\sigma)} \mathrm{d}\sigma \mathrm{d}b.$$

(ii) By taking $\eta(a) \equiv 1$, we can define the R_0 of **isolated mixing case**:

$$R_{0i} = \max_{a\in[0,\hat{a}]} R_{0i}(a), \tag{3.10}$$

where $R'_{0i}(a), R''_{0i}(a), R'''_{0i}(a)$ is the R_0 corresponding to the no vaccine ($p=0$), with vaccine but no vaccine wane ($p \neq 0$, $\theta_2(a) \equiv 0$) with vaccine and vaccine wane ($p \neq 0$, $\theta_2(a) \neq 0$) as

$$R'_{0i}(a) = (1-\epsilon k_1)(1-\theta_1 p)\delta\beta \int_0^{\hat{a}} d(a,b) \int_0^b \int_0^\sigma c(\tau) e^{-\delta(\sigma-\tau)} \mathrm{d}\tau e^{-\gamma(b-\sigma)} \mathrm{d}\sigma \mathrm{d}b,$$

$$R''_{0i}(a) = (1-\epsilon k_2)\theta_1 p \theta_4 \delta\beta\theta_3 \int_0^{\hat{a}} d(a,b) \int_0^b \int_0^\sigma c(\tau) e^{-\int_0^\tau \theta_2(w)\mathrm{d}w} e^{-\delta(\sigma-\tau)} \mathrm{d}\tau e^{-\gamma(b-\sigma)} \mathrm{d}\sigma \mathrm{d}b,$$

$$R'''_{0i}(a) = (1-\epsilon k_2)\theta_1 p \theta_4 \delta\beta \int_0^{\hat{a}} d(a,b) \int_0^b \int_0^\sigma c(\tau) \left(1 - e^{-\int_0^\tau \theta_2(w)\mathrm{d}w}\right) e^{-\delta(\sigma-\tau)} \mathrm{d}\tau e^{-\gamma(b-\sigma)} \mathrm{d}\sigma \mathrm{d}b,$$

and

$$R_{0i}(a) = R'_{0i}(a) + R''_{0i}(a) + R'''_{0i}(a).$$

(iii) The R_0 of **heterogeneous preferential mixing case** can be defined by the combination of $R_{0i}(a)$ and R_{0p}:

$$R_0 = \max_{a\in[0,\hat{a}]} [\eta(a) R_{0i}(a) + (1-\eta(a)) R_{0p}]. \tag{3.11}$$

The explicit formula of R_0 that given by (3.11) is essential for the uniqueness and stability of ESS. The numerical calculations in Section 6 are a further proof of the conclusion.

4. Existence and stability of the DFSS

For the DFSS (disease-free steady state) E^0 of system (2.6), it is clear that

$$e^0 = \hat{e}^0 = i^0 = \hat{i}^0 = l^0 = \hat{l}^0 = r^0 = 0,$$

so $\lambda = 0$. $s^0, v^0, \hat{s}^0$ are can be solved as

$$s^0 = (1-\theta_1 p), v^0 = \theta_1 p \mathrm{e}^{-\int_0^a \theta_2(\sigma)\mathrm{d}\sigma}, \hat{s}^0 = \theta_1 p \left(1 - \mathrm{e}^{-\int_0^a \theta_2(\sigma)\mathrm{d}\sigma}\right),$$

so the DFSS E^0 exists and is unique.

4.1. Local stability of the DFSS

To give the local stability of DFSS, we make a translation transformation to system (2.6) by

$$x = \overline{x} + x^*, x = (s, v, \widehat{s}, e, \widehat{e}, i, \widehat{i}, l, \widehat{l}, r).$$

Then linearizing the system at E^0 to obtain

$$\begin{cases} (\frac{\partial}{\partial a} + \frac{\partial}{\partial t})\overline{s} = -(1-\theta_1 p)\overline{\lambda}, \\ (\frac{\partial}{\partial a} + \frac{\partial}{\partial t})\overline{v} = -\theta_2(a)\overline{v} - \theta_1 p e^{-\int_0^a \theta_2(\sigma)d\sigma}(\theta_2(a) + \theta_3\overline{\lambda}), \\ (\frac{\partial}{\partial a} + \frac{\partial}{\partial t})\overline{\widehat{s}} = \theta_2(a)\left(\overline{v} + \theta_1 p e^{-\int_0^a \theta_2(\sigma)d\sigma}\right) - \overline{\lambda}\theta_1 p\left(1 - e^{-\int_0^a \theta_2(\sigma)d\sigma}\right), \\ (\frac{\partial}{\partial a} + \frac{\partial}{\partial t})\overline{e} = (1-\theta_1 p)\overline{\lambda} - \delta\overline{e}, \\ (\frac{\partial}{\partial a} + \frac{\partial}{\partial t})\overline{\widehat{e}} = \theta_1 p\overline{\lambda}\left(1 + (\theta_3 - 1)e^{-\int_0^a \theta_2(\sigma)d\sigma}\right) - \delta\overline{\widehat{e}}, \\ (\frac{\partial}{\partial a} + \frac{\partial}{\partial t})\overline{i} = k_1\delta\overline{e} - \gamma\overline{i}, \\ (\frac{\partial}{\partial a} + \frac{\partial}{\partial t})\overline{\widehat{i}} = k_2\delta\overline{\widehat{e}} - \gamma\overline{\widehat{i}}, \\ (\frac{\partial}{\partial a} + \frac{\partial}{\partial t})\overline{l} = (1-k_1)\delta\overline{e} - \gamma\overline{l}, \\ (\frac{\partial}{\partial a} + \frac{\partial}{\partial t})\overline{\widehat{l}} = (1-k_2)\delta\overline{\widehat{e}} - \gamma\overline{\widehat{l}}, \end{cases} \tag{4.1}$$

with boundary conditions

$$\overline{x}(t, 0) = 0, x = (s, v, \widehat{s}, e, \widehat{e}, i, \widehat{i}, l, \widehat{l}),$$

where

$$\overline{\lambda}(t, a) = \beta c(a)\int_0^{\hat{a}} K(a, u)\overline{h}(t, u)du, \qquad \overline{h} = (1-\epsilon)\overline{i} + (1-\epsilon)\theta_4\overline{\widehat{i}} + \overline{l} + \theta_4\overline{\widehat{l}}.$$

Let

$$\overline{x} = \overline{x}(a)e^{\xi t}, x = (s, v, \widehat{s}, e, \widehat{e}, i, \widehat{i}, l, \widehat{l}, r).$$

Substituting it into system (4.1) yields

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}a}\bar{s}+\xi\bar{s}=-(1-\theta_1 p)\bar{\lambda}, \\ \frac{\mathrm{d}}{\mathrm{d}a}\bar{v}+\xi\bar{v}=-\theta_2(a)\bar{v}-\theta_1 p \mathrm{e}^{-\int_0^a \theta_2(\sigma)\mathrm{d}\sigma}(\theta_2(a)+\theta_3\bar{\lambda}), \\ \frac{\mathrm{d}}{\mathrm{d}a}\bar{\hat{s}}+\xi\bar{\hat{s}}=\theta_2(a)\left(\bar{v}+\theta_1 p \mathrm{e}^{-\int_0^a \theta_2(\sigma)\mathrm{d}\sigma}\right)-\bar{\lambda}\theta_1 p\left(1-\mathrm{e}^{-\int_0^a \theta_2(\sigma)\mathrm{d}\sigma}\right), \\ \frac{\mathrm{d}}{\mathrm{d}a}\bar{e}+\xi\bar{e}=(1-\theta_1 p)\bar{\lambda}-\delta\bar{e}, \\ \frac{\mathrm{d}}{\mathrm{d}a}\bar{\hat{e}}+\xi\bar{\hat{e}}=\theta_1 p\bar{\lambda}\left(1+(\theta_3-1)\mathrm{e}^{-\int_0^a \theta_2(\sigma)\mathrm{d}\sigma}\right)-\delta\bar{\hat{e}}, \\ \frac{\mathrm{d}}{\mathrm{d}a}\bar{i}+\xi\bar{i}=k_1\delta\bar{e}-\gamma\bar{i}, \\ \frac{\mathrm{d}}{\mathrm{d}a}\bar{\hat{i}}+\xi\bar{\hat{i}}=k_2\delta\bar{\hat{e}}-\gamma\bar{\hat{i}}, \\ \frac{\mathrm{d}}{\mathrm{d}a}\bar{l}+\xi\bar{l}=(1-k_1)\delta\bar{e}-\gamma\bar{l}, \\ \frac{\mathrm{d}}{\mathrm{d}a}\bar{\hat{l}}+\xi\bar{\hat{l}}=(1-k_2)\delta\bar{\hat{e}}(a)-\gamma\bar{\hat{l}}, \end{cases} \tag{4.2}$$

with initial value conditions

$$\bar{x}(0)=0, x=(s,v,\hat{s},e,\hat{e},i,\hat{i},l,\hat{l}),$$

where

$$\bar{\lambda}(a)=\beta c(a)\int_0^{\hat{a}}K(a,u)\bar{h}(u)\mathrm{d}u, \qquad \bar{h}=(1-\epsilon)\bar{i}+(1-\epsilon)\theta_4\bar{\hat{i}}+\bar{l}+\theta_4\bar{\hat{l}}.$$

The solutions $\bar{i},\bar{\hat{i}},\bar{l},\bar{\hat{l}}$ of system (4.2) can be solved as

$$\begin{cases} \bar{i}=(1-\theta_1 p)k_1\delta\int_0^a\int_0^\sigma\bar{\lambda}(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma, \\ \bar{\hat{i}}=\theta_1 p k_2\delta\int_0^a\left(1+(\theta_3-1)\mathrm{e}^{-\int_0^\sigma\theta_2(w)\mathrm{d}w}\right)\int_0^\sigma\bar{\lambda}(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma, \\ \bar{l}=(1-\theta_1 p)(1-k_1)\delta\int_0^a\int_0^\sigma\bar{\lambda}(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma, \\ \bar{\hat{l}}=\theta_1 p(1-k_2)\delta\int_0^a\left(1+(\theta_3-1)\mathrm{e}^{-\int_0^\sigma\theta_2(w)\mathrm{d}w}\right)\int_0^\sigma\bar{\lambda}(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma, \end{cases}$$

then

$$\begin{aligned} \bar{h}(a) &= (1-\epsilon k_1)(1-\theta_1 p)\delta\int_0^a\int_0^\sigma\bar{\lambda}(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma, \\ &+(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\int_0^a \mathrm{e}^{-\int_0^\sigma\theta_2(w)\mathrm{d}w}\int_0^\sigma\bar{\lambda}(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma \\ &\quad+\int_0^a\left(1-\mathrm{e}^{-\int_0^\sigma\theta_2(w)\mathrm{d}w}\right)\int_0^\sigma\bar{\lambda}(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma\Big]. \end{aligned}$$

Letting $\Lambda(a)\triangleq\int_0^{\hat{a}}K(a,\sigma)\bar{h}(\sigma)\mathrm{d}\sigma$, we have $\bar{\lambda}(a) = \beta c(a)\Lambda(a)$. Substituting the $\bar{h}(a)$ into $\Lambda(a)$, we have

$$\Lambda(a) = (1-\epsilon k_1)(1-\theta_1 p)\delta\beta\int_0^{\hat{a}} K(a,b)\int_0^a\int_0^\sigma c(u)\Lambda(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b,$$
$$+(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\beta\int_0^{\hat{a}} K(a,b)\int_0^a \mathrm{e}^{-\int_0^\sigma \theta_2(w)\mathrm{d}w}\int_0^\sigma c(u)\Lambda(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b$$
$$+\int_0^{\hat{a}} K(a,b)\int_0^a\Big(1-\mathrm{e}^{-\int_0^\sigma \theta_2(w)\mathrm{d}w}\Big)\int_0^\sigma c(u)\Lambda(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\Big].$$

Taking maximum both sides and dividing by $\widehat{\Lambda}\triangleq\max_{a\in[0,\hat{a}]}\Lambda(a)>0$, we get the following characteristic equation of ξ

$$G(\xi)-1=0,$$

where

$$G(\xi)\triangleq(1-\epsilon k_1)(1-\theta_1 p)\delta\beta\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^a\int_0^\sigma c(u)\frac{\Lambda(u)}{\widehat{\Lambda}}\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b,$$
$$+(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\beta\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^a \mathrm{e}^{-\int_0^\sigma \theta_2(w)\mathrm{d}w}\int_0^\sigma c(u)\frac{\Lambda(u)}{\widehat{\Lambda}}\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b$$
$$+\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^a\Big(1-\mathrm{e}^{-\int_0^\sigma \theta_2(w)\mathrm{d}w}\Big)\int_0^\sigma c(u)\frac{\Lambda(u)}{\widehat{\Lambda}}\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\Big].$$

Theorem 4.1. *The DFSS of system (2.6) is locally asymptotically stable if $R_0<1$.*

Proof. $G(\xi)$ is monotone decreasing function of ξ and

$$\lim_{\xi\to-\infty}G(\xi)=+\infty,\lim_{\xi\to+\infty}G(\xi)=0.$$

We get $G(0)\le R_0$, since $\frac{\Lambda(u)}{\widehat{\Lambda}}\le 1$. The $R_0<1$ implies that $G(\xi)=1$ has a unique negative real root ξ^* in $(-\infty,0)$, see Fig. 2.

Next, we prove that all the complex solutions of $G(\xi)=1$ have negative real parts when $\xi^*<0$. Let $z=u+iv(v\neq 0)$ be an arbitrary complex solution of $G(\xi)=1$. Notice that

$$G(\xi^*)=1=|G(u+iv)|\le G(u), u<\xi^*<0.$$

4.2. Global stability of the DFSS

We discuss global stability of the DFSS.

Theorem 4.2. *The DFSS E^0 of system (2.6) is globally asymptotically stable if $R_0<1$.*

Proof. Integrating equations $\bar{i},\bar{\hat{i}},\bar{l},\bar{\hat{l}}$ of system (2.6) along the characteristic lines $t-a=constant$, gives

$$\begin{cases} i=k_1\delta\int_0^a\int_0^u\lambda(t-u+w,w)s(t-u+w,w)\mathrm{e}^{-\delta(u-w)}\mathrm{d}w\mathrm{e}^{-\gamma(a-u)}\mathrm{d}u,\\ \hat{i}=k_2\delta\int_0^a\int_0^u\lambda(t-u+w,w)(\theta_3 v(t-u+w,w)+\hat{s}(t-u+w,w))\mathrm{e}^{-\delta(u-w)}\mathrm{d}w\mathrm{e}^{-\gamma(a-u)}\mathrm{d}u\\ l=(1-k_1)\delta\int_0^a\int_0^u\lambda(t-u+w,w)s(t-u+w,w)\mathrm{e}^{-\delta(u-w)}\mathrm{d}w\mathrm{e}^{-\gamma(a-u)}\mathrm{d}u,\\ \hat{l}=(1-k_2)\delta\int_0^a\int_0^u\lambda(t-u+w,w)(\theta_3 v(t-u+w,w)+\hat{s}(t-u+w,w))\mathrm{e}^{-\delta(u-w)}\mathrm{d}w\mathrm{e}^{-\gamma(a-u)}\mathrm{d}u.\end{cases}\tag{4.3}$$

where

$$\lambda(t,a)=\beta c(a)\int_0^{\hat{a}}K(a,u)h(t,u)\mathrm{d}u,\tag{4.4}$$

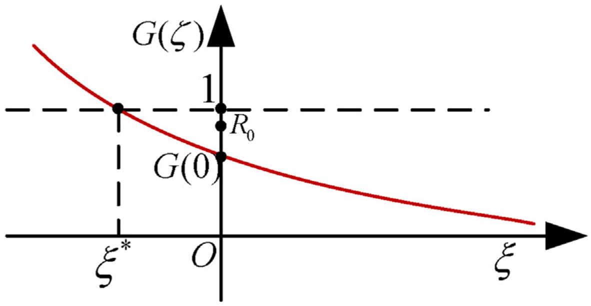

Fig. 2. Local stability of DFSS.

and

$$h = (1-\epsilon)i + (1-\epsilon)\theta_4\widehat{i} + l + \theta_4\widehat{l}.$$

Define a function

$$g(t,a) \triangleq \int_0^{\hat{a}} K(a,b)h(t,b)\mathrm{d}b,$$

then $\lambda(t, a) = \beta c(a)g(t, a)$ in (2.7). By putting (4.3) into (4.4), using the fact

$$s \le 1-\theta_1 p, v \le \theta_1 p \mathrm{e}^{-\int_0^u \theta_2(w)\mathrm{d}w}, \widehat{s} \le \theta_1 p\left(1-\mathrm{e}^{-\int_0^u \theta_2(w)\mathrm{d}w}\right),$$

and taking maximum both sides of the equation, we obtain

$$\widehat{g}(t) \le \beta\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]} K(a,u)h(t,u)\mathrm{d}u, \tag{4.5}$$

where $\widehat{g}(t) \triangleq \max_{a\in[0,\hat{a}]} g(t,a)$ and

$$\begin{aligned} h &= (1-\epsilon k_1)(1-\theta_1 p)\delta\int_0^a\int_0^\sigma c(\sigma)g(t-\sigma+u)e^{-\delta(\sigma-u)}\mathrm{d}\sigma e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\int_0^a\int_0^\sigma c(\sigma)g(t-\sigma+u)e^{-\int_0^u \theta_2(w)\mathrm{d}w} e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ &+\int_0^a\int_0^\sigma c(\sigma)g(t-\sigma+u)\left(1-e^{-\int_0^u \theta_2(w)\mathrm{d}w}\right)e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma\Big], \end{aligned}$$

Applying the supermum limit on both sides of (4.5), we get

$$\lim_{t\to+\infty}\sup\,\widehat{g}(t) \le R_0 \lim_{t\to+\infty}\sup\,\widehat{g}(t). \tag{4.6}$$

As we supposed that $R_0 < 1$, the only way inequality (4.1) holds is

$$\lim_{t\to+\infty}\sup\,\widehat{g}(t) = 0.$$

This means $\lim_{t\to+\infty} h(t, a) = 0$, that is $i = \widehat{i} = l = \widehat{l} = 0$, hence proves the theorem.

5. Uniqueness and stability of ESS

We investigate the uniqueness and local stability of the ESS of system (2.6).

5.1. Existence and uniqueness of the ESS

Assume that $M*$ is the ESS of system (2.6) and the fixed point problem (3.6) holds. Define function

$$q^*(a) \triangleq \int_0^{\hat{a}} K(a,\sigma)h^*(\sigma)\mathrm{d}\sigma,$$

then, by equation (3.2), we get $\lambda*(a) = \beta c(a)q*(a)$. Substituting it into equation (3.5), we have

$$q^*(a) = \int_0^{\hat{a}} K(a,\sigma)H_1(q^*)(\sigma)\mathrm{d}\sigma, \tag{5.1}$$

where

$$\begin{aligned} H_1(q^*)(a) \triangleq & (1-\epsilon k_1)(1-\theta_1 p)\delta\beta \int_0^a \int_0^\sigma c(u)q^*(u)e^{-\beta\int_0^u c(w)q^*(w)\mathrm{d}w} e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ & +(1-\epsilon k_2)\theta_1 p\theta_4\delta\beta\Big[\theta_3\int_0^a\int_0^\sigma c(u)q^*(u)e^{-\int_0^u(\theta_2(w)+\beta\theta_3 c(w)q^*(w))\mathrm{d}w} e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ & +\int_0^a\int_0^\sigma c(u)q^*(u)\int_0^u\theta_2(w)e^{-\int_0^w(\theta_2(\rho)+\theta_3\beta c(\rho)q^*(\rho))\mathrm{d}\rho} e^{-\beta\int_w^u\beta c(\rho)q^*(\rho)\mathrm{d}\rho}\mathrm{d}w e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma\Big]. \end{aligned} \tag{5.2}$$

Taking maximum both side of (5.1) and dividing by $\hat{q} \triangleq \max_{a\in[0,\hat{a}]}q^*(a) > 0$, we get the following equation

$$H(q^*) - 1 = 0, \tag{5.3}$$

where

$$H(q^*) \triangleq \beta \int_0^{\hat{a}} \max_{a\in[0,\hat{a}]} K(a,\sigma)H_2(q^*)(\sigma)\mathrm{d}\sigma, \tag{5.4}$$

and

$$\begin{aligned} H_2(q^*)(a) \triangleq & (1-\epsilon k_1)(1-\theta_1 p)\delta\beta \int_0^a \int_0^\sigma c(u)\frac{q^*(u)}{\hat{q}}e^{-\beta\int_0^u c(w)q^*(w)\mathrm{d}w} e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ & +(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\int_0^a\int_0^\sigma c(u)\frac{q^*(u)}{\hat{q}}e^{-\int_0^u(\theta_2(w)+\theta_3\beta c(w)q^*(w))\mathrm{d}w} e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\ & +\int_0^a\int_0^\sigma c(u)\frac{q^*(u)}{\hat{q}}\int_0^u\theta_2(w)e^{-\int_0^w(\theta_2(\rho)+\theta_3\beta c(\rho)q^*(\rho))\mathrm{d}\rho} e^{-\beta\int_w^u c(\rho)q^*(\rho)\mathrm{d}\rho}\mathrm{d}w e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma\Big]. \end{aligned}$$

It can be observed that the existence and uniqueness of ESS is associated with the positive solution $q*$ of equation (5.3).

Theorem 5.1. If $R_0 > 1$, *then normalized system (2.6) has at most one ESS, while there is no ESS if* $R_0 \le 1$.

Proof. By the definition of $q*(a)$, we know that $q*(a)$ is continuous distribution function of a. The distribution of $q*(a)$ can be seen as a uniform distribution $\hat{q}$ (maximum value of the $q*(a)$) approximately, see Fig. 3(A). Then $H(q*)$ can be written as

$$H(\hat{q}) = \max_{a\in[0,\hat{a}]}\int_0^{\hat{a}} K(a,\sigma)\widehat{H}(\hat{q})(\sigma)\mathrm{d}\sigma,$$

where

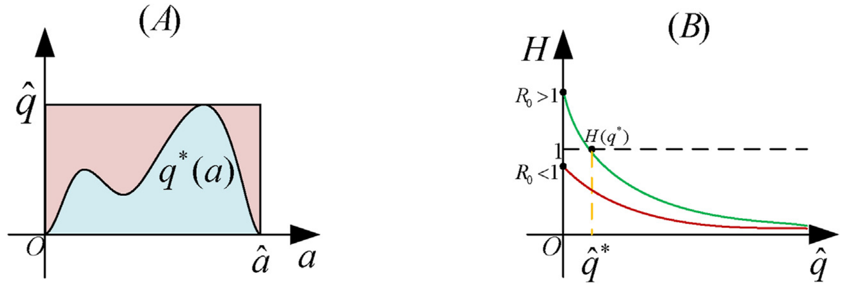

Fig. 3. Existence and uniqueness of ESS. (A) Uniform distribution $q^*(a)$; (B) Existence and uniqueness of $\widehat{q}^*$.

$$
\begin{aligned}
\widehat{H}(\widehat{q})(a) \quad & \triangleq (1-\epsilon k_1)(1-\theta_1 p)\delta\beta \int_0^a \int_0^\sigma c(u) e^{-\beta\hat{q}\int_0^u c(w)\mathrm{d}w} e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\
& +(1-\epsilon k_2)\theta_1 p\theta_4\delta\Big[\theta_3\beta \int_0^a \int_0^\sigma c(u) e^{-\int_0^u (\theta_2(w)+\beta\theta_3 c(w)\hat{q})\mathrm{d}w} e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma \\
& +\int_0^a \int_0^\sigma c(u) \int_0^u \theta_2(w) e^{-\int_0^w (\theta_2(\rho)+\beta\theta_3 c(\rho)\hat{q})\mathrm{d}\rho} e^{-\beta\hat{q}\int_w^u c(\rho)\mathrm{d}\rho} \mathrm{d}w e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(a-\sigma)}\mathrm{d}\sigma\Big].
\end{aligned}
$$

$H(\widehat{q})$ is monotone decreasing about $\widehat{q}$ and

$$\lim_{\hat{q}\to+\infty} H(\widehat{q}) = 0, \qquad \lim_{\hat{q}\to 0} H(\widehat{q}) = R_0.$$

Thus, $R_0 > 1$ implies that there exists unique positive number $\widehat{q}^*$ (see Fig. 3(B)) such that

$$H(\widehat{q}^*) - 1 = 0. \tag{5.5}$$

On the contrary, $H(\widehat{q}) - 1 = 0$ has no positive solutions if $R_0 \leq 1$ (see Fig. 3(B)). That is, there is no ESS if $R_0 \leq 1$.

5.2. Forwardly bifurcates of the ESS

In this subsection, we obtain following bifurcation results of ESS of system (2.6) using a similar approach introduced in (Okuwa et al., 2019).

Theorem 5.2. *The DFSS of system (2.6) is unstable if $R_0 > 1$ and the ESS forwardly bifurcates from the DFSS when R_0 crosses unity.*

Proof. Let $r > 0$ be a bifurcation parameter and suppose that $c(a) = rc_1(a)$, where the $c_1(a)$ is chosen such as $R_0 = 1$ if $r = 1$. Then, by the (3.8), it holds that

$$
\begin{aligned}
1 \quad & = (1-\epsilon k_1)(1-\theta_1 p)\delta\beta \int_0^{\hat{a}} \max_{a\in[0,\hat{a}]} K(a,b) \int_0^b \int_0^\sigma c_1(u) e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b \\
& +(1-\epsilon k_2)\theta_1 p\theta_4\delta\beta\Big[\theta_3 \int_0^{\hat{a}} \max_{a\in[0,\hat{a}]} K(a,b) \int_0^b \int_0^\sigma c_1(u) e^{-\int_0^u \theta_2(w)\mathrm{d}w} e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b \\
& +\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]} K(a,b) \int_0^b \int_0^\sigma c_1(u)\Big(1-e^{-\int_0^u \theta_2(w)\mathrm{d}w}\Big) e^{-\delta(\sigma-u)}\mathrm{d}u e^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\Big].
\end{aligned}
$$

The continuous distribution of $q^*(a)$ can be seen as uniform distribution $\widehat{q}$ as before (see Fig. 3(A)), then equation (5.3) can be rewritten as

$$\Psi(\widehat{q}, r) \triangleq \beta \max_{a\in[0,\hat{a}]} \int_0^{\hat{a}} K(a,\sigma)\Psi_1(\widehat{q}, r)(\sigma)\mathrm{d}\sigma - 1 = 0,$$

where

$$\Psi_1(\widehat{q},r) \qquad (a)\triangleq(1-\epsilon k_1)(1-\theta_1 p)\delta r\int_0^a\int_0^\sigma c_1(u)e^{-\beta\hat{q}r\int_0^u c_1(w)\mathrm{d}w}e^{-\delta(\sigma-u)}\mathrm{d}ue^{-\gamma(a-\sigma)}\mathrm{d}\sigma$$
$$+(1-\epsilon k_2)\theta_1 p\theta_4\delta r\Big[\theta_3\int_0^a\int_0^\sigma c_1(u)e^{-\int_0^u(\theta_2(w)+\beta\theta_3 rc_1(w)\hat{q})\mathrm{d}w}e^{-\delta(\sigma-u)}\mathrm{d}ue^{-\gamma(a-\sigma)}\mathrm{d}\sigma$$
$$+\int_0^a\int_0^\sigma c_1(u)\int_0^u\theta_2(w)e^{-\int_0^w(\theta_2(\rho)+\beta\theta_3 rc_1(\rho)\hat{q})\mathrm{d}\rho}e^{-\beta\hat{q}c\int_w^u c_1(\rho)\mathrm{d}\rho}\mathrm{d}we^{-\delta(\sigma-u)}\mathrm{d}ue^{-\gamma(a-\sigma)}\mathrm{d}\sigma.\Big]$$

$\Psi(\widehat{q},r)$ satisfies $\Psi(0,1)=R_0-1=0$ and $\frac{\partial\Psi}{\partial\widehat{q}}(0,1)<0$.

By the implicit function theorem, $\widehat{q}=\widehat{q}(r)$ is the function of r and $\widehat{q}(1)=0$. It can be observed that

$$\frac{\partial\Psi}{\partial r}(0,1)=R_0=1,$$

and

$$\widehat{q}'(1)=-\left(\frac{\partial\Psi}{\partial\widehat{q}}(0,1)\right)^{-1}\frac{\partial\Psi}{\partial r}(0,1)>0,$$

which implies that $\widehat{q}(r)>0$ if $|r-1|$ is small enough. Then ESS forwardly bifurcates as $r=R_0=1$.

5.3. Local stability of the ESS

In this subsection, we provide the local stability of ESS of system (2.6). Let

$$x=\widetilde{x}+x^*, x=(s,v,\widehat{s},e,\widehat{e},i,\widehat{i},l,\widehat{l},r),$$

then the normalized system (2.6) is

$$\begin{cases}
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widetilde{s}=-\lambda^*\widetilde{s}-s^*\widetilde{\lambda},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widetilde{v}=-(\theta_2(a)+\theta_3\lambda^*)\widetilde{v}-\theta_3 v^*\widetilde{\lambda},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widetilde{\widehat{s}}=\theta_2(a)\widetilde{v}-\lambda^*\widetilde{\widehat{s}}-\widehat{s}^*\widetilde{\lambda},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widetilde{e}=\lambda^*\widetilde{s}+s^*\widetilde{\lambda}-\delta\widetilde{e},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widetilde{\widehat{e}}=\lambda^*(\theta_3\widetilde{v}+\widetilde{\widehat{s}})+(\theta_3 v^*+\widehat{s}^*)\widetilde{\lambda}-\delta\widetilde{\widehat{e}},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widetilde{i}=k_1\delta\widetilde{e}-\gamma\widetilde{i},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widetilde{\widehat{i}}=k_2\delta\widetilde{\widehat{e}}-\gamma\widetilde{\widehat{i}},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widetilde{l}=(1-k_1)\delta\widetilde{e}-\gamma\widetilde{l},\\
(\frac{\partial}{\partial a}+\frac{\partial}{\partial t})\widetilde{\widehat{l}}=(1-k_2)\delta\widetilde{\widehat{e}}-\gamma\widetilde{\widehat{l}},
\end{cases} \tag{5.6}$$

with

$$\widetilde{x}(t,0)=0, x=(s,v,\widehat{s},e,\widehat{e},i,\widehat{i},l,\widehat{l},r),$$

where

$$\lambda^*(a)=\beta c(a)\int_0^{\hat{a}}K(a,u)h^*(u)\mathrm{d}u, \quad h^*=(1-\epsilon)i^*+(1-\epsilon)\theta_4\widehat{i}^*+l^*+\theta_4\widehat{l}^*,$$

and

$$\widetilde{\lambda}(t,a)=\beta c(a)\int_0^{\hat{a}}K(a,u)\widetilde{h}(t,u)\mathrm{d}u, \qquad \widetilde{h}=(1-\epsilon)\widetilde{i}+(1-\epsilon)\theta_4\widetilde{\widehat{i}}+\widetilde{l}+\theta_4\widetilde{\widehat{l}}.$$

Let

$$\widetilde{x}=\widetilde{x}(a)\mathrm{e}^{\xi t}, x=(s,v,\widehat{s},e,\widehat{e},i,\widehat{i},l,\widehat{l},r).$$

Substituting it into system (5.6) yields

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{s}+\xi\widetilde{s}=-\lambda^*\widetilde{s}-s^*\widetilde{\lambda}, \\ \dfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{v}+\xi\widetilde{v}=-(\theta_2(a)+\theta_3\lambda^*)\widetilde{v}-\theta_3 v^*\widetilde{\lambda}, \\ \dfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{\widehat{s}}+\xi\widetilde{\widehat{s}}=\theta_2(a)\widetilde{v}-\widehat{s}^*\widetilde{\lambda}-\lambda^*\widetilde{\widehat{s}}, \\ \dfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{e}+\xi\widetilde{e}=\lambda^*\widetilde{s}+s^*\widetilde{\lambda}-\delta\widetilde{e}, \\ \dfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{\widehat{e}}+\xi\widetilde{\widehat{e}}=\lambda^*(\theta_3\widetilde{v}+\widetilde{\widehat{s}})+(\theta_3 v^*+\widehat{s}^*)\widetilde{\lambda}-\delta\widetilde{\widehat{e}}, \\ \dfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{i}+\xi\widetilde{i}=k_1\delta\widetilde{e}-\gamma\widetilde{i}(a), \\ \dfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{\widehat{i}}+\xi\widetilde{\widehat{i}}=k_2\delta\widetilde{\widehat{e}}-\gamma\widetilde{\widehat{i}}, \\ \dfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{l}+\xi\widetilde{l}=(1-k_1)\delta\widetilde{e}-\gamma\widetilde{l}, \\ \dfrac{\mathrm{d}}{\mathrm{d}a}\widetilde{\widehat{l}}+\xi\widetilde{\widehat{l}}=(1-k_2)\delta\widetilde{\widehat{e}}-\gamma\widetilde{\widehat{l}}, \end{cases} \tag{5.7}$$

with

$$\widetilde{x}(0)=0, x=(s,v,\widehat{s},e,\widehat{e},i,\widehat{i},l,\widehat{l},r),$$

where

$$\widetilde{\lambda}(a)=\beta c(a)\int_0^{\hat{a}}K(a,u)\widetilde{h}(u)\mathrm{d}u.$$

The solutions of system (5.7) is

$$\begin{cases}
\widetilde{s} = -\int_0^a s^*(\sigma)\widetilde{\lambda}(\sigma)\mathrm{e}^{-\xi(a-\sigma)}\mathrm{e}^{-\int_\sigma^a \lambda^*(\theta)d\theta}\mathrm{d}\sigma \\
\widetilde{v} = -\theta_3 \int_0^a v^*(\sigma)\widetilde{\lambda}(\sigma)\mathrm{e}^{-\xi(a-\sigma)}\mathrm{e}^{-\int_\sigma^a (\theta_2(\theta)+\theta_3\lambda^*(\theta))d\theta}\mathrm{d}\sigma \\
\widetilde{\widetilde{s}} = -\int_0^a [\widehat{s}^*(\sigma)\widetilde{\lambda}(\sigma) - \theta_2(\sigma)\widetilde{v}(\sigma)]\mathrm{e}^{-\xi(a-\sigma)}\mathrm{e}^{-\int_\sigma^a \lambda^*(\theta)d\theta}\mathrm{d}\sigma \\
\widetilde{e} = \int_0^a [\lambda^*(\sigma)\widetilde{s}(\sigma) + s^*(\sigma)\widetilde{\lambda}(\sigma)]\mathrm{e}^{-(\xi+\delta)(a-\sigma)}\mathrm{d}\sigma, \\
\widetilde{\widetilde{e}} = \int_0^a [\lambda^*(\sigma)(\theta_3\widetilde{v}(\sigma) + \widetilde{\widetilde{s}}(\sigma)) + (\theta_3 v^*(\sigma) + \widehat{s}^*(\sigma))\widetilde{\lambda}(\sigma)]\mathrm{e}^{-(\xi+\delta)(a-\sigma)}\mathrm{d}\sigma, \\
\widetilde{i} = k_1\delta\int_0^a \widetilde{e}(\sigma)\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma, \\
\widetilde{\widetilde{i}} = k_2\delta\int_0^a \widetilde{\widetilde{e}}(\sigma)\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma, \\
\widetilde{l} = (1-k_1)\delta\int_0^a \widetilde{e}(\sigma)\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma, \\
\widetilde{\widetilde{l}} = (1-k_2)\delta\int_0^a \widetilde{\widetilde{e}}(\sigma)\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma.
\end{cases}$$

Substituting these solutions into expression of $\widetilde{h}(a)$, we have

$$\begin{aligned}
\widetilde{h}(a) \quad &= (1-\epsilon k_1)\delta\int_0^a\int_0^\sigma [\lambda^*(u)\Lambda_1(u) + s^*(u)\widetilde{\lambda}(u)]\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma \\
&+(1-\epsilon k_2)\theta_4\delta\Big[\theta_3\int_0^a\int_0^\sigma [\lambda^*(u)\Lambda_2(u) + v^*(u)\widetilde{\lambda}(u)]\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma \\
&\quad + \int_0^a\int_0^\sigma [\lambda^*(u)\Lambda_3(u) + \widehat{s}^*(u)\widetilde{\lambda}(u)]\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-(\xi+\gamma)(a-\sigma)}\mathrm{d}\sigma\Big],
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 &\triangleq -\int_0^u s^*(w)\widetilde{\lambda}(w)\mathrm{e}^{-\xi(u-w)}\mathrm{e}^{-\int_w^u \lambda^*(\theta)d\theta}\mathrm{d}w, \\
\Lambda_2 &\triangleq -\theta_3\int_0^u v^*(w)\widetilde{\lambda}(w)\mathrm{e}^{-\xi(u-w)}\mathrm{e}^{-\int_w^u (\theta_2(\theta)+\theta_3\lambda^*(\theta))d\theta}\mathrm{d}w, \\
\Lambda_3 &\triangleq -\int_0^u [\widehat{s}^*(w)\widetilde{\lambda}(w) - \theta_2(w)\Lambda_2(w)]\mathrm{e}^{-\xi(u-w)}\mathrm{e}^{-\int_w^u \lambda^*(\theta)d\theta}\mathrm{d}w.
\end{aligned}$$

Denoting $\Lambda(a) \triangleq \int_0^{\hat{a}} K(a,u)\widetilde{h}(u)\mathrm{d}u$, we have $\widetilde{\lambda}(a) = \beta c(a)\Lambda(a)$. Substituting $\widetilde{h}(a)$ into $\Lambda(a)$, taking maximum both side and dividing by $\widehat{\Lambda} \triangleq \max_{a\in[0,\hat{a}]}\Lambda(a) > 0$, we get the following characteristic equation of ξ,

$$Q(\xi) - 1 = 0,$$

where

$$Q(\xi) = (1-\epsilon k_1)\delta\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma[\lambda^*(u)\frac{\Lambda_1(u)}{\widehat{\Lambda}}+s^*(u)\beta c(u)\frac{\Lambda(u)}{\widehat{\Lambda}}]\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b$$
$$+(1-\epsilon k_2)\theta_4\delta\Bigg[\theta_3\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma[\lambda^*(u)\frac{\Lambda_2(u)}{\widehat{\Lambda}}+v^*(u)\beta c(u)\frac{\Lambda(u)}{\widehat{\Lambda}}]\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b$$
$$+\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma[\lambda^*(u)\frac{\Lambda_3(u)}{\widehat{\Lambda}}+\widehat{s}^*(u)\beta c(u)\frac{\Lambda(u)}{\widehat{\Lambda}}]\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\Bigg],$$

and

$$\frac{\Lambda_1}{\widehat{\Lambda}} = -\beta\int_0^u s^*(w)c(w)\frac{\Lambda(w)}{\widehat{\Lambda}}\mathrm{e}^{-\xi(u-w)}\mathrm{e}^{-\int_w^u\lambda^*(\theta)d\theta}\mathrm{d}w,$$
$$\frac{\Lambda_2}{\widehat{\Lambda}} = -\theta_3\beta\int_0^u v^*(w)c(w)\frac{\Lambda(w)}{\widehat{\Lambda}}\mathrm{e}^{-\xi(u-w)}\mathrm{e}^{-\int_w^u(\theta_2(\theta)+\theta_3\lambda^*(\theta))d\theta}\mathrm{d}w,$$
$$\frac{\Lambda_3}{\widehat{\Lambda}} = -\int_0^u[\widehat{s}^*(w)\beta c(w)\frac{\Lambda(w)}{\widehat{\Lambda}}-\theta_2(w)\frac{\Lambda_2(w)}{\widehat{\Lambda}}]\mathrm{e}^{-\xi(u-w)}\mathrm{e}^{-\int_w^u\lambda^*(\theta)d\theta}\mathrm{d}w.$$

Theorem 5.3. *The ESS of system (2.6) is locally asymptotically stable if $R_0 > 1$.*

Proof. The $Q(0)$ can be split into two parts as

$$Q(0) = J_1 + J_2,$$

where

$$J_1 \triangleq (1-\epsilon k_1)\delta\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma\lambda^*(u)\frac{\Lambda_1(u)}{\widehat{\Lambda}}\mathrm{e}^{-\delta(\sigma-u)}\mathrm{d}u\mathrm{e}^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b$$
$$+(1-\epsilon k_2)\theta_4\delta\Bigg[\theta_3\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma\lambda^*(u)\frac{\Lambda_2(u)}{\widehat{\Lambda}}\mathrm{e}^{-\delta(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b$$
$$+\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma\lambda^*(u)\frac{\Lambda_3(u)}{\widehat{\Lambda}}\mathrm{e}^{-\delta(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\Bigg],$$

and

$$J_2 \triangleq (1-\epsilon k_1)\delta\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma s^*(u)\beta c(u)\frac{\Lambda(u)}{\widehat{\Lambda}}\mathrm{e}^{-\delta(\sigma-u)}\mathrm{d}u\mathrm{e}^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b$$
$$+(1-\epsilon k_2)\theta_4\delta\Bigg[\theta_3\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma v^*(u)\beta c(u)\frac{\Lambda(u)}{\widehat{\Lambda}}\mathrm{e}^{-\delta(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b$$
$$+\int_0^{\hat{a}}\max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma\widehat{s}^*(u)\beta c(u)\frac{\Lambda(u)}{\widehat{\Lambda}}\mathrm{e}^{-\delta(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\Bigg].$$

The $J_1 < 0$ since

$$\frac{\Lambda_1(u)}{\widehat{\Lambda}} = -\beta\int_0^u s^*(w)c(w)\frac{\Lambda(w)}{\widehat{\Lambda}}\mathrm{e}^{-\int_w^u \lambda^*(\theta)d\theta}\mathrm{d}w < 0,$$

$$\frac{\Lambda_2(u)}{\widehat{\Lambda}} = -\theta_3\beta\int_0^u v^*(w)c(w)\frac{\Lambda(w)}{\widehat{\Lambda}}\mathrm{e}^{-\int_w^u (\theta_2(\theta)+\theta_3\lambda^*(\theta))d\theta}\mathrm{d}w < 0,$$

$$\frac{\Lambda_3(u)}{\widehat{\Lambda}} = -\int_0^u [\widehat{s}^*(w)\beta c(w)\frac{\Lambda(w)}{\widehat{\Lambda}} - \theta_2(w)\frac{\Lambda_2(w)}{\widehat{\Lambda}}]\mathrm{e}^{-\int_w^u \lambda^*(\theta)d\theta}\mathrm{d}w < 0.$$

According to the expression of $H(\widehat{q}^*)$ in (5.5) and solution of $s^*, v^*, \widehat{s}^*$ in (3.4), we have

$$\begin{aligned} H(\widehat{q}^*) \quad &= (1-\epsilon k_1)\delta\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma s^*(u)\beta c(u)\mathrm{e}^{-\delta(\sigma-u)}\mathrm{d}u\mathrm{e}^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b \\ &+(1-\epsilon k_2)\theta_4\delta\left[\theta_3\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma v^*(u)\beta c(u)\mathrm{e}^{-\delta(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\right. \\ &\left.+\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma \widehat{s}^*(u)\beta c(u)\mathrm{e}^{-\delta(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-\gamma(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\right] = 1 \end{aligned}$$

Then, using the fact $\frac{\Lambda(w)}{\widehat{\Lambda}} \le 1$, we get

$$Q(0) \le 1 + J_1 < 1,$$

and

$$\begin{aligned} Q(\xi) \quad &< (1-\epsilon k_1)\delta\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma s^*(u)\beta c(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}u\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b \\ &+(1-\epsilon k_2)\theta_4\delta\left[\theta_3\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma v^*(u)\beta c(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\right. \\ &\left.+\int_0^{\hat{a}} \max_{a\in[0,\hat{a}]}K(a,b)\int_0^b\int_0^\sigma \widehat{s}^*(u)\beta c(u)\mathrm{e}^{-(\xi+\delta)(\sigma-u)}\mathrm{d}\sigma\mathrm{e}^{-(\xi+\gamma)(b-\sigma)}\mathrm{d}\sigma\mathrm{d}b\right] \triangleq g(\xi). \end{aligned}$$

It can be seen that $g(\xi)$ is a monotone decreasing function of ξ and

$$\lim_{\xi\to-\infty}g(\xi) = +\infty, \lim_{\xi\to+\infty}g(\xi) = 0.$$

Then, one has $Q(\xi) < g(0) = H(\widehat{q}^*) = 1$, $\mathrm{Re}\xi > 0$. $Q(\xi) = 1$ has roots only in the region $\mathrm{Re}\xi < 0$. Then, if $R_0 > 1$, all roots of $Q(\xi) = 1$ have negative real parts. The proof is similar to that of Theorem 4.1 and the approximate graph about $Q(\xi)$ as Fig. 2.

6. Numerical simulations

Assuming that the total number of children is $N = 500000$, the maximum age of the children is $\widehat{a} = 20$, 10% of severely infected individuals are isolated (setting $\epsilon = 0.1$), the average incubation rate of mumps is 19 (15–24) days (setting $\delta = 30/19$), the recovery rate of mumps is 12 (10–15) days (setting $\delta = 30/12$), the 20–40% of mumps infections are asymptomatic (setting $k_1 = 0.7$), the 50% of vaccinated mumps infections are asymptomatic (setting $k_2 = 0.5$), primary vaccine failure is $\theta_1 = 0.9$, vaccine wane per month is $\theta_2 = 0.3/12$, vaccine leakiness $\theta_3 = 0.5$, relative infectivity is $\theta_4 = 0.9$. Consider the physical contact rate as

$$c(a) = \beta_0(3.19 - 1.57\cos(a) + 3.88\sin(a) - 0.98\cos(2a) - 0.5\sin(2a)), a\in[0,\widehat{a}],$$

where β_0 is an undetermined constant, see (Azimaqin et al., 2022). The initial value function is fixed as

$$I_0 = -I^2 + 20a + 20, S_0 = N - I_0, X_0 = 0, X = (V, S^v, E, E^v, I^v, L, L^v, R).$$

6.1. Proportional mixing case

For proportional mixing case ($\eta(a) \equiv 0$), the R_0 can be calculated by formula (3.9) without and with vaccine respectively.

If $p = 0$ (there is no vaccine) and $\beta_0 = 0.438$, we obtain $R_0 \approx 0.9987 < 1$. If $p = 0.5$ (there is a vaccine) and $\beta_0 = 0.545$, we obtain $R_0 \approx 0.9921 < 1$. Figs. 4(A) and 5(A) show that, with time evolution, the infected population tends to zero. In fact, by Theorem 4.1, the DFSS is globally stable if $R_0 < 1$.

On the other hand, if $p = 0$ and $\beta_0 = 0.448$, we obtain $R_0 \approx 1.0215 > 1$. If $p = 0.5$ and $\beta_0 = 0.561$, we obtain $R_0 \approx 1.0212 > 1$. Figs. 4(B) and 5(B) indicate that with the evolution of time, the infected population tends to a positive equilibrium. In fact, by Theorem 5.3, there exists a locally stable ESS if $R_0 > 1$. The numerical results show that the ESS is globally stable.

6.2. Isolated mixing case

In case of isolated mixing ($\eta(a) \equiv 1$), The R_0 can be calculated by formula (3.10) without vaccination. Similar results are can be achieved for cases with vaccines.

If $p = 0$ and $\beta = 0.324$, then we get the figure of $R_0(a)$ and the maximum value $R_0 \approx 0.9921 < 1$, see Fig. 6(A). As time evolves, the infected population tends to zero, see Fig. 7(A). In fact, by Theorem 4.1, the DFSS is globally stable if $R_0 < 1$.

On the other hand, if $p = 0$ and $\beta = 0.334$, then we get the figure of $R_0(a)$ and the maximum $R_0 \approx 1.0212 > 1$, see Fig. 6(B). As time evolves, the infected population tends to positive equilibrium, see Fig. 7(B). In fact, by Theorem 5.3, there exists a locally stable ESS if $R_0 > 1$. The numerical results show that the ESS is globally stable.

6.3. Homogenous preferential mixing case

In case of preferential mixing ($\eta(a) \equiv 0.5$), it is possible to calculate the R_0 using formula (3.11) without vaccination. Similar results are can be achieved for cases with vaccines.

If $p = 0$ and $\beta = 0.37$, then we get the figure of $R_0(a)$ and the maximum value $R_0 \approx 0.9952 < 1$, see Fig. 8(A). As time evolves, the infected population tends to zero, see Fig. 9(A). In fact, by Theorem 4.1, the DFSS is globally stable if $R_0 < 1$.

On the other hand, if $p = 0$ and $\beta = 0.393$, then we get the figure of $R_0(a)$ and the maximum $R_0 \approx 1.0571 > 1$, see Fig. 8(B). As time evolves, the infected population tends to positive equilibrium, see Fig. 9(B). In fact, by Theorem 5.3, there exists a locally stable ESS if $R_0 > 1$. The numerical results show that the ESS is globally stable.

7. Discussion

For the heterogeneous age-structured model, it is not yet known whether the ESS is unique or stable, because it's hard to derive the explicit formula for R_0 (Okuwa et al., 2019). (Huang, Kang, Lu, & et al, 2022) studied the existence and uniqueness of ESS based on the explicit formula the R_0 in the separable mixing case, it is shown that the steady state is locally stable under

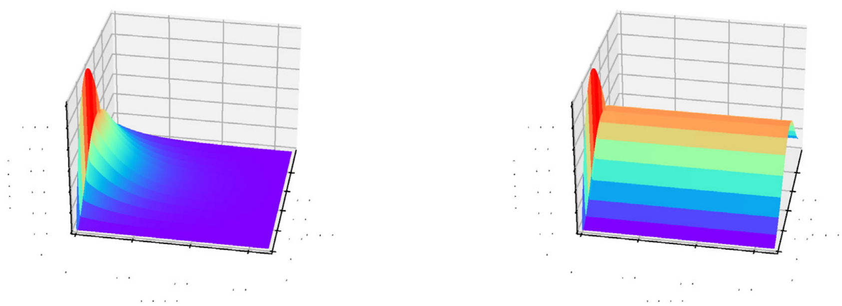

Fig. 4. Proportional mixing, $p = 0$. (A) $\beta_0 = 0.438$, $R_0 \approx 0.9987 < 1$; (B) $\beta_0 = 0.448$, $R_0 \approx 1.0215 > 1$.

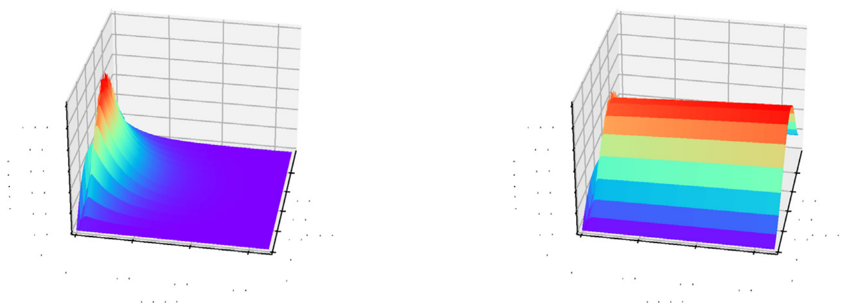

Fig. 5. Proportional mixing, $p = 0.5$. (A) $\beta_0 = 0.545$, $R_0 \approx 0.9921 < 1$; (B) $\beta_0 = 0.561$, $R_0 \approx 1.0212 > 1$.

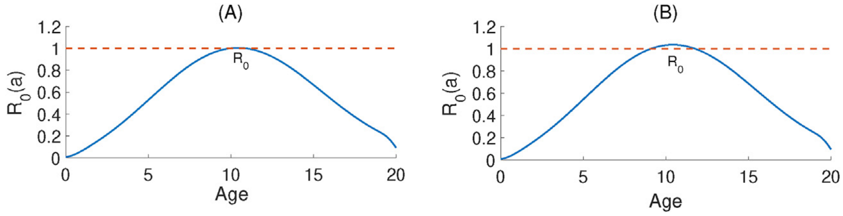

Fig. 6. $R_0(a)$ for isolated mixing. (A) $\beta = 0.324$, $R_0 \approx 0.9921 < 1$; (B) $\beta = 0.334$, $R_0 \approx 1.0212 > 1$.

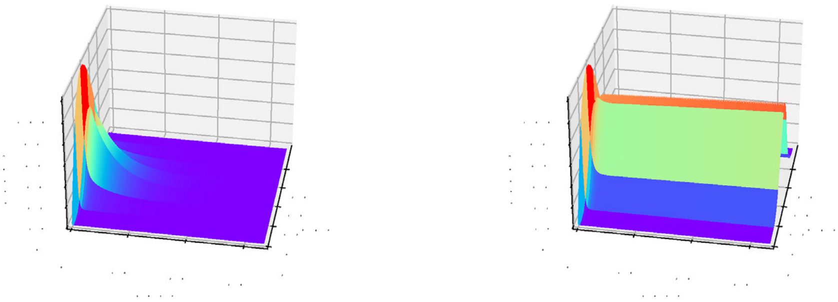

Fig. 7. Isolated mixing, $p = 0$. (A) $\beta_0 = 0.324$, $R_0 \approx 0.9921 < 1$; (B) $\beta_0 = 0.334$, $R_0 \approx 1.0212 > 1$.

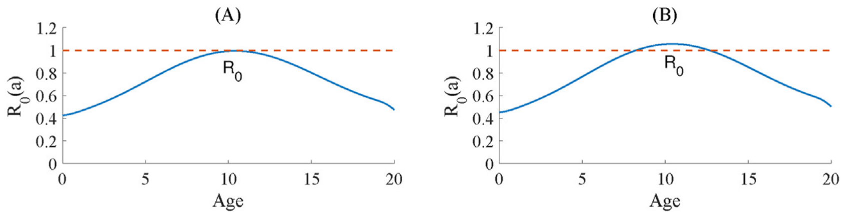

Fig. 8. $R_0(a)$ for preferential mixing. (A) $\beta_0 = 0.37$, $R_0 \approx 0.9952 < 1$; (B) $\beta_0 = 0.393$, $R_0 \approx 1.0571 > 1$.

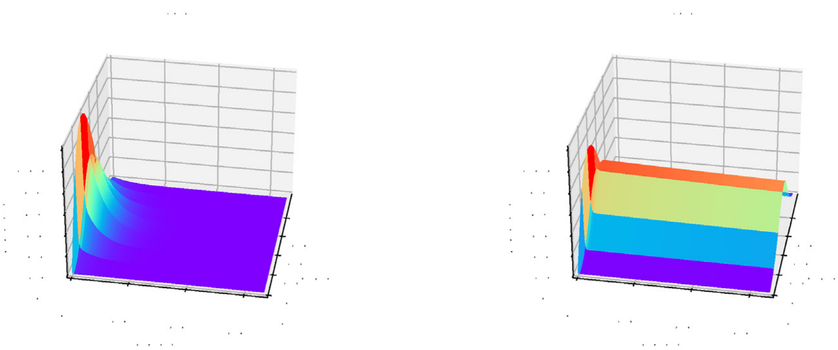

Fig. 9. Preferential mixing, $p = 0$. (A) $\beta_0 = 0.37$, $R_0 \approx 0.9952 < 1$; (B) $\beta_0 = 0.393$, $R_0 \approx 1.0571 > 1$.

some additional conditions. The global stability of the steady states of age-structured model remains an open issue even in the separable mixing case.

In this paper, we defined the explicit formula of R_0 for heterogeneous age-structured model of mumps under the various mixing case (isolation, proportional and heterogeneous) with or without the vaccine. It is shown that the disease free steady states is global stable if $R_0 < 1$, the ESS is unique and locally stable if $R_0 > 1$ without any additional conditions. A number of numerical examples are given to support the theory. Studying on global stability of endemic steady state will be a challenge in the future.

Mumps continues to be one of the greatest public health issues around the world, and vaccine is still the best way to prevent and control it. Four vaccine-related parameters were taken into account in our heterogeneous age-structured model. Mathematical models are applied to improve vaccine policy. For future extension of our model, we present an age-structured model with periodic parameters (seasonality) for mumps, the existence and stability of the periodic solution are also discussed. The age-structured model of mumps provides a detailed framework for understanding the mechanisms by which the disease spreads among different age groups. This model can be applied not only to mumps but also to many other common infectious diseases in children, such as measles, thus revealing the epidemiological characteristics of these diseases and the similarities and differences in prevention strategies. In this way, we can better address childhood infectious diseases and reduce their public health impact.

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Authors declare

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CRediT authorship contribution statement

Nurbek Azimaqin: Writing – original draft, Validation, Software, Methodology, Investigation, Funding acquisition, Conceptualization. **Yingke Li:** Writing – review & editing, Validation, Investigation. **Xianning Liu:** Writing – review & editing, Supervision, Project administration, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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